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Uniform existence of the integrated density of states for random Schrödinger operators on metric graphs over $\mathbb{Z}^d$

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Abstract

We consider ergodic random Schrödinger operators on the metric graph $\mathbb{Z}^d$ with random potentials and random boundary conditions taking values in a finite set. We show that normalized finite volume eigenvalue counting functions converge to a limit uniformly in the energy variable. This limit, the integrated density of states, can be expressed by a closed Shubin–Pastur type trace formula. It supports the spectrum and its points of discontinuity are characterized by existence of compactly supported eigenfunctions. Among other examples we discuss random magnetic fields and percolation models.
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1. Introduction

This paper is devoted to the spectral analysis of certain random ergodic operators defined on metric graphs with a $\mathbb{Z}^d$-structure. More precisely, we consider a Schrödinger operator with an ergodic random potential which takes values in a finite set of potentials, with ergodic random boundary conditions from a finite set, and with ergodic random magnetic fields from a finite set. Such a model may be called random quantum graph. A restriction of such an operator to a finite cube with selfadjoint boundary conditions has purely discrete spectrum, hence it has a

well-defined eigenvalue counting function. As the size of the cube tends to infinity the normalized counting functions converge to the so-called *integrated density of states* (IDS) or *spectral distribution function*. The corresponding result is well established for ergodic Schrödinger operators in the continuum (i.e. on $L^2(\mathbb{R}^2)$) and on the lattice (i.e. on $\ell^2(\mathbb{Z}^d)$), see for instance [1,15,16,26,27]. A modification of the proof to ergodic, random Schrödinger on metric graphs was presented in [12, §7] and yields the existence of the IDS.

However, the abovementioned results concern *pointwise* convergence of the distribution functions, or rather *weak convergence* of the corresponding measures. In the present paper we show that for the class of random operators at hand the convergence holds uniformly in the energy variable.

Two features of the operators considered are helpful to derive such a strong statement about the convergence: Firstly, since the potential and other data take only finitely many values frequencies of finite configuration patterns exist in a nice way. Secondly, since the underlying geometric structure exhibits in many ways one-dimensional properties the relevant perturbation operators turn out to have a uniformly bounded spectral shift function. The latter feature can be understood as a finite-rank property in a generalized sense, see Section 3 for a more precise statement.

These two properties of the considered operators enable us to apply an abstract ergodic theorem for Banach space valued random variables which was proven recently in [22]. There the ergodic theorem was already used to establish uniform convergence for a certain class of random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$. The situation for Schrödinger operators on metric graphs is somewhat more complicated since the operators are unbounded and the IDS is not a probability distribution any more. Still, quantum graph operators are technically much easier to handle than Schrödinger operators in truly multi-dimensional space, i.e. on $L^d(\mathbb{R}^2)$, $d \geqslant 2$.

The paper is organized as follows. In Section 2 we define precisely the operators we want to study and state the main results. In Section 3 we discuss finite rank perturbations of the kind as they appear in our proofs. In Section 4 we show that the ergodic theorem of [22] can be applied to the sequence of spectral shift functions obtained from an exhaustion of the metric graph by cubes. This is then used in Section 5 to prove our main results. In Section 6 we show how magnetic fields on metric graphs manifest themselves in the boundary conditions. Finally, in Section 7 we discuss site- and edge-percolation as an example for our results.

2. Model and results

We define a metric graph G_d over $\mathbb{Z}^d$ in the following way. Let e_j, $j = 1, \ldots, d$, be the standard basis of the real d-dimensional space $\mathbb{R}^d$. The vertex set V_d of the graph consists of the points $\mathbb{Z}^d \subset \mathbb{R}^d$ which have integer coefficients. Now, to each vertex x and $j \in \{1, \ldots, d\}$ we associate the edge $e = [x, x + e_j]$ with starting point $s(e) = x$ and endpoint $r(e) = x + e_j$ given by

$$[x, x + e_j] := \{x + te_j : t \in \,]0, 1[\}.$$

Thus, each edge can be canonically identified with the interval $]0, 1[$. This procedure may seem to induce an orientation on our graph. However, it turns out that all relevant quantities are independent of the choice of orientation.

The set of all edges is denoted by E_d. The union $V_d \cup \bigcup_{e \in E_d} e$ is a closed subset of $\mathbb{R}^d$, hence we can consider it as a topological space with the metric inherited from $\mathbb{R}^d$. We denote this metric subspace by G_d. We will need to consider finite subgraphs G of G_d as well. By a

subgraph we mean a subset of the edges of G_d together with all adjacent vertices. All functions we consider live on the topological space G_d or subgraphs of it. This shows, in particular, that the choice of orientation at each edge is not relevant.

The operators we are interested in will be defined on the Hilbert space

$$L^2(E_d) := \bigoplus_{e \in E_d} L^2(e)$$

and their domains of definition will be subspaces of

$$W^{2,2}(E_d) := \bigoplus_{e \in E_d} W^{2,2}(e),$$

where $W^{2,2}(e)$ is the usual Sobolev space of $L^2(e)$ functions whose (weak) derivatives up to order two are in $L^2(e)$ as well. The restriction of $f \in W^{2,2}(E_d)$ to an edge e is denoted by f_e. For an edge $e = [s(e), r(e)] = [x, x + e_j]$ and g in $W^{2,2}(e)$ the boundary values of g

$$g(s(e)) := \lim_{t \to 0} g(x + te_j), \qquad g(r(e)) := \lim_{t \to 1} g(s(e) + te_j)$$

and the boundary values of g'

$$g'(s(e)) := \lim_{\epsilon \to 0} \frac{g(x + \epsilon e_j) - g(x)}{\epsilon} \quad \text{and} \quad g'(r(e)) := \lim_{\epsilon \to 0} \frac{g(r(e)) - g(r(e) - \epsilon e_j)}{-\epsilon}$$

exist by standard Sobolev type theorems. Note that we have introduced a sign in the derivative at the endpoint of an edge. This makes our definition of derivative canonical, i.e. independent of the choice of orientation of the edge. For $f \in W^{2,2}(E_d)$ and each vertex x we gather the boundary values of $f_e(x)$ over all edges e adjacent to x in a vector $f(x)$. Similarly, we gather the boundary values of $f'_e(x)$ over all edges e adjacent to x in a vector $f'(x)$.

