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Even Liaison Classes Generated by Gorenstein Linkage

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1. INTRODUCTION

Liaison theory has made much progress since the appearance of [29]. Most of the work has been done for subschemes of projective space linked via complete intersections. In this paper we consider the equivalence classes of arbitrary equidimensional subschemes of an arithmetically Gorenstein subscheme generated by linkage via arithmetically Gorenstein subschemes. The purpose of this paper is to show that several standard results are still true in this setting.

To achieve these generalizations we use somewhat different methods. They are mainly algebraic. Thus we have chosen an algebraic language in the body of the paper, i.e., we consider arbitrary unmixed homogeneous ideals in a graded Gorenstein K-algebra R. This way it also becomes clear that all of the results hold still true for ideals in a local Gorenstein ring containing a field if the graded structure is not used explicitly in the statement. On the other hand, it is easy to translate the results into a geometric language.

The set-up is described in Section 2. There we also recall several results and concepts, most notably characterizations of k-syzygies and so-called q-presentations, and we give a cohomological unmixedness criterion for ideals.

Liaison theory is well understood for ideals of codimension 2 thanks to work of Rao [30], (Ballico), Bolondi and Migliore [3], [2], and Martin-Deschamps and Perrin [22]. The results rely mainly on three basic tools; resolutions of E- and N-type, respectively; basic double linkage; and minimal shift. We show that these tools can be generalized to our setting

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in Section 3–5. The main result in Section 3 describes maps Φ and Ψ from the set of even liaison classes of ideals of codimension c to certain $(c+1)$-syzygies and certain reflexive modules, respectively (cf. Theorem 3.10). As a consequence we get results comparing the Hilbert function and cohomology modules of linked ideals.

In the next two sections we show how these tools give the main results of liaison theory in codimension 2: Rao's correspondence and the Lazarsfeld–Rao property, provided R is also an integral domain. The only additional ingredient is a version of Bourbaki's Theorem (cf. Proposition 6.3). Then we show in Section 6 that Rao's correspondence holds true in our setting. It says that the maps Φ and Ψ mentioned above are in fact bijective. Thus they provide parameterizations of all even liaison classes. The Lazarsfeld–Rao property is proved in Section 7. It describes a structure common to every even liaison class that does not contain a complete intersection. Contrary to Rao's correspondence, the Lazarsfeld–Rao property has no analogue for ideals in a local Gorenstein ring, since it makes explicit use of the graded structure. These results are new in this generality, because in [28] the ring R is just a polynomial ring and in [4] Proj R is assumed to be smooth and connected, and the considered subschemes of Proj R are supposed to be locally Cohen–Macaulay.

Extensions of the results of Section 6 and 7 to liaison classes of ideals having codimension ≥ 3 are still missing. We discuss some of the arising problems in Section 8. The results of Sections 3–5 indicate that Gorenstein liaison gives an interesting equivalence relation. We hope that the generality of our results in codimension 2 will be useful for further generalizations of the theory to higher codimensions.

2. PREPARATORY RESULTS

In this section we fix notation and collect some results we will need later on. Among them are duality results, criteria for k-syzygies, q-presentations, and a cohomological unmixedness criterion.

Let R be a ring. If $R = \bigoplus_{i \in \mathbb{N}} R_i$ is graded, then the irrelevant maximal ideal $\bigoplus_{i>0} R_i$ of R is denoted by $\mathfrak{m}_R$ or simply $\mathfrak{m}$. It is always assumed that R_0 is an infinite field K and that the K-algebra R is generated by the elements of R_1. Hence $(R, \mathfrak{m})$ is *local in the sense of [8].

If M is a module over the graded ring R, it is always assumed to be $\mathbb{Z}$-graded. The set of its homogeneous elements of degree i is denoted by M_i or $[M]_i$. All homomorphisms between graded R-modules will be morphisms in the category of graded R-modules, i.e., will be graded of degree zero. If M is assumed to be a graded R-module, it is always understood that R is a graded K-algebra as above. We refer to the context just described as the graded situation. Although we are mainly interested in graded objects, it is useful to have also a local situation in mind where $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$ containing a field. We note that our results also hold true (with the usual modifications) in this local context unless the assertion makes use of the graded structure.

If M is an R-module, $\dim M$ denotes the Krull dimension of M. The symbols rank_R or simply rank are reserved to denote the rank of M in case it has one. For a K-module, rank_K just denotes the vector space dimension over the field K.

There are two types of duals of an R-module M we are going to use. The R-dual of M is $M^* = \operatorname{Hom}_R(M, R)$. If M is graded, then M^* is graded, too. If R is a graded K-algebra, then M is also a K-module, and the K-dual $M^\vee$ of M is defined to be the graded module $\operatorname{Hom}_K(M, K)$, where K is considered as a graded module concentrated in degree zero. $M^\vee$ has a natural structure as a graded R-module. Note that $R^\vee$ is the injective hull of $K^\vee \cong K \cong R/\mathfrak{m}$ in the category of graded R-modules. If $\operatorname{rank}_K [M]_i < \infty$ for all integers i, then there is a canonical isomorphism $M \cong M^{\vee\vee}$.

Duality results

A ring R is said to be Gorenstein if it has finite injective dimension. Over a Gorenstein ring duality theory is particularly simple. We denote the index of regularity of a graded ring by $r(R)$. If R is just the polynomial ring $K[x_0, \dots, x_n]$, then $r(R) = -n$. We will often use the following duality result (cf. [32], [35]) without further mentioning.

PROPOSITION 2.1. *Let S be a graded Gorenstein ring of dimension $n+1$. Let M be a graded R-module where R is a quotient of S. Then we have for all $i \in \mathbb{Z}$ natural isomorphisms of graded S-modules*

$$H^i_{\mathfrak{m}_R}(M)^\vee \cong \operatorname{Ext}_S^{n+1-i}(M, S)(r(S) - 1).$$

Let M be an R-module where $n+1 = \dim R$ and $d = \dim M$. Then

$$K_M = \operatorname{Ext}_R^{n+1-d}(M, R)(r(R) - 1)$$

is said to be the *canonical module* of M. Usually the canonical module is defined as the module representing the functor $H^d_{\mathfrak{m}}(M \otimes_R \_)^\vee$, if such a module exists. If R is Gorenstein, it does, and it is just the module defined above (cf. [32]). Duality theory also relates the cohomology modules of M and K_M. Later on we will need the following result of Schenzel [32], Corollary 3.1.3.

PROPOSITION 2.2. *Let M be a graded module over the Gorenstein ring R. Suppose that $H^i_{\mathfrak{m}}(M)$ has finite length if $i \neq d = \dim M$. Then there are* *canonical isomorphisms for* $i = 2, \ldots, d-1$,

$$H_{\mathfrak{m}}^{d+1-i}(K_M) \cong H_{\mathfrak{m}}^{i}(M)^{\vee}.$$

If the assumption on the cohomology of M in the theorem above is satisfied, then M is said to have *cohomology of finite length*. If R is a quotient of a Gorenstein ring, then an R-module has cohomology of finite length if and only if M is equidimensional and is locally Cohen−Macaulay.

With regard to the regularity index we need the following fact.

LEMMA 2.3. *Let* $\mathfrak{c} \subset R$ *be an ideal generated by an* R*-regular sequence* $\{f_1, \ldots, f_c\}$ *of homogeneous elements of degrees* $d_1, \ldots, d_c$. *Then it holds*

$$r(R/\mathfrak{c}) = d_1 + \cdots + d_c + r(R).$$

Proof. By induction on c it suffices to prove the formula if $c = 1$. Let $f = f_1$ be an R-regular element of degree $d = d_1$. Then multiplication by f induces the exact sequence

$$0 \to R(-d) \xrightarrow{f} R \to R/fR \to 0.$$

Now recall that the regularity index of a graded R-module M is

$$r(M) = \min\{t \in \mathbb{Z} \mid h_M(i) = p_M(i) \text{ for all } i \geq t\}.$$

Hence the exact sequence above implies $r(R/fR) = d + r(R)$. ∎

k-syzygies

If M is an R-module, then $\dim M \leq \dim R$. M is said to be *maximal* if $\dim M = \dim R$. Let Q be the total ring of fractions of R. Then an R-module M is said to be *torsion-free* if the natural map $M \to M \otimes Q$ is injective. The bilinear map $M \times M^* \to R$, $(m, \varphi) \mapsto \varphi(m)$ induces a natural homomorphism $h: M \to M^{**}$. The module M is said to be *torsionless* if h is injective, and M is said to be *reflexive* if h is an isomorphism. We want to compare these notions with the following one.

DEFINITION 2.4. Let M be a graded R-module. Then N is said to be a *k-syzygy* of M if there is an exact sequence of R-modules

$$0 \to N \to F_k \xrightarrow{\varphi_k} F_{k-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} M \to 0,$$

where the modules F_i, $i = 1, \ldots, k$, are free $R-$ modules.

By our standard conventions the morphisms in the exact sequence above are required to be morphisms of degree zero. Moreover, an R-module N

is simply called a k-syzygy if it is a k-syzygy of some R-module. Thus, a $(k+1)$-syzygy is also a k-syzygy.

Let $\varphi\colon F \to M$ be a homomorphism of R-modules where F is free. Then φ is said to be a *minimal* homomorphism if $\varphi \otimes id_{R/\mathfrak{m}}\colon F/\mathfrak{m}F \to M/\mathfrak{m}M$ is the zero map in the case where M is free and an isomorphism in the case where φ is surjective.

In the situation of the definition above, N is said to be a *minimal k-syzygy* of M if the morphisms φ_i, $i = 1, \dots, k$, are minimal. It is uniquely determined up to isomorphism by Nakayama's lemma. If the minimal k-syzygy N happens to be free, then the exact sequence in the definition above is called a minimal free resolution of M. Note that every finitely generated projective R-module is free because of our assumptions on R.

It follows by [1] that for a finitely generated module over a Gorenstein ring, the conditions torsionless, torsion-free, $(\tilde{S}_1)$ and 1-syzygy are all equivalent. The same applies to the condition reflexive, $(\tilde{S}_2)$, and 2-syzygy. It is more difficult to identify third and higher syzygies. For this a condition of Serre type is used. A module M is said to satisfy condition $(\tilde{S}_k)$ if

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min\{k, \dim R_{\mathfrak{p}}\}$$

for all prime ideals $\mathfrak{p} \subset \mathfrak{m}$.

Moreover, we want to consider the cohomological annihilators $\mathfrak{a}_i(M) = \operatorname{Ann}_R H^i_{\mathfrak{m}}(M)$. If R is Gorenstein, then we have $\dim R/\mathfrak{a}_i(M) \leq i$ for all integers i, where we put $\dim M = -\infty$ if $M = 0$. Higher syzygies can be characterized as follows.

PROPOSITION 2.5. *Let R be a Gorenstein ring and let M be a finitely generated R-module. Then the following conditions are equivalent*:

(a) *M is a k-syzygy.*

(b) *M satisfies $(\tilde{S}_k)$.*

(c) $\dim R/\mathfrak{a}_i(M) \leq i - k$ *for all* $i < \dim R$.

Moreover, if $k \geq 3$, then conditions (a)–(c) *are equivalent to the condition that M is reflexive and* $\operatorname{Ext}^i_R(M^*, R) = 0$ *if* $1 \leq i \leq k-2$.

Proof. By duality we know that the annihilators of $\operatorname{Ext}^i_R(M, R)$ and $H^{\dim R - i}_{\mathfrak{m}}(M)$ coincide. Hence we get *grade* $\operatorname{Ext}^i_R(M, R) = \dim R - \dim R/\mathfrak{a}_{\dim R - i}(M)$, and the result follows by [1, Theorem 4.25] and [12, Theorem 3.8]. ∎

The last statement gets particularly simple if the module has cohomology of finite length.

COROLLARY 2.6. *Let R be a Gorenstein ring and let M be a finitely generated, maximal R-module with cohomology of finite length. Then M is a k-syzygy if and only if* depth $M \geq k$.

Following Horrocks [18], a maximal R-module E is said to be an *Eilenberg–MacLane module* of depth t, $0 \leq t \leq n+1 = \dim R$ if

$$H^j_{\mathfrak{m}}(E) = 0 \qquad \text{for all } j \neq t \text{ where } 0 \leq j \leq n.$$

An Eilenberg–MacLane module of depth $n+1$ is Cohen–Macaulay, and thus is a free module if it has finite projective dimension. More generally, a relation between Eilenberg–MacLane modules and syzygy modules is described in the next result, which is proved as Theorem 3.9 in [26].

LEMMA 2.7. *Let E be a reflexive module of depth $t \leq n$. Then E is an Eilenberg–MacLane module with finite projective dimension if and only if E^* is an $(n+2-t)$-syzygy of a module M of dimension $\leq t-2$. In this case it holds*

$$M \cong H^i_{\mathfrak{m}}(E)^{\vee}(1-r).$$

q-presentations

Let q be a positive integer. We want to recall some properties of so-called q-presentations. q-presentations have been used by Evans and Griffith in [13] to study lifting properties of vector bundles and in [26] to characterize, for example, arithmetically Buchsbaum subschemes of projective space. In a local situation these presentations were first considered by Auslander and Bridger [1]. Their uniqueness properties were established by Evans and Griffith [12, 13]. Here we adapt the notion to graded modules over a graded Gorenstein K-algebra R following [26].

