Ain Shams Engineering Journal (2016) xxx, xxx–xxx

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Analysis of an unreliable retrial *G*-queue with working vacations and vacation interruption under Bernoulli schedule

P. Rajadurai *, **V.M. Chandrasekaran, M.C. Saravanarajan**

Department of Mathematics, School of Advanced Sciences, VIT University, Vellore 632014, Tamilnadu, India

Received 10 August 2015; revised 15 February 2016; accepted 2 March 2016

KEYWORDS

Retrial queue;
G-queue;
Working vacations;
Vacation interruption;
Supplementary variable technique

Abstract In this paper, we consider a single server retrial queueing system with working vacations. Further vacation interruption is considering with the regular busy server is subjected to breakdown due to the arrival of negative customers. When the orbit becomes empty at the time of service completion for a positive customer, the server goes for a working vacation. The server works at a lower service rate during working vacation (WV) period. If there are customers in the system at the end of each vacation, the server becomes idle and ready for serving new arrivals with probability p (single WV) or it remains on vacation with probability q (multiple WVs). By using the supplementary variable technique, we found out the steady state probability generating function for the system and its orbit. System performance measures, reliability measures and stochastic decomposition law are discussed. Finally, some numerical examples and cost optimization analysis are presented.

1. Introduction

In queueing theory, vacation queues and retrial queues are active research topics for a long time. We can find general models in vacation queues and retrial queues from Ke et al. [1] and Artalejo and Gomez-Corral [2] respectively. In retrial queueing system, queues with repeated attempts are characterized by an arriving customer who finds the server busy, leaves the service area and repeats its demand after some time. Between trials, the blocked customer joins a pool of unsatisfied customers called orbit. An arbitrary customer in the orbit who repeats the request for service is independent of the rest of the customers in the orbit. Such a retrial queue plays a special role in computers, telecommunication systems, communication protocols and retail shopping queues, etc.

For the last two decades, many researchers studied queueing networks with the concept of positive and negative customers. Queues with negative customers (called *G*-queues) have concerned huge interests due to their extensive applications in computers, communication networks, neural networks and manufacturing systems [3,4]. The named *G*-queue has been adopted for the queue with negative customers in the

* Corresponding author. Tel.: +91 416 2202321; fax: +91 416 2243092.
E-mail addresses: psdurai17@gmail.com (P. Rajadurai), vmcsn@vit.ac.in (V.M. Chandrasekaran), mcsaravanarajan@vit.ac.in (M.C. Saravanarajan).
Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.asej.2016.03.008
2090-4479

Please cite this article in press as: Rajadurai P et al., Analysis of an unreliable retrial *G*-queue with working vacations and vacation interruption under Bernoulli schedule, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.03.008

acknowledgment of Gelenbe, who first introduced this type of queue in [5,6]. Harrison [7] has studied the idea of compositional reversed Markov processes with applications to G-networks. The positive customer arrives into the system and gets service as ordinary queueing customers, but the negative customers enter into the system only at the service time of positive customers. This type of negative customers removes the positive customers being in service from the system and causes the server breakdown and the service channel will fail for a short interval of time. When the server fails, it will be sent to repair immediately. After completion of repair, the server will treat as good as new. Do [8] has presented a survey on queueing systems with G-networks, negative customers and applications. Choudhury and Ke [9], Rajadurai et al. [10,11] and Singh et al. [12] have discussed the retrial queue with the concept of breakdown and repair. Recently, Krishnakumar et al. [13], Gao and Wang [14], Peng et al. [15] and Rajadurai et al. [16,17] have discussed different types of queueing models operating with the simultaneous presence of negative arrivals.

In working vacation period (WV), the server gives service to customer at lower service rate, but the server stops the service completely in the normal vacation period. This queueing system has major applications in providing network service, web service, file transfer service and mail service, etc. In 2002, Servi and Finn [18] have introduced an $M/M/1$ queueing system with working vacations. Wu and Takagi [19] have extended the $M/M/1/\mathrm{WV}$ queue to an $M/G/1/\mathrm{WV}$ queue. Very recently, Arivudainambi et al. [20] have introduced $M/G/1$ retrial queue with single working vacation. Liu and Song [21] have discussed a discrete time retrial queue with non-persistent customers and working vacations. Furthermore if there are customers in the system at the end of a lower speed service, the server can stop the vacation and come back to the normal busy state. This policy is called vacation interruption. Li and Tian [22] have presented an $M/M/1$ queueing model with working vacations and vacation interruption. Some of the authors such as, Zhang and Hou [23], Gao and Liu [24], Gao et al. [25], Zhang and Liu [26], Rajadurai et al. [27,28] have analyzed a single server retrial queue with working vacations and vacation interruptions.

In this paper, we have extended the work of Gao et al. [25] and Zhang and Liu [26] by incorporating the concept of negative customers (G-queues) in both single and multiple working vacations with breakdowns and repair. To the author's best of knowledge, there is no work published in the queueing literature with the combination of retrial queueing system with general retrial times, negative customers, single and multiple working vacations, vacation interruption and breakdowns by using the method of supplementary variable technique. The mathematical results and theory of queues of this model provide to serve a specific and convincing application in the computer processing system. Our model is helpful to managers who can design a system with economic management.

The remainder of this work is given as follows. The detailed mathematical description and practical applications of our model are given in Section 2. The steady state joint distribution of the server state and the number of customers in the system and its orbit are obtained in Section 3. Some system performance measures, the mean busy period, the mean busy cycle and reliability measures are obtained in Section 4. In Section 5, conditional stochastic decomposition is shown good for our model. Important special cases of this model are given in Section 6. Cost optimization analysis is discussed in Section 7. In Section 8, the effects of various parameters on the system performance are analyzed numerically. Conclusion and summary of the work are presented in Section 9.

2. Description of the model

In this section, we consider a single server retrial queueing system with both single and multiple working vacations and vacation interruption, where the regular busy server is subjected to breakdown due to the arrival of negative customers. The detailed description of our model is given as follows:

- ***The arrival process:*** Customers arrive at the system according to a Poisson process with rate λ.
- ***The retrial process:*** If an arriving positive customer finds that the server is free, the customer begins his service immediately. Otherwise, the server is busy or working vacation or breakdown, the arrivals join pool of blocked customers called an orbit in accordance with FCFS discipline, which means that only one customer at the head of the orbit queue is allowed to access the server. We assume that inter-retrial times follow a general random variable R with an arbitrary distribution $R(t)$ having corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.
- ***The Bernoulli working vacation process:*** The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ. During a vacation period if any customer arrives, the server gives service at a lower speed service rate. If any customers in the orbit at a lower speed service completion instant in the vacation period then the server will stop the vacation and come back to the normal busy period which means that vacation interruption happens. If no customers are in the system at the end of the vacation, the server either remains idle to serve a new customer with probability p (single working vacation) in regular mode or leaves for another working vacation with probability $q = 1 - p$ (multiple working vacation). When a vacation ends and if there are customers in the orbit then the server switches to the normal working level. During the working vacation period, the service time follows a general random variable S_v with distribution function $S_v(t)$ having LST $S_v^*(\vartheta)$ and the first moment is given by $S_v^{*\prime}(\theta) = \int_0^\infty x e^{-\theta x} dS_v(x)$.
- ***The regular service process:*** Whenever a new positive customer or retry positive customer arrives at the server idle state then the server immediately starts normal service for the arrivals. The service time follows a general distribution and it is denoted by the random variable S_b with distribution function $S_b(t)$ having LST $S_b^*(\vartheta)$ and the first moment is given by $S_b^{*\prime}(\delta) = \int_0^\infty x e^{-\delta x} dS_b(x)$.
- ***The removal rule and the repair process:*** The negative customers arrive from outside the system according to a Poisson arrival rate δ. These negative customers arrive only at the regular service time of the positive customers. Negative customers cannot accumulate in a queue and do not receive service, will remove the positive customers being in service from the system. These types of negative customer's cause the server breakdown and the service channel will fail for

a short interval of time. When the server fails, it will be sent to repair immediately. After completion of repair, the server will treat as good as new. The repair time (denoted by G) of the server is assumed to be arbitrarily distributed with distribution function $G(t)$ having LST $G^*(\vartheta)$ and the first and second moments are denoted by $g^{(1)}$ and $g^{(2)}$.

- We assume that all the random variables (inter-arrival times, retrial times, service times, working vacation times and repair times) defined above are independent of each other.