Given the boundary values of functions, we can now discuss the concept of boundary condition. Here we use material from [11,19] to which we refer for further details and proofs. A single-vertex boundary condition at $x \in V$ is a choice of subspace L_x of $\mathbb{C}^{4d}$ with dimension $2d$ such that

$$\eta((v, v'), (w, w')) := \langle v', w \rangle - \langle v, w' \rangle$$

vanishes for all $(v, v'), (w, w') \in L_x$. An $f \in W^{2,2}(E_d)$ is said to satisfy the single-vertex boundary condition L_x at x if $(f(x), f'(x))$ belongs to L_x. A field of single-vertex boundary conditions $L := \{L_x : x \in V_d\}$ will be called boundary condition. Given such a field, we obtain a selfadjoint realization Δ_L of the Laplacian Δ on $L^2(E_d)$ by choosing the domain

$$\mathcal{D}(\Delta_L) := \{ f \in W^{2,2}(E_d) : \forall x : (f(x), f'(x)) \in L_x \}.$$

Particularly relevant boundary conditions are Dirichlet boundary conditions with subspace L^D consisting of all those (v, v') with $v = 0$, Neumann conditions with subspace L^N consisting of all those (v, v') with $v' = 0$, and Kirchhoff (also known as free) boundary conditions L^K consisting of all (v, v') with v having all components equal and v' having the sum over its components equal to 0.

Remark 1. More general types of boundary conditions are conceivable. We will restrict ourselves to those just introduced. They may be called graph local boundary conditions, as each boundary condition involves values of f at one vertex only.

Everything discussed so far including existence of limits of functions at the vertices and the notions of boundary condition extends in the obvious way to subgraphs. Moreover, for a subgraph G of G_d with edge set E, we write $W^{2,2}(E) := \bigoplus_{e \in E} W^{2,2}(e)$. The number of edges of a finite subgraph G of G_d is denoted by $|E|$.

In order to define random operators we need some further data including a probability space $(\Omega, \mathbb{P})$ and an action of (a subgroup of) the automorphism group of G_d on Ω and maps L, V from Ω into the space of boundary conditions and potentials, respectively.

For us two groups will be relevant, the full automorphism group Γ_{full} and the group Γ_{trans} of translations by $\mathbb{Z}^d$. Note that Γ_{full} is generated by translations by vectors in $\mathbb{Z}^d$ and a finite set of rotations. We choose Γ to be Γ_{full} or Γ_{trans} and assume that it acts ergodically on $(\Omega, \mathbb{P})$ via measure preserving transformations. To simplify the notation we identify $\gamma \in \Gamma$ with the associated measure preserving transformation.

Let us describe the type of random operators we will consider in this paper:

Assumption 1. Let $(\Omega, \mathbb{P})$ be a probability space and $\Gamma \in \{\Gamma_{\text{full}}, \Gamma_{\text{trans}}\}$ a group acting ergodically on $(\Omega, \mathbb{P})$. Let $\mathcal{B}$ be a finite subset of $L^\infty(0, 1)$ and $\mathcal{L}$ a finite set of boundary conditions. A random potential is a map

$$V : \Omega \to \bigotimes_{e \in E_d} \mathcal{B}, \quad \text{with } V\big(\gamma(\omega)\big)_{\gamma(e)} = V(\omega)_e$$

for all $\gamma \in \Gamma$ and $e \in E_d$. A random boundary condition is a map

$$L : \Omega \to \bigotimes_{v \in V_d} \mathcal{L}, \quad \text{with } L\big(\gamma(\omega)\big)\big(\gamma(x)\big) = L(\omega)(x)$$

for all $\gamma \in \Gamma$ and $v \in V_d$. A family of random operators (H_ω) on $L^2(E_d)$ can be defined with domain of definition

$$\mathcal{D}(H_\omega) := \big\{ f \in W^{2,2}(E_d) : \big(f(x), f'(x)\big) \in L(\omega)(x) \text{ for all } x \in V_d \big\}$$

acting by

$$(H_\omega f)(e) := -f_e'' + V(\omega)_e f_e$$

for each edge e. These are selfadjoint lower bounded operators.

We assume throughout the paper that Assumption 1 holds, and for this reason do not repeat it in every lemma. Note however, that in many statements we will only need part of the structure described in Assumption 1.

Remark 2. While Γ_{full} is not commutative it is a natural object to deal with. In particular, let us note that the Laplacian without potential and boundary conditions in all vertices identical to Kirchhoff conditions is invariant under Γ_{full}.

We will need to consider restrictions of our operators to finite subgraphs. These are finite subgraphs associated to finite subsets of $\mathbb{Z}$. The cardinality of a finite subset Q of $\mathbb{Z}^d$ is denoted by $|Q|$. Let

$$S := \left\{ \sum_{j=1}^{d} t_j e_j \colon 0 < t_j < 1 \right\}.$$

For a finite subset Q of $\mathbb{Z}^d$, we define the associated subgraph G_Q of G_d by

$$G_Q := (Q + \bar{S}) \cap G_d.$$

The vertices and edges of G_d contained in G_Q are denoted by V_Q and E_Q, respectively. Note that $V_Q \supset Q$. The set V_Q^i of inner vertices of G_Q is then given by those vertices of G_Q all of whose adjacent edges (in G_d) are contained in G_Q. The set of inner edges E_Q^i of G_Q is given by those edges whose both endpoints are inner. The vertices of G_Q which are not inner are called boundary vertices. The set of all boundary vertices is denoted by V_Q^∂. Similarly, the set of edges which are not inner, is denoted by E_Q^∂.

The restriction H_ω^Q of the random operator H_ω to G_Q has domain given by

$$\mathcal{D}(H_\omega^Q) := \left\{ f \in W^{2,2}(E_Q) \, \middle| \, \forall x \in V_Q^i \colon \big(f(x), f'(x)\big) \in L(\omega)(x), \right.$$
$$\left. \forall x \in V_Q^\partial \colon \big(f(x), f'(x)\big) \in L^D \right\}.$$

This operator is again selfadjoint, lower bounded, and has purely discrete spectrum. Let us enumerate the eigenvalues of H_ω^Q in ascending order

$$\lambda_1(H_\omega^Q) < \lambda_2(H_\omega^Q) \leqslant \lambda_3(H_\omega^Q) \leqslant \cdots$$

and counting multiplicities. Then, the eigenvalue counting function n_ω^Q on $\mathbb{R}$ defined by

$$n_\omega^Q(\lambda) := \sharp\big\{ n \in \mathbb{N} \, \big| \, \lambda_n(H_\omega^Q) \leqslant \lambda \big\}$$

is monotone increasing and right continuous, i.e. a distribution function, which is associated to a pure point measure, μ_ω^Q. Denote by

$$N_\omega^Q(\lambda) := \frac{1}{|E_Q|} n_\omega^Q(\lambda)$$

the volume-scaled version of $n_\omega^Q(\lambda)$ and note that $|E_Q| = d|Q|$ as the edge to vertex ratio in the graph (V_d, E_d) is equal to d.

A sequence $(Q_l)_{l \in \mathbb{N}}$ of finite subsets of $\mathbb{Z}^d$ is called a *van Hove sequence* in $\mathbb{Z}^d$ if $\lim_{l \to \infty} \frac{|V_{Q_l}^\partial|}{|Q_l|} = 0$. For a finite subgraph H of G_d let χ_H the multiplication operator by the characteristic function of H. Denote the trace on the operators on $L^2(E_d)$ by $\mathrm{Tr}[\ldots]$. We can now state the main result of the paper.