By definition we set the projective dimension of a trivial module equal to zero.

DEFINITION 2.8. Let q be an integer with $1 \leq q \leq \dim R$. An exact sequence of finitely generated graded R-modules,

$$0 \to P \overset{\varphi}{\to} N \to M \to 0,$$

is said to be a *q-presentation* of the finitely generated R-module M if

(i) P has projective dimension $< q$ and

(ii) $H^j_{\mathfrak{m}}(N) = 0$ for all j with $\dim R - q \leq j < \dim R$.

It is said to be a *minimal* q-presentation if there does not exist a nontrivial free R-module F such that F is a direct summand of P and N and φ induces an isomorphism of F onto F.

If the q-presentation is not minimal, we say that P and N have a common direct free summand. If a nontrivial module P has projective dimension $< q$, then it has depth $> \dim R - q$ due to the Auslander–Buchsbaum formula. If R is a polynomial ring, then the converse is also true.

A q-presentation "distributes" the local cohomology modules of M among P and N, as the following observation shows.

LEMMA 2.9. *If* $0 \to P \to N \to M \to 0$ *is a q-presentation of* M, *then*

$$H_{\mathfrak{m}}^j(N) \cong \begin{cases} H_{\mathfrak{m}}^j(M) & \text{if } j < \dim R - q \\ 0 & \text{if } \dim R - q \leq j < \dim R \end{cases}$$

and

$$H_{\mathfrak{m}}^j(P) \cong \begin{cases} 0 & \text{if } j \leq \dim R - q \\ H_{\mathfrak{M}}^{j-1}(M) & \text{if } \dim R - q < j < \dim R. \end{cases}$$

Proof. The claims follows immediately by the definition of a q-presentation and by the long exact cohomology sequence it induces. ∎

Fortunately, q-presentations do exist.

PROPOSITION 2.10. *A finitely generated graded module M over a Gorenstein ring R admits for every q with* $1 \leq q \leq \dim R$ *a minimal q-presentation*

$$0 \to P \to N \to M \to 0.$$

It is uniquely determined up to isomorphisms of exact sequences.

A proof of this result can be found in [26, Theorem II.1.5].

Unmixedness criterion

For the study of liaison we will construct ideals; we want to know if they are unmixed. An ideal $I \subset R$ is said to be of *pure* codimension or unmixed if all of the associated prime ideals of R/I have the same codimension. We say that an R-module M is *locally free in codimension c* if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all primes $\mathfrak{p} \in \operatorname{Proj} R$ having codimension $\leq c$. Later on we will need the following result.

LEMMA 2.11. *Let R be Gorenstein and let $I \subset R$ be a homogeneous ideal of codimension* $c \geq 2$ *with* $(c-1)$*-presentation*

$$0 \to P \to N \to I \to 0.$$

Then the following conditions are equivalent:

(a) *I is unmixed.*

(b) $\dim R/\mathfrak{a}_i(R/I) < i$ *if* $i < \dim R/I$.

(c) *N is a reflexive R-module.*

Moreover, in this case N is locally free in codimension $c - 1$.

Proof. Since $H^i_{\mathfrak{m}}(N) \cong H^{i-1}_{\mathfrak{m}}(R/I)$ if $i \leq \dim R/I$ and $H^i_{\mathfrak{m}}(N) = 0$ if $\dim R/I < i < \dim R$, the equivalence of (b) and (c) follows by Proposition 2.5.

The equivalence of claims (a) and (b) is essentially shown in the proof of Lemma 4 in [25]. For more details we refer to [26, Lemma III.1.3].

It follows by Lemma 2.9 that P is an Eilenberg–MacLane module of depth $n + 3 - c$ and that $\operatorname{Ext}^{c-2}_R(P, R)$ is an R-module of dimension $n + 1 - c$. Therefore its localization at a prime ideal $\mathfrak{p} \subset \operatorname{Proj} R$ of codimension $\leq c - 1$ vanishes. Hence $P_{\mathfrak{p}}$ is a Cohen–Macaulay module of finite projective dimension over $R_{\mathfrak{p}}$, and thus is free by the Auslander–Buchsbaum formula.

Since I has pure codimension c, the localization of the $(c - 1)$-presentation of I at prime ideals of codimension $\leq c - 1$ splits. This shows that N is locally free in codimension $c - 1$. ∎

Remark 2.12. With the notation of the previous statement, N^* is a $(c + 1)$-syzygy due to Proposition 2.5. It follows that N^* and thus also N are even locally free in codimension $c + 1$ if R is regular.

Liaison

Let $\mathfrak{c} \subset R$ be a homogeneous ideal where R is a graded Gorenstein K-algebra of dimension $n + 1$. Then $\mathfrak{c}$ is said to be a *Gorenstein ideal* of codimension c if $\mathfrak{c}$ is a perfect ideal and $R/\mathfrak{c}$ is Gorenstein of dimension $n + 1 - c$. Note that $\mathfrak{c}$ is a Gorenstein ideal of codimension c if and only if $\mathfrak{c}$ has a minimal free graded resolution

$$0 \to F_c \to \cdots \to F_1 \to \mathfrak{c} \to 0,$$

where $F_1, \ldots, F_c$ are finitely generated free R-modules and F_c has rank 1.

The ideal $\mathfrak{c}$ is called a complete intersection of codimension c if it is generated by an R-regular sequence of length c. Every complete intersection is a Gorenstein ideal. If I has codimension two, then the converse is also true as a consequence of the Theorem of Hilbert-Burch. Note that there are Gorenstein ideals of any codimension ≥ 3 that are not a complete intersection.

DEFINITION 2.13. Let $I, J, \mathfrak{c} \subset R$ be homogeneous ideals such that $\mathfrak{c} \subset I \cap J$ and $\mathfrak{c}$ is a Gorenstein ideal of codimension c. Then I and J are said to be (directly) *linked by* $\mathfrak{c}$ if

(i) I and J have pure codimension c

(ii) $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$.

Remark 2.14. (i) Our definition of linkage follows [33]. More precisely, we should call it Gorenstein linkage. Here we just say linkage for simplicity, and because linkage is always understood in the sense above. If R is a polynomial ring and $\mathfrak{c}$ is a complete intersection, then our notion of linkage agrees with the usual definition (cf. [29]).

(ii) If I and J are linked by $\mathfrak{c}$ and have no primary components in common, then it holds that $\mathfrak{c} = I \cap J$ [33, Proposition 2.3]. In this case it is said that I and J are *geometrically linked* by $\mathfrak{c}$. If I and J are geometrically linked, then they are linked.

(iii) Let $\mathfrak{c}$ be a Gorenstein ideal of codimension c and let I be an ideal of pure codimension c containing $\mathfrak{c}$ such that $\mathfrak{c} \neq I$. Put $J = \mathfrak{c} : I$. Then I and J are linked by $\mathfrak{c}$ [33, Proposition 2.2]. In particular, the two relations in condition (ii) of the definition above are equivalent.

(iv) Hartshorne [16] has found an alternative approach to linkage by complete intersections, using his theory of generalized Gorenstein divisors.

We already observed that linkage is symmetric. But it is usually neither reflexive nor transitive. So we need to extend it to get an equivalence relation.

DEFINITION 2.15. *Liaison* is the equivalence relation generated by linkage. The equivalence classes are called *liaison classes*. That is, two ideals I and I' belong to the same liaison class if there are ideals $J_0 = I, J_1, \ldots, J_{s-1}, J_s = I'$ such that J_i and J_{i+1} are linked $(0 \leq i < s)$. If s is even, then it is said that I and I' are *evenly linked*. Otherwise I and I' are *oddly linked*. Notice that even linkage also generates an equivalence relation, and the equivalence classes are called *even liaison classes*.

Some basic questions arise immediately. How many (even) liaison classes exist? Is it possible to parameterize them? What does an even liaison class look like? Can it be given some structure? It is the purpose of the next three sections to present tools to attack these questions. Then it is shown how they can be used to get complete answers for the liaison classes of ideals of codimension 2.

The methods we will use are mainly algebraic. But we consider liaison primarily as a geometric concept. Thus we mention briefly some relations between the algebraic and geometric notions that allow us to switch between the two languages.

Let $Z = \mathit{Proj}(R)$ be a projective scheme over K. For any sheaf $\mathcal{F}$ on Z, we define the cohomology modules $H_*^i(Z, \mathcal{F}) = \oplus_{t \in \mathbb{Z}} H^i(Z, \mathcal{F}(t))$.

There are two functors relating graded R-modules and sheaves of modules over Z. One is the "sheafification" functor, which associates with each graded R-module M the sheaf $\tilde{M}$. This functor is exact.

In the opposite direction there is the "twisted global sections" functor, which associates with each sheaf $\mathcal{F}$ of modules over Z the graded R-module $H_*^0(Z, \mathcal{F})$. This functor is only left exact. If $\mathcal{F}$ is quasi-coherent, then the sheaf $\widetilde{H_*^0(Z, \mathcal{F})}$ is canonically isomorphic to $\mathcal{F}$. However, if M is a graded R-module, then the module $H_*^0(Z, \tilde{M})$ is not isomorphic to M in general. In fact, $H_*^0(Z, \tilde{M})$ even need not be finitely generated if M is finitely generated. Thus the functors $\widetilde{\ }$ and $H_*^0(Z, \_)$ do not establish an equivalence of categories between graded R-modules and quasi-coherent sheaves of modules over Z. However, there is the following comparison result (cf. [35]).

PROPOSITION 2.16. *Let M be a graded R-module. Then there is an exact sequence*

$$0 \to H_{\mathfrak{m}}^0(M) \to M \to H_*^0(Z, \tilde{M}) \to H_{\mathfrak{m}}^1(M) \to 0,$$

and for all $i \geq 1$ there are isomorphisms

$$H_*^i(Z, \tilde{M}) \cong H_{\mathfrak{m}}^{i+1}(M).$$

It follows, for example, that a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is a vector bundle if and only if $H_*^0(\mathcal{E})$ has cohomology of finite length.

If $X \subset Z$ is a projective subscheme, then $I_X = H_*^0(Z, \mathcal{I}_X)$ is the (saturated) homogeneous ideal of X. X is called arithmetically Gorenstein if its homogeneous ideal $I_X \subset R$ is a Gorenstein ideal. It is said that two subschemes $V, W \subset Z$ are linked by an arithmetically Gorenstein subscheme Z if the homogeneous ideals of V and W are linked by I_X.

In the following we will always make the assumption that R is a graded Gorenstein K-algebra of dimension $n + 1$ and that the codimension c of linked ideals satisfies $2 \leq c \leq n$.

3. RESOLUTIONS OF E-TYPE AND N-TYPE

The purpose of this section is to show the existence of maps Φ and Ψ from the set of even liaison classes in the set of stable equivalence classes of certain reflexive modules. This will be achieved by the help of resolutions of E-type and N-type. The name for these resolutions has been introduced in [22]. Note that the $(c - 1)$-presentation of an ideal of codimension c gives rise to a resolution of N-type.

DEFINITION 3.1. Let $I \subset R$ be a homogeneous ideal of codimension $c \geq 2$. Then an *E-type resolution* of I is an exact sequence of finitely generated graded R-modules,

$$0 \to E \to F_{c-1} \to \cdots \to F_1 \to I \to 0,$$

where the modules $F_1, \ldots, F_{c-1}$ are free.

An *N-type resolution* of I is an exact sequence of finitely generated graded R-modules,

$$0 \to G_c \to \cdots \to G_2 \to N \to I \to 0,$$

where $G_2, \ldots, G_c$ are free and $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n + 2 - c \leq i \leq n$.

These resolutions of I are said to be *minimal* if it is not possible to split off free direct summands.

Remark 3.2. (i) A (minimal) E-type resolution of I always exists because it is just the beginning of a (minimal) free resolution of I. It follows that

$$H^i_{\mathfrak{m}}(E) \cong H^{i-c}_{\mathfrak{m}}(R/I) \qquad \text{if } i \leq n.$$

In particular, I is saturated if and only if depth $E > c$. This shows that the sheafification of an E-type resolution gives an $\mathscr{E}$-type resolution of the ideal sheaf $\tilde{I}$ in the sense of [28, Definition 1.5] if and only if I is a saturated ideal.

(ii) Every ideal admits a (minimal) N-type resolution. To see this, consider a minimal $(c-1)$-presentation of I,

$$0 \to P \to N \to I \to 0.$$

Then Lemma 2.9 shows that by resolving P minimally, we obtain a resolution of N-type. Conversely, any N-type resolution gives rise to a $(c-1)$-presentation of I. Observe that we have by Lemma 2.9,

$$H^i_{\mathfrak{m}}(N) \cong \begin{cases} H^{i-1}_{\mathfrak{m}}(R/I) & \text{if } i \leq n + 1 - c \\ 0 & n + 2 - c \leq i \leq n. \end{cases}$$

If I is unmixed, then Lemma 2.11 and Proposition 2.5 imply that N^* is a $(c+1)$-syzygy; in particular, we get depth $N^* > c$. This shows that the sheafification of an N-type resolution gives an $\mathscr{N}$-type resolution of the ideal sheaf $\tilde{I}$ in the sense of [28, Definition 1.13] if and only I is of pure codimension c (cf. also [28, Corollary 1.20]).

