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On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the Region Connection Calculus ☆

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Abstract

The computational properties of qualitative spatial reasoning have been investigated to some degree. However, the question of the boundary between polynomial and NP-hard reasoning problems has not been addressed yet. In this paper we explore this boundary in the "Region Connection Calculus" RCC-8. We extend Bennett's encoding of RCC-8 in modal logic. Based on this encoding, we prove that reasoning is NP-complete in general and identify a maximal tractable subset of the relations in RCC-8 that contains all base relations. Further, we show that for this subset path-consistency is sufficient for deciding consistency. © 1999 Published by Elsevier B.V. All rights reserved.

Keywords: Qualitative spatial reasoning; Computational complexity; Region Connection Calculus; Tractable subclasses; Path-consistency

1. Introduction

When describing a spatial configuration or when reasoning about such a configuration, often it is not possible or desirable to obtain precise, quantitative data. In these cases, qualitative reasoning about spatial configurations may be used [8]. Since space offers a very rich structure, many different aspects of space such as, for example, distance,

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direction, shape, or topology can be treated in a qualitative way. It is therefore the general goal of qualitative spatial representation and reasoning to develop a qualitative theory of space that covers many of the different aspects of space. Naturally, this general theory must grow from different theories focusing on single aspects of space. Apart from a large expressiveness that a general theory of space should have, it is also highly desirable to allow for efficient reasoning. For this it is necessary to study the computational properties of the theory.

In this paper we focus on a particular approach to qualitative spatial reasoning developed by Randell et al. [36], the so-called *Region Connection Calculus* (RCC), which is based on binary topological relations. One variant of this calculus, RCC-8, uses eight mutually exhaustive and pairwise disjoint relations, called base relations, to describe the topological relationship between two regions (see also Egenhofer [12]). These relations can be regarded as the spatial counterpart of Allen's well-known interval relations for temporal reasoning [1].

Some of the computational properties of RCC-8 have been analyzed by Grigni et al. [19] and Nebel [31]. However, no attempt has yet been made to determine the boundary between polynomial and NP-hard fragments of RCC-8, as it has been done for Allen's interval calculus [33] and recently for RCC-5, a sub-calculus of RCC-8 [23]. We address this problem and identify a maximal fragment of RCC-8 that is still tractable and contains all base relations. As in the case of qualitative temporal reasoning, this proof relies on a computer generated case-analysis that cannot be reproduced in a research paper. [2] Further, we show that for this fragment path-consistency is sufficient for deciding consistency.

As topological information is easily accessible, there are many possible and some existing applications of the calculus in areas like spatial information systems [5,20], spatial configuration tasks, (robot) navigation [26], computer vision, natural language processing [26], document analysis, visual languages [18], and qualitative simulation of physical processes [10,34]. Even when the expressive power of the calculus is too weak for a particular application, with the efficient reasoning mechanisms resulting from the work of this paper it can be used as fast preprocessing for a more expressive spatial representation.

The rest of the paper is structured as follows. In Section 2 we introduce RCC-8 and some basic terminology and definitions that are used in the rest of the paper. In Section 3 we introduce and extend Bennett's [3] encoding of RCC-8 in a propositional modal logic. In Section 4, we show that reasoning in RCC-8 is NP-hard by proving that the simpler calculus RCC-5 is already NP-hard. Using the modal encoding of RCC-8, we show in Section 5 how reasoning in RCC-8 can be reduced to satisfiability in propositional logic. Based on that, in Section 6 a tractable subset of RCC-8 is identified which contains only relations representable as Horn clauses. Further, using a computer generated case-analysis we prove that the set is maximal. In Section 7 we discuss the applicability of the path-consistency algorithm, and in Section 8 we give an estimation of the applicability of the maximal tractable subset to the general reasoning problem. In the appendices, we give a concise introduction to modal logic and an enumeration of the tractable fragment of RCC-8.

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[2] The programs can be obtained from the authors.

2. Qualitative spatial reasoning with RCC

The Region Connection Calculus (RCC) is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are non-empty regular subsets of a topological space [36]. Relationships between spatial regions are defined in terms of the relation $\mathsf{C}(a,b)$, read as "a connects with b". In the standard interpretation of the RCC theory, the relation $\mathsf{C}(a,b)$ is true if and only if the closure of region a is connected to the closure of region b, i.e., if the closures of the two regions share a common point. Regions themselves do not have to be internally connected, i.e., a region may consist of different disconnected parts, and regions are allowed to have holes. The domain of *spatial variables* (denoted as X, Y, Z) is the set of all spatial regions of the topological space. The RCC theory is formulated in first-order predicate calculus [36].

RCC-8 [36] is a set of eight jointly exhaustive and pairwise disjoint (JEPD) relations, called *base relations*, definable in the RCC theory, denoted as DC, EC, PO, EQ, TPP, NTPP, TPP^{-1}, and NTPP^{-1}, with the meaning of *DisConnected, Externally Connected, Partial Overlap, EQual, Tangential Proper Part, Non-Tangential Proper Part*, and their converses. Exactly one of these relations holds between any two spatial regions. These relations can be given a straightforward topological interpretation in terms of point-set topology (see Table 1), which is almost the same as the semantics for the topological relations given by Egenhofer [12] (though Egenhofer places stronger constraints on the domain of regions, e.g., regions must be one-piece and are not allowed to have holes). Examples for the RCC-8 relations are shown in Fig. 1.