Theorem 3. *Let Q be a finite subset of $\mathbb{Z}^d$. Then, the function N defined by*

$$N(\lambda) := \frac{1}{|E_Q|} \int_\Omega \mathrm{Tr}\big[\chi_{G_Q}\chi_{]-\infty,\lambda]}(H_\omega)\big] d\mathbb{P}(\omega) \tag{1}$$

does not depend on the choice of Q, is the distribution function of a measure μ, and for any van Hove sequence (Q_l) in $\mathbb{Z}^d$

$$\lim_{l\to\infty} \big\| N_\omega^{Q_l} - N \big\|_\infty = 0$$

for almost every $\omega \in \Omega$. In particular, for almost every $\omega \in \Omega$, $N_\omega^{Q_l}(\lambda)$ converges as $l \to \infty$ pointwise to $N(\lambda)$ for every $\lambda \in \mathbb{R}$.

Remark 4. (a) We chose to define the integrated density of states by a Shubin–Pastur formula and to prove that it coincides with the almost sure limit along van Hove sequences using an ergodic theorem. Alternatively, one could take the point of view that the intuitively relevant objects are the normalised eigenvalue counting functions on finite graphs. Then the IDS would be defined as their limit along a van Hove sequence and the trace formula (1) would be the result of the theorem.

(b) For random Schrödinger operators with more general potentials and Kirchhoff boundary conditions on all vertices the IDS was constructed in [12] as the pointwise almost everywhere limit of the sequence $N_\omega^{Q_l}$.

(c) In the case that Γ is the full automorphism group Γ_{full} the equality

$$N(\lambda) = \frac{1}{|E(G)|} \int_\Omega \mathrm{Tr}\big[\chi_G \chi_{]-\infty,\lambda]}(H_\omega)\big] d\mathbb{P}(\omega) \tag{2}$$

holds for any finite subgraph G with edge set $E(G)$. To see this note that by linearity

$$\int_\Omega \mathrm{Tr}\big[\chi_S \chi_{]-\infty,\lambda]}(H_\omega)\big] d\mathbb{P}(\omega) = \sum_{j=1}^d \int_\Omega \mathrm{Tr}\big[\chi_{[0,e_j]}\chi_{]-\infty,\lambda]}(H_\omega)\big] d\mathbb{P}(\omega)$$

$$= d \int_\Omega \mathrm{Tr}\big[\chi_{[0,e_1]}\chi_{]-\infty,\lambda]}(H_\omega)\big] d\mathbb{P}(\omega).$$

For the second equality we used that the d terms in the sum are all equal, since the group Γ_{full} acts transitively on the edges. For an arbitrary finite graph G one can do a similar calculation, and formula (2) follows.

While the definition of the IDS involves an ergodic theorem, there are other spectral features of H_ω whose almost sure independence of ω uses only the ergodicity of the group action. Prominent examples are the spectrum $\sigma(H_\omega)$ and its subsets $\sigma_{pp}(H_\omega)$, $\sigma_{sc}(H_\omega)$, $\sigma_{ac}(H_\omega)$, $\sigma_{disc}(H_\omega)$, $\sigma_{ess}(H_\omega)$ according to the spectral type. In fact, by applying the general framework of [23] we immediately infer the following theorem.

Theorem 5. *There exist subsets of the real line* Σ, Σ_{pp}, Σ_{sc}, Σ_{ac}, Σ_{disc}, Σ_{ess} *and an* $\Omega' \subset \Omega$ *of full measure such that* $\sigma(H_\omega) = \Sigma$ *and* $\sigma_\bullet(H_\omega) = \Sigma_\bullet$ *for all these spectral types* $\bullet \in \{pp, sc, ac, disc, ess\}$ *and all* $\omega \in \Omega'$.

Our goal here is to establish a relation between the almost sure spectrum of H_ω and its IDS. More precisely, in two corollaries to Theorem 3 we relate the topological supp μ of μ and the set $S_p(\mu) := \{\lambda \in \mathbb{R}: \ \mu(\{\lambda\}) > 0\}$ of atoms of μ to the spectrum of H_ω. Note that the set of discontinuities of the IDS is precisely $S_p(\mu)$.

Corollary 6. Σ *equals the topological support* supp μ *of* μ.

As usual an $f \in W^{2,2}(E_d)$ is said to be compactly supported if $f_e \equiv 0$ for all but finitely many edges e.

Corollary 7. *Denote by* Σ_{cmp} *the set of energies* $\lambda \in \mathbb{R}$ *such that there exists almost surely a compactly supported* L^2-*eigenfunction* f_ω *with* $H_\omega f_\omega = \lambda f_\omega$. *Then*

$$S_p(\mu) = \Sigma_{cmp}. \tag{3}$$

Remark 8. (a) We will prove Eq. (3) using the following ingredients: Theorem 3 and the reasoning presented in [18]. Note, however, that both of the sets in (3) are given in terms of the infinite-volume operator H_ω rather than by its restrictions to finite subgraphs. Indeed, there exists a proof of (3) which makes use neither of finite-graph operators, respectively their eigenvalue counting functions, nor of an ergodic theorem: Once one has established the invariance property of the IDS, as done in e.g. [23], Eq. (3) follows from the mere existence of a van Hove sequence. This strategy is implemented in the proof of Theorem 2.3(i) of [29] for operators on combinatorial graphs and can also be adapted to our situation. Note however, that there are certain situations where it is rather natural to relate the (size of the) jumps of the IDS to those of its finite-graph analogues, see for instance [6,7,24].

(b) There are many examples where the IDS has discontinuities. Indeed, the free Laplacian, i.e. the Schrödinger operator with identically vanishing potential, with Dirichlet, Neumann, or Kirchhoff boundary conditions exhibits compactly supported eigenfunctions. The relevance of Theorem 3 is that one has uniform convergence all the same.

(c) If the randomness entering the potential of the operator is sufficiently strong it is natural to expect a smoothing effect on the IDS. In fact, in [12] for a class of alloy-type random potentials the Lipschitz-continuity of the IDS was established. In [10] we show for a different class of random potentials how one can estimate the modulus of continuity of the IDS.

3. Finite rank perturbations

We will have to deal with changes in the spectral counting function due to changes of boundary conditions and changes of potentials. In particular, we will have to cut graphs into smaller pieces by suitable boundary conditions in order to apply the ergodic theorem of [22]. These changes are finite rank perturbations in an appropriate sense and hence change the spectral counting functions only in a very quantifiable way, which is made precise in this section.