(iii) The discussion above shows that every ideal admits a minimal resolution of E-type and N-type, respectively, which is uniquely determined up to isomorphisms of exact sequences.

(iv) It follows just from the definition that some properties of I are directly related to properties of E and N, respectively. For example, it is easy to see that E (respectively, N) is a maximal Cohen–Macaulay module if and only if R/I is Cohen–Macaulay. If I has finite projective dimension, then R/I is Cohen–Macaulay if and only if E (respectively, N) is a free module.

If I is of pure codimension c, then R/I is locally Cohen–Macaulay if and only if it has cohomology of finite length, and this holds true if and only if E (respectively, N) has cohomology of finite length. It follows that in the case where I has in addition finite projective dimension, Proj(R/I) is (locally) Cohen–Macaulay if and only if $\tilde{E}$ (respectively, $\tilde{N}$) is a vector bundle on Proj(R).

We want to spell out explicitly a consequence of Lemma 2.11. Note that the module E in an E-type resolution of an arbitrary ideal I of codimension c is always a c-syzygy. If I is unmixed, it is even $a(c+1)$-syzygy. More precisely, we have

LEMMA 3.3. *Let $I \subset R$ be a homogeneous ideal of codimension c $(2 \leq c \leq n)$ having E- and N-type resolution as in Definition* 3.1. *Then the following conditions are equivalent*:

(a) *I is of pure codimension c.*

(b) *N is reflexive.*

(c) *E is a $(c+1)$-syzygy.*

Proof. The equivalence of (a) and (b) is due to Lemma 2.11.

To prove the remaining assertions, we note that by Proposition 2.5 E is a $(c+1)$-syzygy if and only if

$$\dim R/\mathfrak{a}_i(E) \leq i - c - 1 \qquad \text{for all } i \leq n.$$

But we have already observed that

$$H^i_{\mathfrak{m}}(E) \cong H^{i-c}_{\mathfrak{m}}(R/I) \qquad \text{for all } i \leq n.$$

By inspection of Lemma 2.11 we conclude the claims. ∎

Next, we want to compare the resolutions of E- and N-type of linked ideals. For this we have to consider the mapping cone construction. The crucial observation of the following result is that sometimes we know a priori that we can split off the last module in the sequence provided by the mapping cone construction.

LEMMA 3.4. *Consider the following exact sequences of finitely generated graded R-modules*:

$$0 \to M' \xrightarrow{f} M \to M'' \to 0$$
$$0 \to B_m \to \cdots \to B_0 \to M' \to 0$$
$$0 \to C_m \to \cdots \to C_0 \to M \to 0.$$

If f can be lifted to a morphism of complexes $f_\bullet$: $B_\bullet \to C_\bullet$, then the mapping cone gives the exact sequence

$$0 \to B_m \to C_m \oplus B_{m-1} \to C_{m-1} \oplus B_{m-2} \to \cdots$$
$$\to C_1 \oplus B_0 \to C_0 \to M'' \to 0.$$

If, in addition, $m \geq 2$ and f_m: $B_m \to C_m$ is split-injective, then there is an exact sequence

$$0 \to C \oplus B_{m-1} \to C_{m-1} \oplus B_{m-2} \to \cdots \to C_1 \oplus B_0 \to C_0 \to M'' \to 0.$$

Here C is the module such that $C_m \cong \operatorname{im} f_m \oplus C$.

Proof. The first claim is fairly standard. Since $B_\bullet$ and $C_\bullet$ are acyclic, the mapping cone $cone(f_\bullet)$ is acyclic, too (cf., for example, [37, 1.5]). To prove the second assertion, we consider the following complexes, derived from the given complexes $B_\bullet$ and $C_\bullet$ in an obvious way:

$$B'_\bullet\colon 0 \to B_{m-1} \to \cdots \to B_0 \to 0$$
$$C'_\bullet\colon 0 \to C \to C_{m-1} \to \cdots \to C_0 \to 0.$$

Then $f_\bullet$ gives rise to a morphism $f'_\bullet$: $B'_\bullet \to C'_\bullet$. Thus we get a long exact homology sequence (cf. [37, 1.5.2]):

$$H_m(cone(f'_\bullet)) \to H_{m-1}(B'_\bullet) \to H_{m-1}(C'_\bullet) \to H_{m-1}(cone(f'_\bullet))$$
$$\xrightarrow{g} H_{m-2}(C'_\bullet) \to \cdots .$$

If $m > 2$, then $H_{m-2}(C'_\bullet) = 0$. If $m = 2$, then $\operatorname{im} g = \ker f = 0$. Hence, we always have $g = 0$. A little diagram chase shows $H_m(cone(f'_\bullet)) = 0$. Therefore the isomorphisms

$$H_{m-1}(B'_\bullet) \cong B_m \cong \operatorname{im} f_m \cong H_{m-1}(C'_\bullet)$$

imply $H_{m-1}(cone(f'_\bullet)) = 0$. It follows that $cone(f'_\bullet)$ is acyclic. This yields the desired exact sequence. ∎

The next result is a slight generalization of [25, Lemma 2]. It involves the canonical module $K_{R/I}$ of R/I as introduced in Section 2.

LEMMA 3.5. *If I and J are linked by $\mathfrak{c}$, then there is an exact sequence*:

$$0 \to K_{R/I}(1 - r(R/\mathfrak{c})) \to R/\mathfrak{c} \to R/J \to 0.$$

Proof. Since $R/\mathfrak{c}$ is Gorenstein by assumption, duality provides

$$\begin{aligned} K_{R/I}(1 - r(R/\mathfrak{c})) &\cong H_{\mathfrak{m}}^{\dim R/I}(R/I)^{\vee}(1 - r(R/\mathfrak{c})) \\ &\cong H_{\mathfrak{m}_{R/\mathfrak{c}}}^{\dim R/\mathfrak{c}}(R/I)^{\vee}(1 - r(R/\mathfrak{c})) \\ &\cong \operatorname{Hom}_{R/\mathfrak{c}}(R/I, R/\mathfrak{c}) \cong \operatorname{Hom}_R(R/I, R/\mathfrak{c}) \\ &\cong (c : I)/c \cong J/\mathfrak{c}. \end{aligned}$$

The assertion follows. ∎

The previous result can be used to compare Hilbert polynomials of linked ideals. We denote the Hilbert function $\operatorname{rank}_K[M]_t$ of a finitely generated graded R-module by $h_M(t)$. The Hilbert polynomial $p_M(t)$ is the polynomial such that $h_M(t) = p_M(t)$ for all sufficiently large t. It can be written in the form

$$p_M(t) = h_0(M)\binom{t}{d-1} + h_1(M)\binom{t}{d-2} + \cdots + h_{d-1}(M),$$

where $h_0(M), \ldots, h_{d-1}(M)$ are integers, $d = \dim M$ and $h_0(M) > 0$. For an ideal I the integer $\deg I = h_0(R/I)$ is called the degree of I. It is just the degree of the subscheme $\operatorname{Proj}(R/I)$. Now we can state the following.

COROLLARY 3.6. *Let $I, J \subset R$ be homogeneous ideals of pure codimension c linked by $\mathfrak{c}$. Then we have*

(a) $\deg J = \deg \mathfrak{c} - \deg I$ *and if* $c < n$,

$$h_1(R/J) = \tfrac{1}{2}(r(R/\mathfrak{c}) - n + c)[\deg I - \deg J] + h_1(R/I).$$

(b) *If R/I is locally Cohen–Macaulay, then we get*

$$p_{R/J}(t) = p_{R/\mathfrak{c}}(t) + (-1)^{n+1-c} p_{R/I}(r(R/\mathfrak{c}) - 1 - t), \qquad t \in \mathbb{Z}.$$

Proof. Using Lemma 3.5 and Lemma 2.11, we can copy the proof of the theorem in [25]. Thus we obtain claim (b),

$$\deg J = \deg \mathfrak{c} - \deg I,$$

and

$$(*) \quad h_1(R/J) = (r(R/\mathfrak{c}) - n + c)\deg I + h_1(R/I) + h_1(R/\mathfrak{c}).$$

Because of the symmetry of linkage, we can interchange I and J in $(*)$. Thus we get

$$h_1(R/\mathfrak{c}) = -\tfrac{1}{2}(r(R/\mathfrak{c}) - n + c)\deg \mathfrak{c}.$$

Plugging this into $(*)$ provides the second formula in (a). ∎

Remark 3.7. (i) The Corollary generalizes the theorem in [25] slightly. Hartshorne (cf. [16, Proposition 4.7]) has also found Corollary 3.6(b) in his more special context.

(ii) Suppose that $\dim R/I = 2$, i.e., $C_1 = \operatorname{Proj}(R/I)$ and $C_2 = \operatorname{Proj}(R/J)$ are curves. Let us denote their arithmetic genus by g_1 and g_2, respectively. Assume also that $R = K[x_0, \dots, x_n]$ is a polynomial ring and that C_1 and C_2 are linked by a complete intersection cut out by hypersurfaces of degree $d_1, \dots, d_{n-1}$. Using Lemma 2.3, we see that Corollary 3.6(a) takes the familiar form (cf. [24, Corollary 4.2.11])

$$g_1 - g_2 = \tfrac{1}{2}(d_1 + \cdots d_{n-1} - n - 1)[\deg C_1 - \deg C_2].$$

Thus one should view claim (a) above as a generalization of this useful formula.

Now we are ready to show that resolutions of E- and N-type are interchanged by linkage.

PROPOSITION 3.8. *Let $I, J \subset R$ be homogeneous ideals of pure codimension c linked by $\mathfrak{c}$. Suppose I has resolutions of E- and N-type as in Definition* 3.1. *Let*

$$0 \to D_c \to \cdots \to D_1 \to \mathfrak{c} \to 0$$

be a minimal free resolution of $\mathfrak{c}$. Put $s = \mathfrak{r}(R/\mathfrak{c}) - r(R)$. Then J has an N-type resolution,

$$0 \to F_1^*(-s) \to D_{c-1} \oplus F_2^*(-s) \to \cdots$$
$$\to D_2 \oplus F_{c-1}^*(-s) \to D_1 \oplus E^*(-s) \to J \to 0,$$

and an E-type resolution,

$$0 \to N^*(-s) \to D_{c-1} \oplus G_2^*(-s) \to \cdots \to D_1 \oplus G_c^*(-s) \to J \to 0.$$

Proof. We proceed in several steps. We begin by showing the first claim, starting with an E-type resolution of I.

(I) First, we will show that the complex

$$0 \to R \to F_1^* \to \cdots \to F_{c-1}^* \to E^* \to \operatorname{Ext}_R^c(R/I, R) \to 0,$$

obtained by dualizing the given E-type resolution, is in fact an exact sequence.

Indeed, resolving E, we get an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & F_{c+1} \arrow[r, "\varphi_{c+1}"] & F_c \arrow[rr, "\varphi_c"] \arrow[dr] & & F_{c-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "\varphi_1"] & I \arrow[r] & 0. \\
& & & E \arrow[ur] \arrow[dr] & & & & & \\
& & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

Dualizing with respect to R gives a complex

$$0 \to R \to F_1^* \to \cdots \to F_{c-1}^* \xrightarrow{\varphi_c^*} F_c^* \xrightarrow{\varphi_{a+1}^*} F_{c+1}^*$$

and an exact sequence

$$0 \to E^* \to F_c^* \xrightarrow{\varphi_{a+1}^*} F_{c+1}^*.$$

It follows that $\ker \varphi_{c+1}^* \cong E^*$; thus $\operatorname{Ext}_R^c(R/I, R) \cong E^*/\operatorname{im} \varphi_c^*$. Moreover, we get by duality that

$$\ker \varphi_{i+1}^*/\operatorname{im} \varphi_i^* \cong \operatorname{Ext}_R^i(R/I, R) \cong H_{\mathfrak{m}}^{n+1-i}(R/I)^\vee (1 - r(R)) = 0 \qquad \text{if } i < c,$$

because $\dim R/I = n + 1 - c$. Therefore we can splice together the two complexes above, and the resulting diagram,

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & F_1^* \arrow[r] & \cdots \arrow[r] & F_{c-1}^* \arrow[rr] \arrow[dr] & & E^* \arrow[r] & \operatorname{Ext}_R^c(R/I, R) \arrow[r] & 0, \\
& & & & & \operatorname{im} \varphi_c^* \arrow[ur] \arrow[dr] & & & \\
& & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is exact.

(II) Next, we want to show that

$$D_c \cong R(-s) \quad \text{and} \quad D_{c-i}^* \cong D_i(s) \qquad \text{for all } i \text{ with } 1 \le i < c.$$

Since $R/\mathfrak{c}$ is Gorenstein, D_c must have rank one, i.e., $D_c \cong R(-t)$ for some t. Dualizing the given minimal resolution of $\mathfrak{c}$, we get the minimal resolution of $\operatorname{Ext}_R^c(R/\mathfrak{c}, R)$:

$$0 \to R \to D_1^* \to \cdots \to D_{c-1}^* \to R(t) \to \operatorname{Ext}_R^c(R/\mathfrak{c}, R) \to 0.$$

We also have $\operatorname{Ext}_R^c(R/\mathfrak{c}, R) \cong K_{R/\mathfrak{c}}(1 - r(R))$ and $K_{R/\mathfrak{c}} \cong R/\mathfrak{c}(r(R/\mathfrak{c}) - 1)$. It follows that $\operatorname{Ext}_R^c(R/\mathfrak{c}, R) \cong R/\mathfrak{c}(s)$ and $t = s$. The remaining assertion of the claim is a consequence of the uniqueness properties of minimal resolutions.

