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Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule

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Abstract In this paper, we consider a single server retrial queueing system with working vacations. Further vacation interruption is considering with the regular busy server is subjected to breakdown due to the arrival of negative customers. When the orbit becomes empty at the time of service completion for a positive customer, the server goes for a working vacation. The server works at a lower service rate during working vacation (WV) period. If there are customers in the system at the end of each vacation, the server becomes idle and ready for serving new arrivals with probability p (single WV) or it remains on vacation with probability q (multiple WVs). By using the supplementary variable technique, we found out the steady state probability generating function for the system and its orbit. System performance measures, reliability measures and stochastic decomposition law are discussed. Finally, some numerical examples and cost optimization analysis are presented.
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1. Introduction

In queueing theory, vacation queues and retrial queues are active research topics for a long time. We can find general models in vacation queues and retrial queues from Ke et al. [1] and Artalejo and Gomez-Corral [2] respectively. In retrial queueing system, queues with repeated attempts are characterized by an arriving customer who finds the server busy, leaves the service area and repeats its demand after some time. Between trials, the blocked customer joins a pool of unsatisfied customers called orbit. An arbitrary customer in the orbit who repeats the request for service is independent of the rest of the customers in the orbit. Such a retrial queue plays a special role in computers, telecommunication systems, communication protocols and retail shopping queues, etc.

For the last two decades, many researchers studied queueing networks with the concept of positive and negative customers. Queues with negative customers (called G-queues) have concerned huge interests due to their extensive applications in computers, communication networks, neural networks and manufacturing systems [3,4]. The named G-queue has been adopted for the queue with negative customers in the

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Acknowledgment of Gelenbe, who first introduced this type of queue in [5,6]. Harrison [7] has studied the idea of compositional reversed Markov processes with applications to G-networks. The positive customer arrives into the system and gets service as ordinary queuing customers, but the negative customers enter into the system only at the service time of positive customers. This type of negative customers removes the positive customers being in service from the system and the server breakdown and the service channel will fail for a short interval of time. When the server fails, it will be sent to repair immediately. After completion of repair, the server will treat as good as new. Do [8] has presented a survey on queuing systems with G-networks, negative customers and applications. Choudhry and Ke [9], Rajadurai et al. [10,11] and Singh et al. [12] have discussed the retrial queue with the concept of breakdown and repair. Recently, Krishnakumar et al. [13], Gao and Wang [14], Peng et al. [15] and Rajadurai et al. [16,17] have discussed different types of queuing models operating with the simultaneous presence of negative arrivals.

In working vacation period (WV), the server gives service to customer at lower service rate, but the server stops the service completely in the normal vacation period. This queuing system has major applications in providing network service, web service, file transfer service and mail service, etc. In 2002, Servi and Finn [18] have introduced an $M/M/1$ queueing system with working vacations. Wu and Takagi [19] have extended the $M/M/1/WV$ queue to an $M/G/1/WV$ queue. Very recently, Arivudainambi et al. [20] have introduced $M/G/1$ retrial queue with single working vacation. Liu and Song [21] have discussed a discrete time retrial queue with non-persistent customers and working vacations. Furthermore if there are customers in the system at the end of a lower speed service, the server can stop the vacation and make the normal busy state. This policy is called vacation interruption. Li and Tian [22] have presented an $M/M/1$ queueing model with working vacations and vacation interruption. Some of the authors such as, Zhang and Hou [23], Gao and Liu [24], Gao et al. [25], Zhang and Liu [26], Rajadurai et al. [27,28] have analyzed a single server retrial queue with working vacations and vacation interruptions.

In this paper, we have extended the work of Gao et al. [25] and Zhang and Liu [26] by incorporating the concept of negative customers (G-queues) in both single and multiple working vacations with breakdowns and repair. To the author's best of knowledge, there is no work published in the queuing literature with the combination of retrial queuing system with general retrial times, negative customers, single and multiple working vacations, vacation interruption and breakdowns by using the method of supplementary variable technique. The mathematical results and theory of queues of this model provide to serve a specific and convincing application in the computer processing system. Our model is helpful to managers who can design a system with economic management.

The remainder of this work is given as follows. The detailed mathematical description and practical applications of our model are given in Section 2. The steady state joint distribution of the server state and the number of customers in the system and its orbit are obtained in Section 3. Some system performance measures, the mean busy period, the mean busy cycle and reliability measures are obtained in Section 4. In Section 5, conditional stochastic decomposition is shown good for our model. Important special cases of this model are given in Section 6. Cost optimization analysis is discussed in Section 7. In Section 8, the effects of various parameters on the system performance are analyzed numerically. Conclusion and summary of the work are presented in Section 9.

2. Description of the model

In this section, we consider a single server retrial queuing system with both single and multiple working vacations and vacation interruption, where the regular busy server is subjected to breakdown due to the arrival of negative customers. The detailed description of our model is given as follows:

- **The arrival process**: Customers arrive at the system according to a Poisson process with rate $\lambda$.
- **The retrial process**: If an arriving positive customer finds that the server is free, the customer begins his service immediately. Otherwise, the server is busy or working vacation or breakdown, the arrivals join pool of blocked customers called an orbit in accordance with FCFS discipline, which means that only one customer at the head of the orbit queue is allowed to access the server. We assume that inter-retrial times follow a general random variable $R$ with an arbitrary distribution $R(t)$ having corresponding Laplace Steltjes Transform (LST) $R'(\phi)$.
- **The Bernoulli working vacation process**: The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter $\theta$. During a vacation period if any customer arrives, the server gives service at a lower speed service rate. If any customers in the orbit at a lower speed service completion instant in the vacation period then the server will stop the vacation and come back to the normal busy period which means that vacation interruption happens. If no customers are in the system at the end of the vacation, the server either remains idle to serve a new customer with probability $p$ (single working vacation) in regular mode or leaves for another working vacation with probability $q = 1 - p$ (multiple working vacation). When a vacation ends and if there are customers in the orbit then the server switches to the normal working level. During the working vacation period, the service time follows a general random variable $S_i$ with distribution function $S_i(t)$ having LST $S_i^*(\theta)$ and the first moment is given by $S_i'(\theta) = \int_0^\infty xe^{-\theta x}dS_i(x)$.
- **The regular service process**: Whenever a new positive customer or retry positive customer arrives at the server idle state then the server immediately starts normal service for the arrivals. The service time follows a general distribution and it is denoted by the random variable $S_i$ with distribution function $S_i(t)$ having LST $S_i^*(\theta)$ and the first moment is given by $S_i'(\theta) = \int_0^\infty xe^{-\theta x}dS_i(x)$.
- **The removal rule and the repair process**: The negative customers arrive from outside the system according to a Poisson arrival rate $\delta$. These negative customers arrive only at the regular service time of the positive customers. Negative customers cannot accumulate in a queue and do not receive service, will remove the positive customers being in service from the system. These types of negative customers cause the server breakdown and the service channel will fail for
a short interval of time. When the server fails, it will be sent to repair immediately. After completion of repair, the server will treat as good as new. The repair time (denoted by \( G \)) of the server is assumed to be arbitrarily distributed with distribution function \( G(t) \) having LST \( G^*(\theta) \) and the first and second moments are denoted by \( g^{(1)} \) and \( g^{(2)} \).

- We assume that all the random variables (inter-arrival times, retrial times, service times, working vacation times and repair times) defined above are independent of each other.

2.1. Practical application of the proposed model

The potential practical application of this model is in the operational model of stochastic production and inventory systems with a multipurpose production facility. The production facility performs other additional tasks using the time between succeeding productions. We consider a practical problem related to a production to order system. In production order system, the customer orders for the product (positive customers) and some customers cancel orders (negative customers) due to financial crisis or disaster, etc. Let us assume that order system follows the Poisson processes. At the time of arriving customer orders the production facility is busy with other order, and a new arriving order will form a waiting line which corresponds to the retrial queue. Otherwise, the order is processed immediately. After the completion of order, the management policy sets up the facility and begins product to the next order on the list; otherwise, another external order arrives before the order is made. The order time is assumed to be generally distributed (general retrial time). During the service time of ordering, if any canceling order exists then the major production will be stopped (i.e. server breakdown). To enhance the production facility performance, the management policy is to set up the optional job facility (working vacation). After completion of service for last order and there is no order in system (system empty), the facility stops the major production and is available to perform optional jobs (single working vacation) with lower production rate (lower speed service rate). After the result of disaster or an optional job completion, if there are no orders, the production facility will continue to perform the optional jobs at lower production rate (multiple working vacations). Otherwise, it performs the major production. If there are orders at the instant of the optional jobs completion, the production facility will perform the major production (vacation interruption). This type of system is very useful to increase the performance of the production facility and to stop the production facility from becoming overloaded.

Another practical application of this model is in the area of computer processing system. In a computer processing system, the buffer size (orbit) used to store messages is finite and the messages (customers) arrive into the system one by one, and the processor (server) is in charge of processing messages. The working mail server may be affected by virus (negative customers), and the system may be subjected to electronic fails (breakdowns) during service period and receive repair immediately. If the processor is available indicating that it is not currently working on a task and then a message is processed. The messages are temporarily stored in a buffer to be served some time later (retrial time) according to FCFS if the processor is unavailable. To enhance the computer performance, whenever all messages are processed and no new messages arrive, the processor will perform a sequence of maintenance jobs, such as virus scan (working vacations). During the maintenance period, the processor can deal with the messages at the slower rate to economize the cost (working vacation period). Upon completion of the each maintenance, the processor checks the messages and decides whether or not to resume the normal service rate (single working vacation). At this moment, if no message is in the system then the processor may decide to go for another maintenance activity (multiple working vacations). This type of working vacation discipline is a good approximation of such computer processing system.

3. Steady state analysis

In this section, we develop the steady state difference–differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times, the elapsed working vacation times and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states and for the number of customers in the system and orbit.

3.1. The steady state equations

In steady state, we assume that \( R(0) = 0, \ R(\infty) = 1, \ S_b(0) = 0, \ S_b(\infty) = 1, \ S_v(0) = 0, \ S_v(\infty) = 1, \ G(0) = 0, \ G(\infty) = 1 \) are continuous at \( x = 0 \). So that the function \( a(x), \mu_b(x), \mu_v(x) \) and \( \xi(x) \) are the conditional completion rates (hazard rate) for retrial, normal service, lower rate service and repair respectively.

\[
\begin{align*}
0, & \quad \text{if the server is free and in working vacation period}, \\
1, & \quad \text{if the server is free and in regular service period}, \\
2, & \quad \text{if the server is busy and in regular service period at time } t, \\
3, & \quad \text{if the server is busy and in working vacation period at time } t, \\
4, & \quad \text{if the server is under repair period at time } t.
\end{align*}
\]
Thus the supplementary variables $R^i(t)$, $S^i_0(t)$, $S^i_1(t)$ and $G^i(t)$ are introduced in order to obtain a bivariate Markov process $(X(t), N(t); t \geq 0)$, where $X(t)$ denotes the server state and the server is free on both regular busy period and working vacation period, regular busy, on working vacation and under repair. If $X(t) = 1$ and $N(t) > 0$, then $R^i(t)$ represent the elapsed retial time, if $X(t) = 2$ and $N(t) \geq 0$ then $S^i_0(t)$ corresponding to the elapsed time of the customer being served in regular busy period. If $X(t) = 3$ and $N(t) > 0$, then $S^i_1(t)$ corresponding to the elapsed time of the customer being served in lower rate service period. If $X(t) = 4$ and $N(t) \geq 0$, then $G^i(t)$ corresponding to the elapsed time of the server being repaired.

