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NOTES

Range Inclusion and Operator Equations

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Let A and D be positive operators on a complex Hilbert space H. In this work we show that the operator equation $lDX + X^*A = A$ $(l > 0)$ has a unique solution C_l satisfying certain conditions, if there exists a positive operator B such that $AB + (D - B)A \geq 0$. This implies several previously known results in special cases on this class of operator equations. Also, along with some definitions, a counterexample to a previous conjecture will be given. © 1996 Academic Press, Inc.

INTRODUCTION

The purpose of this paper is two-fold.

(a) Describe very briefly the previous research on the range inclusion in order to find the solutions of the class of operator equations

$$lDX + X^*A = A \qquad \text{for all real } l > 0.$$

This implies several previously known results on solution of the class of operator equations in the special case.

(b) Give a counterexample to a previous conjecture related to range inclusion. Throughout this work, H will denote a complex Hilbert space, and A, B, and D will be positive bounded linear operators on H. Recall that $A \geq B$ means that $A - B$ is positive and $A : B$ denotes the parallel sum of A and B. Also, the range of A will be denoted by $R(A)$.

In Section 1, we survey some results about the range inclusion and give an example which establishes that a previously known conjuncture is false.

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In Section 2, we give a sufficient condition for the operator equation

$$lDX + X^*A = A$$

to have a positive solution. This class of operator equations arises in various practical situations including the studies of superconductivity, boundary value problems, sensitivity analysis, and optimal control.

1. RANGE INCLUSION

The concept of range inclusion has been studied by the author [8–10] and Green and Morley [6, 7]. We begin with a principle of symmetry for range inclusion.

THEOREM 1.1 [8]. *If A and B are linear transformations on a vector space X, then $R(A) \subseteq R(A+B)$ if and only if $R(B) \subseteq R(A+B)$.*

LEMMA 1.1 (Green and Morley [6, 7]). *Let A and B be positive operators. If $AB = BA$, then $R(A) \subseteq R(A+B)$.*

THEOREM 1.2 [9]. *If $AB + BA + B^2 \geq 0$, then $R(A + lB) \supseteq R(A)$, for all $l \geq 1$.*

THEOREM 1.3 [10]. *If $AB + BA \geq 0$, then $R(A + lB) \supseteq R(A)$ for all $l \geq 0$.*

Note that Theorem 1.3 implies Theorem 1.2 and Lemma 1.1.

Now we give an example which establishes that the following conjecture is false.

Conjecture. If $R(A + lB) \supseteq R(A)$ for all $l \geq 0$, then $AB + BA \geq 0$.

EXAMPLE 1.1. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4/3 & 1 \\ 1 & 1 \end{pmatrix}$ act on C^2. Clearly $R(A)$, $R(B)$, and $R(A, lB)$ are closed for all $l \geq 0$. Thus, $R(A + lB) = R((A + lB)^{1/2}) = R(A^{1/2}) + R((lB)^{1/2}) = R(A^{1/2}) + R(B^{1/2}) = R(A) + R(B) \supseteq R(A)$, *and hence* $R(A) \subseteq R(A + lB)$ *for all* $l \geq 0$. *Also* $AB + BA = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ *which is not positive.*

The following theorem generalizes some of the work done by Green and Morley [6, 7], Bunce [3], and the author [9].

THEOREM 1.4 [10]. *Let A, B, and C be positive operators. If $AB + CA \geq 0$, then for a fixed $l \geq 0$,*

(1) $R[A + l(B + C)] = R(A) + R(B + C)$;

(2) $R[A + l(B + C)] \supseteq R(A)$;

(3) $R[A + l(B + C)] \supseteq R(B + C)$;

(4) $\|A[A + l(B + C) + \epsilon]^{-1}\| \leq M_1$ *for some real* M_1 *and all* $\epsilon > 0$;

(5) $\|(B + C)[A + l(B + C) + \epsilon]^{-1}\| \leq M_2$ *for some real* M_2 *and* $\epsilon > 0$. *Moreover*, (1), (2), (3), (4), *and* (5) *are equivalent*.

2. SOLUTION OF $lDX + X^*A = A$

In this section, it is shown that if there exists a positive operator B such that $AB + (D - B)A \geq 0$, then the operator equation $lDX + X^*A = A$ $(l > 0)$ has a unique solution C_l satisfying condition (a), (b), and (c) of the following theorem.

THEOREM 2.1 [10]. *If*, $R(A + B) \supseteq R(A)$, *then the operator equation* $AX + X^*B = A$ *has a unique solution* C *on* H *so that*

(a) $\|C\|^2 = \inf\{\mu/A_2 \leq \mu(A + B)^2\}$;

(b) $\ker(A) = \ker(c)$, *and*

(c) $R(A + B)^{-} \supseteq R(D)$.

Moreover, C *is positive if and only if* $AB = BA$.

THEOREM 2.2. *Suppose there exists a positive operator* B *such that* $AB + (D - B)A \geq 0$, *then operator equation* $lDX + X^*A = A$ $(l > 0)$ *has a unique bounded solution* C_l *on* H *such that*

(a) $\|C_l\|^2 = \inf\{\mu/A^2 \leq \mu(A + D)^2\}$

(b) $\text{Ker}(A) = \text{Ker}(C_l)$; *and*

(c) $\text{Range}(A + lD)^{-} \supseteq \text{Range}(C_l)$

Moreover, C_l *is positive if and only if* $AD = DA$.

Proof. If $AB + (D - B)A \geq 0$, then $AB + (D - B)A + BA + A(D - B) = AD + DA \geq 0$. It follows from Theorem 1.3 that $R(A + lD) \supseteq R(A)$. Note that for each real $l \geq 0$, there exists a unique bounded operator C_l satisfying the desired condition such that
$A: lD = lDC_l$ and $lD: A = A(I - C_l)$. Moreover, since $(lD : A^*) = (lD : A)$, one can conclude that $lDC_l = (I - C_l^*)A$. This shows that $lDC_l + C_l^*A = A$. Also, since $A = (A + lD)C_l$ and $\langle C_l^*(A + lD)x, (A + lD)x\rangle = \langle Ax, (A + lD)x\rangle = \langle x, (A^2 + lAD)x\rangle$ for all x. Thus C_l^* is positive if and only if $AD = DA$, because $A^2 + lAD \geq 0$ if and only if $AD = DA$ and the proof is complete.

Note that if $AD + DA \geq 0$, then we can select B as $D/_2$ and hence we have $AB + (D - B)A = AD/_2 + (D - D/_2)A = AD/_2 + D/_2A \geq 0$. Thus the operator equation $lDX + X^*A = A$ has a unique solution C_l

satisfying the desired condition. This shows that the above theorem generalizes Lemma 2.1 of [10].

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