(III) According to Lemma 3.5 and step (II), we have the following diagram with exact rows and column:

$$\begin{tikzcd}[column sep=small]
 & & & & & & & 0 \arrow[d] & \\
 & & & & & & & K_{R/J}(1 - r(R/\mathfrak{c})) \arrow[d] & \\
0 \arrow[r] & R(-s) \arrow[r] & D_{c-1} \arrow[r] & \cdots \arrow[r] & D_1 \arrow[r] & R \arrow[r] & & R/\mathfrak{c} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & E \arrow[r] & F_{c-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & R \arrow[r] & R/I \arrow[r] \arrow[d] & 0 \\
 & & & & & & & 0. &
\end{tikzcd}$$

Since the modules $D_1, \ldots, D_{c-1}$ are free, the epimorphism $R/\mathfrak{c} \to R/I$ lifts to a morphism of complexes. Thus, using step (II), we get by dualizing the commutative exact diagram:

$$\begin{tikzcd}[column sep=small]
 & & & & & & \operatorname{Ext}_R^{c-1}(K_{R/J}, R)(1 - r(R/c)) \arrow[d] & \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & F_1^* \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{c-1}^* \arrow[r] \arrow[d] & E^* \arrow[r] \arrow[d] & \operatorname{Ext}_R^c(R/I, R) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & R \arrow[r] & D_{c-1}(s) \arrow[r] & \cdots \arrow[r] & D_1(s) \arrow[r] & R(s) \arrow[r] & \operatorname{Ext}_R^c(R/\mathfrak{c}, R) \arrow[r] & 0.
\end{tikzcd}$$

Since $K_{R/J}$ has dimension $n + 1 - c$, we obtain by duality $\operatorname{Ext}_R^{c-1}(K_{R/J}, R) = 0$. Moreover, we have already seen that $\operatorname{Ext}_R^c(R/I, R) \cong K_{R/I}(s + 1 - r(R/\mathfrak{c}))$ and $\operatorname{Ext}_R^c(R/\mathfrak{c}, R) \cong R/\mathfrak{c}(s)$. Therefore it follows (cf. Lemma 3.5) that α is injective and $\operatorname{coker} \alpha \cong R/J(s)$. Then Lemma 3.4 gives the N-type resolution of J as claimed.

(IV) The proof for the E-type resolution of J is similar. We only sketch it. Replacing the E-type resolution of I by the given N-type resolution in the first diagram of step (III) after dualizing, we get the following exact commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & & & & & & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & N^* \arrow[r] \arrow[d] & G_2^* \arrow[r] \arrow[d] & \cdots \arrow[r] & G_c^* \arrow[r] \arrow[d] & K_{R/I}(s + 1 - r(R/\mathfrak{c})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] & D_{c-1}(s) \arrow[r] & D_{c-2}(s) \arrow[r] & \cdots \arrow[r] & R(s) \arrow[r] & R/\mathfrak{c}(s) \arrow[r] \arrow[d] & 0 \\
 & & & & & & R/J(s) \arrow[d] & \\
 & & & & & & 0. &
\end{tikzcd}$$

Again Lemma 3.4 gives the desired E-type resolution of J. ∎

Remark 3.9. Proposition 3.8 generalizes the corresponding results in [28, Section 1]. In fact, in the special case, where $\mathfrak{c}$ is generated by an R-regular sequence of elements of degrees $d_1, \dots, d_c$, we get $s = d_1 + \cdots + d_c$, due to Lemma 2.3.

Now we have to introduce some more notation.

Following Bruns (cf. [6] and [7]), here a finitely generated R-module M is said to be *orientable* if it has a rank and is locally free in codimension 1, and there is a homomorphism $\bigwedge^{\operatorname{rank} M} M \to R$ whose image has codimension at least 2. Note that M is orientable if it is locally free in codimension 1 and either R is factorial or M has finite projective dimension.

Recall that two graded maximal R-modules M and N are said to be *stably equivalent* if there are free R-modules F, G and an integer c such that

$$M \oplus F \cong N(c) \oplus G.$$

It is clear that stable equivalence is an equivalence relation.

Let I be an ideal of pure codimension c. We have seen in Remark 3.2 that the minimal E- and N-type resolutions of I are uniquely determined. Hence, according to Lemma 3.3 there is a well-defined map φ from the set of homogeneous ideals of pure codimension c $(2 \le c \le n)$ into the set of isomorphism classes of finitely generated $(c+1)$-syzygies where $\varphi(I)$ is just the last module in a minimal E-type resolution of I. It follows by [7, Proposition 2.8] that $\varphi(I)$ is orientable.

Similarly, because of Lemma 3.3 we get a well-defined map ψ from the set of homogeneous ideals of pure codimension c $(2 \le c \le n)$ into the set of isomorphism classes of finitely generated reflexive modules N with $H^i_{\mathfrak{m}}(N) = 0$ if $n - c + 2 \le i \le n$, by defining $\psi(I) = N$, if I has the minimal N-type resolution

$$0 \to G_c \to \cdots \to G_2 \to N \to I \to 0.$$

Thus $\psi(I)$ is an orientable module, too.

Now we state the main result of this section.

THEOREM 3.10. *The map φ induces a well-defined map Φ from the set of even liaison classes of ideals of pure codimension c into the set of stable equivalence classes of finitely generated, orientable $(c+1)$-syzygies being locally free in codimension $c - 1$.*

The map ψ induces a well-defined map Ψ from the set of even liaison classes of ideals of pure codimension c into the set of stable equivalence classes of finitely generated, orientable, reflexive modules N that satisfy $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n - c + 2 \le i \le n$ and are locally free in codimension $c - 1$.

Proof. Proposition 3.8 shows that the maps Φ and Ψ are well defined. Hence the first claim follows by Lemma 3.3, and the second one by Lemma 2.11. ∎

If R is just a polynomial ring over K then the statement has a simpler form.

COROLLARY 3.11. *Let R be a polynomial ring over K. Then the map φ induces a well-defined map Φ from the set of even liaison classes of ideals in R of pure codimension c into the set of stable equivalence classes of finitely generated $(c+1)$-syzygies.*

The map ψ induces a well-defined map Ψ from the set of even liaison classes of ideals in R of pure codimension c into the set of stable equivalence classes of finitely generated reflexive modules N that satisfy $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n - c + 2 \leq i \leq n$.

Proof. Since every finitely generated graded R-module has finite projective dimension, it is orientable (cf. [6]). Moreover, if M is a $(c+1)$-syzygy, it must be locally free in codimension $c+1$. Hence Theorem 3.10 implies the claim (cf. also Remark 2.12) ∎

The first part of the corollary is a generalization of [28, Proposition 2.4]. Now we consider again an arbitrary Gorenstein K-algebra.

Remark 3.12. Let M_c be the set of even liaison classes of ideals of codimension c, let M_E be the set of stable equivalence classes of finitely generated $(c+1)$-syzygies being locally free in codimension $c-1$ and having a rank, and let M_N be the set of stable equivalence classes of finitely generated reflexive modules N that are locally free in codimension $c-1$, have a rank, and satisfy $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n - c + 2 \leq i \leq n$. Then there are the following commutative diagrams:

$$\begin{array}{ccc} M_c & \xrightarrow{\Phi} & M_E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ M_c & \xrightarrow{\Psi} & M_N \end{array}$$

and

$$\begin{array}{ccc} M_c & \xrightarrow{\Psi} & M_N \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ M_c & \xrightarrow{\Phi} & M_E \end{array}$$

where α is induced by linkage and β is induced by dualization with respect to R.

Note that β is well defined because of Proposition 2.5.

Theorem 3.10 implies, for example, that in the case where $\varphi(I)$ and $\varphi(J)$ are not stably equivalent, the ideals $I, J \subset R$ do not belong to the same even liaison class. According to Remark 3.2, we know, with the cohomology modules of one of the modules R/I, $\varphi(I)$ and $\psi(I)$, the cohomology modules of all of them. Therefore Theorem 3.10 gives also necessary cohomological conditions for two ideals being in the same even liaison class.

COROLLARY 3.13. *Let I and I' be homogeneous ideals of pure codimension c. Then it holds that*

(a) *If I and I' belong to the same even liaison class, then there is an integer s such that*

$$H^i_{\mathfrak{m}}(R/I) \cong H^i_{\mathfrak{m}}(R/I')(s) \qquad \text{if } i \leq n - c.$$

(b) *If I and J are oddly linked and locally Cohen–Macaulay, then there is an integer s such that*

$$H^i_{\mathfrak{m}}(R/I) \cong H^{n+1-c-i}_{\mathfrak{m}}(R/J)^{\vee}(s) \qquad \text{if } 1 \leq i \leq n - c.$$

Moreover, if I and J are (directly) linked by $\mathfrak{c}$, then $s = 1 - r(R/\mathfrak{c})$.

Proof. According to Theorem 3.10 and Remark 3.2, it only remains to show the second claim of (b). Put $E = \varphi(I)$. Then we have by Proposition 3.8

$$H^i_{\mathfrak{m}}(E) \cong H^{i-c}_{\mathfrak{m}}(R/I) \qquad \text{if } i \leq n$$

and

$$H^i_{\mathfrak{m}}(E^*) \cong H^{i-1}_{\mathfrak{m}}(R/J) \qquad \text{if } i \leq n - c + 1.$$

Thus the claim is a consequence of

$$H^i_{\mathfrak{m}}(E^*) \cong H^{n+2-i}_{\mathfrak{m}}(E)^{\vee}(1 - r(R)) \qquad \text{if } 2 \leq i \leq n,$$

which follows by Proposition 2.2 ∎

Remark 3.14. Part (b) of the previous statement has been shown in [33] by different means. Other proofs have been given by Migliore [23] and Hartshorne [16, Proposition 4.7].

Note that (a) is a consequence of (b) if I is locally Cohen–Macaulay. Without this hypothesis part (a) was known before only for subschemes of projective space linked by complete intersections (cf. [16, Proposition 4.5]).

We have just seen that properties of evenly linked ideals are more closely related than those of oddly linked ideals. The next result also supports this philosophy. It strengths [16, Proposition 4.5(b)] and should be compared with Corollary 3.6.

LEMMA 3.15. *Suppose $I, J, I' \subset R$ are homogeneous ideals such that I and J are linked by $\mathfrak{c}$ and J and I' are linked by $\mathfrak{c}'$. Then it holds for all integers j that*

$$h_{R/I'}(j) = h_{R/I}(j + r(R/\mathfrak{c}) - r(R/\mathfrak{c}')) + h_{R/\mathfrak{c}'}(j) - h_{R/\mathfrak{c}}(j + r(R/\mathfrak{c}) - r(R/\mathfrak{c}')).$$

Proof. According to Lemma 3.5 we have the following exact sequences:

$$0 \to K_{R/J}(1 - r(R/\mathfrak{c})) \to R/\mathfrak{c} \to R/I \to 0$$
$$0 \to K_{R/J}(1 - r(R/\mathfrak{c}')) \to R/\mathfrak{c}' \to R/I' \to 0.$$

The claim follows. ∎

4. BASIC DOUBLE LINKAGE

Given an unmixed ideal I, basic double linkage is a technique that allows one to produce new ideals in the even liaison class of I. In fact, Hartshorne [16, Proposition 4.4] has shown that in the case of liaison by complete intersections, two ideals belong to the same even liaison class if and only if one of them can be obtained from the other ideal by a finite number of basic double links.

For Gorenstein liaison a more general notion of basic double linkage seems to be necessary. Here we propose one that still has some of the nice properties of the basic double linkage used so far in liaison theory.

Basic double linkage is related to liaison addition. The latter has been introduced by Schwartau [34] for ideals of grade 2 and has been generalized recently by Geramita and Migliore [15] (cf. also [4, Section 3]). Using ideas of [15], we can show the following result.

PROPOSITION 4.1. *Let $I_1, \ldots, I_c \subset R$ be homogeneous ideals such that $I_1, \ldots, I_s$ have codimension c and $I_{s+1} = \cdots = I_c = R$, where $1 \le s \le c$ and $2 \le c \le n$. Let $\{f_1, \ldots, f_c\}$ be an R-regular sequence of homogeneous ele-*

ments of degrees $d_1, \dots, d_c$ *satisfying*

$$(*) \qquad f_i \in \bigcap_{\substack{1 \le j \le c \\ j \ne i}} I_j \qquad (1 \le i \le c).$$

Then $J = f_1 I_1 + \cdots + f_c I_c$ *is an ideal of codimension* c *having the following properties*:

(a) $H^i_{\mathfrak{m}}(J) \cong \oplus_{j=1}^s H^i_{\mathfrak{m}}(I_j)(-d_j)$ *if* $i \le n - c + 1$.