We analyze the ergodicity of the embedded Markov chain at departure, vacation or repair epochs. Let $\{t_n; n = 1, 2, \ldots\}$ be the sequence of epochs of either normal service completion times or working vacation completion times or repair period ends. The sequence of random vectors $Z_n = (X(t_n)+, N(t_n)+)$ forms a Markov chain which is embedded in the retrial queueing system.

**Theorem 3.1.** The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < R^i(\lambda)$ for our system to be stable, where $\rho = (\lambda/\delta)(1 - S^i_0(\delta))(1 + \delta g^{(1)})$.

**Proof.** To prove the sufficient condition of ergodicity, it is very convenient to use Foster’s criterion (see Pakes [29]), which states that the chain $\{Z_n; n \in N\}$ is irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j), j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{j+1}) - f(z_j)]/\varepsilon$, is finite for all $j \in N$ and $\psi_j \leq -\varepsilon$ for all $j \in N$, except perhaps for a finite number $J$.

In our case, we consider the function $f(j) = j$. Then we have

$$\psi_j = \begin{cases} \rho - 1, & \text{if } j = 0, \\ \rho - R^i(\lambda), & \text{if } j = 1, 2, \ldots \end{cases}$$

Clearly the inequality $\rho < R^i(\lambda)$ is sufficient condition for Ergodicity.

To prove the inequality condition, as noted in Sennott et al. [30], if the Markov chain $\{Z_n; n \geq 1\}$ satisfies Kaplan’s condition, namely, $\psi_j < \infty$ for all $j > 0$ and there exits $j_0 \in N$ such that $\psi_j \geq 0$ for $j \geq j_0$. Notice that, in our case, Kaplan’s condition is satisfied because there is a $k$ such that $m_j = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n; n \in N\}$. Then $\rho > R^i(\lambda)$ implies the non-Ergodicity of the Markov chain.

Let us define the limiting probabilities $P_0(t) = P\{X(t) = 0, N(t) = 0\}; P_1(t) = P\{X(t) = 1, N(t) = 0\}$ and the probability densities are

$$P_n(x, t)dx = P\{X(t) = 1, N(t) = n, x \leq R^i(t) < x + dx\},$$

for $t \geq 0$, $x > 0$ and $n \geq 1$.

$$\Pi_{n,0}(x, t)dx = P\{X(t) = 2, N(t) = n, x \leq S^i_0(t) < x + dx\},$$

for $t \geq 0$, $x > 0$, $n \geq 0$.

$$Q_{n,0}(x, t)dx = P\{X(t) = 3, N(t) = n, x \leq S^i_1(t) < x + dx\},$$

for $t \geq 0$, $x > 0$ and $n \geq 0$.

$$R_n(x, t)dx = P\{X(t) = 4, N(t) = n, x \leq G^i(t) < x + dx\},$$

for $t \geq 0$, $x > 0$ and $n \geq 0$.

We assume that the stability condition is fulfilled in the sequel and so that we can set $Q_0 = \lim_{t \to \infty} Q_n(t)$, $P_0 = \lim_{t \to \infty} P_n(t)$ and limiting densities for $t \geq 0$, $x \geq 0$ and $n \geq 1$.

$$P_n(x) = \lim_{t \to \infty} P_n(x, t), \quad \Pi_{n,0}(x) = \lim_{t \to \infty} \Pi_{n,0}(x, t),$$

$$Q_{n,0}(x) = \lim_{t \to \infty} Q_{n,0}(x, t) \quad \text{and} \quad R_n(x) = \lim_{t \to \infty} R_n(x, t).$$

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$\dot{\lambda}P_0 = 0$$

(3.1)

$$(\lambda + \theta)Q_0 = 0, n \geq 1$$

(3.2)

$$\dot{\Pi}_{n,0} + (\lambda + \delta + \mu_1)\Pi_{n,0} = 0, n = 0$$

(3.3)

$$\dot{Q}_{n,0} + (\lambda + \delta + \mu_1)Q_{n,0} = 0, n = 0$$

(3.4)

$$\dot{\Pi}_{n,0} + (\lambda + \delta + \mu_1)\Pi_{n,0} = \lambda \Pi_{n-1,0}(x), n \geq 1,$$

(3.5)

$$\dot{Q}_{n,0} + (\lambda + \theta + \mu_1)Q_{n,0} = \lambda Q_{n-1,0}(x), n \geq 1$$

(3.6)

$$\dot{R}_0 + (\lambda + \xi)R_0 = 0, n = 0$$

(3.7)

$$\dot{R}_n + (\lambda + \xi)R_n = \lambda R_{n-1}(x), n \geq 1.$$
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\[ R_n(0) = \delta \int_0^\infty \Pi_b h(x)dx, \quad n \geq 0 \quad (3.14) \]

The normalizing condition is

\[ P_0 + Q_0 + \sum_{n=1}^{\infty} \int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty} \int_0^\infty \Pi_b h(x)dx \]
\[ + \int_0^\infty Q_n(x)dx + \int_0^\infty R_n(x)dx = 1 \quad (3.15) \]

3.2. The steady state solution

The steady state solution of the retrial queueing model is obtained by using the probability generating function technique. To solve the above equations, the PGFs are defined for \( |z| \leq 1 \) as follows:

\[ P(x, z) = \sum_{n=0}^{\infty} P_n(z)x^n; \quad P(0, z) = \sum_{n=0}^{\infty} P_n(0)x^n; \]
\[ \Pi_b(z) = \sum_{n=0}^{\infty} \Pi_b h(n)x^n; \quad \Pi_b(0, z) = \sum_{n=0}^{\infty} \Pi_b h(n)x^n; \]
\[ Q_n(x, z) = \sum_{n=0}^{\infty} Q_n h(n)x^n; \quad Q_n(0, z) = \sum_{n=0}^{\infty} Q_n h(n)x^n; \]
\[ R(x, z) = \sum_{n=0}^{\infty} R_n h(n)x^n \quad \text{and} \quad R(0, z) = \sum_{n=0}^{\infty} R_n(0)x^n \]