RCC-5 is a set of five JEPD relations [2] definable in the RCC theory on a coarser level of granularity than RCC-8. For RCC-5 the boundary of a region is not taken into account, i.e., one does not distinguish between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 base relations DR for *DiscRete* and PP for *Proper Part*, respectively. Thus, RCC-5 contains the five base relations DR, PO, PP, PP^{-1}, and EQ. In this work we will focus on RCC-8, but most of our results can easily be applied to RCC-5.

Table 1
Topological interpretation of the eight base relations of RCC-8. $i(\cdot)$ specifies the topological interior of a spatial region, $\overline{\cdot}$ the topological closure

RCC-8 relation	Topological constraints
$\mathsf{DC}(a,b)$	$\overline{a} \cap \overline{b} = \emptyset$
$\mathsf{EC}(a,b)$	$i(\overline{a}) \cap i(\overline{b}) = \emptyset, \overline{a} \cap \overline{b} \neq \emptyset$
$\mathsf{PO}(a,b)$	$i(\overline{a}) \cap i(\overline{b}) \neq \emptyset, \overline{a} \not\subseteq \overline{b}, \overline{b} \not\subseteq \overline{a}$
$\mathsf{TPP}(a,b)$	$\overline{a} \subset \overline{b}, \overline{a} \not\subseteq i(\overline{b})$
$\mathsf{TPP}^{-1}(a,b)$	$\overline{b} \subset \overline{a}, \overline{b} \not\subseteq i(\overline{a})$
$\mathsf{NTPP}(a,b)$	$\overline{a} \subset i(\overline{b})$
$\mathsf{NTPP}^{-1}(a,b)$	$\overline{b} \subset i(\overline{a})$
$\mathsf{EQ}(a,b)$	$\overline{a} = \overline{b}$

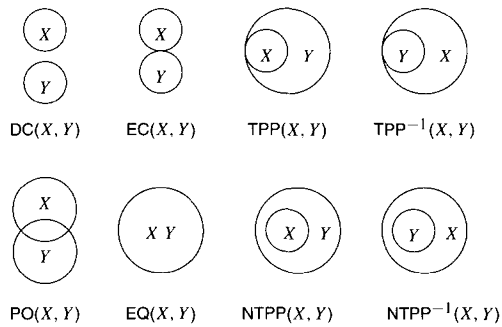

Fig. 1. Two-dimensional examples for the eight base relations of RCC-8.

Spatial constraints or RCC-8-*constraints* are written as XRY or $R(X, Y)$, where R is an RCC-8 relation. Sometimes it is not known which of the eight base relations of RCC-8 holds between two regions, but it is possible to exclude some of them. In order to represent this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in 2^8 different RCC-8 relations, [3] including the union of all base relations, which is called *universal relation*. In the following we will write sets of base relations to denote these unions. Using this notation, DR, e.g., is identical to {DC, EC}. A *refinement* of an RCC-8 relation R is any sub-relation of R, e.g., {DC, PO} is a refinement of {DC, EC, PO}. Apart from union ($\cup$), other operations are defined, namely, converse ($\smile$), intersection ($\cap$), and composition ($\circ$) of relations. The formal definitions of these operations are:

$$\begin{aligned}
&\forall X, Y\colon\ X(R \cup S)Y \leftrightarrow XRY \vee XSY,\\
&\forall X, Y\colon\ X(R \cap S)Y \leftrightarrow XRY \wedge XSY,\\
&\forall X, Y\colon\ \qquad XR^{\smile}Y \leftrightarrow YRX,\\
&\forall X, Y\colon\ X(R \circ S)Y \leftrightarrow \exists Z\colon (XRZ \wedge ZSY).
\end{aligned}$$

Converse, intersection and union of relations can easily be obtained by performing the corresponding set theoretic operations. Composition of base relations has to be computed using the formal definitions of the relations [2,35]. The compositions of the eight base relations are shown in Table 2. Every entry in the composition table specifies the relation obtained by composing the base relation of the corresponding row with the base relation of the corresponding column. Composition of two arbitrary RCC-8 relations can be obtained by computing the union of the composition of the base relations. Note that the composition table only corresponds to the given extensional definition of composition if the universal region is not permitted [6].

A spatial configuration can be described by a set Θ of spatial constraints. One important computational problem is deciding *consistency* of Θ, i.e., deciding whether Θ has a

[3] In some papers the set of all possible unions of base relations is denoted as 2^{RCC8}. We will, however, use RCC-8 to refer to the set of all possible unions of base relations and $\mathcal{B}$ to refer to the set of base relations of RCC-8.