(b) $h_{R/J}(k) = h_{R/\mathfrak{c}}(k) + \sum_{j=1}^s h_{R/I_j}(k - d_j)$ *for all integers* k *where* $\mathfrak{c} = (f_1, \dots, f_c)$. *In particular, it holds* $J \subset I_1 \cap \cdots \cap I_s \cap \mathfrak{c}$ *and* $\deg J = \deg I_1 + \cdots + \deg I_s + \deg \mathfrak{c}$.

(c) J *is saturated if and only if all ideals* $I_1, \dots, I_s$ *are saturated.*

(d) *Let* $N_j = \psi(I_j)$ $(1 \le j \le s)$. *Then* J *has an* N*-type resolution of the form*

$$0 \to G_c \to \cdots \to G_2 \to G_1 \oplus \bigoplus_{j+1}^{s} N_j(-d_j) \to J \to 0,$$

where the modules $G_1, \dots, G_c$ *are free*.

(e) J *is of pure codimension* c *if and only if all ideals* $I_1, \dots, I_s$ *are of pure codimension* c.

(f) *Let* (P) *be one of the following properties*: *Cohen–Macaulay, locally Cohen–Macaulay, or finite projective dimension. Then* R/J *has property* (P) *if and only if* R/I_i *has the property* (P) *for all* $i = 1, \dots, s$.

Proof. Since c is generated by an R-regular sequence, the Koszul complex $K_\bullet(\mathfrak{c}; R)$ provides a minimal free resolution,

$$0 \to F_c \to \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} \bigoplus_{j=1}^{c} R(-d_j) \xrightarrow{(f_1, \dots, f_c)} \mathfrak{c} \to 0.$$

Assumption $(*)$ provides that the ideals generated by $f_1, \dots, f_c$ but f_i are contained in I_i. Thus using the well-known matrix that describes φ_2, it follows that $\operatorname{im} \varphi_2 \subset \oplus_{j=1}^c I_j(-d_j)$. Hence, the Koszul complex above gives rise to an exact sequence:

$$0 \to F_c \to \cdots \xrightarrow{\varphi_3} F_2 \to \bigoplus_{j=1}^{c} I_j(-d_j) \to J \to 0.$$

Put $M = \operatorname{coker} \varphi_3$. Then we obtain from the Koszul resolution of $\mathfrak{c}$:

$$H^i_m(M) \cong H^{i-2}_m(R/\mathfrak{c}) = 0 \qquad \text{if} \quad i \le n \quad \text{and} \quad i \ne n - c + 3$$

and

$$h_M(k) = \sum_{j=1}^{c} h_R(k - d_j) - h_{\mathfrak{c}}(k), \qquad k \in \mathbb{Z}.$$

Hence the exact sequence

$$0 \to M \to \bigoplus_{j=1}^{c} I_j(-d_j) \to J \to 0 \qquad (**)$$

implies claims (a)–(c).

Moreover, the mapping cone procedure applied to $(**)$ using the free resolution of M provided by $K_\bullet(\mathfrak{c}; R)$ and the N-type resolutions of the various ideals I_j implies the existence of an exact sequence as asserted in (d). Now the module $\oplus_{j=1}^{s} N_j(-d_j)$ is reflexive is reflexive if and only if all of the modules N_j $(1 \leq j \leq s)$ are reflexive. Therefore claim (e) follows by Lemma 2.11. The last assertion is a consequence of (a) and (d). ∎

Remark 4.2. (i) Claim (e) seems to be new, even in the special situation considered in [15].

(ii) For possible variations of the statement, we refer to [15]. By now it should be clear how the corresponding proofs given there can be adapted.

It is said that the ideal J in Proposition 4.1 is obtained by liaison addition of the ideals $I_1, \ldots, I_c$. Claim (a) of the statement explains the expression "addition." The cohomology of the new ideal is the sum of the cohomologies of the ideals we started with. A relation to liaison will become transparent by the following construction.

DEFINITION 4.3. Let $I \subset R$ be a homogeneous ideal of pure codimension c where $2 \leq c \leq n$. Let $\mathfrak{d} = (f_2, \ldots, f_s) \subset I$ be a Gorenstein ideal of codimension $c - 1$, and let $f \in R$ be a homogeneous element of degree h such that $\mathfrak{d} : f = \mathfrak{d}$. Then the ideal $I' = fI + \mathfrak{d}$ is said to be a *basic double link* of I (*of shift* h).

Martin-Deschamps and Perrin [22] have proposed the notion basic double link of height h. But h has noting to do with the height of the ideal J. Thus we want to use the modification above.

Our proposal for basic double linkage is more general than the one used so far in liaison theory. In the case in which $\mathfrak{d}$ is a complete intersection, our notion of basic double linkage agrees with the standard one, and basic double linkage is a special case of liaison addition.

The next result also justified the terminology "basic double link."

PROPOSITION 4.4. *If I' is a basic double link of I of shift h such that $I' = fI + \mathfrak{d}$, then I' is evenly linked to I. Moreover, if I has resolutions of E- and N-type as in Definition 3.1 and $\mathfrak{c} = fR + \mathfrak{d}$ has the minimal free resolution*

$$0 \to P_c \to \cdots \xrightarrow{\varphi_3} P_2 \xrightarrow{\varphi_2} R(-h) \oplus P_1 \to \mathfrak{c} \to 0,$$

then I' has resolutions of E- and N-type:

$$0 \to E(-h) \oplus P_c \to F_{c-1}(-h) \oplus P_{c-1} \to \cdots \to F_1(-h) \oplus P_1 \to I' \to 0$$
$$0 \to G_c(-h) \oplus P_c \to \cdots \to G_2(-h) \oplus P_2 \to N(-h) \oplus P_1 \to I' \to 0.$$

Moreover, the Hilbert function increases, i.e., it holds

$$h_{R/I'}(k) \geq h_{R/I}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

Proof. Since $\mathfrak{d}$ is a Gorenstein ideal and the image of f in $R/\mathfrak{d}$ is $R/\mathfrak{d}$-regular, the ideal $\mathfrak{c} = fR + \mathfrak{d}$ is a Gorenstein ideal of codimension c. Let us denote the minimal generators of $\mathfrak{d}$ by $f_2, \ldots, f_s$ and their degrees by $d_2, \ldots, d_s$. Let $r_1, \ldots, r_s \in R$ be elements such that $r_1 f + \sum_{j=2}^{s} r_j f_j = 0$. It follows that $r_1 \in \mathfrak{d}: f = \mathfrak{d} \subset I$; thus we get $\operatorname{im} \varphi_2 \subset I(-h) \oplus P_1 = I(-h) \oplus \bigoplus_{j=2}^{s} R(-d_j)$, and we can proceed as in the proof of Proposition 4.1. We obtain an exact sequence,

$$0 \to M \to I(-h) \oplus \bigoplus_{j=2}^{s} R(-d_j) \to I' \to 0,$$

where $M = \operatorname{coker} \varphi_3$. Hence the mapping cone procedure gives the asserted E- and N-type resolution of I'.

By the resolution of $\mathfrak{c}$ we have

$$h_M(k) = h_R(k-h) + \sum_{j=2}^{s} h_R(k-d_j) - h_{\mathfrak{c}}(k), \qquad k \in \mathbb{Z}.$$

Thus the short exact sequence above provides

$$h_{R/I'}(k) = h_{R/\mathfrak{c}}(k) + h_{R/I}(k-h), \qquad k \in \mathbb{Z}. \qquad (*)$$

Since $\operatorname{codim} I > \operatorname{codim} \mathfrak{d}$, we can find a homogeneous element $g \in I$ such that $\mathfrak{d} : g = \mathfrak{d}$ and $\mathfrak{c}' = gR + \mathfrak{d}$ is properly contained in I. Then we also get $\mathfrak{d} : fg = \mathfrak{d}$. Hence $\mathfrak{c}'$ and $\mathfrak{c}'' = fgR + \mathfrak{d}$ are Gorenstein ideals of codimension c and $I'' = \mathfrak{c}'' : (\mathfrak{c}' : I)$ is evenly linked to I. We want to show $I' = I''$.

According to Lemma 2.3 we obtain $r(R/\mathfrak{c}') - r(R/\mathfrak{c}'') = -h$. Therefore Lemma 3.15 provides

$$h_{R/I''}(k) = h_{R/I}(k-h) + h_{R/\mathfrak{c}''}(k) - h_{R/\mathfrak{c}'}(k-h).$$

Since the residue classes of f, g, fg are $R/\mathfrak{d}$-regular, we get

$$h_{R/\mathfrak{c}}(k) = h_{R/\mathfrak{c}''}(k) - h_{R/\mathfrak{c}'}(k-h).$$

Using $(*)$ we obtain

$$h_{R/I'}(k) = h_{R/I''}(k) \qquad \text{for all } k \in \mathbb{Z}.$$

On the other hand, we have $\mathfrak{c}'' \subset I''$ and

$$fI \subset (gf, f_2, \ldots, f_s)I : (g, f_2, \ldots, f_s) = \mathfrak{c}''I : c' \subset I''.$$

It follows that $I' = fI + \mathfrak{c}'' \subset I''$. Since the Hilbert functions of I' and I'' coincide, we finally get $I' = I''$.

Our final claim follows because $I' \subset I$. ∎

If the ground field K is infinite and the ideal $\mathfrak{d} \subset I$ is chosen, it is always possible to find an element f of degree 1 to perform a double link.

5. MINIMAL SHIFT

To study a single liaison class $\mathscr{L}$, we note that the map ψ introduced before Theorem 3.10 can be used to give $\mathscr{L}$ a $\mathbb{Z}$-graded structure. It gives rise to the notions of minimal shift and minimal elements of $\mathscr{L}$.

For a finitely generated module M, we denote the stable equivalence class containing M by $[M]$. If M is not free, we can write $M \cong M_0 \oplus F$, where F is a free module and $M_0 \neq 0$ does not have a nontrivial free direct summand. Then every element of $[M]$ is of the form $M_0(t) \oplus G$ for some integer t and some free module G.

Let I be a homogeneous ideal of pure codimension c that is not perfect. We denote the even liaison class it generates by $\mathscr{L}_I$. Put $N = \psi(I)$. Then we have $\Psi(\mathscr{L}_I) = [N] = [N_0]$, that is, for all $I' \in \mathscr{L}_I$ there is a unique integer t and a free module F such that

$$\psi(I') = N_0(t) \oplus F.$$

Lemma 4.4 implies that for any integer $h \geq 0$ there is a free module F and an ideal $I' \in \mathscr{L}_I$ such that $N_0(t-h) \oplus F = \psi(I')$. The next result shows that in the case where $N_0(t) \oplus F \in \Psi(\mathscr{L}_I)$, the integer t cannot be arbitrarily large.

PROPOSITION 5.1. *Let $I \subset R$ be a homogeneous ideal of pure codimension c $(2 \le c \le n)$ with $(c-1)$-presentation*

$$0 \to P \to N_0(t) \oplus F \xrightarrow{\gamma} I \to 0,$$

where F is a free module and N_0 does not have a free direct summand. Suppose that I is not a perfect ideal. Then it holds that

$$t \le e\left(H_{\mathfrak{m}}^{n+1}(N_0)\right) - r(R) + 1.$$

If N^* has finite projective dimensin, then we even have

$$t \le e\left(H_{\mathfrak{m}}^{n+1}(N_0)\right) - r(R).$$

Proof. Let $\gamma_0 \colon N_0(t) \to I$ and $\gamma_1 \colon F \to I$ be the maps induced by γ.
Suppose that γ_0 is the zero map. Then we get

$$P = \ker \gamma = N_0(t) \oplus \ker \gamma_1.$$

This is impossible if R/I is Cohen–Macaulay. Indeed, I is not perfect by assumption. It follows that I has infinite projective dimension, and thus the same applies to N_0 by the given $(c-1)$-presentation of I. Hence, N_0 cannot be a direct summand of P, because P has projective dimension $c-2$.

If R/I is not Cohen–Macaulay, then we obtain by Lemma 2.9 that depth $N_0 =$ depth $R/I + 1 \le \dim R/I = n + 1 - c$. This is again a contradiction to the identity above, since depth $P = n + 3 - c$.

Therefore we have shown that $\gamma_0 \neq 0$. Now, according to Lemma 2.11 the module N_0 is reflexive. It follows that the dual map $\gamma_0^* \colon R \to N_0^*(-t)$ cannot be zero. It follows that

$$0 \ge a\left(N_0^*(-t)\right) = a(N_0^*) + t.$$

Thus the claim is furnished by duality.

Assume now that N^* has finite projective dimension. Let $y \in [N_0^*]_{-t}$ be the image of the unit element of R. Since N_0 is reflexive, the order ideal of y is contained in I, and thus it has codimension $\le c$. Since N_0^* is a $(c+1)$-syzygy due to Proposition 2.5, it follows by [12, Theorem 3.14] that y is not a minimal generator of $N_0^*(-t)$. Therefore we obtain

$$0 > a\left(N_0^*(-t)\right) = a(N_0^*) + t$$

and conclude as above. ∎

It is not clear (to the author) if the improved bound always holds true. At least we will see in Example 5.5 below that it cannot be further improved in general.

The last result shows that the following notions are well defined.