2.1. Practical application of the proposed model

The potential practical application of this model is in the operational model of stochastic production and inventory systems with a multipurpose production facility. The production facility performs other additional tasks using the time between succeeding productions. We consider a practical problem related to a *production to order system*. In production order system, the customer orders for the product (positive customers) and some customers cancel orders (negative customers) due to financial crisis or disaster, etc. Let us assume that order system follows the Poisson processes. At the time of arriving customer orders the production facility is busy with other order, and a new arriving order will form a waiting line which corresponds to the retrial queue. Otherwise, the order is processed immediately. After the completion of order, the management policy sets up the facility and begins product to the next order on the list; otherwise, another external order arrives before the order is made. The order time is assumed to be generally distributed (general retrial time). During the service time of ordering, if any canceling order exists then the major production will be stopped (i.e. server breakdown). To enhance the production facility performance, the management policy is to set up the optional job facility (working vacation). After completion of service for last order and there is no order in system (system empty), the facility stops the major production and is available to perform optional jobs (single working vacation) with lower production rate (lower speed service rate). After the result of disaster or an optional job completion, if there are no orders, the production facility will continue to perform the optional jobs at lower production rate (multiple working vacations). Otherwise, it performs the major production. If there are orders at the instant of the optional jobs completion, the production facility will perform the major production (vacation interruption). This type of system is very useful to increase the performance of the production facility and to stop the production facility from becoming overloaded.

Another practical application of this model is in the area of computer processing system. In a computer processing system, the buffer size (orbit) used to store messages is finite and the messages (customers) arrive into the system one by one, and the processor (server) is in charge of processing messages. The working mail server may be affected by virus (negative customers), and the system may be subjected to electronic fails (breakdowns) during service period and receive repair immediately. If the processor is available indicating that it is not currently working on a task and then a message is processed. The messages are temporarily stored in a buffer to be served some time later (retrial time) according to FCFS if the processor is unavailable. To enhance the computer performance, whenever all messages are processed and no new messages arrive, the processor will perform a sequence of maintenance jobs, such as virus scan (working vacations). During the maintenance period, the processor can deal with the messages at the slower rate to economize the cost (working vacation period). Upon completion of the each maintenance, the processor checks the messages and decides whether or not to resume the normal service rate (single working vacation). At this moment, if no message is in the system then the processor may decide to go for another maintenance activity (multiple working vacations). This type of working vacation discipline is a good approximation of such computer processing system.

3. Steady state analysis

In this section, we develop the steady state difference–differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times, the elapsed working vacation times and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states and for the number of customers in the system and orbit.

3.1. The steady state equations

In steady state, we assume that $R(0) = 0$, $R(\infty) = 1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $S_v(0) = 0$, $S_v(\infty) = 1$, $G(0) = 0$, $G(\infty) = 1$ are continuous at $x = 0$. So that the function $a(x)$, $\mu_b(x)$, $\mu_v(x)$ and $\xi(x)$ are the conditional completion rates (hazard rate) for retrial, normal service, lower rate service and repair respectively.

i.e., $a(x)dx = \dfrac{dR(x)}{1-R(x)}$; $\quad \mu_b(x)dx = \dfrac{dS_b(x)}{1-S_b(x)}$;
$\mu_v(x)dx = \dfrac{dS_v(x)}{1-S_v(x)}$; $\quad \xi(x)dx = \dfrac{dG(x)}{1-G(x)}$.

In addition, let $R^0(t)$, $S_b^0(t)$, $S_v^0(t)$ and $G^0(t)$ be the elapsed retrial time, elapsed normal service time, elapsed working vacation time and elapsed repair time respectively at time t. Further, we introduce the random variable,

$$X(t) = \begin{cases} 0, & \text{if the server is free and in working vacation period,} \\ 1, & \text{if the server is free and in regular service period,} \\ 2, & \text{if the server is busy and in regular service period at time } t, \\ 3, & \text{if the server is busy and in working vacation period at time } t, \\ 4, & \text{if the server is under repair period at time } t. \end{cases}$$

Thus the supplementary variables $R^0(t)$, $S_b^0(t)$, $S_v^0(t)$ and $G^0(t)$ are introduced in order to obtain a bivariate Markov process $\{X(t), N(t); t \geqslant 0\}$, where $X(t)$ denotes the server state $(0, 1, 2, 3, 4)$ depending on the server is free on both regular busy period and working vacation period, regular busy, on working vacation and under repair. If $X(t) = 1$ and $N(t) > 0$, then $R^0(t)$ represent the elapsed retrial time, if $X(t) = 2$ and $N(t) \geqslant 0$ then $S_b^0(t)$ corresponding to the elapsed time of the customer being served in regular busy period. If $X(t) = 3$ and $N(t) \geqslant 0$, then $S_v^0(t)$ corresponding to the elapsed time of the customer being served in lower rate service period. If $X(t) = 4$ and $N(t) \geqslant 0$, then $G^0(t)$ corresponding to the elapsed time of the server being repaired.

We analyze the ergodicity of the embedded Markov chain at departure, vacation or repair epochs. Let $\{t_n; n = 1, 2, \ldots\}$ be the sequence of epochs of either normal service completion times or working vacation completion times or repair period ends. The sequence of random vectors $Z_n = \{X(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 3.1. *The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$ for our system to be stable, where $\rho = (\lambda/\delta)(1 - S_b^*(\delta))(1 + \delta g^{(1)})$*

Proof. To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (see Pakes [29]), which states that the chain $\{Z_n; n \in N\}$ is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j), j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{n+1}) - f(z_n)/z_n = j]$ is finite for all $j \in N$ and $\psi_j \leqslant -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} \rho - 1, & \text{if } j = 0, \\ \rho - R^*(\lambda), & \text{if } j = 1, 2, \ldots \end{cases}$$

Clearly the inequality $\rho < R^*(\lambda)$ is sufficient condition for Ergodicity.

To prove the necessary condition, as noted in Sennott et al. [30], if the Markov chain $\{Z_n; n \geqslant 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \geqslant 0$ and there exits $j_0 \in N$ such that $\psi_j \geqslant 0$ for $j \geqslant j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n; n \in N\}$. Then $\rho \geqslant R^*(\lambda)$ implies the non-Ergodicity of the Markov chain. □

Let us define the limiting probabilities $Q_0(t) = P\{X(t) = 0, N(t) = 0\}$; $P_0(t) = P\{X(t) = 1, N(t) = 0\}$ and the probability densities are

$$P_n(x,t)dx = P\{X(t) = 1, N(t) = n, x \leqslant R^0(t) < x + dx\},$$
for $t \geqslant 0$, $x \geqslant 0$ and $n \geqslant 1$.
$$\Pi_{b,n}(x,t)dx = P\{X(t) = 2, N(t) = n, x \leqslant S_b^0(t) < x + dx\},$$
for $t \geqslant 0$, $x \geqslant 0$, $n \geqslant 0$.
$$Q_{v,n}(x,t)dx = P\{X(t) = 3, N(t) = n, x \leqslant S_v^0(t) < x + dx\},$$
for $t \geqslant 0$, $x \geqslant 0$ and $n \geqslant 0$.
$$R_n(x,t)dx = P\{X(t) = 4, N(t) = n, x \leqslant G^0(t) < x + dx\},$$
for $t \geqslant 0$, $x \geqslant 0$ and $n \geqslant 0$.

We assume that the stability condition is fulfilled in the sequel and so that we can set $Q_0 = \lim_{t\to\infty} Q_0(t)$; $P_0 = \lim_{t\to\infty} P_0(t)$ and limiting densities for $t \geqslant 0$, $x \geqslant 0$ and $n \geqslant 1$.

$$P_n(x) = \lim_{t\to\infty} P_n(x,t); \quad \Pi_{b,n}(x) = \lim_{t\to\infty} \Pi_{b,n}(x,t);$$
$$Q_{v,n}(x) = \lim_{t\to\infty} Q_{v,n}(x,t) \quad \text{and} \quad R_n(x) = \lim_{t\to\infty} R_n(x,t).$$

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$\lambda P_0 = \theta p Q_0 \tag{3.1}$$

$$(\lambda + \theta) Q_0 = \theta q Q_0 + \int_0^\infty \Pi_{b,0}(x)\mu_b(x)dx + \int_0^\infty Q_{v,0}(x)\mu_v(x)dx + \int_0^\infty R_0(x)\xi(x)dx \tag{3.2}$$

$$\frac{dP_n(x)}{dx} + (\lambda + a(x))P_n(x) = 0, \quad n \geqslant 1 \tag{3.3}$$

$$\frac{d\Pi_{b,0}(x)}{dx} + (\lambda + \delta + \mu_b(x))\Pi_{b,0}(x) = 0, \quad n = 0, \tag{3.4}$$

$$\frac{d\Pi_{b,n}(x)}{dx} + (\lambda + \delta + \mu_b(x))\Pi_{b,n}(x) = \lambda \Pi_{b,n-1}(x), \quad n \geqslant 1, \tag{3.5}$$

$$\frac{dQ_{v,0}(x)}{dx} + (\lambda + \theta + \mu_v(x))Q_{v,0}(x) = 0, \quad n = 0 \tag{3.6}$$

$$\frac{dQ_{v,n}(x)}{dx} + (\lambda + \theta + \mu_v(x))Q_{v,n}(x) = \lambda Q_{v,n-1}(x), \quad n \geqslant 1 \tag{3.7}$$

$$\frac{dR_0(x)}{dx} + (\lambda + \xi(x))R_0(x) = 0, \quad n = 0. \tag{3.8}$$

$$\frac{dR_n(x)}{dx} + (\lambda + \xi(x))R_n(x) = \lambda R_{n-1}(x), \quad n \geqslant 1. \tag{3.9}$$