Table 2
Composition table for the eight base relations of RCC-8, where $*$ specifies the universal relation

$\circ$	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	EQ
DC	$*$	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC	DC	DC
EC	DC,EC PO,TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP NTPP	EC,PO TPP NTPP	PO TPP NTPP	DC,EC	DC	EC
PO	DC,EC PO,TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	$*$	PO TPP NTPP	PO TPP NTPP	DC,EC PO,TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	PO
TPP	DC	DC,EC	DC,EC PO,TPP NTPP	TPP NTPP	NTPP	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP^{-1} $NTPP^{-1}$	TPP
NTPP	DC	DC	DC,EC PO,TPP NTPP	NTPP	NTPP	DC,EC PO,TPP NTPP	$*$	NTPP
TPP^{-1}	DC,EC PO,TPP^{-1} $NTPP^{-1}$	EC,PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO,EQ TPP TPP^{-1}	PO TPP NTPP	TPP^{-1} $NTPP^{-1}$	$NTPP^{-1}$	TPP^{-1}
$NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO,TPP^{-1} TPP,NTPP $NTPP^{-1}$,EQ	$NTPP^{-1}$	$NTPP^{-1}$	$NTPP^{-1}$
EQ	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	EQ

solution, which is an assignment of regions of some topological space to variables of Θ in a way that all relations hold. [4] We call this problem RSAT. When only relations of a specific set S are used in Θ, the corresponding reasoning problem is denoted by RSAT(S). RSAT is a Constraint Satisfaction Problem (CSP) [28], where variables are nodes and relations are arcs of the constraint graph and the domain of the variables are subsets of a topological space. So RSAT can be solved using the standard methods developed for CSP's with infinite domains (see, e.g., [24]).

A partial method for determining inconsistency of a CSP is the *path-consistency method* which enforces path-consistency on a CSP [27,30]. A CSP is *path-consistent* if and only if for any consistent instantiation of any two variables, there exists an instantiation of any

[4] Here, the dimension of the topological space is not considered. Renz [37], however, found that whenever a set of constraints over RCC-8 has a solution in a topological space of some dimension, it has a solution in topological spaces of any dimension. This is not the case if regions must be one-piece [19]. See Section 9 for a further discussion of this topic.

third variable such that the three values taken together are consistent. It is necessary but not sufficient for the consistency of a CSP that path-consistency can be enforced, i.e., a CSP where path-consistency cannot be enforced is not consistent, but a CSP is not necessarily consistent when path-consistency can be enforced. A naive way to enforce path-consistency on a CSP is to strengthen relations by successively applying the following operation until a fixed point is reached:

$$\forall k\colon\ R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj}),$$

where i, j, k are nodes and R_{ij} is the relation between i and j. The resulting CSP is equivalent to the original CSP, i.e., it has the same set of solutions. If the empty relation occurs while performing this operation the CSP is inconsistent, otherwise the resulting CSP is path-consistent. More advanced algorithms enforce path-consistency in times $O(n^3)$ where n is the total number of nodes in the graph [29].

Other useful computational problems include RMIN, the problem of finding the minimal relations, i.e., the strongest implied relation for each pair of spatial regions, and RENT, the problem of whether a spatial constraint is entailed by Θ. As it was shown for the corresponding temporal problems [17,42], these problems are equivalent to RSAT under polynomial Turing reductions.

3. Encoding of RCC-8 in modal logic

Another way of solving problems concerning RCC-8 is using the encoding of the relations in first-order predicate logic. However, such an encoding does not lead to efficient decision procedures. In order to overcome this problem, Bennett [2,3] used different encodings of RCC-8 in propositional intuitionistic and modal logic. In this work we will use Bennett's encoding of RCC-8 in propositional modal logic [3]. An introduction to modal logics is given in Appendix A.

Bennett obtained the modal encoding by analyzing the relationship of regions to the universe $\mathcal{U}$. For the modal encoding we are using, Bennett restricted his analysis to closed regions that are connected if they share a point and overlap if they share an interior point. [5] If, e.g., X and Y are disconnected, the complement of the intersection of X and Y is equal to the universe. Further, both regions must not be empty, i.e., the complements of both X and Y are not equal to the universe. In the same way all topological constraints corresponding to the RCC-8 relations (see Table 1) can be written as constraints of the form $(m = \mathcal{U})$ and $(e \neq \mathcal{U})$, where m and e are set-theoretic expressions, denoted as *model constraints* and *entailment constraints*, respectively [2]. In the above example, $\overline{X \cap Y}$ is the model constraint and $\overline{X}$ and $\overline{Y}$ are the entailment constraints. Any model constraint must hold, whereas no entailment constraint must hold [2].

For some of the constraints it is necessary to refer to the interior of regions. For this purpose the topological interior operator i is used. This operator must satisfy the following axiom schemata for arbitrary sets $\phi, \psi \subseteq \mathcal{U}$ [3]:

[5] There is also a modal encoding based on open regions which is not as simple as the encoding based on closed regions [3].