Let H_1 and H_2 be selfadjoint lower bounded operators with discrete spectra. Thus the corresponding counting functions n_{H_1} and n_{H_2} are well defined. Then, the spectral shift function is the difference of two counting functions, i.e.

$$\xi_{H_1, H_2}(\lambda) = n_{H_2}(\lambda) - n_{H_1}(\lambda).$$

We will make use of the min–max principle in the form

$$n_H(\lambda) = \max\{\dim X \mid X \leqslant \mathcal{D}(H), \ H|_X \leqslant \lambda\},$$

where X runs over all linear subspaces (denoted by $\leqslant$) of the domain of H on which H is bounded above by λ in the sense that $\langle f, Hf \rangle \leqslant \lambda \langle f, f \rangle$ for all $f \in X$. The following lemma will be useful for dealing with several types of perturbations:

Lemma 9. *Let H_1, H_2 be selfadjoint lower bounded operators with discrete spectra on a separable Hilbert space. Assume that $\mathcal{D}_0 \leqslant \mathcal{D}(H_1) \cap \mathcal{D}(H_2)$ such that $\mathcal{D}_0$ has finite index k in $\mathcal{D}(H_2)$, i.e. $\dim \mathcal{D}(H_2)/\mathcal{D}_0 = k$. If $H_2|_{\mathcal{D}_0} \geqslant H_1|_{\mathcal{D}_0}$ then*

$$n_{H_2}(\lambda) \leqslant n_{H_1}(\lambda) + k, \quad i.e. \quad \xi_{H_1, H_2}(\lambda) \leqslant k.$$

Proof. By the min–max principle for the counting functions we have:

$$
\begin{aligned}
n_{H_2}(\lambda) &= \max\{\dim X \mid X \leqslant \mathcal{D}(H_2), \ H_2|_X \leqslant \lambda\} \\
&\leqslant \max\{\dim X \mid X \leqslant \mathcal{D}_0, \ H_2|_X \leqslant \lambda\} + k \\
&\leqslant \max\{\dim X \mid X \leqslant \mathcal{D}_0, \ H_1|_X \leqslant \lambda\} + k \\
&\leqslant \max\{\dim X \mid X \leqslant \mathcal{D}(H_1), \ H_1|_X \leqslant \lambda\} + k \\
&= n_{H_1}(\lambda) + k
\end{aligned}
$$

and the proof is finished. $\square$

This easily leads to the familiar boundedness of the spectral shift function for finite rank perturbations:

Corollary 10. *Let H be selfadjoint with discrete spectrum and K selfadjoint and bounded. Let Y_+ (respectively Y_-) be the strictly positive (respectively negative) spectral subspace of K. If $k_+ := \dim Y_+ < \infty$ then $n(H, \lambda) - k_+ \leqslant n(H + K, \lambda)$. If $k_- := \dim Y_- < \infty$ then $n(H + K, \lambda) \leqslant n(H, \lambda) + k_-$. In particular, the spectral shift function for finite rank perturbations is bounded by the rank of the perturbation.*

Proof. Applying the previous lemma to $H_2 = H + K$, $H_1 = H$ (such that $\mathcal{D}(H_2) = \mathcal{D}(H_1) = \mathcal{D}(H_1)$), choosing $\mathcal{D}_0 = \mathcal{D}(H) \cap Y_+^\perp$, yields one half of the claimed estimate. The other half follows if we replace K by $-K$ and H by $H + K$. $\square$

The perturbations in our applications will be of finite rank only in the following general sense:

Corollary 11. *Let H_0 be densely defined, closed, symmetric and lower bounded with deficiency index k. Let H_1, H_2 be two selfadjoint extensions of H_0 with discrete spectrum. Then,*

$$|\xi_{H_1,H_2}| \leqslant k. \tag{4}$$

Proof. This is a direct consequence of the previous lemma with $\mathcal{D}_0 = D(H_0)$. $\square$

Remark 12. Alternatively this corollary could be proven using the invariance principle for the spectral shift function and the second resolvent identity. In fact, this would not need the assumption of discrete spectra.

Let us now turn to the operators of interest for our applications. These are the restrictions of $-\Delta$ to a finite subgraph of G_d, with selfadjoint boundary conditions and possibly with an additional L^∞-potential.

Lemma 13. *Let G be a subgraph of G_d. Let Δ and Δ' be two selfadjoint realizations of the Laplacian on G. Set $\mathcal{D}_0 = \mathcal{D}(-\Delta) \cap \mathcal{D}(-\Delta')$ and $k := \dim \mathcal{D}(-\Delta)/\mathcal{D}_0$. Then $k \leqslant 2|E|$, and $|\xi_{-\Delta,-\Delta'}| \leqslant k$.*

Proof. Set $m := |E|$. Obviously, $W_0^{2,2}(E) := \{f \in W^{2,2}(E): (f(x), f'(x)) = 0 \; \forall x \in V(G)\}$ is a subspace contained in all possible domains, and all of them are contained in $W^{2,2}(E)$. Because of $\dim W^{2,2}(E)/W_0^{2,2}(E) = 4m$ and the rank $2m$ difference for the domains, $\mathcal{D}_0$ has index at most $2m$, and we can apply Lemma 11. $\square$

We are particularly interested in Dirichlet respectively Neumann conditions on the edges of a subgraph G. The corresponding operators $-\Delta_D^G, -\Delta_N^G$ decompose completely as direct sums over the edges, and the counting functions are given by $n_{-\Delta_D^G}(\lambda) = |E|n_D(\lambda)$ and $n_{-\Delta_N^G}(\lambda) = |E|n_N(\lambda)$. Here,

$$n_D(\lambda) = \begin{cases} \lfloor \sqrt{\lambda}/\pi \rfloor & \text{if } \lambda \geqslant 0, \\ 0 & \text{otherwise,} \end{cases} \qquad n_N(\lambda) = n_D(\lambda) + \chi_{[0,\infty[}(\lambda)$$

are the counting functions for the Dirichlet and Neumann Laplacians on $]0, 1[$. $\lfloor x \rfloor$ is defined for $x \in \mathbb{R}$ by $\lfloor x \rfloor \in \mathbb{Z}$ and $x - 1 < \lfloor x \rfloor \leqslant x$. Together with the previous lemma, this implies the following proposition.

Proposition 14. *Let G be a finite subgraph of G_d and Δ any selfadjoint realization of the Laplacian on G. Then, $|n_{-\Delta}(\lambda) - |E|n_D(\lambda)| \leqslant 2|E|$.*

In fact, looking at the quadratic forms one sees that the Dirichlet form dominates the forms of all other selfadjoint extensions; therefore, $|E|n_D$ is a lower bound to the counting functions of all other selfadjoint extensions.

Of course, everything said so far about selfadjoint extensions remains true when we add bounded perturbations. The spectral shift is determined in the following lemma.