To solve the Eqs. (3.2)–(3.9), the steady state boundary conditions at $x = 0$ are followed,

$$P_n(0) = \int_0^\infty \Pi_{b,n}(x)\mu_b(x)dx + \int_0^\infty R_n(x)\xi(x)dx + \int_0^\infty Q_{v,n}(x)\mu_v(x)dx, \quad n \geqslant 1 \tag{3.10}$$

$$\Pi_{b,0}(0) = \left(\int_0^\infty P_1(x)a(x)dx + \theta \int_0^\infty Q_{v,0}(x)dx + \lambda P_0\right), \quad n = 0, \tag{3.11}$$

$$\Pi_{b,n}(0) = \left(\int_0^\infty P_{n+1}(x)a(x)dx + \lambda \int_0^\infty P_n(x)dx + \theta \int_0^\infty Q_{v,n}(x)dx\right), \quad n \geqslant 1, \tag{3.12}$$

$$Q_{v,n}(0) = \begin{cases} \lambda Q_0, & n = 0 \\ 0, & n \geqslant 1 \end{cases} \tag{3.13}$$

$$R_n(0) = \delta \int_0^\infty \Pi_{b,n}(x)dx, \quad n \geqslant 0 \tag{3.14}$$

The normalizing condition is

$$P_0 + Q_0 + \sum_{n=1}^{\infty} \int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty} \left(\int_0^\infty \Pi_{b,n}(x)dx + \int_0^\infty Q_{v,n}(x)dx + \int_0^\infty R_n(x)dx \right) = 1 \tag{3.15}$$

3.2. The steady state solution

The steady state solution of the retrial queueing model is obtained by using the probability generating function technique. To solve the above equations, the PGFs are defined for $|z| \leqslant 1$ as follows:

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n; \quad P(0,z) = \sum_{n=1}^{\infty} P_n(0)z^n;$$

$$\Pi_b(x,z) = \sum_{n=0}^{\infty} \Pi_{b,n}(x)z^n; \quad \Pi_b(0,z) = \sum_{n=0}^{\infty} \Pi_{b,n}(0)z^n;$$

$$Q_v(x,z) = \sum_{n=0}^{\infty} Q_{v,n}(x)z^n; \quad Q_v(0,z) = \sum_{n=0}^{\infty} Q_{v,n}(0)z^n;$$

$$R(x,z) = \sum_{n=0}^{\infty} R_n(x)z^n \quad \text{and} \quad R(0,z) = \sum_{n=0}^{\infty} R_n(0)z^n$$

On multiplying the Eqs. (3.2)–(3.14) by z^n and summing over n, ($n = 0, 1, 2, \ldots$), we get

$$\frac{\partial P(x,z)}{\partial x} + (\lambda + a(x))P(x,z) = 0 \tag{3.16}$$

$$\frac{\partial \Pi_b(x,z)}{\partial x} + (\lambda(1-z) + \delta + \mu_b(x))\Pi_b(x,z) = 0, \tag{3.17}$$

$$\frac{\partial Q_v(x,z)}{\partial x} + (\lambda(1-z) + \theta + \mu_v(x))Q_v(x,z) = 0 \tag{3.18}$$

$$\frac{\partial R(x,z)}{\partial x} + (\lambda(1-z) + \xi(x))R(x,z) = 0 \tag{3.19}$$

$$P(0,z) = \int_0^\infty \Pi_b(x,z)\mu_b(x)dx + \int_0^\infty Q_v(x,z)\mu_v(x)dx + \int_0^\infty R(x,z)\xi(x)dx - ((\lambda+\theta)Q_0 - \theta q P_0) \tag{3.20}$$

$$\Pi_b(0,z) = \frac{1}{z} \int_0^\infty P(x,z)a(x)dx + \lambda \int_0^\infty P(x,z)dx + \theta \int_0^\infty Q_v(x,z)dx, \tag{3.21}$$

$$Q_v(0,z) = \lambda Q_0 \tag{3.22}$$

$$R(0,z) = \delta \int_0^\infty \Pi_b(x,z)dx \tag{3.23}$$

Solving the partial differential Eqs. (3.20)–(3.23), it follows that

$$P(x,z) = P(0,z)[1 - R(x)]e^{-\lambda x} \tag{3.24}$$

$$\Pi_b(x,z) = \Pi_b(0,z)[1 - S_b(x)]e^{-A_b(z)x}, \tag{3.25}$$

$$Q_v(x,z) = Q_v(0,z)[1 - S_v(x)]e^{-A_v(z)x}, \tag{3.26}$$

$$R(x,z) = R(0,z)[1 - G(x)]e^{-b(z)x}, \tag{3.27}$$

where $A_b(z) = (\delta + \lambda(1-z))$, $A_v(z) = (\theta + \lambda(1-z))$ and $b(z) = \lambda b(1-z)$.

Inserting Eqs. (3.24)–(3.27) in (3.21) and make some manipulation, finally we get,

$$\Pi_b(0,z) = \frac{P(0,z)}{z}[R^*(\lambda) + z(1 - R^*(\lambda))] + \lambda P_0 + \lambda Q_0 V(z) \tag{3.28}$$

where $V(z) = \frac{\theta}{\theta + \lambda(1-z)}\left[1 - S_v^*(A_v(z))\right]$.

Using Eqs. (3.25)–(3.27) in (3.20), we get

$$P(0,z) = \Pi_b(0,z)S_b^*(A_b(z)) + Q_v(0,z)S_v^*(A_v(z)) + R(0,z)G^*(b(z)) - (\lambda + p\theta)Q_0 \tag{3.29}$$

Using Eqs. (3.25)–(3.27) in (3.23), we get

$$R(0,z) = \delta \Pi_b(0,z)\left(\frac{1 - S_b^*(A_b(z))}{A_b(z)}\right), \tag{3.30}$$

Using Eqs. (3.22), (3.28), and (3.30) in (3.29), we get

$$P(0,z) = \frac{Nr(z)}{Dr(z)} \tag{3.31}$$

$$Nr(z) = zQ_0\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right) + (\lambda V(z) + \theta p) \times \left(S_b^*(A_b(z)) + \frac{\delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)}{A_b(z)}\right)\right\}$$

$$Dr(z) = \left\{z - (R^*(\lambda) + z(1 - R^*(\lambda))) \left(S_b^*(A_b(z)) + \frac{\delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)}{A_b(z)}\right)\right\}$$

Using Eq. (3.21) in (3.28), we get

$$\Pi_b(0,z) = Q_0\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right\}/Dr(z) \tag{3.32}$$

Using Eq. (3.32) in (3.30), we get

$$R(0,z) = \delta Q_0\left(1 - S_b^*(A_b(z))\right)\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right\}/A_b(z) \times Dr(z), \tag{3.33}$$

Using the Eqs. (3.22) and (3.31)–(3.33) in (3.24)–(3.27), then we get the results for the following PGFs $P(x,z)$, $\Pi_b(x,z)$, $Q_v(x,z)$ and $R(x,z)$. Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 3.2. *The marginal probability distributions of the number of customers in the orbit when server being idle, busy, on working vacation and under repair are given by*

$$P(z) = \frac{Nr(z)}{Dr(z)}$$

$$Nr(z) = zQ_0\left(\frac{1 - R^*(\lambda)}{\lambda}\right)\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right) + (\lambda V(z) + \theta p)\left(S_b^*(A_b(z)) + \frac{\delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)}{A_b(z)}\right)\right\}$$

$$Dr(z) = \left\{z - (R^*(\lambda) + z(1 - R^*(\lambda)))\left(S_b^*(A_b(z)) + \frac{\delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)}{A_b(z)}\right)\right\} \tag{3.34}$$

$$\Pi_b(z) = Q_0\left(1 - S_b^*(A_b(z))\right)\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right\} / A_b(z) \times Dr(z) \quad (3.35)$$

$$Q_v(z) = \{\lambda Q_0 V(z)/\theta\} \quad (3.36)$$

$$R(z) = \delta Q_0 \left\{ \begin{array}{l} \left(\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right) \\ \times\left(1 - S_b^*(A_b(z))\right)(1 - G^*(b(z))) \end{array} \right\} \Big/ b(z) \times A_b(z) \times Dr(z), \quad (3.37)$$

where

$$Q_0 = \frac{R^*(\lambda) - \rho}{\left\{(\lambda/\theta)\left(1 - S_v^*(\theta)\right) + R^*(\lambda)(1 + (\theta p/\lambda)) - \rho S_v^*(\theta)\right\}} \quad (3.38)$$

$$P_0 = \frac{\theta p(R^*(\lambda) - \rho)}{\lambda\left\{(\lambda/\theta)\left(1 - S_v^*(\theta)\right) + R^*(\lambda)(1 + (\theta p/\lambda)) - \rho S_v^*(\theta)\right\}} \quad (3.39)$$

$\rho = (\lambda/\delta)\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)$; $A_b(z) = (\delta + \lambda(1 - z))$; $A_v(z) = (\theta + \lambda(1 - z))$ and $b(z) = \lambda b(1 - z)$