Table 3
Bennetts encoding of the eight base relations in modal logic [3]

Relation	Model constraints	Entailment constraints
$\mathsf{DC}(X,Y)$	$\neg(X \wedge Y)$	$\neg X, \neg Y$
$\mathsf{EC}(X,Y)$	$\neg(\mathbf{I}X \wedge \mathbf{I}Y)$	$\neg(X \wedge Y), \neg X, \neg Y$
$\mathsf{PO}(X,Y)$	—	$\neg(\mathbf{I}X \wedge \mathbf{I}Y), X \to Y, Y \to X, \neg X, \neg Y$
$\mathsf{TPP}(X,Y)$	$X \to Y$	$X \to \mathbf{I}Y, Y \to X, \neg X, \neg Y$
$\mathsf{TPP}^{-1}(X,Y)$	$Y \to X$	$Y \to \mathbf{I}X, X \to Y, \neg X, \neg Y$
$\mathsf{NTPP}(X,Y)$	$X \to \mathbf{I}Y$	$Y \to X, \neg X, \neg Y$
$\mathsf{NTPP}^{-1}(X,Y)$	$Y \to \mathbf{I}X$	$X \to Y, \neg X, \neg Y$
$\mathsf{EQ}(X,Y)$	$X \to Y, Y \to X$	$\neg X, \neg Y$

$$i(\phi) \subseteq \phi, \tag{1}$$

$$i(i(\phi)) = i(\phi), \tag{2}$$

$$i(\mathcal{U}) = \mathcal{U}, \tag{3}$$

$$i(\phi \cap \psi) = i(\phi) \cap i(\psi). \tag{4}$$

The model and entailment constraints can be encoded in modal logic, where regions correspond to propositional atoms, the interior operator i corresponds to a modal operator $\mathbf{I}$ (see Table 3), and the universe $\mathcal{U}$ corresponds to the set of all worlds W [3]. The axiom schemata for i must also hold for the modal operator $\mathbf{I}$, which results in the following axiom schemata [3] for arbitrary modal formulas ϕ, ψ:

$$\mathbf{I}\phi \to \phi, \tag{5}$$

$$\mathbf{II}\phi \leftrightarrow \mathbf{I}\phi, \tag{6}$$

$$\mathbf{I}\top \leftrightarrow \top \text{ (for any tautology } \top), \tag{7}$$

$$\mathbf{I}(\phi \wedge \psi) \leftrightarrow \mathbf{I}\phi \wedge \mathbf{I}\psi. \tag{8}$$

Axiom schemata 1 and 2 correspond to the modal axioms **T** and **4** and axiom schemata 3 and 4 already hold for any modal logic K, so **I** is a modal S4-operator (see Appendix A).

The 4 axiom schemata specified by Bennett are not sufficient to exclude non-closed regions as it was intended. In order to account for that, we add one formula for each atom X, which corresponds to the topological property of regular closed regions: A regular closed region is the closure of an open region. $\neg X$ specifies the complement of X, and, thus, $\neg\mathbf{I}\neg X$ the closure of X.

$$X \leftrightarrow \neg\mathbf{I}\neg\mathbf{I}X. \tag{9}$$

Note that the S4 encoding can be used to reason about any kind of open or closed regions. Both the non-emptiness constraint, i.e., the entailment constraint $\neg X$, and the regularity constraint (9) are optional and can be regarded as properties of regions definable in the modal representation. They are needed to make the representation conform to the intended interpretation of the original RCC theory.

In order to combine the different model and entailment constraints, Bennett [3] uses another modal operator $\Box$. $\Box\varphi$ is interpreted as $\varphi = \mathcal{U}$ and $\neg\Box\varphi$ as $\varphi \neq \mathcal{U}$. Since m is a model constraint if $m = \mathcal{U}$ holds, any model constraint m can be written as $\Box m$ and any entailment constraint e as $\neg\Box e$. If $\Box X$ is true in a world w of a model $\mathcal{M}$, written as $\mathcal{M}, w \Vdash \Box X$, then X must be true in any world of $\mathcal{M}$. So $\Box$ is an S5-operator with the constraint that all worlds are mutually accessible. Therefore Bennett calls it a strong S5-operator [3]. Now all model and all entailment constraints containing the strong S5-operator can be conjunctively combined to a single modal formula. So the modal encoding of RCC-8 is made with an S4-operator that corresponds to the topological interior operator and a strong S5-operator that is used to obtain a single modal formula.

4. Computational properties of RCC-8

In this section we prove that reasoning with RCC-8 is NP-hard by showing that reasoning with a subset of RCC-8 is already NP-hard. We will then show how complexity results for subsets of RCC-8 can be carried over to other subsets of RCC-8, and, using this result, give NP-hardness proofs for a number of different subsets $\mathcal{S}$ of RCC-8. All of these proofs use a transformation from a propositional satisfiability problem to RSAT($\mathcal{S}$) by constructing a set of spatial constraints Θ for every instance $\mathcal{I}$ of the propositional satisfiability problem, such that Θ is consistent if and only if $\mathcal{I}$ is a positive instance. The propositional satisfiability problems we use are 3SAT, the problem of deciding whether there is a truth assignment for a set of clauses where each clause has exactly three literals, as well as two variants of 3SAT where truth assignments of particular types are required [15]. These variants are NOT-ALL-EQUAL-3SAT, the problem of deciding whether there is a truth assignment such that for every clause at least one literal is assigned *true* and one literal is assigned *false*, and ONE-IN-THREE-3SAT, the problem of deciding whether there is a truth assignment such that for every clause exactly one literal in every clause is assigned *true*. All three decision problems are NP-hard [40].