Lemma 15. *Let* $-\Delta$ *be a selfadjoint realization of the Laplacian on a subgraph* G *of* G_d *and let* W_1, W_2 *be bounded selfadjoint operators on* $L^2(E)$. *Set* $H_j = -\Delta + W_j$, $j = 1, 2$. *Then there is a constant* C *depending only on the norms of* W_1, W_2 *such that*

$$|\xi_{H_1, H_2}|(\lambda) \leqslant C|E|.$$

Proof. Set $m = |E|$. According to Proposition 14,

$$m(-2 + n_D(\lambda)) \leqslant n_{-\Delta}(\lambda) \leqslant m(2 + n_D(\lambda)) \quad \text{and therefore}$$
$$m(-2 + n_D(\lambda - \|W\|)) \leqslant n_{-\Delta + W}(\lambda) \leqslant m(2 + n_D(\lambda + \|W\|))$$

for any bounded selfadjoint W because $-\|W\| \leqslant W \leqslant \|W\|$ implies $n_H(\lambda - \|W\|) \leqslant n_{H+W}(\lambda) \leqslant n_H(\lambda + \|W\|)$ for any selfadjoint H. Consequently, the explicit form of n_D given above leads to

$$m(-3 - \sqrt{\|W\|}/\pi) \leqslant n_{-\Delta + W}(\lambda) - \sqrt{\lambda}/\pi \leqslant m(2 + \sqrt{\|W\|}/\pi)$$

and subtracting the estimates for H_1 and H_2 we finally get

$$|\xi_{H_1, H_2}|(\lambda) \leqslant m(5 + \sqrt{\|W_1\|}/\pi + \sqrt{\|W_2\|}/\pi).$$

Using the fact that the Dirichlet counting function is a lower bound would allow us to replace 5 by 3. $\quad\square$

Remark 16. The considerations of this section hold for general finite metric graphs with all edges of equal length.

4. An abstract ergodic theorem

In this section, we discuss the abstract ergodic theorem of [22] and show how it can be applied to give convergence of a suitably defined spectral shift function of a family of random operators. See [21] for earlier one-dimensional results of the same type. This is the crucial step in the proof of our main result.

Let $\mathcal{A}$ be a finite set and choose $\Gamma \in \{\Gamma_{\text{trans}}, \Gamma_{\text{full}}\}$. Then, we can proceed as follows. The set of all finite subsets of $\mathbb{Z}^d$ is denoted by $\mathcal{F}$. A map $\mathcal{V}: \mathbb{Z}^d \to \mathcal{A}$ is called an $\mathcal{A}$-*colouring* of $\mathbb{Z}^d$. A map $P: Q(P) \to \mathcal{A}$ with $Q(P) \in \mathcal{F}$ is called an $\mathcal{A}$-*pattern*. For $M \in \mathbb{N}$ we denote by C_M the cube at the origin with sidelength $M - 1$, i.e.

$$C_M := \{x \in \mathbb{Z}^d: 0 \leqslant x_j \leqslant M - 1, \ j = 1, \ldots, d\}.$$

The set of patterns P with $Q(P) = C_M$ is denoted by $\mathcal{P}_0^B(M)$. The set of all C_M, $M \in \mathbb{N}$, is denoted by $\mathcal{C}$, and a pattern with $Q(P) \in \mathcal{C}$ is called a cube pattern. For a pattern P and $Q \in \mathcal{F}$ with $Q \subset Q(P)$ we define the restriction $P \cap Q$ of P to Q in the obvious way by $P \cap Q: Q \to \mathcal{A}, x \mapsto P(x)$. For a pattern P and $\gamma \in \Gamma$ we define the shifted map γP by $\gamma P: \gamma Q(P) \to \mathcal{A}, \gamma(y) \mapsto P(y)$. On the set of all patterns we define an equivalence relation by

$P \sim P'$ if and only if there exists a $\gamma \in \Gamma$ with $\gamma P = P'$. For a cube pattern P and an arbitrary pattern P' we define the number of occurrences of the pattern P in P' by

$$\sharp_P^\Gamma P' := \sharp\{x \in Q(P') \colon\; P' \cap (x + Q(P)) \sim P\}.$$

Definition 17. A map $b \colon \mathcal{F} \to [0, \infty)$ is called a *boundary term* if $b(Q) = b(t + Q)$ for all $t \in \mathbb{Z}^d$ and $Q \in \mathcal{F}$, $\lim_{j \to \infty} |Q_j|^{-1} b(Q_j) = 0$ for any van Hove sequence (Q_j), and there exists $D > 0$ with $b(Q) \leqslant D|Q|$ for all $Q \in \mathcal{F}$.

Definition 18. Let $(X, \|\cdot\|)$ be a Banach space and $F \colon \mathcal{F} \to X$ be given. Let $\Gamma \in \{\Gamma_{\text{trans}}, \Gamma_{\text{full}}\}$ be given.

(a) The function F is said to be *almost-additive* if there exists a boundary term b such that

$$\left\| F\left(\bigcup_{k=1}^m Q_k\right) - \sum_{k=1}^m F(Q_k) \right\| \leqslant \sum_{k=1}^m b(Q_k)$$

for all $m \in \mathbb{N}$ and all pairwise disjoint sets $Q_k \in \mathcal{F}$, $k = 1, \ldots, m$.
(b) Let $\mathcal{V} \colon \mathbb{Z}^d \to \mathcal{A}$ be a colouring. The function F is said to be *Γ–$\mathcal{V}$-invariant* if

$$F(Q) = F(\gamma Q)$$

whenever $\gamma \in \Gamma$ and $Q \in \mathcal{F}$ obey $\gamma(\mathcal{V} \cap Q) = \mathcal{V} \cap (\gamma Q)$. In this case there exists a function $\widetilde{F}$ on the cubes $\mathcal{C}$ with values in X such that

$$F(\gamma Q) = \widetilde{F}\big(\gamma^{-1}\big(\mathcal{V} \cap (\gamma Q)\big)\big)$$

for cubes $Q \in \mathcal{C}$ and $\gamma \in \Gamma$.
(c) The function F is said to be *bounded* if there exists a finite constant $C > 0$ such that

$$\|F(Q)\| \leqslant C|Q|$$

for all $Q \in \mathcal{F}$.

Theorem 19. *Let $\mathcal{A}$ be a finite set, $\mathcal{V} \colon \mathbb{Z}^d \to \mathcal{A}$ an $\mathcal{A}$-colouring and $(X, \|\cdot\|)$ a Banach-space. Let $\Gamma \in \{\Gamma_{\text{full}}, \Gamma_{\text{trans}}\}$ be given. Let $(Q_j)_{j \in \mathbb{N}}$ be a van Hove sequence such that for every pattern P the frequency $\nu_P = \lim_{j \to \infty} |Q_j|^{-1} \sharp_P^\Gamma(\mathcal{V} \cap Q_j)$ exists. Let $F \colon \mathcal{F} \to X$ be a Γ–$\mathcal{V}$-invariant, almost-additive bounded function. Then the limits*

$$\overline{F} := \lim_{j \to \infty} \frac{F(Q_j)}{|Q_j|} = \lim_{M \to \infty} \sum_{P \in \mathcal{P}_0^B(M)} \nu_P \frac{\widetilde{F}(P)}{|C_M|}$$

exist and are equal.

Remark 20.

(a) The theorem is proven in [22] for $\Gamma = \Gamma_{\text{trans}}$. However, the proof carries over verbatim to $\Gamma = \Gamma_{\text{full}}$.
(b) We have explicit bounds on speed of convergence in terms of speed of convergence of the frequencies. For details we refer to [22].