Proof. Integrating the Eqs. (3.24)–(3.27) with respect to x, we define the PGFs as, $P(z) = \int_0^\infty P(x, z)dx$, $\Pi_b(z) = \int_0^\infty \Pi_b(x, z)dx$, $Q_v(z) = \int_0^\infty Q_v(x, z)dx$, $R(z) = \int_0^\infty R(x, z)dx$. By using the normalized condition, we can determine the probability that the server is idle (P_0). Thus, by setting $z = 1$ in (3.34)–(3.37) and applying de l'Hôpital's rule whenever necessary and we get $P_0 + Q_0 + P(1) + \Pi_b(1) + Q_v(1) + R(1) = 1$. □

Theorem 3.3. *The probability generating function of number of customers in the system and orbit size distribution at stationary point of time is*

$$K_s(z) = \frac{Nr_s(z)}{Dr_s(z)} = P_0 + Q_0 + P(z) + z(\Pi_b(z) + Q_v(z)) + R(z) \quad (3.40)$$

$$Nr_s(z) = Q_0 \left\{ \begin{array}{l} (1 - z) \left[\begin{array}{l} ((\lambda/\theta)(\theta + \lambda z V(z)) + \theta p)\left\{zA_b(z) - (R^*(\lambda) + z(1 - R^*(\lambda)))\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\} \\ +z(1 - R^*(\lambda)) \left\{ \begin{array}{l} (\lambda V(z) + \theta p)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right) \\ +A_b(z)\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right) \end{array} \right\} \end{array} \right] \\ +\left(1 - S_b^*(A_b(z))\right)(zb(z) + \delta(1 - G^*(b(z))))\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right\} \end{array} \right\}$$

$$Dr_s(z) = b(z) \times \left\{zA_b(z) - (R^*(\lambda) + z(1 - R^*(\lambda)))\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\}$$

$$K_o(z) = \frac{Nr_o(z)}{Dr_s(z)} = P_0 + Q_0 + P(z) + \Pi_b(z) + Q_v(z) + R(z). \quad (3.41)$$

$$Nr_0(z) = Q_0 \left\{ \begin{array}{l} (1 - z) \left[\begin{array}{l} ((\lambda/\theta)(\theta + \lambda V(z)) + \theta p)\left\{zA_b(z) - (R^*(\lambda) + z(1 - R^*(\lambda)))\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\} \\ +z(1 - R^*(\lambda)) \left\{ \begin{array}{l} (\lambda V(z) + \theta p)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right) \\ +A_b(z)\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right) \end{array} \right\} \end{array} \right] \\ +\left(1 - S_b^*(A_b(z))\right)(b(z) + \delta(1 - G^*(b(z))))\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + z(\lambda V(z) + \theta p)\right\} \end{array} \right\}$$

where Q_0 is given in Eq. (3.38).

Proof. The PGF of the number of customer in the system ($K_s(z)$) and orbit ($K_o(z)$) are obtained by using $K_s(z) = P_0 + Q_0 + P(z) + z(\Pi_b(z) + Q_v(z)) + R(z)$ and $K_o(z) = P_0 + Q_0 + P(z) + \Pi_b(z) + Q_v(z) + R(z)$. Substituting the Eqs. (3.34)–(3.39) in the above results, then the Eqs. (3.40) and (3.41) can be obtained by direct calculation. □

4. System performance measures

In this section, we derive some system probabilities, mean number of customers in the system and its orbit, reliability analysis, mean busy period and mean busy cycle of this model.

4.1. System state probabilities

From Eqs. (3.34)–(3.37), by setting $z \to 1$ and applying l'Hôpital's rule whenever necessary, then we get the following results,

(i) The probability that the server is idle during the retrial

$$P = Q_0(1 - R^*(\lambda)) \times \left\{ \frac{(\lambda/\theta)\left(1 - S_v^*(\theta)\right) + \left(\lambda\left(1 - S_v^*(\theta)\right) + \theta p\right)\left(\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)}{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)} \right\}$$

(ii) The probability that the server is regular busy,

$$\Pi_b = Q_0\left(\left(1 - S_b^*(\delta)\right)/\delta\right)\left\{ \frac{\left(\lambda^2/\theta\right)\left(1 - S_v^*(\theta)\right) + R^*(\lambda)\left(\lambda\left(1 - S_v^*(\theta)\right) + \theta p\right)}{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)} \right\}$$

(iii) The probability that the server is on working vacation

$$Q_v = \{\lambda Q_0(1 - S_v^*(\theta))/\theta\}$$

(iv) The probability that the server is under repair,

$$R = Q_0 g^{(1)}(1 - S_b^*(\delta))\left\{\frac{(\lambda^2/\theta)(1 - S_v^*(\theta)) + R^*(\lambda)(\lambda(1 - S_v^*(\theta)) + \theta p)}{R^*(\lambda) - (\lambda(1 - S_b^*(\delta))(1 + \delta g^{(1)})/\delta)}\right\}$$

4.2. Mean system size and orbit size

If the system is in steady state condition,

(i) The expected number of customers in the orbit (L_q) is obtained by differentiating (3.41) with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z) = Q_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_q''(1) = -2\lambda(1 - S_b^*(\delta))(1 + \delta g^{(1)})\{(\theta - \lambda)p - \lambda S_v^*(\theta)\} - 2(\delta\lambda^2/\theta)(1 - R^*(\lambda))(1 - S_v^*(\theta))$$

$$\begin{aligned}Nr_q'''(1) = {} & 3\lambda\tau S_v^*(\theta) - 2(\lambda^3/\theta)(1 + \delta g^{(1)})(1 - S_v^*(\theta) \\ & + \theta R^*(\lambda)S_v^{*\prime}(\theta))((\lambda/\theta)(1 - S_v^*(\theta)) + 1) \\ & - 6(\delta\lambda g^{(1)}(1 - S_b^*(\delta)) - \lambda S_b^*(\delta))\{(\lambda V''(1)) \\ & - (1 - R^*(\lambda))((\lambda/\theta)(\theta + \lambda(1 - S_v^*(\theta))) + \theta p)\} \\ & - 6\lambda(1 - S_b^*(\delta))(1 - R^*(\lambda))(1 + \delta g^{(1)})(\theta p \\ & + \lambda(1 - S_v^*(\theta))) + 6\lambda((\lambda/\theta)(\theta + \lambda(1 - S_v^*(\theta))) + \theta p) \\ & - 6\lambda^2((\delta/\theta)(V'(1) + (1 - R^*(\lambda))(1 - S_v^*(\theta))) + \lambda S_v^{*\prime}(\theta))\end{aligned}$$

$$Dr_q''(1) = -2\lambda(R^*(\lambda) - \rho)$$

$$Dr_q'''(1) = 3\lambda(\tau + 2\lambda\delta\rho) - 6\lambda R^*(\lambda)(\delta\lambda g^{(1)}(1 - S_b^*(\delta)) - \lambda S_b^*(\delta))$$

where $V'(1) = \frac{\lambda}{\theta}(1 - S_v^*(\theta) + \theta S_v^{*\prime}(\theta))$; $\rho = (\lambda/\delta)(1 - S_b^*(\delta))(1 + \delta g^{(1)})$; $S_v^{*\prime}(\theta) = \int_0^\infty xe^{-\theta x}dS_v(x)$; $S_b^{*\prime}(\delta) = \int_0^\infty xe^{-\delta x}dS_b(x)$.

$$\tau = \lambda^2(2S_b^{*\prime}(\delta)(1 + \delta g^{(1)}) + \delta g^{(2)}(1 - S_b^*(\delta)))$$

(ii) The expected number of customers in the system (L_s) is obtained by differentiating (3.40) with respect to z and evaluating at $z = 1$

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z) = Q_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

where

$$Nr_s'''(1) = Nr_q'''(1) - 6\lambda R^*(\lambda)(1 - S_b^*(\delta))(\theta p + \lambda(1 - S_v^*(\theta))) + 6\delta g^{(1)}(\lambda^3/\theta)(1 - S_b^*(\delta))(1 - S_v^*(\theta))$$

4.3. Reliability measures

In the queueing system with unreliable server, the reliability measures will provide the information which is required for the improvement of the system. To justify and validate the analytical results of this model, the availability measure (A_v) and failure frequency (F_f) are obtained as follows:

(i) The steady state availability A_v, which is the probability that the server is either working for a positive customer or in an idle period such that the steady state availability of the server is given by

$$\begin{aligned}A_v &= 1 - \lim_{z\to 1}(R(z)) = 1 - R(1) \\ &= 1 - \left\{\frac{g^{(1)}(1 - S_b^*(\delta))\left\{\left(\frac{\lambda^2}{\theta}\right)(1 - S_v^*(\theta)) + R^*(\lambda)(\lambda(1 - S_v^*(\theta)) + \theta p)\right\}}{\{(\lambda/\theta)(1 - S_v^*(\theta)) + R^*(\lambda)(1 + (\theta p/\lambda)) - \rho S_v^*(\theta)\}}\right\}\end{aligned}$$