The different transformations we use in this section as well as in Section 6 have in common that every variable v of the propositional satisfiability problem is transformed to two RCC-8-constraints $X_v\{R_t, R_f\}Y_v$ and $X_{\neg v}\{R_t, R_f\}Y_{\neg v}$ corresponding to the positive and the negative literal of v, where R_t and R_f are RCC-8-relations with $R_t \cap R_f = \emptyset$. v is assigned *true* if and only if $X_v\{R_t\}Y_v$ holds and assigned *false* if and only if $X_v\{R_f\}Y_v$ holds. Since the two literals corresponding to a variable need to have opposite assignments, we have to make sure that $X_v\{R_t\}Y_v$ holds if and only if $X_{\neg v}\{R_f\}Y_{\neg v}$ holds, and *vice versa*, for which additional "polarity constraints" are required. In addition, every literal occurrence l of the propositional satisfiability problem is transformed to the RCC-8-constraint $X_l\{R_t, R_f\}Y_l$, where $X_l\{R_t\}Y_l$ holds if and only if l is assigned *true*. In order to assure the correct assignment of positive and negative literal occurrences with respect to the corresponding variable, polarity constraints are required again. For instance, if the variable v is assigned *true*, i.e., $X_v\{R_t\}Y_v$ holds, then $X_p\{R_t\}Y_p$ must hold for every positive literal occurrence p of v, and $X_n\{R_f\}Y_n$ must hold for every negative literal occurrence n of v. Further, "clause constraints" have to be added to assure that the clause requirements of the specific propositional satisfiability problem are satisfied. For example, if $\{i, j, k\}$

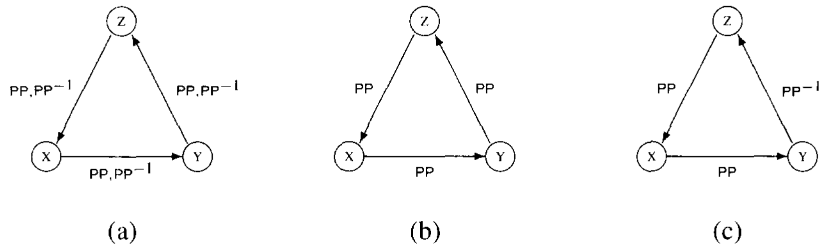

Fig. 2. Property 1: (a) is the original configuration, (b) is an impossible, (c) is a possible refinement of the relations to base relations.

is a clause of an instance of ONE-IN-THREE-3SAT, then exactly one of the constraints $X_i\{R_t\}Y_i$, $X_j\{R_t\}Y_j$, and $X_k\{R_t\}Y_k$ must hold. In the constraint graphs displayed in the figures of this section the relation {PO} is symbolized as a dotted line, spatial variables are symbolized as circles.

In the following we will prove NP-hardness for RCC-5, since the result can be immediately transferred to RCC-8. For this proof we will use NOT-ALL-EQUAL-3SAT and set R_t to PP and R_f to PP^{-1}. The polarity and the clause constraints of the transformation are based on the following two properties that can be verified using the composition table (Table 2).

Property 1. *Let X, Y, Z be spatial variables, where the relation* {PP, PP^{-1}} *holds between all of them. Then any refinement of these relations to base relations such that a path in the constraint graph starting and ending at X, and passing Y and Z contains only* {PP} *or only* {PP^{-1}} *is inconsistent (see Fig. 2).*

Property 2. *Let X, Y, Z be spatial variables, where X{PO}Y, Z{PP, PP^{-1}}X, and Z{PP, PP^{-1}}Y hold. Then a refinement of these relations to base relations is only consistent, if the relations between (Z, X) and between (Z, Y) are refined to the same base relation (see Fig. 3).*

With Property 1 it can be ruled out that all constraints are refined to the same base relation (not-all-equal). Using Property 2 a refinement of one constraint can be propagated to another constraint.

Theorem 3. RSAT(RCC-5) *is* NP-*hard.*

Proof. Transformation of NOT-ALL-EQUAL-3SAT to RSAT(RCC-5) (see also Grigni et al. [19]). Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of variables and $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses of an arbitrary instance of NOT-ALL-EQUAL-3SAT with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$, where $l_{i,j}$ are literal occurrences over variables of $\mathcal{V}$. We will construct a set of spatial constraints Θ, such that Θ is satisfiable if and only if $\mathcal{C}$ is a positive instance of NOT-ALL-EQUAL-3SAT using the following three transformation steps:

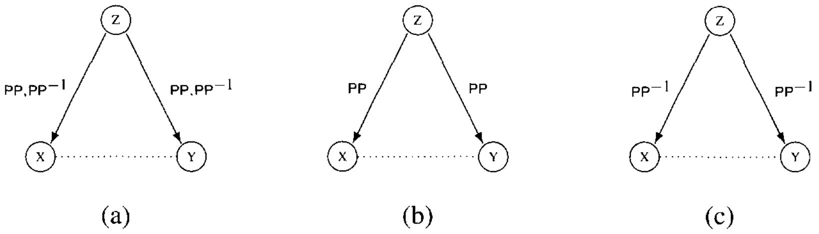

Fig. 3. Property 2: (a) is the original configuration, (b) and (c) are the only possible refinements of the relations to base relations.