Now we turn to the case where the colouring is given by random potentials and boundary conditions. Then each $\omega \in \Omega$ induces a colouring $\mathcal{V}(\omega)$ with values in $\mathcal{A} := (\bigoplus_{j=1}^d \mathcal{B}) \times \mathcal{L}$ given by

$$\mathcal{V}(\omega)(x) := \big(V\big([x, x + e_j]\big)_j, L_x\big).$$

Lemma 21. *Let (Q_j) be an arbitrary van Hove sequence. Then, for almost every $\omega \in \Omega$ the frequency $\nu_P = \lim_{j \to \infty} |Q_j|^{-1} \sharp_P^{\Gamma}(\mathcal{V}(\omega) \cap Q_j)$ exists and is independent of ω for every cube pattern P.*

Proof. For a fixed pattern P the frequency exists for almost every ω by a standard ergodic theorem. As there are only countably many P the statement follows. $\quad\square$

In our setting we also have a family of random operators (H_ω) along with the Dirichlet Laplacian Δ_D. As discussed above, these operators can be restricted to the subgraphs G_Q induced by finite sets Q of $\mathbb{Z}^d$. This yields the operator H_ω^Q and Δ_D^Q with spectral counting functions n_ω^Q and n_D^Q respectively. Now, n_D^Q decomposes as a direct sum of operators. Thus, denoting the eigenvalue counting function of the negative Dirichlet Laplacian on $]0, 1[$ by $n_D(\lambda)$ as above, we have $n_D^Q = |E_Q| n_D = d|Q| n_D$. The associated spectral shift function is given as

$$\xi_\omega^Q(\lambda) := n_\omega^Q(\lambda) - d|Q| n_D(\lambda) = d|Q|\big(N_\omega^Q(\lambda) - n_D(\lambda)\big).$$

The crucial point is that ξ_ω falls into the framework of almost additive F introduced above. This is shown next.

Lemma 22. *Let $(\mathcal{D}, \|\cdot\|_\infty)$ be the Banach space of right continuous bounded functions on $\mathbb{R}$. Then, for each $\omega \in \Omega$ the function $\xi_\omega : \mathcal{F} \to \mathcal{D}$, $Q \mapsto \xi_\omega^Q$, is a bounded, $\mathcal{V}(\omega)$ invariant almost additive function.*

Proof. *Almost additivity*: Obviously, $Q \mapsto n_D^Q$ is almost additive with boundary term equal to zero, as Dirichlet boundary conditions decouple everything. As for n_ω, we note that $Q = \bigcup_{k=1}^m Q_k$ with pairwise disjoint Q_k, $k = 1, \ldots, m$, induces a decomposition of G_Q into G_{Q_k}, $k = 1, \ldots, m$. The restriction H_ω^Q of H_ω to G_Q differs from the direct sum $\bigoplus_{k=1}^m H_\omega^{Q_k}$ only by boundary conditions on the boundary of the G_{Q_k}, $k = 1, \ldots, m$. These boundary conditions yield boundary terms in the corresponding eigenvalues counting functions by Lemma 13. Thus, n_ω is almost additive as well. Hence, ξ_ω is almost additive as it is a difference of almost additive functions.

$\mathcal{V}(\omega)$-*invariance*: This is clear from the definitions.

Boundedness: By Lemma 13, changing all boundary conditions to Dirichlet conditions introduces an error term of the order $2|E|$. On the other hand, the operator with potential and only Dirichlet conditions can easily be compared to the Dirichlet operator without any potential by Lemma 15. □

Remark 23. The need to use a spectral shift function, i.e. the difference between n_ω and n_D, in the above proof comes exclusively from the last step in the proof. This use of a spectral shift function to define the IDS bears some similarity to how the integrated density of surface states is defined, see for instance [2,3,8,9,20].

The key result is now the following proposition.

Proposition 24. *There is a bounded right continuous function* $\Xi : \mathbb{R} \to \mathbb{R}$ *such that for a given van Hove sequence* (Q_l) *for almost every* $\omega \in \Omega$ *the uniform convergence*

$$\lim_{l \to \infty} \left\| \frac{\xi_\omega^{Q_l}}{|E_{Q_l}|} - \Xi \right\|_\infty = 0$$

holds.

Proof. Almost sure existence of the limit is a direct consequence of the previous two lemmas and the first theorem of this section. In fact, this theorem gives an explicit formula for the limit in terms of frequencies ν_P and $\widetilde{F}$. This formula shows that the limit does not depend on ω. □

5. Proofs of the main results

In this section we gather the material of the previous sections in order to prove Theorem 3 and its two corollaries.

Proof of Theorem 3. We only consider the case $\Gamma = \Gamma_{\text{trans}}$. The case $\Gamma = \Gamma_{\text{full}}$ is similar and even simpler.

We first show independence of N of the choice of Q: Set $G_x = x + \bar{S}$. The invariance assumption on the random operators, the invariance of the trace under unitary conjugation and the invariance of $\mathbb{P}$ under translations easily show that the expression

$$\int_\Omega \text{tr}\big(\chi_{G_x} \chi_{]\infty,\lambda]}(H_\omega)\big) \, d\mathbb{P}(\omega)$$

does not depend on $x \in \mathbb{Z}^d$. Now, the claimed independence of Q follows easily.

We now show convergence of the $|E_{Q_l}|^{-1} n_\omega^{Q_l}$, $l \in \mathbb{N}$. Proposition 24 yields almost sure convergence of

$$\frac{1}{|E_{Q_l}|} \xi_\omega^Q(\cdot) = \frac{1}{|E_{Q_l}|} n_\omega^Q(\cdot) - n_D(\cdot)$$

to a limit Ξ with respect to the supremum norm. As the subtracted term $n_D(\cdot)$ does not depend on l we obtain the desired convergence of $\frac{1}{|E_{Q_l}|} \xi_\omega^Q(\cdot) + n_D$. The limit is $\widetilde{N} := \Xi + n_D$.

Finally, we have to show that $N = \widetilde{N}$. To do so it suffices to show that the measures associated to N and $\widetilde{N}$ respectively are equal. This in turn follows once we show vague convergence. By a variant of the Stone/Weierstrass theorem it suffices therefore to show that

$$\frac{1}{|E_{Q_l}|} \operatorname{tr}\big(\chi_{G_{Q_l}} f(H_\omega) - f\big(H_\omega^{Q_l}\big)\big) \to 0, \quad l \to \infty, \tag{$*$}$$

for all f of the form $f(t) = (t - z)^{-1}$ for z with non-vanishing imaginary part. For $Q = Q_l$ we can split the graph G_d into the two components G_Q and $G_d \setminus G_Q$. Then, H_ω and $H_\omega^{G_Q} \oplus H_\omega^{G_d \setminus G_Q}$ differ only by boundary conditions on the set V_Q^∂. Thus, by the second resolvent identity

$$D := f(H_\omega) - f\big(H_\omega^{G_Q} \oplus H_\omega^{G_d \setminus G_Q}\big)$$

is an operator of rank at most $4d|V_Q^\partial|$. Moreover, D is obviously bounded by $2|\mathrm{Im}(z)|^{-1}$. This gives

$$\big|\operatorname{tr}\big(\chi_{G_{Q_l}} f(H_\omega) - f\big(H_\omega^{Q_l}\big)\big)\big| = \big|\operatorname{tr}\big(\chi_{G_{Q_l}}\big(f(H_\omega) - f\big(H_\omega^{G_Q} \oplus H_\omega^{G_d \setminus G_Q}\big)\big)\big)\big|$$

$$\leqslant 8d \frac{1}{|\mathrm{Im}(z)|} |V_{Q_l}^\partial|.$$

As (Q_l) is van Hove we obtain $(*)$. $\quad\square$

Proof of Corollary 7. This is a variant of the proof of Theorem 2 in [18].