(ii) The steady state failure frequency is obtained as

$$\begin{aligned}F_f &= \delta \times \Pi_b(1) \\ &= \left\{\frac{Q_0(1 - S_b^*(\delta))\{(\lambda^2/\theta)(1 - S_v^*(\theta)) + R^*(\lambda)(\lambda(1 - S_v^*(\theta)) + \theta p)\}}{R^*(\lambda) - (\lambda(1 - S_b^*(\delta))(1 + \delta g^{(1)})/\delta)}\right\}\end{aligned}$$

4.4. Mean busy period and busy cycle

Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions. The results follow directly by applying the argument of an alternating renewal process [9] which leads to

$$P_0 = \frac{E(T_0)}{E(T_b) + E(T_0)};\ E(T_b) = \frac{1}{\lambda}\left(\frac{1}{P_0} - 1\right) \quad \text{and}$$

$$E(T_c) = \frac{1}{\lambda P_0} = E(T_0) + E(T_b) \tag{4.1}$$

where T_0 is length of the system in empty state and $E(T_0) = (1/\lambda)$. Substituting the Eq. (3.39) into (4.1) and use the above results, then we can get

$$E(T_b) = \left\{\frac{(\lambda/\theta)(1 - S_v^*(\theta)) + R^*(\lambda) - ((1 - S_b^*(\delta))/\delta)(1 + \delta g^{(1)})(\lambda S_v^*(\theta) + \theta p)}{\theta p(R^*(\lambda) - \rho)}\right\} \tag{4.2}$$

$$E(T_c) = \left\{\frac{(\lambda/\theta)(1 - S_v^*(\theta)) + R^*(\lambda)(1 + (\theta p/\lambda)) - S_v^*(\theta)(\lambda(1 - S_b^*(\delta))/\delta)(1 + \delta g^{(1)})}{\theta p(R^*(\lambda) - \rho)}\right\} \tag{4.3}$$

5. Conditional stochastic decomposition property

In this section, we study the stochastic decomposition property of the system size distribution. The number of customers in the system is distributed as the sum of two independent random variables. In particular, in the context our system, we will discuss the conditional stochastic decomposition of the number of customers in the orbit given that the server is busy. Let N_b is the conditional orbit size of our retrial queuing system given that server is busy and N_0 is the conditional orbit size of the M/G/1 retrial queueing system with negative customers is given that the server is busy which is discussed in Theorem 5.1,

Theorem 5.1. *The conditional orbit size N_b is given that the server is busy can be decomposed into the sum of two independent random variables*

$N_b = N_0 + N_c$

where N_0 has the generating function $N_0(z)$ as follows,

$$N_0(z) = \frac{\delta\left(1 - S_b^*(A_b(z))\right)(z-1)\left\{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)\right\}}{\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\}\left(1 - S_b^*(\delta)\right)}$$

and N_c is the additional queue length due to vacations with the probability generating function $N_c(z)$ as follows,

$$N_c(z) = \frac{\left\{\begin{array}{l}(\lambda/\theta)V(z)\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\} \\ +\left(1 - S_b^*(A_b(z))\right)\left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)\left(R^*(\lambda) + z(1 - R^*(\lambda))\right) + z(\lambda V(z) + \theta p)\right\}\end{array}\right\} \times \left(1 - S_b^*(\delta)\right)}{\delta\left(1 - S_b^*(A_b(z))\right)(z-1)\left\{R^*(\lambda)\left((\lambda/\theta)\left(1 - S_v^*(\theta)\right) + \left(1 - S_b^*(\delta)\right)\left(\lambda\left(1 - S_v^*(\theta)\right) + \theta p\right)\right) - \left(\lambda^2/\theta\right)g^{(1)}\left(1 - S_b^*(\delta)\right)\left(1 - S_v^*(\theta)\right)\right\}}$$

Proof. The mathematical version of the stochastic decomposition law is $N_b(z) = N_0(z)N_c(z)$.

In an M/G/1 retrial queueing system with negative customers, the marginal function of the number of customers in the orbit when the server is busy is given by

$$\Phi(z) = \frac{\lambda P_0 R^*(\lambda)\left(1 - S_b^*(A_b(z))\right)(z-1)}{\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\}}$$

and the probability that the server is busy is given by

$$\Phi(1) = \left\{\frac{\lambda P_0 R^*(\lambda)\left(1 - S_b^*(\delta)\right)}{\delta\left\{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)\right\}}\right\},$$

then for the probability generating function $N_0(z)$, we have

$$N_0(z) = \frac{\Phi(z)}{\Phi(1)} = \frac{\delta\left(1 - S_b^*(A_b(z))\right)(z-1)\left\{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)\right\}}{\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\}\left(1 - S_b^*(\delta)\right)}$$

From the Eqs. (3.35) and (3.36), we know that for our retrial system the probability generating function of N_b is given by

$$N_b(z) = \frac{\Pi_b(z) + Q(z)}{\Pi_b(1) + Q(1)}$$

$$= \frac{\left\{\begin{array}{l}(\lambda/\theta)V(z)\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\} + \left(1 - S_b^*(A_b(z))\right) \\ \left\{\left(\lambda\left(S_v^*(A_v(z)) - 1\right) - \theta p\right)\left(R^*(\lambda) + z(1 - R^*(\lambda))\right) + z(\lambda V(z) + \theta p)\right\}\left\{R^*(\lambda) - \left(\lambda\left(1 - S_b^*(\delta)\right)\left(1 + \delta g^{(1)}\right)/\delta\right)\right\}\end{array}\right\}}{\left\{\begin{array}{l}\left\{zA_b(z) - \left(R^*(\lambda) + z(1 - R^*(\lambda))\right)\left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right\} \\ \times\left\{R^*(\lambda)\left((\lambda/\theta)\left(1 - S_v^*(\theta)\right) + \left(1 - S_b^*(\delta)\right)\left(\lambda\left(1 - S_v^*(\theta)\right) + \theta p\right)\right) - \left(\lambda^2/\theta\right)g^{(1)}\left(1 - S_b^*(\delta)\right)\left(1 - S_v^*(\theta)\right)\right\}\end{array}\right\}}$$

$$N_b(z) = N_0(z) \times N_c(z)$$

From above stochastic decomposition law, we observe that $N_b(z) = N_0(z) \times N_c(z)$ which conform that the decomposition results of Gao et al. [25], also valid for this special vacation system. □

6. Special cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

Case (i): *No retrial and Multiple working vacations*
Let $R^*(\lambda) \to 1$, our model can be reduced to an $M/G/1$ G-queue with working vacations and vacation interruption. In this case, $K_s(z)$ can be found as follows,

$$K_s(z) = P_0 \frac{\left\{ \begin{array}{l} (1-z)\left[\left(zA_b(z) - \left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right)((\lambda z V(z)/\theta) + 1)\right] \\ +\left(1 - S_b^*(A_b(z))\right)\left[\left(S_v^*(A_v(z)) - 1\right) + zV(z)\right](zb(z) + \delta(1 - G^*(b(z)))) \end{array} \right\}}{(1-z)\left(zA_b(z) - \left(A_b(z)S_b^*(A_b(z)) + \delta G^*(b(z))\left(1 - S_b^*(A_b(z))\right)\right)\right)}$$

This coincides with the result of Zhang and Liu [26].

Case (ii): *No negative arrival, No vacation interruption and Single working vacation*
Let $(\delta,\theta) \to (0,0)$, our model can be reduced to $M/G/1$ retrial queue with single working vacation. In this case, $K_s(z)$ coincides with the result of Arivudainambi et al. [20] as follows,

$$K_s(z) = \left\{ \frac{S_v^*(\lambda)\left(R^*(\lambda) - \lambda\beta^{(1)}\right)}{\lambda E(S_v) - R^*(\lambda)S_v^*(\lambda)} \right\} \left\{ \frac{\left\{\left[\left(S_v^*(\lambda - \lambda z) - 1\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + (1-z)R^*(\lambda)S_v^*(\lambda)\right]\right\} S_b^*(\lambda - \lambda z)}{S_v^*(\lambda)\left(z - (R^*(\lambda) + z(1 - R^*(\lambda)))S_b^*(\lambda - \lambda z)\right)} \right\}$$

Case (iii): *No retrial, No negative arrival and Multiple working vacation*
Let $\delta \to 0$ and $R^*(\lambda) \to 1$, suppose that there is no retrial time in the system then we get an $M/G/1$ queue with working vacations and vacation interruption. In this case, $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{\left(1 - \lambda\beta^{(1)}\right)\left\{(1-z)\left[\left(z - S_b^*(A_b(z))\right)((\lambda z V(z)/\theta) + 1)\right] + z\left(1 - S_b^*(A_b(z))\right)\left[S_v^*(A_v(z)) + zV(z) - 1\right]\right\}}{\left(1 + (\lambda/\theta)\left(1 - S_v^*(\theta)\right) - \lambda\beta^{(1)}S_v^*(\theta)\right)(1-z)\left(z - S_b^*(A_b(z))\right)}$$

This coincides with the result of Zhang and Hou [23].