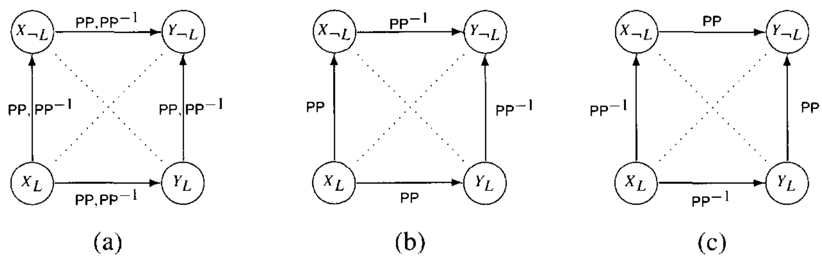

Fig. 4. The polarity constraints (a) for the transformation of NOT-ALL-EQUAL-3SAT assure that positive and negative literals of the same variable have opposite assignments: (b) and (c) are the only possible refinements of the relations to base relations.

(1) For each variable $v_L \in \mathcal{V}$ the spatial variables X_L, Y_L, $X_{\neg L}$ and $Y_{\neg L}$ are introduced by adding the spatial constraints $X_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_L$ and $X_{\neg L}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L}$ to Θ. Additionally, the following polarity constraints are added to Θ (see Fig. 4(a)):

$$X_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{\neg L}, \quad Y_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_L\{\mathsf{PO}\}Y_{\neg L}, \quad Y_L\{\mathsf{PO}\}X_{\neg L}.$$

(2) For each literal occurrence $l_{i,j}$ the spatial variables $X_{i,j}$ and $Y_{i,j}$ are introduced by adding the spatial constraint $X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{i,j}$ to Θ. Depending on whether the literal occurrence is positive or negative, different polarity constraints have to be added to Θ.
(a) $l_{i,j} \equiv v_L$:

$$X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{\neg L}, \quad Y_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_{i,j}\{\mathsf{PO}\}Y_{\neg L}, \quad Y_{i,j}\{\mathsf{PO}\}X_{\neg L}.$$

(b) $l_{i,j} \equiv \neg v_L$:

$$X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_L, \quad Y_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_L,$$
$$X_{i,j}\{\mathsf{PO}\}Y_L, \quad Y_{i,j}\{\mathsf{PO}\}X_L.$$

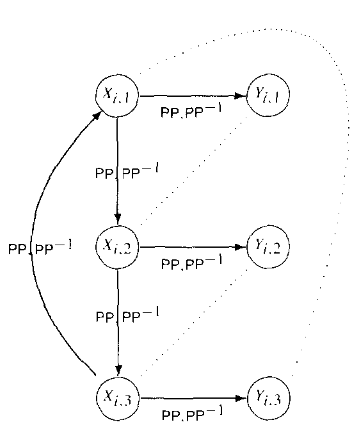

Fig. 5. Transformation of a not-all-equal clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ to spatial constraints.

(3) For each clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ the following clause constraints are added to Θ (see Fig. 5):

$$X_{i,1}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,2}, \quad X_{i,2}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,3}, \quad X_{i,3}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,1},$$
$$X_{i,1}\{\mathsf{PO}\}Y_{i,3}, \quad X_{i,2}\{\mathsf{PO}\}Y_{i,1}, \quad X_{i,2}\{\mathsf{PO}\}Y_{i,3}.$$

With this transformation for every literal as well as for every literal occurrence two spatial variables X and Y (with the appropriate indices) are introduced. When a literal occurrence or a literal is assigned *true*, the corresponding spatial variables hold the relation $X\{\mathsf{PP}\}Y$, when a literal occurrence or a literal is assigned *false*, the corresponding spatial variables hold the relation $X\{\mathsf{PP}^{-1}\}Y$.

Transformation step (1) introduces the spatial variables corresponding to the positive and the negative literal of each variable. Property 2 assures that positive and negative literals have opposite assignments. This is shown in Fig. 4. Transformation step (2) introduces spatial variables for every literal occurrence. Again, Property 2 assures correct assignments. Finally, transformation step (3) together with Property 1 makes sure that the not-all-equal condition of the literals of every clause is also fulfilled by the corresponding spatial variables. We now have to show that an instance of NOT-ALL-EQUAL-3SAT has a solution if and only if the set of spatial constraints Θ obtained by the given transformation is consistent.

(RSAT $\Rightarrow$ NOT-ALL-EQUAL-3SAT): Suppose that the set of spatial constraints Θ obtained by transformation from a given instance Σ of NOT-ALL-EQUAL-3SAT is consistent, and suppose that θ is a consistent instantiation of Θ. Then an assignment σ that satisfies Σ can be obtained in the following way: For every variable $v_L \in \mathcal{V}$, if $\theta(X_L)\{\mathsf{PP}\}\theta(Y_L)$ holds, then $\sigma(v_L)$ is *true*, otherwise $\sigma(v_L)$ is *false*.

(NOT-ALL-EQUAL-3SAT $\Rightarrow$ RSAT): Suppose that Σ is a positive instance of NOT-ALL-EQUAL-3SAT, and suppose that σ is an assignment that satisfies Σ. Then the set of spatial constraints Θ obtained by the transformation from Σ with respect to σ is consistent. We will show this by constructing a spatial configuration that satisfies all relations of Θ.