On multiplying the Eqs. (3.2)–(3.14) by \( z^n \) and summing over \( n \), \( (n = 0, 1, 2, \ldots) \), we get

\[ \frac{\partial P(x, z)}{\partial x} + (\lambda + a(x))P(x, z) = 0 \quad (3.16) \]
\[ \frac{\partial \Pi_b(z)}{\partial x} + (\lambda(1 - z) + \delta + \mu_b h(x))\Pi_b(z) = 0, \quad (3.17) \]
\[ \frac{\partial Q_n(z, x)}{\partial x} + (\lambda(1 - z) + \theta + \mu_h x)Q_n(z, x) = 0 \quad (3.18) \]
\[ \frac{\partial R(x, z)}{\partial x} + (\lambda(1 - z) + \xi(x))R(x, z) = 0 \quad (3.19) \]

\[ P(0, z) = \int_0^\infty \Pi_b h(x)P_0(x)dx + \int_0^\infty Q_1 h(x)P_0(x)dx \]
\[ + \int_0^\infty R h(x)Q_0 h(x)dx - ((\lambda + \theta)Q_0 - \theta P_0) \quad (3.20) \]
\[ \Pi_b(0, z) = \frac{1}{z} \int_0^\infty P(x, z)a(x)dx + \lambda \int_0^\infty P(x, z)dx \]
\[ + \theta \int_0^\infty Q_1 h(x)dx, \quad (3.21) \]
\[ Q_1(0, z) = \lambda Q_0 \quad (3.22) \]
\[ R(0, z) = \delta \int_0^\infty \Pi_b h(x)dx \quad (3.23) \]

Solving the partial differential Eqs. (3.20)–(3.23), it follows that

\[ P(x, z) = \frac{Q(x, z)[1 - R(x)]e^{-\lambda x}}{z} \quad (3.24) \]
\[ \Pi_b h(x) = \frac{P(0, x)[1 - S_b h(x)]e^{-\lambda x}}{\lambda}, \quad (3.25) \]
\[ Q_n h(x) = \frac{Q_n(0, x)[1 - S_n h(x)]e^{-\lambda x}}{\lambda}, \quad (3.26) \]

\[ R(x, z) = \frac{R(0, x)[1 - G(x)]e^{-\lambda x}}{z} \quad (3.27) \]

where \( A_b = (\delta + \lambda(1 - z)), \quad A_1 = (\theta + \lambda(1 - z)) \) and \( b(z) = \lambda \beta(1 - z). \)

Inserting Eqs. (3.24)–(3.27) in (3.21) and make some manipulation, finally we get,

\[ \Pi_b h(0, z) = \frac{P(0, z)[1 - G(z)]e^{-\lambda x}}{z} \quad (3.28) \]

Using Eqs. (3.25)–(3.27) in (3.20), we get

\[ P(0, z) = \Pi_b h(0, z)S_n h(A_b(0)) + Q_1(0, z)S_n h(A_1(0)) \]
\[ + R(0, z)G(z) - (\lambda + \theta)Q_0 \quad (3.29) \]

Using Eqs. (3.25)–(3.27) in (3.23), we get

\[ R(0, z) = \delta \Pi_b h(0, z) \left( \frac{1 - S_n h(A_b(0))}{A_b(0)} \right) \quad (3.30) \]

Using Eqs. (3.22), (3.28), and (3.30) in (3.29), we get

\[ P(0, z) = \frac{N_r(z)}{Dr(z)} \quad (3.31) \]

\[ N_r(z) = zQ_0 \left\{ \left( -\lambda S_n h(A_b(0)) - 1 \right) - \theta P \right\} \]
\[ + \lambda V(z) \theta \left( \frac{S_n h(A_b(0))}{A_b(0)} \right) \quad (3.32) \]

Using Eq. (3.21) in (3.28), we get

\[ \Pi_b h(0, z) = \frac{\theta P(1 - S_n h(A_b(0))}{\lambda} \left\{ \left( -\lambda S_n h(A_b(0)) - 1 \right) - \theta P \right\} \]
\[ + z(\lambda V(z) + \theta P) / Dr(z) \quad (3.33) \]

Using Eq. (3.32) in (3.30), we get

\[ R(0, z) = \delta \Pi_b h(0, z) \left( \frac{1 - S_n h(A_b(0))}{A_b(0)} \right) \]
\[ \times \lambda V(z) + \theta P) / Dr(z) \quad (3.34) \]

Using the Eqs. (3.22) and (3.31)–(3.33) in (3.24)–(3.27), then we get the results for the following PFGs \( P(x, z), \Pi_b h(x, z), Q_1 h(x, z) \) and \( R(x, z) \). Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

**Theorem 3.2.** The marginal probability distributions of the number of customers in the orbit when server being idle, busy, on working vacation and under repair are given by

\[ P(z) = \frac{N_r(z)}{Dr(z)} \]
\[ N_r(z) = zQ_0 \left( \frac{1 - R(z)}{\lambda} \right) \left\{ \left( -\lambda S_n h(A_b(0)) - 1 \right) - \theta P \right\} \]
\[ + (\lambda V(z) + \theta P) \left( \frac{S_n h(A_b(0))}{A_b(0)} \right) \left( \frac{\delta G(z)}{z} \left( 1 - S_n h(A_b(0)) \right) \right) \]
\[ Dr(z) = \left( z - (R(z) + z(1 - R(z)) \right) \left( \frac{S_n h(A_b(0))}{A_b(0)} \right) \left( \frac{\delta G(z)}{z} \left( 1 - S_n h(A_b(0)) \right) \right) \]