Case (iv): *No negative arrival and Multiple working vacations*
Let $\delta = p = 0$, our model can be reduced to a single server retrial queueing system with working vacations. In this case, $K_s(z)$ can be yielded as follows,

$$K_s(z) = P_0 \frac{\left\{ (1-z)\left[\begin{array}{l} \left(z - (R^*(\lambda) + z(1 - R^*(\lambda)))S_b^*(A_b(z))\right)((\lambda V(z)/\theta) + 1) \\ +z(1 - R^*(\lambda))\left[\left(S_v^*(A_v(z)) + V(z)S_b^*(A_b(z))\right) - 1\right] \end{array} \right] + \left(1 - S_b^*(A_b(z))\right)\left[\left(S_v^*(A_v(z)) - 1\right)(R^*(\lambda) + z(1 - R^*(\lambda))) + zV(z)\right] \right\}}{(1-z)\left(z - (R^*(\lambda) + z(1 - R^*(\lambda)))S_b^*(A_b(z))\right)}$$

This coincides with the result of Gao et al. [25].

Case (v): *Single working vacation*
Let $p = 1$, our model can be reduced to a single server retrial queueing system with negative customers, single working vacation and vacation interruption.

Case (vi): *Multiple working vacations*
Let $p = 0$, our model can be reduced to a single server retrial queueing system with negative customers, multiple working vacation and vacation interruption.

7. Cost optimization analysis

In order to carry out cost analysis, the optimum design of a retrial queueing system is to determine the optimal system parameters, such as optimal mean service rate or optimal number of servers (see in [31,32]). In this section, the optimal design of the single server retrial G-queue with working vacations and vacation interruption under Bernoulli schedule is addressed. Based on the definitions of cost elements (C_h, C_o, C_s and C_a) and cost structure listed below, the total expected cost function per unit time is given by

Please cite this article in press as: Rajadurai P et al., Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.03.008

Table 1 Effects of (C_h, C_o) on the expected cost function TC with C_s = \$1000 and C_a = \$100.

(C_h, C_o)	(5, 100)	(5, 110)	(5, 120)	(10, 100)	(15, 100)
TC	301.7446	303.3765	305.0084	303.4891	305.2337

$$TC = C_hL_s + C_o\frac{E(T_b)}{E(T_c)} + C_s\frac{1}{E(T_c)} + C_a\frac{E(T_0)}{E(T_c)}$$
$$= C_hL_s + C_o(1-P_0) + C_s\lambda + C_aP_0$$

where C_h is the holding costs per unit time for each customer present in the system, C_o is the cost per unit time for keeping the server on and in operations, C_s is setup cost per busy cycle and C_a is the startup cost per unit time for the preparation work of the server before starting the service.

If we assume exponential retrial times, service times, working vacation times and repair times then for the following values of the cost elements and other parameters like: $\lambda = 1$; $\mu_b = 5$; $\mu_v = 2$; $a = 2$; $\xi = 3$; $\theta = 3$; $\delta = 0.1$; $p = 0.5$; $C_h = 5$, $C_o = 100$, $C_s = 1000$ and $C_a = 100$, we find the total expected cost per unit of time $\boldsymbol{TC}$ **= 301.7446;** Also, in this case, the steady-state availability of the server $\boldsymbol{A_v}$ **= 98.94%** while the steady-state failure frequency of the server is $\boldsymbol{F_f}$ **= 3.18%.**

Moreover, we can examine the behavior of the expected cost function under different values of the cost parameters. Let us fix the system parameters values as follows: $\lambda = 1$; $\mu_b = 5$; $\mu_v = 2$; $a = 2$; $\xi = 3$; $\theta = 3$; $\delta = 0.1$; $p = 0.5$. Tables 1–3 illustrate the effects of (C_h, C_o), (C_o, C_a) and (C_s, C_a) on the expected cost function, respectively. It can be see that the expected cost function shows a linearly increasing trend with increasing cost parameters.

Similarly, a sensitivity analysis of some of the parameters on the system can be conducted. Fixing the base values given above, one parameter can be varied at a time and the corresponding objective function value can be computed. The graphs (from Figs. 1–4) show the effect of some of the system parameters (λ, δ, a, p) on the total expected cost per unit of time.

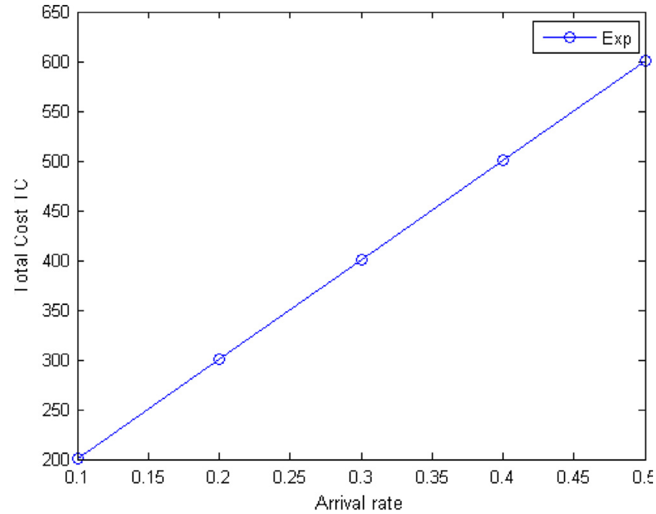


Figure 1 TC versus λ.

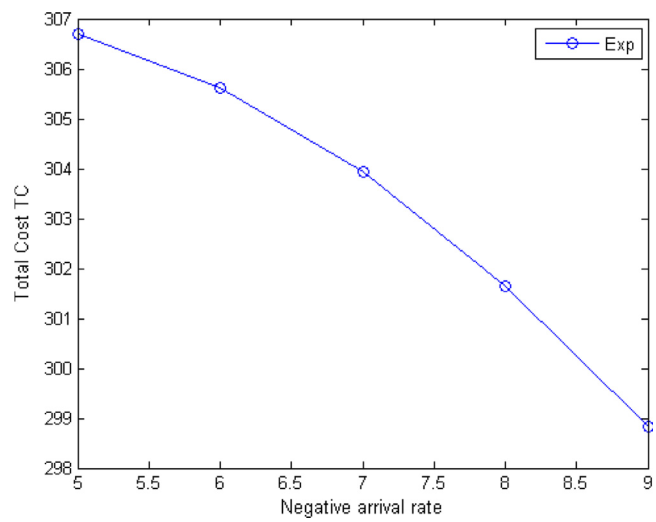


Figure 2 TC versus δ.

8. Numerical examples

In this section, we present some numerical examples to study the effect of various parameters in the system performance measures of our system where all retrial times, service times, working vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied. MATLAB software has been used to illustrate the results numerically. Note that the exponential distribution is $f(x) = \upsilon e^{-\upsilon x}, x > 0$, Erlang-2 stage distribution is $f(x) = \upsilon^2 xe^{-\upsilon x}, x > 0$ and hyper-exponential distribution is $f(x) = c\upsilon e^{-\upsilon x} + (1-c)\upsilon^2 e^{-\upsilon^2 x}, x > 0$.

Table 2 Effects of (C_o, C_a) on the expected cost function TC with C_h = \$5 and C_s = \$1000.

(C_o, C_a)	(100, 100)	(125, 100)	(150, 100)	(100, 110)	(100, 120)
TC	301.7446	305.8243	309.9041	310.1127	318.4808

Table 3 Effects of (C_a, C_s) on the expected cost function TC with C_h = \$5 and C_o = \$100.

(C_a, C_s)	(100, 1000)	(110, 1000)	(120, 1000)	(100, 1050)	(100, 1100)
TC	301.7446	310.1127	318.4808	311.7446	321.7446

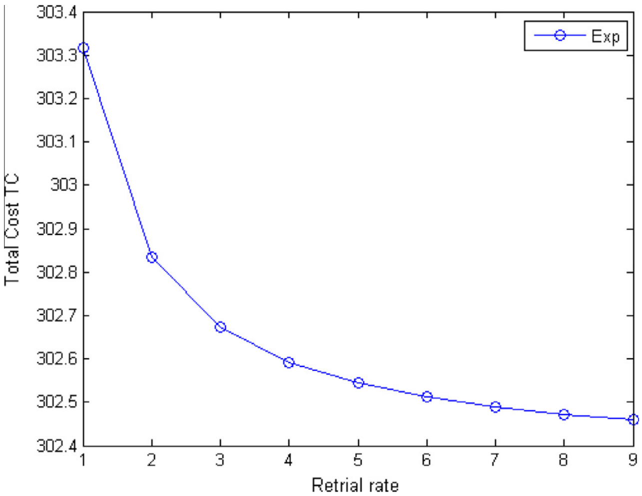


Figure 3 *TC* versus *a*.