Before that we will point out some properties of Θ. Let Θ' be the set of spatial constraints transformed from Σ with transformation steps (1) and (2) with respect to σ, i.e., if, for example, $\sigma(v_L) = \mathit{true}$ then $X_L\{\mathsf{PP}\}Y_L$ holds. Θ' has the following properties:

(i) Since transformation step (3) introduces no spatial variable, Θ' contains the same spatial variables as Θ.

(ii) For any variable $v_L \in \mathcal{V}$ the four corresponding spatial variables are related as shown in Fig. 4(b) or (c).

(iii) For any literal occurrence $l_{i,j}$ the two corresponding spatial variables together with two spatial variables of the affiliated variable are in a form as shown in Fig. 4(b) or (c).

(iv) Only the relations $\{\mathsf{PP}\}, \{\mathsf{PP}^{-1}\}, \{\mathsf{PO}\}$ and $\{*\}$ occur in Θ'.

(v) If $X\{\mathsf{PP}\}Y$ holds for a spatial variable X and a spatial variable Y, then there is no spatial variable Z in Θ' with $Z\{\mathsf{PP}\}X$.

(vi) If $X\{\mathsf{PP}^{-1}\}Y$ holds for a spatial variable X and a spatial variable Y, then there is no spatial variable Z in Θ' with $Z\{\mathsf{PP}^{-1}\}X$.

Because of (v) and (vi), the spatial variables can be divided into two sets. The "small" set S contains all spatial variables that are proper parts of other spatial variables, the "big" set B contains all spatial variables that are converse proper parts of other spatial variables. The relation PO only holds between regions of the same set. All other relations are unspecified. For proving that Θ' is consistent, we will give a spatial configuration M' that holds all the specified relations and therefore is a model for Θ'. In M' every spatial variable X_i of the small set S is instantiated by a "small region" S'_i, every spatial variable X_i of the big set B is instantiated by a "big region" B'_i.

- Every small region $S'_i \in M'$ consists of two parts s' and s'_i, where s' is common to all small regions. $s'_i\{\mathsf{DR}\}s'_j$ for all $i \neq j$ and $s'_i\{\mathsf{DR}\}s'$ for all $i = 1, \ldots, |S|$.
- Every big region $B'_i \in B$ consists of two parts b' and b'_i, where b' is common to all big regions. $b'_i\{\mathsf{DR}\}b'_j$ for all $i \neq j$ and $b'_i\{\mathsf{DR}\}b'$ for all $i = 1, \ldots, |B|$.
- For all $i = 1, \ldots, |S|$: $s'_i\{\mathsf{PP}\}b'$ and $s'\{\mathsf{PP}\}b'$.

As every small region is a proper part of every big region and all small regions as well as all big regions partially overlap, Θ' is consistent. Θ results from Θ' after applying transformation step (3). The spatial configuration M' is no model for Θ, because two relations of every clause do not hold. This can be seen in Fig. 6 which shows two possible refinements of a clause and the instantiations of the spatial variables with small and big regions. In every clause there is either one small region which is a proper part of another small region or a big region which is a proper part of another big region. In Fig. 6 this is the case for $B_1\{\mathsf{PP}^{-1}\}B_2$ and $S_4\{\mathsf{PP}^{-1}\}S_5$. Also there is one small region in every clause that partially overlaps one big region. In Fig. 6 this is the case for $B_2\{\mathsf{PO}\}S_1$ and $B_5\{\mathsf{PO}\}S_4$.

With a few changes to M', we can construct a spatial configuration M that is a model for Θ. In M every spatial variable X_i of the small set is instantiated by S_i, every spatial variable X_i of the big set is instantiated by B_i. Apart from the following exceptions S_i is equal to S'_i and B_i is equal to B'_i:

- For any $i, j = 1, \ldots, |S|$: If $S_i\{\mathsf{PP}^{-1}\}S_j$ must hold then $S_i := S'_i \cup s'_j$.
- For any $i, j = 1, \ldots, |B|$: If $B_i\{\mathsf{PP}^{-1}\}B_j$ must hold then $B_i := B'_i \cup b'_j$.
- For any $i = 1, \ldots, |B|$, $j = 1, \ldots, |S|$: If $B_i\{\mathsf{PO}\}S_j$ must hold then $B_i := B'_i \setminus s'_j$.

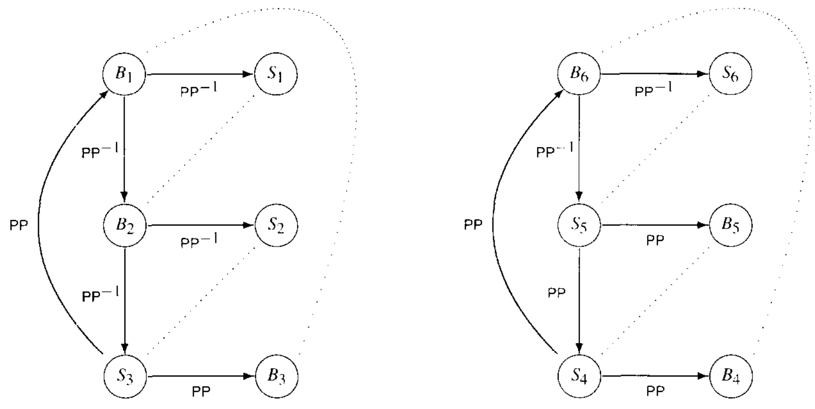

Fig. 6. Two different refinements of not-all-equal clauses to base relations.