DEFINITION 5.2. Let $\mathscr{L}$ be an even liaison class of nonperfect ideals, and let $N_0 \in \Psi(\mathscr{L})$ be a module that does not have a nontrivial free direct summand. Then the greatest integer t_0 such that there is an ideal $I \in \mathscr{L}$ and a free module F with $\psi(I) = N_0(t_0) \oplus F$ is said to be the *minimal shift* of $\mathscr{L}$.

For an integer $h \geq 0$ we denote by $\mathscr{L}^h$ the set of ideals $I \in \mathscr{L}$ such that $\psi(I) = N_0(t_0 - h) \oplus F'$ for some free module F'. If $I \in \mathscr{L}^h$, we say that I *occurs in shift* h. The elements of $\mathscr{L}^0$ are called *minimal elements* of their even liaison class $\mathscr{L}$.

The notation above is also used for a subscheme $X \subset \operatorname{Proj} R$ by identifying X with its homogeneous ideal $I(X) \subset R$. It extends the one introduced in [4] for subschemes of codimension 2 to arbitrary codimension. Indeed, we will see in the next section that an ideal of codimension 2 is not contained in the liaison class of a complete intersection if and only if it is not perfect. Thus Proposition 5.1 may be viewed as a generalization and strengthening of [4, Lemma 4.1].

If R/I is not Cohen–Macaulay, its cohomology groups can be used to determine the shift in which I occurs. This is the approach taken in [3, Proposition 1.4], where the existence of a minimal shift was first established.

COROLLARY 5.3. *Let $\mathscr{L}$ be an even liaison class of ideals I such that R/I is not Cohen–Macaulay. Let $I \in \mathscr{L}^0$ be a minimal element and let $I' \in \mathscr{L}$ be another ideal. Then there is an integer $h \geq 0$ such that*

$$(*) \qquad H^i_{\mathfrak{m}}(I) \cong H^i_{\mathfrak{m}}(I')(-h) \qquad \text{for all } i \leq n + 1 - c.$$

Moreover, I' occurs in shift h if and only if $(*)$ holds.

Proof. Let

$$0 \to F_c \to \cdots \to F_2 \to N_0(t_0) \oplus F_1 \to I \to 0$$

be an N-type resolution of I where N_0 does not have a free direct summand. Then by Theorem 3.10 I' has an IN-type resolution,

$$0 \to F'_c \to \cdots \to F'_2 \to N_0(t) \oplus F'_1 \to I' \to 0,$$

where $t \leq t_0$ because I is minimal. Because of Lemma 2.9, it holds for all $i \leq n + 1 - c$ that

$$H^i_{\mathfrak{m}}(I') \cong H^i_{\mathfrak{m}}(N_0)(t) \cong H^i_{\mathfrak{m}}(I)(t - t_0),$$

showing our assertions. ∎

The last two results say that the cohomology modules of an unmixed ideal I such that R/I is not Cohen–Macaulay cannot be shifted indefinitely leftward. The ideals whose associated modules attain the leftmost possible shift are the minimal elements justifying the terminology minimal shift.

In [26] and [27] arithmetically Buchsbaum subschemes of projective space have been characterized by admitting a so-called Omega-resolution, and the occurring Omega-resolutions have been studied. These results imply a sufficient condition for an arithmetically Buchsbaum subscheme being minimal.

COROLLARY 5.4. *Let $X \subset \mathbb{P}^n$ be an arithmetically Buchsbaum subscheme of codimension c. Then X has a minimal N-type resolution of the form*

$$0 \to \mathscr{F}_c \to \cdots \to \mathscr{F}_2 \to \mathscr{F}_1 \oplus \bigoplus_j \left(\Omega^{p_j}_{\mathbb{P}^n}(-e_j)\right)^{s_j} \to \mathscr{I}_X \to 0,$$

where $\mathscr{F}_1, \ldots, \mathscr{F}_c$ are direct sums of line bundles. If $\min\{p_j + e_j\} = 1$, then X is minimal in its even liaison class.

Proof. According to Proposition II.3.12 in [26] (cf. also [27]), it holds for all integers j that $p_j + e_j \geq 1$. Thus the claim follows by Corollary 5.3. ∎

EXAMPLE 5.5 (cf. [3, Example 1.6]). Let $X \subset \mathbb{P}^4$ be a general projection of the Veronese surface in $\mathbb{P}^5$. Then X has an N-type resolution,

$$0 \to \left(\mathscr{O}_{\mathbb{P}^4}(-3)\right)^3 \to \Omega_{\mathbb{P}^4} \to \mathscr{I}_X \to 0.$$

Hence X is minimal in its even liaison class by the last result. Since $e(H^4_*(\Omega_{\mathbb{P}^4})) = -4$, we could have also deduced this fact by Proposition 5.1. This shows that the bound in that result is optimal for X.

6. RAO'S CORRESPONDENCE

In this section we consider exclusively liaison classes of codimension 2 ideals of an integral domain. The main result is that the maps Φ and Ψ are bijective. The resulting bijections between liaison classes and the stable equivalence classes of reflexive sheafs described in Theorem 3.10 are usually called Rao's correspondence. Rao [31] was the first to establish it (in a special case).

Throughout the rest of the paper we will assume that besides being Gorenstein, R is also an integral domain.

We first prove injectivity of the maps Φ and Ψ by using just the tools described in the previous sections. The method is essentially due to Rao [31]. Unfortunately, it does not work in codimension ≥ 3.

LEMMA 6.1. *Let I be a homogeneous ideal of pure codimension* 2 *with a minimal N-type resolution*,

$$0 \to F \to N \xrightarrow{\gamma} I \to 0.$$

If $N \cong N' \oplus R(-d)$, then we can link I twice to obtain an ideal $I' \in \mathscr{L}_I$ with N-type resolution

$$0 \to F' \to N'(h) \to I' \to 0$$

for some integer h.

Proof. Let f be the generator of the image of $\gamma|_{R(-d)}$. Since the given N-type resolution is minimal, f is a minimal generator of I of degree d. Thus f is a nonzero divisor of the domain R. We choose an $f' \in I$ such that $\{f, f'\}$ is an R-regular sequence. Put $d' = deg\ f'$, $\mathfrak{c} = (f, f')$ and $J = \mathfrak{c} : I$. The embedding $\mathfrak{c} \to I$ gives rise to a map $R(-d) \oplus R(-d') \to N \cong N' \oplus R(-d)$ whose restriction to $R(-d)$ is an isomorphism. Hence, we obtain by Proposition 3.8 an E-type resolution

$$0 \to N^*(-d-d') \xrightarrow{\delta} R(-d) \oplus R(-d') \oplus F^*(-d-d') \to J \to 0,$$

where δ maps $N^*(-d-d')$ onto $R(-d)$. Thus we can split off $R(-d)$ and get the exact sequence

$$0 \to (N')^*(-d-d') \to R(-d') \oplus F^*(-d-d') \to J \to 0.$$

Now we choose minimal generators g, g' of J of degree e and e', respectively, such that $\mathfrak{c}' = (g, g')$ is a complete intersection. Thus $R(-e) \oplus R(-e')$ is a direct summand of $R(-d') \oplus F^*(-d-d')$. Hence, we can split off $R(-e) \oplus R(-e')$ in the N-type resolution of $I' = \mathfrak{c}' : J$ obtained by Proposition 3.8, that is, I' has an N-type resolution of the form

$$0 \to F' \to N'(d + d' - e - e') \to I \to 0. \quad \blacksquare$$

The next result shows that for ideals of pure codimension 2, the map Ψ is injective.

PROPOSITION 6.2. *Let $I, I' \subset R$ be homogeneous ideals of pure codimension* 2 *with N-type resolutions of the form*

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \to N \to I \to 0$$

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \to N(h) \to I' \to 0.$$

Then I and I' belong to the same even liaison class.

Proof. If N is a free module, then I and I' are perfect ideals. In this case it is well known that we can link both ideals in $s-1$ steps to a complete intersection. Moreover, all complete intersections belong to the same liaison class, which is, in fact, an even liaison class because a complete intersection can be linked to itself.

Thus we assume now that N is not free. Applying Lemma 6.1 repeatedly, we may even assume that N does not have a free direct summand.

Let m_i $(1 \leq i \leq s)$ be the generator of the image of $R(-a_i)$ in N, and let n_i be the generator of the image of $R(-b_i)$ in N. Suppose that $m_i = n_i$ for $i < t \leq s$. We want to show that we can find ideals I_1 and I_1' in the even linkage class of I and I', respectively, with resolution where $m_i = n_i$ for $i \leq t$.

For this, we choose elements $u, v \in [N]_p$ whose images f_1, f_2 in I and g_1, g_2 in I' generate complete intersection $\mathfrak{c}$ and $\mathfrak{c}'$, respectively. If p is sufficiently large, this is certainly possible. According to Proposition 3.8 the ideals $J = \mathfrak{c} : I$ and $j' = \mathfrak{c}' : I'$ have E-type resolutions as follows:

$$0 \to N^*(-2p) \to R^2(-p) \oplus \bigoplus_{i=1}^{s} R(a_i - 2p) \to J \to 0,$$

$$0 \to N^*(2h - 2p) \to R^2(h-p) \oplus \bigoplus_{i=1}^{s} R(b_i + 2h - 2p) \to J' \to 0.$$

Let $f \in J$ be the generator o the image of $R(a_t - 2p)$ in J, and let $g \in J'$ be the generator of the image of $R(b_t + 2h - 2p)$; f and g are nonzero divisors because N does not have a free direct summand. Since $\{f_1, f_2\}$ and $\{g_1, g_2\}$ are regular sequences, it is possible to find $\lambda, \mu \in K$ such that $\mathfrak{d} = (f, f')$ and $\mathfrak{d}' = (g, g')$ are complete intersections where $f' = \lambda f_1 + \mu f_2$ and $g' = \lambda g_1 + \mu g_2$. Put $I_1 = \mathfrak{d} : J$ and $I_1' = \mathfrak{d}' : J'$. Since N does not have a free direct summand, the E-type resolutions of J and J' above must be minimal. It follows that f, f' are minimal generators of J and that g, g' are minimal generators of J'. Therefore we can split off $R(-p) \oplus R(a_t - 2p)$ (respectively, $R(h-p) \oplus R(b_t + 2h - 2p)$) in the N-type resolution of I_1 (respectively, I_1') given by Proposition 3.8. The resulting resolutions are

$$0 \to \bigoplus_{i \neq t} R(a_t - a_i - p) \oplus R(a_t - 2p) \xrightarrow{\alpha} N(a_t - p) \to I_1 \to 0,$$

$$0 \to \bigoplus_{i \neq t} R(b_t - b_i + h - p) \oplus R(b_t + 2h - 2p) \xrightarrow{\alpha'} N(b_t + h - p)$$
$$\to I_1' \to 0,$$

where the image of α is generated by $m_1, \dots, m_{t-1}, m_{t+1}, \dots, m_s, \lambda u + \mu v$, and the image of α' is generated by $n_1, \dots, n_{t-1}, n_{t+1}, \dots, n_s, \lambda u + \mu v$. This means that we have replaced m_t and n_t by $\lambda u + \mu v$. Repeating this process, we finally get ideals I_s and I'_s with resolutions where the images of the corresponding maps coincide. It follows that $I_s = I'_s$, completing the proof. ∎

Now we want to study the image of the map ψ. This requires a result not contained in the previous sections. It is usually referred to as the Theorem of Bourbaki and is related to the existence of basic elements.

Let M be a finitely generated R-module. A submodule $M' \subset M$ is said to be *w-fold basic* in M at a prime ideal $\mathfrak{p} \subset R$ if the minimal number of generators of $(M/M')_{\mathfrak{p}}$ is at most the number of minimal generators of $M_{\mathfrak{p}}$ minus w. We denote by $\mathscr{C}$ the set of homogeneous prime ideals $\mathfrak{p} \subset R$ having codimension ≤ 1. Then we call $M' \subset M$ w-fold basic in M at $\mathscr{C}$ if it is w-fold basic in M at all primes $\mathfrak{p} \in \mathscr{C}$.

PROPOSITION 6.3. *Let M be a torsion-free module of rank $s+1$ that is locally free in codimension* 1. *Then it holds that*

(a) *There are homogeneous elements $m_1, \dots, m_s \in M$ generating a submodule M' that is s-fold basic in M at $\mathscr{C}$.*

(b) *If M is orientable and $M' \subset M$ is any submodule generated by s elements of degree $a_1, \dots, a_s$ that is s-fold basic in M at $\mathscr{C}$, then there is an exact sequence*

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \xrightarrow{\alpha} M \xrightarrow{\nu} I(h) \to 0,$$

where $\operatorname{im} \alpha = M'$ *and $I \subset R$ is a homogeneous ideal of codimension* ≥ 2.

If M^* is not a free R-module, then $\operatorname{codim} I = 2$.