$\Rightarrow$: Let λ be a point of discontinuity of μ. Thus, $\delta := \mu(\{\lambda\}) > 0$. By the uniform convergence proven in Theorem 3 this means that the multiplicity of the eigenvalue λ of $H_\omega^{Q_l}$ is at least $\delta/2|E_{Q_l}|$ for l large. On the other hand, within the eigenspace of $H_\omega^{Q_l}$ to the eigenvalue λ the space of f with $(f(v), f'(v)) = 0$ for all $v \in V_{Q_l}^\partial$ has codimension at most $2d|V_{Q_l}^\partial|$. As (Q_l) is van Hove, we have

$$2d\big|V_{Q_l}^\partial\big| < \delta/2|E_{Q_l}|$$

for l large enough and we obtain a compactly supported eigenfunction of $H_\omega^{Q_l}$ to λ all of whose boundary values are 0. Thus, this function can be extended by 0 to give a compactly supported eigenfunction.

$\Leftarrow$: Choose the van Hove sequence to consist of cubes centered around the origin with increasing sidelengths. For $l \in \mathbb{N}$, define the function χ_l on Ω by $\chi_l(\omega) = 1$ if there exists a compactly supported eigenfunction of H_ω to λ which is supported in Q_l and vanishes identically on all edges $e \in E_{Q_l}^\partial$ and $\chi_l(\omega) = 0$ otherwise. Then

$$0 < \delta_l := \int_\Omega \chi_l(\omega) \, d\mathbb{P}(\omega)$$

for all l large enough. Fix such an l and set $\chi := \chi_l$, $\delta := \delta_l$ and R to be the diameter of Q_l. Then, the ergodic theorem yields

$$\delta = \lim_{k \to \infty} \frac{1}{|Q_k|} \sum_{x \in Q_k} \chi(T^x \omega)$$

almost surely. Choose k_0 with $\frac{1}{|Q_k|} \sum_{x \in Q_k} \chi(T^x \omega) \geqslant \delta/2$ for $k \geqslant k_0$. For each $k \geqslant k_0$ we consider a maximal system M_K of points in Q_k such that $\chi(T^x \omega) = 1$ and points in M_k have distance at least $2R$. A direct combinatorial argument then shows that

$$|M_k| \geqslant \frac{\delta |Q_k|}{2|B_{2R}|}.$$

By construction different points in M_k yield orthogonal compactly supported eigenfunctions of H_ω. This yields

$$\mu_\omega^{Q_k}(\{\lambda\}) \geqslant \frac{\delta |Q_k|}{2|B_{2R}|}$$

for $k > k_0$ and the statement follows from the uniform convergence. $\quad\square$

Proof of Corollary 6. Due to ergodicity and the Pastur–Shubin trace formula the corollary follows by standard arguments. $\quad\square$

6. Operators with magnetic fields

Our setup is general enough to include magnetic fields as well. Since this is not quite obvious from the definitions and since there is a special interest in these operators we devote a section to the question how magnetic fields enter through boundary conditions.

To this end, let G be a metric graph and L a choice of boundary conditions as in Sections 2 and 3. The most general symmetric first order perturbation of $-\frac{d^2}{dt^2}$ on an edge $e \in E$ is, up to zeroth order terms, given by

$$H(a)_e := -\left(\frac{d}{dt} - \iota a_e\right)^2$$

for arbitrary real-valued $a_e \in C^1(\bar{e})$, where $\bar{e}$ is the closure of the edge e, i.e. identified with the closed interval $[0, 1]$. If G is embedded, e.g. in $\mathbb{R}^d$ such as in our case, it is natural to fix a so-called vector potential $A \in (C^1(\mathbb{R}^d))^d$ and to define a_e to be the component A_j of A if e is embedded with euclidean direction e_j. This corresponds to a magnetic field 2-form $B = \sum_{ij} \frac{\partial A_j}{\partial x_i} dx_i \wedge dx_j$ on $\mathbb{R}^d$. We will work in the general setting, though.

The selfadjoint realization of $H(a)$ corresponding to L is then given by the domain

$$\mathcal{D}(H_L(a)) = \left\{ f \in W^{2,2}(E) : \forall x \in V : (f(x), f'(x) - \iota(af)(x)) \in L_x \right\}$$

as the usual partial integration argument shows; i.e. one has to specify mixed Dirichlet and (magnetic) Neumann boundary conditions as expected.

Now define the phase along the edge e by $\varphi_e(t) = \int_0^t a_e(s)\,\mathrm{d}s$. Since each a_e is real-valued, multiplication by $e^{\iota\varphi_e}$ on each edge e defines a unitary operator U on $L^2(E)$ which maps $W^{2,2}(E)$ into itself. A simple calculation shows that

$$\left(U^* H(a) U\right)_e = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$$

on $W^{2,2}(E)$. Therefore, $H_L(a)_e$ is unitarily equivalent to $H_e = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ with domain

$$\begin{aligned}
\mathcal{D}(H_{\tilde{L}}) &= \left\{ f \in W^{2,2}(E) \colon Uf \in \mathcal{D}\big(H_L(a)\big) \right\} \\
&= \left\{ f \in W^{2,2}(E) \colon \forall x \in V \colon \big((Uf)(x), (Uf)'(x) - \iota(aUf)(x)\big) \in L_x \right\}
\end{aligned}$$

for some graph local boundary condition $\tilde{L}$ since multiplication by U preserves graph locality. In order to determine $\tilde{L}$ we calculate, using the notation from Section 2:

$$\begin{aligned}
(Uf)_e\big(s(e)\big) &= f_e\big(s(e)\big), \\
(Uf)_e\big(r(e)\big) &= e^{\iota\varphi(r(e))} f_e\big(r(e)\big), \\
(Uf)'_e\big(s(e)\big) - (aUf)\big(s(e)\big) &= f'_e\big(s(e)\big), \\
(Uf)'_e\big(r(e)\big) - (aUf)\big(r(e)\big) &= e^{\iota\varphi(r(e))} f'_e\big(r(e)\big).
\end{aligned}$$

Hence we define a unitary diagonal matrix u on $\mathbb{C}^{4d}$ by setting a diagonal entry to $e^{\iota\varphi(s(e))} = 1$ if it corresponds to a boundary value at some $s(e)$ and to $e^{\iota\varphi(r(e))}$ if it corresponds to a boundary value at some $r(e)$. Then $\tilde{L} = u^* L$.