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\[\Pi_{k}(z) = Q_{b}(1 - S_{k}(A_{k}(z))) \left\{ (\lambda + z(1 - R^{*}(\lambda))) \right\} / A_{k}(z) \times Dr(z) \] (3.35)

\[Q_{r}(z) = (\lambda Q_{b}V(z)/\theta) \] (3.36)

where

\[Q_{b} = \frac{R^{*}(\lambda) - \rho}{(\lambda/\theta)(1 - S_{c}(\theta)) + R^{*}(\lambda)(1 + (\theta p/\lambda)) - \rho S_{c}(\theta)} \] (3.38)

\[P_{b} = \frac{\theta p(R^{*}(\lambda) - \rho}{\lambda(\lambda/\theta)(1 - S_{c}(\theta)) + R^{*}(\lambda)(1 + (\theta p/\lambda)) - \rho S_{c}(\theta)} \] (3.39)

\[\rho = (\lambda/\theta)(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \]; \[A_{b}(z) = (\delta + \lambda(1 - z));\]

\[A_{r}(z) = (\theta + \lambda(1 - z)) \quad \text{and} \quad b(z) = \lambda(1 - z) \]

**Proof.** The PGF of the number of customers in the system \((K_{b}(z))\) and orbit \((K_{r}(z))\) are obtained by using\(K_{b}(z) = P_{b} + Q_{b} + P(z) + z(P_{b}(z) + Q_{b}(z) + R(z)) \) and \(K_{r}(z) = P_{b} + Q_{b} + p(z) + \Pi_{b}(z) + Q_{b}(z) + R(z). \) Substituting the Eqs. (3.34)–(3.37) in the above results, then the Eqs. (3.40) and (3.41) can be obtained by direct calculation. □

4. **System performance measures**

In this section, we derive some system probabilities, mean number of customers in the system and its orbit, reliability analysis, mean busy period and mean busy cycle of this model.

4.1. **System state probabilities**

From Eqs. (3.34)–(3.37), by setting \(z \to 1\) and applying l'Hôpital's rule whenever necessary, then we get the following results,

(i) The probability that the server is idle during the retrial

\[
P = Q_{b} \left( 1 - R^{*}(\lambda) \right) \\
\times \left\{ \lambda(1 - S_{c}(\theta)) + (\lambda(1 - S_{c}(\theta)) + \theta p) ((1 - S_{c}(\theta))(1 + \delta g^{(1)})/\delta) \right\} \\
\times \left\{ (\lambda/\theta)(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\}
\]

(ii) The probability that the server is regular busy,

\[
\Pi_{b} = Q_{b} \left( 1 - S_{c}(\theta) /\delta \right) \left\{ (\lambda/\theta)(1 - S_{c}(\theta)) + (\lambda(1 - S_{c}(\theta)) + \theta p) R^{*}(\lambda) \right\}
\]

**Theorem 3.3.** The probability generating function of number of customers in the system and orbit size distribution at stationary point of time is

\[
K_{b}(z) = \frac{N_{r}(z)}{Dr(z)} = P_{b} + Q_{b} + P(z) + z(\Pi_{b}(z) + Q_{b}(z)) + R(z)
\] (3.40)

\[
N_{r}(z) = Q_{b} \left\{ (1 - z) \left\{ (\lambda/\theta)(1 + z(1 - R^{*}(\lambda))) \left\{ A_{b}(z)(1 - S_{c}(A_{b}(z))) + \delta G^{*}(b(z))(1 - S_{b}(A_{b}(z))) \right\} \right\} + \right\}
\]

\[
\times \left\{ (\lambda(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\} \left\{ \lambda(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\}
\]

\[Dr(z) = b(z) \times \left\{ (\lambda(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\}
\]

\[
K_{r}(z) = \frac{N_{r}(z)}{Dr(z)} = P_{b} + Q_{b} + P(z) + \Pi_{b}(z) + Q_{b}(z) + R(z).
\] (3.41)

\[
N_{r}(z) = Q_{b} \left\{ (1 - z) \left\{ (\lambda/\theta)(1 + z(1 - R^{*}(\lambda))) \left\{ A_{b}(z)(1 - S_{c}(A_{b}(z))) + \delta G^{*}(b(z))(1 - S_{b}(A_{b}(z))) \right\} \right\} + \right\}
\]

\[
\times \left\{ (\lambda(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\} \left\{ \lambda(1 - S_{c}(\theta)) + (1 + \delta g^{(1)}) \right\}
\]

where \(Q_{b}\) is given in Eq. (3.38).
Analysis of an unreliable retrial G-queue

If the system is in steady state condition, then we can get

$$Q_q = \{\lambda Q_q (1 - S_q (\theta)) / \theta\}$$

(i) The expected number of customers in the orbit ($L_q$) is obtained by differentiating (3.41) with respect to $z$ and evaluating at $z = 1$

$$L_q = K_q (1) = \lim_{\gamma \to 1} \frac{\partial}{\partial z} K_q (z) = Q_q \left[ \frac{N_q (1) D_q (1) - D_q (1) N_q (1)}{3 D_q (1)} \right]$$

(ii) The expected number of customers in the orbit ($L_q$) is obtained by differentiating (3.40) with respect to $z$ and evaluating at $z = 1$

$$L_q = K_q (1) = \lim_{\gamma \to 1} \frac{\partial}{\partial z} K_q (z) = Q_q \left[ \frac{N_q (1) D_q (1) - D_q (1) N_q (1)}{3 D_q (1)} \right]$$