Table 4 shows that when retrial rate (a) increases, then the idle probability (P_0) increases, the mean orbit size (L_q) decreases and probability that server is idle during retrial time (P) also decreases for the values of $\lambda = 1$; $\theta = 2$; $\mu = 9$; $p = 0.7$; $\xi = 5$; $\delta = 0.2$; $\mu_v = 4$; $c = 0.7$. Table 5 shows that when negative arrival rate (δ) increases, the mean orbit size (L_q) increases, probability that server is idle during retrial time (P) increases and the servers failure frequency (F_f) also increases for the values of $\lambda = 1$; $\theta = 2$; $\mu = 9$; $p = 0.7$; $\xi = 5$; $a = 2$; $\mu_v = 4$; $c = 0.7$. Table 6 shows that when vacation rate (θ) increases, the idle probability (P_0) increases, then the mean orbit size (L_q) decreases and probability that server is under working vacation (Q_v) also decrease for the values of $\lambda = 1$; $\delta = 0.2$; $\mu = 9$; $p = 0.7$; $\xi = 5$; $a = 2$; $\mu_v = 4$; $c = 0.7$. Table 7 shows that when lower speed service rate (μ_v) increases, the idle probability (P_0) increases, then the mean orbit size (L_q) decreases and probability that server is on working vacation (Q_v) also decrease for the values of $\lambda = 1$; $\delta = 0.2$; $\mu = 9$; $p = 0.7$; $\xi = 5$; $a = 2$; $\theta = 2$; $c = 0.7$.

For the effect of the parameters $\lambda, a, p, \delta, \theta, \mu$, and μ_v on the system performance measures, three dimensional graphs are illustrated in Figs. 5–8. In Fig. 5, the surface displays an upward trend as expected for increasing the value of arrival rate (λ) and negative arrival rate (δ) against the mean orbit size

Table 4 The effect of retrial rate (a) on P_0, Lq and P.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-exponential		
a	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
Retrial rate									
2.00	0.4583	0.3400	0.0715	0.2394	0.7152	0.3276	0.4103	0.4323	0.1187
3.00	0.4715	0.2269	0.0479	0.3033	0.4580	0.2100	0.4322	0.2913	0.0794
4.00	0.4782	0.1699	0.0361	0.3337	0.3438	0.1540	0.4432	0.2207	0.0596
5.00	0.4822	0.1356	0.0289	0.3514	0.2791	0.1215	0.4498	0.1783	0.0477

Table 5 The effect of negative arrival rate (δ) on Lq, P and F_f.

Retrial distribution	Exponential			Erlang-2 stage			Hyper-exponential		
δ	L_q	P	F_f	L_q	P	F_f	L_q	P	F_f
Negative arrival rate									
0.30	0.3575	0.0718	0.0076	0.8118	0.3323	0.0176	0.4570	0.1188	0.0103
0.40	0.3749	0.0721	0.0133	0.9138	0.3369	0.0309	0.4816	0.1189	0.0181
0.50	0.3922	0.0725	0.0206	1.0218	0.3413	0.0476	0.5063	0.1190	0.0279
0.60	0.4094	0.0728	0.0294	1.1363	0.3456	0.0676	0.5308	0.1191	0.0396

Table 6 The effect of vacation rate (θ) on P_0, Lq and Q_v.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-exponential		
θ	P_0	L_q	Q_v	P_0	L_q	Q_v	P_0	L_q	Q_v
Vacation rate									
1.00	0.5421	0.0561	0.0282	0.4355	0.1140	0.0416	0.5163	0.0777	0.0336
2.00	0.6708	0.0558	0.0150	0.5381	0.0992	0.0206	0.6390	0.0720	0.0173
3.00	0.7285	0.0524	0.0095	0.5845	0.0845	0.0124	0.6940	0.0650	0.0107
4.00	0.7612	0.0487	0.0066	0.6111	0.0730	0.0083	0.7253	0.0589	0.0073

Please cite this article in press as: Rajadurai P et al., Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.03.008

Table 7 The effect of lower speed service rate (μ_v) on P_0, Lq and Q_v.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-exponential		
μ_v	P_0	L_q	Q_v	P_0	L_q	Q_v	P_0	L_q	Q_v
Lower speed service rate									
2.00	0.1111	0.3721	0.1389	0.0489	0.8189	0.0916	0.0962	0.4802	0.1383
3.00	0.1184	0.3572	0.1184	0.0540	0.8174	0.0864	0.1035	0.4658	0.1221
4.00	0.1239	0.3473	0.1032	0.0588	0.8053	0.0817	0.1092	0.4592	0.1092
5.00	0.1281	0.3410	0.0915	0.0631	0.7910	0.0773	0.1139	0.4581	0.0988

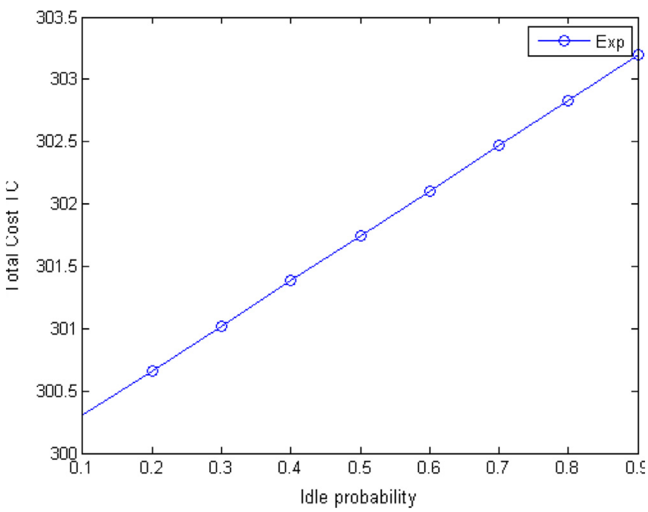


Figure 4 TC versus p.

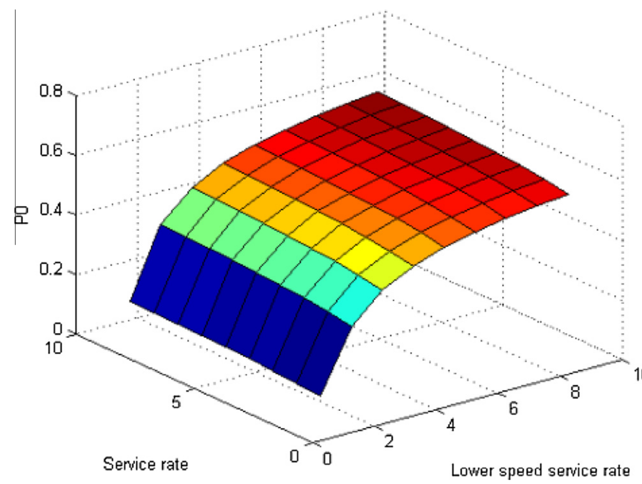


Figure 6 P_0 versus μ_v and μ_b.

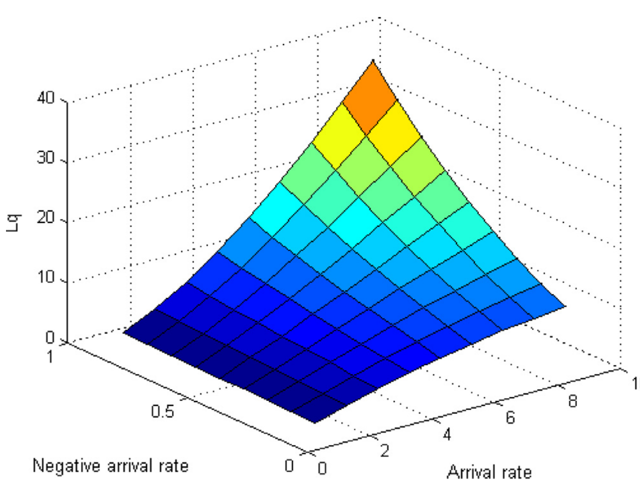


Figure 5 L_q versus λ and δ.

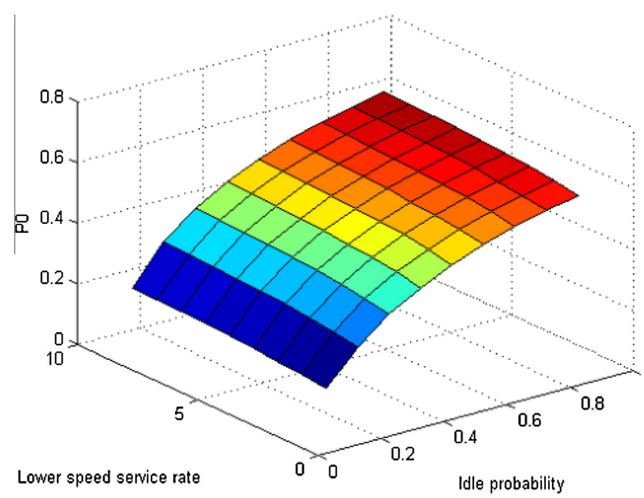


Figure 7 P_0 versus p and θ.

(L_q). In Fig. 6, we examine the behavior of the idle probability (P_0) increases for increasing the value of the lower service rate (μ_v) and regular service rate (μ_b). Fig. 7 shows that the idle probability (P_0) increases for increasing the value of single working vacation idle probability (p) and vacation rate (θ). In Fig. 8, we examine the behavior of the mean orbit size (L_q) decreases for increasing the value of lower speed service rate (μ_v) and retrial rate (a).