Note that Dirichlet and (magnetic) Neumann conditions for the magnetic operator $H(a)$ transform under u into Dirichlet and (non-magnetic) Neumann conditions for H. This is to be expected because these boundary conditions decouple, and any magnetic field can be gauged away in strictly one-dimensional systems. In contrast, (magnetic) Kirchhoff conditions for $H(a)$ transform into completely different boundary conditions: At a vertex x, only the values at outgoing edges (those with $s(e) = x$) coincide; they, in turn, coincide with the values $e^{\iota\varphi(r(e))} f_e(r(e))$ at the incoming edges so that f need not be continuous at the vertex. Similarly, the derivative part of the Kirchhoff condition is modified to a weighted condition.

Finally, let us note that the results of this section imply that the results of Section 2 for Schrödinger operators with random potentials and random (or fixed) boundary conditions lead to the same results for magnetic Schrödinger operators with random potentials and random (or fixed) magnetic fields, specified through the phases $\varphi_e(r(e))$ at the endpoints.

7. Percolation on G_d

In this section we discuss two models which can be seen as versions of site and edge percolation on G_d, respectively. A third model corresponds to percolation on the junctions between vertices and edges. These models are based solely on random boundary conditions. The potential of the operators is identically equal to zero. Unlike in the percolation models on combinatorial graphs "deleted" edges are not removed completely from the graph but only cut off by Dirichlet boundary conditions. The reason is that removing edges would mean to remove

infinite-dimensional subspaces from our Hilbert space. This would result in a spectral distribution function which is not comparable to the one of the Laplacian with the concerned edge included. Hamilton operators on percolation subgraphs of combinatorial graphs have been considered the literature in theoretical physics [4,5,14], computational physics [13] (and references therein), and mathematical physics [17,25,28–30].

Before giving details we would like to emphasize the following: The examples below include cases in which the graphs contain many finite components giving rise to compactly supported eigenfunctions. In particular, the integrated density of states has many discontinuities. In fact, in the subcritical phase the IDS is a step function. However, despite all these jumps our result on uniform convergence does hold!

We first discuss a site percolation model. The percolation process is defined by the following procedure: toss a (possibly biased) coin at each vertex and—according to the outcome—put either a Dirichlet or a Kirchhoff boundary condition on this vertex. Do this at every vertex independently of all the others. To be more precise, let $p \in (0, 1)$ and $q = 1 - p$ be given. Let $\mathcal{A} := \{L^D, L^K\}$ and the probability measure $\nu := p\delta_{L^K} + (1 - p)\delta_{L^D}$ on $\mathcal{A}$ be given. Define Ω as the cartesian product space $\times_{x \in V_d} \mathcal{A}$ with product measure $\mathbb{P} := \bigotimes_{x \in V_d} \nu$. Let L be the stochastic process with coordinate maps $L(\omega)(x) := \omega(x)$. These data yield a family of random operators $-\Delta_\omega := H_\omega$ acting like the free Laplacian with domain given by

$$D(\Delta_\omega) = \left\{ f \in W^{2,2}(E) : \left(f(x), f'(x) \right) \in \omega(x) \; \forall x \in V_d \right\}.$$

Intuitively, placing a Dirichlet boundary condition at a vertex means "removing" it from the metric graph. The $2d$ formerly adjacent edges have now "loose ends." A fundamental result of percolation theory tell us that for sufficiently small values of p the percolation graph consists entirely of finite components almost surely. For these values of p our Laplace operators decouple completely into sums of operators of the form $-\Delta^G$ for finite connected subgraphs G of G_d. Here, Δ^G acts like the free Laplacian and has Dirichlet boundary conditions on its deleted vertices (boundary vertices) and Kirchhoff boundary conditions in its vertices which have not been deleted by the percolation process (interior vertices). Denote the eigenvalue counting function of $-\Delta^G$ by n^G. Defining the frequency ν_G of a finite subgraph of G in ω in the obvious way, we obtain as integrated density of states for the family H_ω

$$N = \sum_G \nu_G n^G,$$

where the sum runs over all finite connected subgraphs of G_d. Thus, the integrated density of states is a pure point measure in this case with many jumps. More interestingly, all these jumps remain present (even if their height is diminished) when we start increasing p. This yields models in which the operators are not given as a direct sum of finite graph operators but still have lots of jumps in their integrated density of states. Related phenomena for combinatorial Laplacians have been studied e.g. in [4,29,30].

We now discuss an edge percolation model. The basic idea is to decide for each edge independently whether Dirichlet boundary conditions are put on both ends or not. All other boundary conditions are Kirchhoff type. The problem when defining this edge percolation model is that our stochastic processes are indexed by vertices rather than edges. We thus have to relate edges to vertices. This is done by going to each vertex and then tossing a (biased) coin for each $j = 1, \ldots, d$ to decide how to deal with the edge $[x, x + e_j]$.

Here are the details. Let $p_0 \in (0, 1)$ and $p_1 = 1 - p_0$ be given. Let $\mathcal{A}$ consist of all maps S from $\{1, \dots, d\}$ to $\{0, 1\}$. Put a probability measure ν on $\mathcal{A}$ by associating the value $\prod_{j=1}^{d} p_{S(j)}$ to the element S. Now, Ω is the cartesian product space $\times_{x \in V_d} \mathcal{A}$ with product measure $\mathbb{P} := \bigotimes_{x \in V_d} \nu$. To each $\omega \in \Omega$ we associate the operator $-\Delta_\omega = H_\omega$ which acts like the free Laplacian and has boundary conditions as follows: The edge $e = [x, x + e_j]$ has Dirichlet boundary conditions on both ends if the random variable associated to the vertex x has the jth component equal to 1. Otherwise the boundary condition is chosen to be Kirchhoff. Here, again the operator decouples completely into operators on finite clusters for small enough values of p_0.

Similarly one can consider a percolation process indexed by pairs (x, e) of adjacent vertices and edges. As in the last model consider a colouring $\mathcal{A}$ consisting of all maps S from $\{1, -1, 2, -2, \dots, d, -d\}$ to $\{0, 1\}$. The probability space and measure are defined similarly as before. Each ω gives rise to a Laplace operator with the following boundary conditions: if the $-j$th component of the random variable associated to the vertex x has the value one, then the edge $[x - e_j, x]$ is decoupled from x by a Dirichlet boundary condition. If the jth component of the same random variable has value one then the edge $[x, x + e_j]$ is decoupled from x by a Dirichlet boundary condition. Conversely, those components of the random variable which are zero correspond to Kirchhoff boundary conditions.

Acknowledgments

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