4.2 Mean system size and orbit size

(i) The expected number of customers in the orbit ($L_q$) is obtained by differentiating (3.41) with respect to $z$ and evaluating at $z = 1$

$$L_q = K_q (1) = \lim_{\gamma \to 1} \frac{\partial}{\partial z} K_q (z) = Q_q \left[ \frac{N_q (1) D_q (1) - D_q (1) N_q (1)}{3 D_q (1)} \right]$$

4.3 Reliability measures

In the queuing system with unreliable server, the reliability measures will provide the information which is required for the improvement of the system. To justify and validate the analytical results of this model, the availability measure ($A_v$) and failure frequency ($F_q$) are obtained as follows:

(i) The steady state availability $A_v$, which is the probability that the server is either working for a positive customer or in an idle period such that the steady state availability of the server is given by

$$A_v = 1 - \lim_{\gamma \to 1} (R (\gamma)) = 1 - R (1)$$

(ii) The steady state failure frequency is obtained as

$$F_q = \delta \times I_q (1)$$

4.4 Mean busy period and busy cycle

Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions. The results follow directly by applying the argument of an alternating renewal process [9] which leads to

$$P_0 = \frac{E(T_b)}{E(T_c) + E(T_0)}; E(T_c) = \frac{1}{\lambda} (\frac{1}{P_0} - 1)$$

and

$$E(T_c) = \frac{1}{\lambda P_0} = E(T_c) + E(T_0)$$

4.5 Conditional stochastic decomposition property

In this section, we study the stochastic decomposition property of the system size distribution. The number of customers in the system is distributed as the sum of two independent random variables. In particular, in the context our system, we will discuss the conditional stochastic decomposition of the number of customers in the orbit given that the server is busy. Let $N_q$ is the conditional orbit size of our retrial queuing system given that server is busy and $N_0$ is the conditional orbit size of the M/G/1 retrial queuing system with negative customers is given that the server is busy which is discussed in Theorem 5.1.
Theorem 5.1. The conditional orbit size $N_b$ is given that the server is busy can be decomposed into the sum of two independent random variables

$$N_b = N_0 + N_c$$

where $N_0$ has the generating function $N_0(z)$ as follows,

$$N_0(z) = \frac{\delta(1 - S_0(A_b(z)))(z - 1)\{R'(\hat{\lambda}) - (\hat{\lambda}(1 - S_0(\delta))(1 + \delta g^{(1)}/\delta)\}}{[zA_0(z) - (R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda})))A_0(z)S_0(A_b(z)) + \delta G'(b(z))(1 - S_0(A_b(z)))\}}(1 - S_0(\delta))$$

and $N_c$ is the additional queue length due to vacations with the probability generating function $N_c(z)$ as follows,

$$N_c(z) = \frac{\{\hat{\lambda}/\theta V(z)\{zA_0(z) - (R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda})))A_0(z)S_0(A_b(z)) + \delta G'(b(z))(1 - S_0(A_b(z)))\}\} + (1 - S_0(A_b(z)))\{\hat{\lambda}(S_0'(A_b(z)) - 1) - \theta p\}(R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda}))) + z(\hat{\lambda}V(z) + \theta p)\}}{\delta(1 - S_0(A_b(z)))(z - 1)\{R'(\hat{\lambda})((\hat{\lambda}/\theta)(1 - S_0(\theta)) + (1 - S_0(\delta))(\hat{\lambda}(1 - S_0(\theta)) + \theta p)) - (\hat{\lambda}z/\theta)g^{(1)}(1 - S_0(\delta))(1 - S_0(\theta))\}}$$

Proof. The mathematical version of the stochastic decomposition law is $N_b(z) = N_0(z)N_c(z)$.

In an M/G/1 retrial queueing system with negative customers, the marginal function of the number of customers in the orbit when the server is busy is given by

$$\Phi(z) = \frac{\hat{\lambda}P_0R'(\hat{\lambda})(1 - S_0(A_b(z)))(z - 1)}{zA_0(z) - (R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda})))A_0(z)S_0(A_b(z)) + \delta G'(b(z))(1 - S_0(A_b(z)))}$$

and the probability that the server is busy is given by

$$\Phi(1) = \frac{\hat{\lambda}P_0R'(\hat{\lambda})(1 - S_0(\delta))}{\delta\{R'(\hat{\lambda}) - (\hat{\lambda}(1 - S_0(\delta))(1 + \delta g^{(1)}/\delta)\}}$$

then for the probability generating function $N_0(z)$, we have

$$N_0(z) = \frac{\Phi(z)}{\Phi(1)} = \frac{\delta(1 - S_0(A_b(z)))(z - 1)\{R'(\hat{\lambda}) - (\hat{\lambda}(1 - S_0(\delta))(1 + \delta g^{(1)}/\delta)\}}{[zA_0(z) - (R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda})))A_0(z)S_0(A_b(z)) + \delta G'(b(z))(1 - S_0(A_b(z)))\}}(1 - S_0(\delta))$$

From the Eqs. (3.35) and (3.36), we know that for our retrial system the probability generating function of $N_c$ is given by

$$N_c(z) = \frac{\hat{\lambda}V(z)\{zA_0(z) - (R'(\hat{\lambda}) + z(1 - R'(\hat{\lambda})))A_0(z)S_0(A_b(z)) + \delta G'(b(z))(1 - S_0(A_b(z)))\}}{\{(\hat{\lambda}S_0'(A_b(z)) - 1) - \theta p\}(R'(\hat{\lambda}) + z(1 - R'(\hat{lambda}))) + z(\hat{\lambda}V(z) + \theta p)\}}N_0(z)(1 - S_0(\delta)(\hat{\lambda}(1 - S_0(\theta)) + \theta p)) - (\hat{\lambda}z/\theta)g^{(1)}(1 - S_0(\delta))(1 - S_0(\theta))\}}$$
This coincides with the result of Zhang and Liu [26].