(c) *Suppose M is a reflexive module that is not free and there is an embedding* $\alpha\colon \oplus_{i=1}^{s} R(-a_i) \to M$. *Then* $\operatorname{coker} \alpha$ *is isomorphic to a homogeneous ideal of pure codimension* 2 *if and only if* $\operatorname{im} \alpha$ *is s-fold basic in M at $\mathscr{C}$ and M is orientable.*

Proof. Claim (a) and the first part of (b) are essentially a special case of results of Bruns. Following [5], it is possible to find homogeneous elements m_i successively such that m_i is 1-fold basic in $M/(\sum_{j=1}^{i-1} m_j R(-a_j))$ at $\mathscr{C}$. It follows that the module M' generated by $m_1, \dots, m_s$ is s-fold basic in M at $\mathscr{C}$ and that M/M' satisfies condition $(\tilde{S}_1)$. Hence (cf. Proposition 2.5) M/M' is a graded torsion-free module of rank 1, and thus it is isomorphic to a homogeneous ideal I that has codimension ≥ 2 due to [7, Proposition 2.8].

Now assume $\operatorname{codim} I \geq 3$. Then after dualizing we obtain the exact sequence

$$0 \to R(-h) \to M^* \to \bigoplus_{i=1}^{s} R(a_i) \to 0,$$

which splits. This gives a contradiction if M^* is not a free module, completing the proof of assertion (b).

To show claim (c), we assume first that $\operatorname{coker} \alpha$ is isomorphic to an ideal I of pure codimension 2. Then M is orientable, and localization of the exact sequence

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \xrightarrow{\alpha} M \to I(h) \to 0$$

at prime ideals of codimension ≤ 1 gives an exact sequence that splits, showing that $\operatorname{im} \alpha$ is s-fold basic in M at $\mathscr{C}$.

For the converse let, as in (b), $I(h) \cong \operatorname{coker} \alpha$ be an ideal of codimension 2. We have

$$H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(I) \cong H^{i-1}_{\mathfrak{m}}(R/I) \qquad \text{if } i < n.$$

Since M is reflexive it follows by Proposition 2.5 and Lemma 2.11 that I has pure codimension 2. ∎

Now our generalization of Rao's correspondence follows easily.

THEOREM 6.4. *The map Φ gives a bijective correspondence between the even liaison classes of homogeneous ideals of pure codimension 2 and the set of stable equivalence classes of finitely generated, orientable 3-syzygies.*

The map Ψ is a bijection between this set of even liaison classes and the set of stable equivalence classes of finitely generated, orientable, reflexive modules N satisfying $H^n_{\mathfrak{m}}(N) = 0$.

Proof. First we consider the map Ψ. By Proposition 6.2 we know that Ψ is injective and Proposition 6.3 implies that it is surjective. Thus Theorem 3.10 shows the second claim. It also proves the first one by Remark 3.12. ∎

Remark 6.5. Using Corollary 3.11 we see that the theorem simplifies if R is just a polynomial ring. This special case gives Theorem 2.11 and Theorem 2.12 of [28]. In fact, using Proposition 2.5 and duality, it is not too difficult to see that E is a 3-syzygy if and only if $\mathscr{E}xt(\tilde{E}^*, \mathscr{O}_{\mathbb{P}^n}) = 0$ and $H^1_*(\mathbb{P}^n, \tilde{E}) = 0$, and that $H^n_{\mathfrak{m}}(N) = 0$ if and only if $\mathscr{E}xt(\tilde{N}, \mathscr{O}_{\mathbb{P}^n}) = 0$ and $H^1_*(\mathbb{P}^n, \tilde{N}^*) = 0$.

Sometimes there is a cohomological criterion for deciding if two ideals are evenly linked. The corresponding result generalizes one known before for locally perfect ideals. It applies, for example, to projectively normal surfaces.

COROLLARY 6.6. *Let $I, J \subset R$ be homogeneous ideals of pure codimension 2. Suppose I and J have finite projective dimension and there is an integer t such that for all $i < n$ it holds that $H^i_{\mathfrak{m}}(I) = 0$ if $i \neq t$. Then I and J belong to the same even liaison class if and only if there is an integer a such that*

$$H^i_{\mathfrak{m}}(I) \cong H^i_{\mathfrak{m}}(J)(a) \qquad \textit{for all } i < n.$$

Proof. The assumption on I implies that $\psi(I) = N$ is a reflexive Eilenberg–MacLane module of finite projective dimension. Hence the claim follows by Theorem 6.4 and Lemma 2.7. ∎

EXAMPLE 6.7. Suppose that R is a Gorenstein ring of dimension 4 that is not a regular ring. Let N be a minimal 2-syzygy of the residue field $K \cong R/\mathfrak{m}$. According to Proposition 6.3, there is an unmixed ideal $I \subset R$ of codimension 2 such that $\psi(I) = N(t)$ for some integer t. It follows that

$$K \cong H^2_{\mathfrak{m}}(N) \cong H^1_{\mathfrak{m}}(R/I)(-t).$$

Let J be an ideal that is (directly) linked to I. Then we obtain by Corollary 3.13 for some integer s,

$$H^1_{\mathfrak{m}}(R/J)(s) \cong K^{\vee} \cong K \cong H^1_{\mathfrak{m}}(R/I)(-t),$$

i.e., the ideals I and J define curves in $\operatorname{Proj} R$ having (up to shift) isomorphic Hartshorne–Rao modules. Nevertheless, I and J do not belong to the same even liaison class. Indeed, since R is not regular by assumption, K and thus N have infinite projective dimension. But $N^*(u) = \varphi(J)$ and thus also $\psi(J)$ have finite projective dimension. It follows that N and $\psi(J)$ cannot be stably equivalent. Hence Theorem 6.4 yields that I and J are in different even liaison classes.

This example shows that we cannot skip the assumption on the projective dimension in the previous corollary. It also shows that the comparison of local cohomology modules of two ideals as in Corollary 3.13 gives strictly weaker necessary conditions than Theorem 3.10 for two ideals being in the same even liaison class.

All of the arguments used in the proof of Theorem 6.4 are also available in the local context. Thus adapting suitable notation, it follows that the previous theorem has the following local analogue.

THEOREM 6.8. *Let R be a local Gorenstein ring containing an infinite field. Suppose* $\dim R = n + 1 \geq 3$. *Then there is a bijective correspondence* (*induced by resolutions of E-type*) *between the set of even liaison classes of ideals in R having pure codimension* 2 *and stable equivalence classes of finitely generated, orientable* 3-*syzygies.*

There is also a bijective correspondence (*induced by resolutions of N-type*) *between this set of even liaison classes and the set of stable equivalence classes of finitely generated, orientable reflexive modules N satisfying $H^n_{\mathfrak{m}}(N) = 0$.*

7. LAZARSFELD–RAO PROPERTY

We continue our study of liaison in codimension 2. Having described all even liaison classes, we are now going to investigate the structure of an even liaison class that does not contain a complete intersection. This structure has been first established for certain even liaison classes by Lazarsfeld and Rao in [21]. The term "Lazarsfeld–Rao property" was introduced in [3]. The method of proving it is essentially due to Ballico, Bollondi, and Migliore (cf. [2]).

Suppose we are given exact sequences

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \to N \to I \to 0$$

and

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \to N \to I'(h) \to 0,$$

where I and I' are homogeneous ideals of codimension 2 and N is not free. Then, using Hilbert polynomials, it follows that $\sum_{i=1}^{s} b_i - \sum_{i=1}^{s} a_i = h$. The crucial observation is that in the case where I and I' both occur in the minimal shift of their even liaison class, it follows not just that $\sum_{i=1}^{s} b_i = \sum_{i=1}^{s} a_i$, but even $a_i = b_i$ for all $i = 1, \dots, s$. To show this we need the following.

LEMMA 7.1. *Let M be a reflexive module of rank $s + 1$ that is locally free in codimension* 1. *Suppose that M is not free and that there are exact sequences*

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \xrightarrow{\alpha} M \to I \to 0,$$

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \xrightarrow{\beta} M \to I'(h) \to 0,$$

where I and I' have pure codimension 2. *Order the degree shifts such that $a_1 \le a_2 \le \cdots \le a_s$ and $b_1 \le b_2 \le \cdots \le b_s$. Then there is an exact sequence*

$$0 \to \bigoplus_{i=1}^{s} R(-c_i) \to M \to J(h') \to 0,$$

where $c_i = \min\{a_i, b_i\}$ and J is an ideal of pure codimension 2.

Proof. Let $\operatorname{im}\alpha$ be generated by $m_1, \ldots, m_s$, and let $\operatorname{im}\beta$ be generated by $n_1, \ldots, n_s$, where $\deg m_i = a_i$ and $\deg n_i = b_i$. We may assume $a_i = b_i$ for all $i < t \le s$ and $a_t < b_t$. We want to show:

Claim. There is an exact sequence

$$0 \to \bigoplus_{i=1}^{s} R(-c_i) \to M \to J'(h'') \to 0,$$

where J' is an ideal of pure codimension 2 and

$$c_i = \begin{cases} a_i & \text{if } i \le t \\ b_i & \text{if } i > t. \end{cases}$$

To prove this, we first note that, replacing m_i and n_i by a general linear combination, we may assume $m_i = n_i$ if $i < t$. Put $M' = M/(\sum_{j=1}^{t-1} m_j R(-a_j))$. Since I' is of pure codimension 2, the image P of the submodule $\sum_{j=t}^{s} n_j R(-b_j)$ in M' is $(s-t+1)$-fold basic in M' at $\mathscr{C}$. Thus the same applies to any submodule of M' containing P because of the following fact. If $\gamma\colon N \to N'$ is an epimorphism of finitely generated modules over a local ring, then N' has not more minimal generators than N. Therefore the image of $m_t R(-a_t) + \sum_{j=t}^{s} n_j R(-b_j)$ is $(s-t+1)$-fold basic in M' at $\mathscr{C}$. Since I is also of pure codimension 2, it follows that the image of $\sum_{j=t}^{s} n_j R(-b_j)$ is $(s-t)$-fold basic in $M'' = M'/m_t R(-a_t)$ at $\mathscr{C}$. Hence the image of $\sum_{j=t}^{v} n_j R(-b_j)$ in M'' is $(v-t)$-fold basic in M'' at $\mathscr{C}$ for all v where $t+1 \le v \le s$. Therefore, because of the fact that K is infinite and $b_t \le b_{t+1} \le \cdots \le b_s$ we can successively find homogeneous elements $n'_v = n_v + \sum_{j=t}^{v-1} r_{vj} n_j$ $(t+1 \le v \le s)$ where the $r_{vj} \in R$ of degree $b_j - b_t$ are sufficiently general such that n'_v is 1-fold basic in $M''/(\sum_{j=t+1}^{v-1} n'_j R(-b_j))$ at $\mathscr{C}$ (cf. [14]). It follows that the submodule generated by $m_1, \ldots, m_t, n'_{t+1}, \ldots, n'_s$ is s-fold basic in M at $\mathscr{C}$. According to Proposition 6.3, this proves our claim.

A repeated application of the claim implies our assertion. ▮

Remark 7.2. Let us consider the partial ordering on s-tuples of integers defined by $(a_1, \ldots, a_s) \le (b_1, \ldots, b_s)$ if $a_j \le b_j$ for all j. Then the previous statement says that for a module M meeting the assumptions of Lemma 7.1 there is a minimal s-tuple $(a_1, \dots, a_s)$ of integers such that there is an embedding $\alpha: \bigoplus_{i=1}^{s} R(-a_i) \to M$ where coker α is isomorphic to an ideal of pure codimension 2. This minimal s-tuple is an interesting invariant of M. In principle it can be determined as follows:

Choose the elements $m_i \in M$ successively such that the image of m_i in $M/(\sum_{j=1}^{i-1} m_j R(-a_j))$ is one of the elements of minimal degree, say a_i, which are 1-fold basic in $M/(\sum_{j=1}^{i-1} m_j R(-a_j))$ at $\mathscr{C}$. Then $(a_1, \dots, a_s)$ is the minimal s-tuple of M. This follows easily by Proposition 6.3 and the argument used to show the claim in the proof of the previous lemma.

Martin-Deschamps and Perrin determine the minimal s-tuple of M by means of their function q (cf. [22, Theorems IV.3.4 and IV.3.7]).

The structure of an even liaison class of ideals with pure codimension 2 described in the next statement is called the *Lazarsfeld–Rao property*.

THEOREM 7.3. *Let $I \subset R$ be a homogeneous ideal of pure codimension 2 that is not perfect. Suppose I occurs in shift h of its even liaison class $\mathscr{L}$, and let $I_0 \in \mathscr{L}^0$ be any minimal element of this class. Then there exists a sequence of ideals $I_0, I_1, \dots, I_v$ such that for all i, $1 \le i \le v$, I_i is a basic double link of I_{i-1} of positive shift and I is a deformation of I_v through ideals all in $\mathscr{L}^h$.*

Proof. (I) According to Rao's correspondence we may assume (possibly after adding free direct summands) that I and I_0 have N-type resolutions of the form

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \xrightarrow{\alpha} N \to I_0 \to 0$$

and

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \xrightarrow{\beta} N \to I(h) \to 0.$$

We may also assume that $a_i \le \cdots \le a_s$ and $b_1 \le \cdots \le b_s$.

(II) First we want to show that $a_i \le b_i$ for all $i = 1, \dots, s$.