From the above numerical examples, we observed that the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

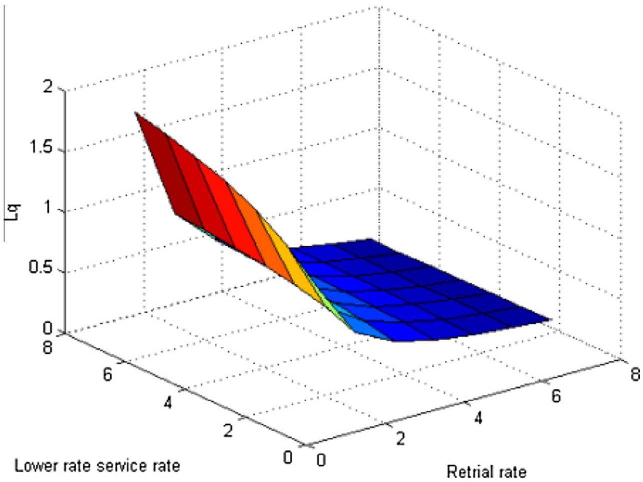


Figure 8 L_q versus μ_v and a.

9. Conclusion

In this work, we have investigated a single server retrial queueing system with negative customers under both single and multiple working vacations and vacation interruption, where the server is subjected to breakdown and repair. The necessary and sufficient condition for the system to be stable is obtained. By using the probability generating function approach and the method of supplementary variable technique, the probability generating functions for the numbers of customers in the system and its orbit when it is free, busy, on working vacation, under repair are derived. Various system's performance measures, reliability measures and conditional stochastic decomposition law are discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples and cost optimization analysis are presented to study the impact of the system parameters and cost elements. The novelty of this investigation is the introduction of both single and multiple working vacations in presence of retrial G-queues and server breakdown. This proposed model has potential practical real life application in production to order system to enhance the performance of the production facility and to stop the production facility from becoming overloaded, in computer processing system and telephone consultation of medical service systems. Moreover, our model can be considered as a generalized version of many existing queueing models (Gao et al. [25] and Zhang and Liu [26]) equipped with many features and associated with many practical situations. Hopefully, this investigation will be great help to the system managers who can design a system with economic management and to make decisions regarding the size of the system and other factors in a well-to-do manner.

Acknowledgments

The authors are thankful to the reviewers for their valuable comments and suggestions for the improvement of the paper.

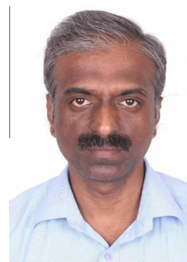
References

[1] Ke JC, Wu CU, Zhang ZG. Recent developments in vacation models: a short survey. Int J Operat Res 2010;7:3–8.
[2] Artalejo J, Gomez-Corral A. Retrial queueing systems. Berlin (Germany): Springer; 2008.
[3] Chao X, Miyazawa M, Pinedo M. Queueing networks: customers, signals and product form solutions. John Wiley and Sons; 1999.
[4] Harrison PG, Marin A. Product forms in multi-way synchronizations. Comput J 2014;57(11):1693–710.
[5] Gelenbe E. Random neural networks with negative and positive signals and product form solution. Neural Comput 1989;1(4):502–10.
[6] Gelenbe E. Product-form queueing networks with negative and positive customers. J Appl Probab 1991;28:656–63.
[7] Harrison PG. Compositional reversed Markov processes, with applications to G-networks. Perform Eval 2004;57:379–408.
[8] Do TV. Bibliography on G-networks, negative customers and applications. Math Comp Model 2011;53:205–12.
[9] Choudhury G, Ke JC. A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. Appl Math Model 2012;36:255–69.
[10] Rajadurai P, Saravanarajan MC, Chandrasekaran VM. Analysis of an $M^{[X]}/(G_1,G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption. Ain Shams Eng J 2014;5(3):935–50.
[11] Rajadurai P, Varalakshmi M, Saravanarajan MC, Chandrasekaran VM. Analysis of $M^{[X]}/G/1$ retrial queue with two phase service under Bernoulli vacation schedule and random breakdown. Int J Math Operat Res 2015;7(1):19–41.
[12] Singh CJ, Jain M, Kumar B. $M^X/G/1$ unreliable retrial queue with option of additional service and Bernoulli vacation. Ain Shams Eng J 2015. http://dx.doi.org/10.1016/j.asej.2015.05.006.
[13] Krishnakumar B, Madheswari SP, Anantha Lakshmi SR. An $M/G/1$ Bernoulli feedback retrial queueing system with negative customers. Oper Res Int J 2013;13:187–210.
[14] Gao S, Wang J. Performance and reliability analysis of an $M/G/1-G$ retrial queue with orbital search and non-persistent customers. Euro J Operat Res 2014;236:561–72.
[15] Peng Y, Liu Z, Wu J. An $M/G/1$ retrial G-queue with preemptive resume priority and collisions subject to the server breakdowns and delayed repairs. J Appl Math Comput 2014;44:187–213.
[16] Rajadurai P, Chandrasekaran VM, Saravanarajan MC. Steady state analysis of batch arrival feedback retrial queue with two phases of service, negative customers, Bernoulli vacation and server breakdown. Int J Math Operat Res 2015;7(5):519–46.
[17] Rajadurai P, Chandrasekaran VM, Saravanarajan MC. Analysis of an $M^{[X]}/G/1$ unreliable retrial G-queue with orbital search and feedback under Bernoulli vacation schedule. OPSEARCH 2015. http://dx.doi.org/10.1007/s12597-015-0226-5.
[18] Servi LD, Finn SG. $M/M/1$ queues with working vacations. Perform Eval 2002;50:41–52.
[19] Wu D, Takagi H. M/G/1 queue with multiple working vacations. Perform Eval 2006;63:654–81.
[20] Arivudainambi D, Godhandaraman P, Rajadurai P. Performance analysis of a single server retrial queue with working vacation. OPSEARCH 2014;51:434–62.
[21] Liu Z, Song Y. Geo/Geo/1 retrial queue with non-persistent customers and working vacations. J Appl Math Comput 2013;42:103–15.
[22] Li J, Tian N. The M/M/1 queue with working vacations and vacation interruption. J Syst Sci Syst Eng 2007;16:121–7.
[23] Zhang M, Hou Z. M/G/1 queue with single working vacation. J Appl Math Comput 2012;39:221–34.
[24] Gao S, Liu Z. An $M/G/1$ queue with single working vacation and vacation interruption under Bernoulli schedule. Appl Math Model 2013;37:1564–79.

[25] Gao S, Wang J, Li W. An $M/G/1$ retrial queue with general retrial times, working vacations and vacation interruption. Asia–Pacific J Oper Res 2014;31:6–31.
[26] Zhang M, Liu Q. An $M/G/1$ G-queue with server breakdown, working vacations and vacation interruption. OPSEARCH 2015;52:256–70.
[27] Rajadurai P, Yuvarani S, Saravanarajan MC. Performance analysis of preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption. Songkl J Sci Tech [in press]. Available from: http://rdo.psu.ac.th/sjstweb/ArticleInPress.php.
[28] Rajadurai P, Saravanarajan MC, Chandrasekaran VM. Analysis of an $M/G/1$ retrial queue with balking, negative customers, working vacations and server breakdown. Int J Appl Eng Res 2015;10(55):4130–5.
[29] Pakes AG. Some conditions for Ergodicity and recurrence of Markov chains. Operat Res 1969;17:1058–61.
[30] Sennott LI, Humblet PA, Tweedi RL. Mean drifts and the non-Ergodicity of Markov chains. Operat Res 1983;31:783–9.
[31] Tadj L, Chodhury G. Optimal design and control of queues. Top 2005;13:359–414.
[32] Yang DY, Wu CH. Cost-minimization analysis of a working vacation queue with N-policy and server breakdowns. Comp Indust Eng 2015;82:151–8.

P. Rajadurai holds his Master of Science degree in Mathematics from Anna University, Chennai. He has completed his Master of Philosophy in VIT University. At present, he is doing his Ph.D. research program under the guidance of Dr. V.M. Chandrasekaran and Dr. M.C. Saravanarajan in School of Advanced Sciences at VIT University. He has published more than 25 research papers in international journals and international conferences. His area of specialization is Stochastic processes and its applications, in particular Queueing theory.

V.M. Chandrasekaran is working as Senior Professor in Department of Mathematics, School of Advanced Sciences, VIT University. He was awarded Ph.D. degree by Madras University in the year 1997. He has 20 years of teaching and research experience. His area of research is algebra and queueing theory. He has published more than 60 research articles in reputed national and international journals. He has guided four Ph.D. and five M.Phil. scholars. At present, He is guiding six Ph.D scholars.



M.C. Saravanarajan is working as an Assistant Professor (Senior) in Department of Mathematics, School of Advanced Sciences, VIT University. He has 14 years of teaching experience. He has completed his Ph.D. degree in VIT University under the guidance of Dr. V.M. Chandrasekaran in queueing theory. He has published more than 30 research papers in international journals and international conferences. He is currently guiding four Ph.D scholars. His area of specialization is Stochastic processes and its applications, in particular Queueing theory.