**Case (ii): No negative arrival, No vacation interruption and Single working vacation**

Let $(\delta, \theta) \to (0, 0)$, our model can be reduced to $M/G/1$ retrial queue with single working vacation. In this case, $K_i(z)$ coincides with the result of Arivudainambi et al. [20] as follows,

$$K_i(z) = P_0 \left\{ \frac{1 - z \left( z A_0(z) - (A_0(z) S_1(A_0(z))) \right) \left[ (1 - \delta G^{*}(b(z))) (1 - S_1'(A_0(z))) \right] ((\dot{z} V(z)/\theta) + 1)}{(1 - z)(z A_0(z) - (A_0(z) S_1(A_0(z))) + \delta G^{*}(b(z))(1 - S_1'(A_0(z))))} \right\}$$

**Case (iii): No retrial, No negative arrival and Multiple working vacation**

Let $\delta \to 0$ and $R'(\lambda) \to 1$, suppose that there is no retrial time in the system then we get an $M/G/1$ queue with working vacations and vacation interruption. In this case, $K_i(z)$ can be obtained as follows,

$$K_i(z) = \frac{S_1'(\delta)) \left( R'(\dot{\lambda} - \lambda \delta) - \lambda \beta(\lambda) \right)}{z E(S_1) - R'(\dot{\lambda}) S_1'(\lambda)} \left\{ \frac{\left\{ z S_1'(A_0(z)) \left[C_0 A_0(z) + z V(z) \right]\right\} S_1'(\lambda) - \left[C_0 R'(\dot{\lambda}) S_1'(\lambda) \right]}{S_1'(\lambda) - \left[C_0 R'(\dot{\lambda}) S_1'(\lambda) \right]} \right\}$$

This coincides with the result of Zhang and Hou [23].

**Case (iv): No negative arrival and Multiple working vacations**

Let $\delta = p = 0$, our model can be reduced to a single server retrial queueing system with working vacations. In this case, $K_i(z)$ can be yielded as follows,

$$K_i(z) = P_0 \left\{ \frac{\left\{ z - (R'(\dot{\lambda}) + z(1 - R'(\dot{\lambda})) S_1'(A_0(z))) \left[C_0 A_0(z) + z V(z) \right]\right\} S_1'(\lambda) - \left[C_0 R'(\dot{\lambda}) S_1'(\lambda) \right]}{z (1 - R'(\dot{\lambda}) + z(1 - R'(\dot{\lambda})) S_1'(A_0(z))) \left[C_0 A_0(z) + z V(z) \right]\right\}$$

This coincides with the result of Gao et al. [25].

**Case (vi): Single working vacation**

Let $p = 1$, our model can be reduced to a single server retrial queueing system with negative customers, single working vacation and vacation interruption.

**Case (vi): Multiple working vacations**

Let $p = 0$, our model can be reduced to a single server retrial queueing system with negative customers, multiple working vacation and vacation interruption.

7. Cost optimization analysis

In order to carry out cost analysis, the optimum design of a retrial queueing system is to determine the optimal system parameters, such as optimal mean service rate or optimal number of servers (see in [31,32]). In this section, the optimal design of the single server retrial $G$-queue with working vacations and vacation interruption under Bernoulli schedule is addressed. Based on the definitions of cost elements ($C_a, C_v, C_s$ and $C_b$) and cost structure listed below, the total expected cost function per unit time is given by

$$K_i(z) = P_0 \left\{ \frac{\left\{ z - (R'(\dot{\lambda}) + z(1 - R'(\dot{\lambda})) S_1'(A_0(z))) \left[C_0 A_0(z) + z V(z) \right]\right\} S_1'(\lambda) - \left[C_0 R'(\dot{\lambda}) S_1'(\lambda) \right]}{z (1 - R'(\dot{\lambda}) + z(1 - R'(\dot{\lambda})) S_1'(A_0(z))) \left[C_0 A_0(z) + z V(z) \right]\right\}$$
\[ TC = C_b L_s + C_o \frac{E(T_b)}{E(T)} + C_s \frac{1}{E(T)} + C_a \frac{E(T_a)}{E(T)} \]

\[ = C_b L_s + C_o (1 - P_b) + C_s \lambda + C_a P_b \]

where \( C_b \) is the holding costs per unit time for each customer present in the system, \( C_o \) is the cost per unit time for keeping the server on and in operations, \( C_s \) is setup cost per busy cycle and \( C_a \) is the startup cost per unit time for the preparation work of the server before starting the service.

If we assume exponential retrial times, service times, working vacation times and repair times then for the following values of the cost elements and other parameters like:

- \( k = 1 \)
- \( \lambda_b = 5 \)
- \( \lambda_v = 2 \)
- \( a = 2 \)
- \( \zeta = 3 \)
- \( \theta = 3 \)
- \( \delta = 0.1 \)
- \( p = 0.5 \)
- \( C_b = 5 \)
- \( C_o = 100 \)
- \( C_s = 1000 \)
- \( C_a = 100 \)

we find the total expected cost per unit of time \( TC = 301.7446 \); Also, in this case, the steady-state availability of the server \( A_v = 98.94\% \) while the steady-state failure frequency of the server is \( F_f = 3.18\% \).

Moreover, we can examine the behavior of the expected cost function under different values of the cost parameters. Let us fix the system parameters values as follows: \( \lambda = 1 \)
- \( \mu_b = 5 \)
- \( \mu_v = 2 \)
- \( a = 2 \)
- \( \zeta = 3 \)
- \( \theta = 3 \)
- \( \delta = 0.1 \)
- \( p = 0.5 \)
- \( C_b = 5 \)
- \( C_o = 100 \)
- \( C_s = 1000 \)
- \( C_a = 100 \)
we find the total expected cost per unit of time \( TC = 301.7446 \); Also, in this case, the steady-state availability of the server \( A_v = 98.94\% \) while the steady-state failure frequency of the server is \( F_f = 3.18\% \).