Assuming the contrary, we get for the integers $c_i = \min\{a_i, b_i\}$ that $h' = \sum_{i=1}^{s} c_i - \sum_{i=1}^{s} a_i < 0$. By Lemma 7.1 there is an exact sequence

$$0 \to \bigoplus_{i=1}^{s} R(-c_i) \to N - J(h') \to 0,$$

where J is an ideal of pure codimension 2. Hence I_0 and J belong to the same even liaison class by Theorem 6.4, but $h' < 0$ shows that I_0 cannot be in the minimal shift. This contradiction proves our claim.

(III) By step (II) we get $h = \sum_{i=1}^{s} b_i - \sum_{i=1}^{s} a_i \geq 0$. Making a basic double link, we want to replace I_0 by an ideal I_1 such that the corresponding difference gets smaller.

Denote the minimal generators of im α by $m_1, \ldots, m_s$, where $\deg m_i = a_i$, and the minimal generators of im β by $n_1, \ldots, n_s$, where $\deg n_i = b_i$.

Suppose $h > 0$. Then there is a $t = \max\{i \in \{1, \ldots, s\} \mid b_i > a_i\}$. Let $v = \max\{i \in \{1, \ldots, s\} \mid b_i = b_t\}$. We want to show:

Claim. $[\operatorname{im} \beta]_{b_v}$ is not contained in im α.

Assume the contrary. Then we obtain $\sum_{i=1}^{v} n_i R(-b_i) \subset \sum_{i=1}^{v} m_i R(-a_i)$. Since im β is s-fold basic in N at $\mathscr{C}$ (cf. Proposition 6.3), the image of $\sum_{i=v+1}^{s} n_i R(-b_i)$ is $s-v$-fold basic in $N/(\sum_{i=1}^{v} n_i R(-b_i))$ at $\mathscr{C}$. Thus the image of $\sum_{i=v+1}^{s} n_i R(-b_i)$ is $s-v$-fold basic in $N/(\sum_{i=1}^{v} m_i R(-a_i))$ at $\mathscr{C}$, too. Hence $F = \sum_{i=1}^{v} m_i R(-a_i) + \sum_{i=v+1}^{s} n_i R(-b_i)$ is s-fold basic in N at $\mathscr{C}$. Comparison with the 1-presentation of I_0 and Proposition 6.3 shows that N/F is isomorphic to an ideal J of codimension 2. Furthermore, we obtain an epimorphism $I(h) \to J$ of torsion-free modules of rank 1. Thus it must be an isomorphism.

But $h > 0$ and $\operatorname{codim} I = \operatorname{codim} J = 2$ imply $\operatorname{rank}_K [I(h)]_j > \operatorname{rank}_K [J]_j$ for all sufficiently large integers j. This contradiction shows our claim.

The claim implies that N contains an element of degree b_v whose residue class modulo im α gives rise to an element $g \neq 0$ of the ideal I_0. Now we choose an element f of degree $b_t - a_t$ such that we can perform a basic double link by putting $I_1 = fI_0 + gR$. According to Proposition 4.4, I_1 has an N-type resolution as follows:

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \oplus R(-b_t) \to N \oplus R(-a_t) \to I_1(b_t - a_t) \to 0.$$

A comparison with the N-type resolution of I,

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \oplus R(-a_t) \to N \oplus R(-a_t) \to I(h) \to 0,$$

shows that we are exactly in the situation we had at the beginning of step (III), with I_1 playing the role of I_0. In fact the degree shifts of $\oplus_{i=1}^{s} R(-b_i) \oplus R(-a_t)$ (once ordered) are greater than or equal to the degree shifts of $\oplus_{i=1}^{s} R(-a_i) \oplus R(-b_t)$, and the difference of their corresponding sums is $h_1 = h - b_t + a_t < h$.

(IV) In the case where $h_1 > 0$, we can repeat step (III) until we get an ideal I_v such that I_v and I have N-type resolutions

$$0 \to F \xrightarrow{\alpha_v} N \oplus G \to I_v(h) \to 0$$

and

$$0 \to F \xrightarrow{\beta'} N \oplus G \to I(h) \to 0,$$

where F, G are free modules. Abusing notation slightly we denote the minimal generators of im α_v again by $m_1, \ldots, m_s$ and those of im β' by $n_1, \ldots, n_s$, such that $\deg m_i = \deg n_i$ for all i. For $\lambda \in K$ we consider the submodule M_λ generated by $\lambda m_i + (1 - \lambda) n_i$. We claim that M_λ is s-fold basic in $N \oplus G$ at $\mathscr{C}$ for general $\lambda \in K$. Indeed, if we choose an $f \in I_v \cap I$ and localize the 1-presentations of I_v and I at $\{1, f, f^2, \ldots\}$, the resulting sequences split. Thus $(\operatorname{im} a_v)_f$ and $(\operatorname{im} \beta')_f$ are free direct summands of the free R_f-module $N_f \oplus G_f$. Hence the same applies to $(M_\lambda)_f$ for all but finitely many $\lambda \in K$. It follows that M_λ is s-fold basic in $N \oplus G$ at a prime ideal $\mathfrak{p}$ for all of these λ if $\mathfrak{p}$ does not contain f. Now let $\mathfrak{p}$ be one of the finitely many prime ideals in $\mathscr{C}$ that contain f. Since im α_v and im β' are s-fold basic in $N \oplus G$ at $\mathfrak{p}$, this is also true for M_λ, except for finitely many $\lambda \in K$. All together we see that for all but finitely many $\lambda \in K$, M_λ is s-fold basic in $N \oplus G$ at $\mathscr{C}$, and thus $(N \oplus G)/M_\lambda$ is isomorphic to an ideal $I_\lambda(h)$ of pure codimension 2 by Proposition 6.3. The ideals I_λ fit together to form a flat family of ideals parameterized by a Zariski open subset $U \subset \mathbb{A}^1$ containing 0 and 1. Hence I is a deformation of I_v through ideals of $\mathscr{L}^h$ due to Theorem 6.4. ∎

The result implies in particular that all minimal elements of an even liaison class have the same Hilbert function. Since the Hilbert function increases by basic double linkage (cf. Proposition 4.4), the elements of $\mathscr{L}^0$ have minimal Hilbert function among the elements of their even liaison class. This gives another justification for the name minimal elements.

8. EXAMPLES AND OPEN PROBLEMS

The results of the last section show that linkage can effectively be used for investigating ideals of codimension 2. The problem of establishing a similarly powerful theory for the (even) liaison classes of ideals of codimension ≥ 3 is widely open. We want to discuss some of the arising difficulties by means of some examples.

With the notation of Definition 2.13 one gets different notions of linkage and the corresponding liaison classes by varying the requirements on the ideal $\mathfrak{c}$. The classical case is that $\mathfrak{c}$ is assumed to be a complete intersection. We call the resulting equivalence relation liaison via complete intersections. If we want to stress the fact that we are using our Definition 2.13 as it is, we say that we are considering liaison via Gorenstein ideals. For ideals of codimension 2, liaison via complete intersections is the same as liaison via Gorenstein ideals. The situation changes in higher codimension.

EXAMPLE 8.1. Let $X \subset \mathbb{P}^n$ $(n \geq 3)$ be a set of $n+1$ linearly independent points. Then X has a linear minimal free resolution. Hence [19, Corollary 5.13] implies that the liaison class of X via complete intersections does not contain a complete intersection.

Now let $P \in \mathbb{P}^n$ be a point such that $Y = X \cup P$ is a set of $n+2$ points in linearly general position, i.e., any subset of $n+1$ points is linearly independent. Then Y is arithmetically Gorenstein (cf., for example, [17, Theorem C]). It follows that X is geometrically linked to the complete intersection P by the Gorenstein ideal $I(Y)$.

Remark 8.2. (i) The example shows that in general, liaison classes via Gorenstein ideals are strictly large than liaison classes via complete intersections.

(ii) In our notation Rao's correspondence says that the maps Φ and Ψ introduced in Theorem 3.10 are bijective. If this were true in arbitrary codimension, it would follow that all perfect ideals belong to the same liaison class. The example above shows that this fails for liaison via complete intersections.

Next we want to use a construction that has been introduced in [10] for a different purpose.

LEMMA 8.3. *Let $H \subset \mathbb{P}^n$ be a hyperplane containing a curve C, i.e., C is a subscheme of pure dimension* 1. *Then there is a line L meeting C in a* (*reduced*) *point. Put $C' = C \cup L$ and let $I(C)$ and $I(C')$ be the homogeneous ideals of C and C' in $R = K[x_0, \dots, x_n]$. Then the R-modules $\psi(I(C))$ and $\psi(I(C'))$ are isomorphic up to direct free summands. In particular, they are stably equivalent.*

Proof. In the proof of [10, Proposition 3.4] it is shown that

$$H^1_{\mathfrak{m}}(R/I(C)) \cong H^1_{\mathfrak{m}}(R/I(C')).$$

It follows that the modules $N = \psi(I(C))$ and $N' = \psi(I(C'))$ are reflexive Eilenberg–MacLane modules of depth 2 where

$$H^2_{\mathfrak{m}}(N) \cong H^2_{\mathfrak{m}}(N').$$

Therefore Lemma 2.7 implies the claim. ∎

EXAMPLE 8.4. Let $C \subset \mathbb{P}^4$ be the union of two skew lines. It is well known that C is arithmetically Buchsbaum with $H^1_*(\mathcal{I}_C) \cong K$. It has a minimal N-type resolution resolution,

$$0 \to \mathcal{O}^2_{\mathbb{P}^4}(-3) \to \mathcal{O}^6_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \Omega_{\mathbb{P}^4} \to \mathcal{I}_C \to 0.$$

According to Corollary 5.4, C is minimal in its even liaison class.

Since C is contained in a hyperplane of $\mathbb{P}^4$, we can apply Lemma 8.3. Thus there is a line $L \subset \mathbb{P}^4$ meeting C in a point, and the curve $C' = C \cup L$ is arithmetically Buchsbaum, too. It has a minimal N-type resolution,

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-4) &\to \mathcal{O}^4_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \\ &\to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \Omega_{\mathbb{P}^4} \to \mathcal{I}_{C'} \to 0. \end{aligned}$$

Thus C' is also minimal in its even liaison class.

The curves C and C' belong to the same even liaison class. Indeed, C' can be linked to a union C'' of two skew lines by an arithmetically Gorenstein curve D (cf. [24, Example 5.4.8]). Using similar arithmetically Gorenstein curves like D, we can link C'' in an even number of steps to a pair C''' of skew lines that is contained in the hyperplane $H \cong \mathbb{P}^3$ spanned by C. Since C and C''' have the same self-dual Hartshorne–Rao module, we can link C and C''' in an odd number of steps by using complete intersections contained in H. Hence we get two minimal elements in an even liaison class with different Hilbert function. Such a phenomenon does not occur in even liaison classes of ideals of codimension 2 according to the Lazarsfeld–Rao property.

Continuing in the fashion of the last example, we can think of the curves C, C' as embedded in $\mathbb{P}^5$ and take C'' as the union of C' with a suitable line such that Lemma 8.3 applies. Then we get arithmetically Buchsbaum curves of codimension 4 and degrees 2, 3, and 4, respectively, which are all minimal in their even liaison class. More generally, starting with a union C of two skew lines in $\mathbb{P}^3$, we can produce $c - 1$ curves of codimension c that are all minimal in their even liaison class (cf. Corollary 5.4) and have degree $\deg C, \dots, \deg C + c - 2$.

As a consequence, we see that in higher codimension we cannot have complete analogues of Rao's correspondence and the Lazarsfeld–Rao property at the same time. However, it is conceivable that for liaison via Gorenstein ideals, the following are true:

(a) The maps Φ and Ψ are (at least) injective.

(b) If I is a nonperfect ideal of pure codimension $c \geq 3$ occurring in shift $h \geq 0$ of its even liaison class $\mathcal{L}$, then there is a sequence of ideals

$I_0, I_1, \ldots, I_v \in \mathscr{L}$ such that $I_0 \in \mathscr{L}^0$, $I_v \in \mathscr{L}^h$, and for all i, $1 \leq i \leq v$, I_i is basic double link of I_{i-1} and I is a deformation of I_v through ideals all in $\mathscr{L}^h$.

Note that (b) is a weaker version of the Lazarsfeld–Rao property because it is not required that I_0 is an arbitrary minimal element of its even liaison class.

The generality of our results in codimension 2 allows the following approach to these questions. Suppose I and J are ideals of pure codimension $c > 2$. Choose a Gorenstein prime ideal $\mathfrak{d} \subset R$ of codimension $c - 2$ that is contained in I and J. Then $\bar{I} = I/\mathfrak{d}$ and $\bar{J} = J/\mathfrak{d}$ have codimension 2 in $R/\mathfrak{d}$. Thus the results of Sections 6 and 7 apply to $\bar{I}$ and $\bar{J}$. Note that I and J are linked if $\bar{I}$ and $\bar{J}$ are linked. Migliore has applied this strategy in [23] to study degenerate subschemes of projective space.

After this paper was finished, new evidence that (a) is true was obtained (cf. [20]). Moreover, in [20] the importance of Gorenstein liaison is stressed by applying it to geometric questions like, for example, nonobstructedness of a given subscheme.

One might also consider even more general linkage. In this respect Walter [36] has shown that allowing the ideal $\mathfrak{c}$ in Definition 2.13 to be an arbitrary perfect ideal does not give a meaningful notion, because one gets only one liaison class for any codimension.

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