8. Numerical examples

In this section, we present some numerical examples to study the effect of various parameters in the system performance measures of our system where all retrial times, service times, working vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied. MATLAB software has been used to illustrate the results numerically. Note that the exponential distribution is \( f(x) = e^{-\lambda x}, x > 0 \), Erlang-2 stage distribution is \( f(x) = x^2 e^{-2\lambda x}, x > 0 \) and hyper-exponential distribution is \( f(x) = c e^{-\lambda x} + (1-c) e^{-\lambda x}, x > 0 \).

### Table 1
<table>
<thead>
<tr>
<th>((C_b, C_o))</th>
<th>((5,100))</th>
<th>((5,110))</th>
<th>((5,120))</th>
<th>((10,100))</th>
<th>((15,100))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(TC)</td>
<td>301.7446</td>
<td>303.3765</td>
<td>305.0084</td>
<td>303.4891</td>
<td>305.2337</td>
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### Table 2
<table>
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<tr>
<th>((C_o, C_a))</th>
<th>((100,100))</th>
<th>((125,100))</th>
<th>((150,100))</th>
<th>((100,110))</th>
<th>((100,120))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(TC)</td>
<td>301.7446</td>
<td>305.8243</td>
<td>309.9041</td>
<td>310.1127</td>
<td>318.4808</td>
</tr>
</tbody>
</table>

### Table 3
<table>
<thead>
<tr>
<th>((C_a, C_s))</th>
<th>((100,100))</th>
<th>((110,100))</th>
<th>((120,100))</th>
<th>((100,1050))</th>
<th>((100,1100))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(TC)</td>
<td>301.7446</td>
<td>310.1127</td>
<td>318.4808</td>
<td>311.7446</td>
<td>321.7446</td>
</tr>
</tbody>
</table>

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Table 4 shows that when retrial rate \((a)\) increases, then the idle probability \((P_0)\) increases, the mean orbit size \((L_q)\) decreases and probability that server is idle during retrial time \((P)\) also decreases for the values of \(\lambda = 1; \theta = 2; \mu = 9; p = 0.7; \xi = 5; \delta = 0.2; \mu_c = 4; c = 0.7\). Table 5 shows that when negative arrival rate \((\delta)\) increases, the mean orbit size \((L_q)\) increases, probability that server is idle during retrial time \((P)\) increases and the servers failure frequency \((F_f)\) also increases for the values of \(\lambda = 1; \theta = 2; \mu = 9; p = 0.7; \xi = 5; a = 2; \mu_c = 4; c = 0.7\). Table 6 shows that when vacation rate \((\theta)\) increases, the idle probability \((P_0)\) increases, then the mean orbit size \((L_q)\) decreases and probability that server is on working vacation \((Q_v)\) also decreases for the values of \(\lambda = 1; \delta = 0.2; \mu = 9; p = 0.7; \xi = 5; a = 2; \theta = 2; c = 0.7\).

For the effect of the parameters \(\lambda, a, \delta, \theta, \mu, \) and \(c\) on the system performance measures, three dimensional graphs are illustrated in Figs. 5–8. In Fig. 5, the surface displays an upward trend as expected for increasing the value of arrival rate \((\lambda)\) and negative arrival rate \((\delta)\) against the mean orbit size \((L_q)\).
In Fig. 6, we examine the behavior of the idle probability \( P_0 \) increases for increasing the value of the lower service rate \( l_v \) and regular service rate \( l_b \). Fig. 7 shows that the idle probability \( P_0 \) increases for increasing the value of single working vacation idle probability \( p \) and vacation rate \( h \).

From the above numerical examples, we observed that the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

<table>
<thead>
<tr>
<th>Vacation distribution</th>
<th>Exponential</th>
<th>Erlang-2 stage</th>
<th>Hyper-exponential</th>
</tr>
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<tr>
<td>( \mu_v )</td>
<td>( P_0 )</td>
<td>( L_q )</td>
<td>( Q_v )</td>
</tr>
<tr>
<td>Lower speed service rate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>0.1111</td>
<td>0.3721</td>
<td>0.1389</td>
</tr>
<tr>
<td>3.00</td>
<td>0.1184</td>
<td>0.3572</td>
<td>0.1184</td>
</tr>
<tr>
<td>4.00</td>
<td>0.1239</td>
<td>0.3473</td>
<td>0.1032</td>
</tr>
<tr>
<td>5.00</td>
<td>0.1281</td>
<td>0.3410</td>
<td>0.0915</td>
</tr>
</tbody>
</table>

(\( L_q \)). In Fig. 6, we examine the behavior of the idle probability \( (P_0) \) increases for increasing the value of the lower service rate \( (\mu_v) \) and regular service rate \( (\mu_b) \). Fig. 7 shows that the idle probability \( (P_0) \) increases for increasing the value of single working vacation idle probability \( (p) \) and vacation rate \( (\theta) \). In Fig. 8, we examine the behavior of the mean orbit size \( (L_q) \) decreases for increasing the value of lower speed service rate \( (\mu_v) \) and retrial rate \( (a) \).
9. Conclusion

In this work, we have investigated a single server retrial queueing system with negative customers under both single and multiple working vacations and vacation interruption, where the server is subjected to breakdown and repair. The necessary and sufficient condition for the system to be stable is obtained. By using the probability generating function approach and the method of supplementary variable technique, the probability generating functions for the numbers of customers in the system and its orbit when it is free, busy, on working vacation, under repair are derived. Various system's performance measures, reliability measures and conditional stochastic decomposition law are discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples and cost optimization analysis are presented to study the impact of the system parameters and cost elements. The novelty of this investigation is the introduction of both single and multiple working vacations and multiple retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. Appl Math Model 2012;36:255–69.


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References


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