As exactly two exceptions occur in every clause and the literal occurrences of every clause correspond to different regions, no region will be changed more than once. The regions of M hold all relations of Θ.

The transformation takes time linear in the number of clauses, so RSAT(Θ) is NP-hard. □

Since RCC-5 is a subset of RCC-8, this result can be easily applied to RCC-8. When {PP} is replaced by {TPP, NTPP} and the same for their converse, Properties 1 and 2 hold accordingly, and the same proof can be carried out.

Corollary 4. RSAT(RCC-8) *is* NP-*hard.*

Since we now know that RSAT is NP-hard, we want to find out whether there are subsets of RCC-8 for which the consistency problem can be decided in polynomial time, and ideally identify the borderline between tractability and intractability. In order to identify this borderline, one has to examine all subsets of RCC-8. We limit ourselves to subsets containing all base relations, because these subsets still allow one to express definite knowledge, if it is available. Additionally, we require the universal relation to be in the subset, so that it is possible to express complete ignorance. This reduces the number of subsets we have to analyze from 2^{256} to 2^{247}. Fortunately, we can reduce the number of subsets further by noting that the computational complexity associated with an arbitrary subset $\mathcal{S}$ is identical to the complexity associated with the closure of this subset under composition, intersection, and converse, denoted by $\widehat{\mathcal{S}}$—an observation that has also been used in determining a maximal tractable subset of Allen's interval calculus [33, Theorem 14]. Instead of reproving this fact for the RCC-8 relations, we will prove a more general result.

Let $\mathcal{S}$ be a set of binary relations over a domain $\mathcal{D}$, then we define the *constraint satisfaction problem* CSPSAT($\mathcal{S}$) as follows:

Given: A finite set Θ of constraints (XRY), where $R \in \mathcal{S}$ and X, Y are variables for values in $\mathcal{D}$.

Question: Is there an instantiation of all variables in Θ such that all constraints are satisfied?

Theorem 5. *Let $\mathcal{C}$ be a set of binary relations that is closed under composition, intersection, and converse. Then for any subset $\mathcal{S} \subseteq \mathcal{C}$ that contains the universal relation, the problem* CSPSAT($\widehat{\mathcal{S}}$) *can be polynomially reduced to* CSPSAT($\mathcal{S}$).

Proof. Let $\mathcal{T} = \widehat{\mathcal{S}} \setminus \mathcal{S}$. Every element $R \in \mathcal{T}$ can be expressed by successive application of composition, intersection and converse of elements of $\mathcal{S}$. Let n be the maximal number of operations needed for a single element, where for each element the minimal number of operations is considered.

We will show by induction that for any set Θ of constraints over $\widehat{\mathcal{S}}$ we can construct a set Θ' of constraints over $\mathcal{S}$ with $|\Theta'| \leqslant 2^n \times |\Theta|$ and Θ consistent if and only if Θ' consistent (induction hypothesis). Since n is fixed, the transformation is polynomial.

Base step ($n = 1$): Θ' contains all constraints $(XSY) \in \Theta$ with $S \in \mathcal{S}$. For any constraint $(XRY) \in \Theta$ with $R \in \mathcal{T}$ one of the following cases applies:

(i) $R = S^{\smile}$ and $S \in \mathcal{S}$. Add (YSX) to Θ'.

(ii) $R = S \circ T$ and $S, T \in \mathcal{S}$. Add (XSZ) and (ZTY) to Θ', where Z is a fresh variable.

(iii) $R = S \cap T$ and $S, T \in \mathcal{S}$. Add (XSY) and (XTY) to Θ'.

Then Θ' is consistent if and only if Θ is consistent, and $|\Theta'| \leqslant 2^1 \times |\Theta|$.

Inductive step: Suppose that $n = k + 1$ and that the induction hypothesis holds for $n = k$. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the set of relations that can only be composed of relations of $\mathcal{S}$ using $k + 1$ operations. Then $\widehat{\mathcal{S}} \setminus \mathcal{T}'$ is the set of relations that can be composed of $\mathcal{S}$ using a maximal number of k operations. So the induction hypothesis is valid for $\widehat{\mathcal{S}} \setminus \mathcal{T}'$. Any relation $R \in \mathcal{T}'$ can be composed of relations of $\widehat{\mathcal{S}} \setminus \mathcal{T}'$ using exactly one operation, so R can be treated as specified in the base step. □

Note that Theorem 5 holds only if there exists an infinite supply of fresh variables; this is not always the case (e.g., bounded variable problems which are studied in logic and model theory). Since RSAT is a special case of CSPSAT, Theorem 5 can also be applied to RSAT.

Corollary 6. *Let $\mathcal{S}$ be a subset of* RCC-8.

(i) RSAT($\widehat{\mathcal{S}}$) $\in$ P *if and only if* RSAT($\mathcal{S}$) $\in$ P.

(ii) RSAT($\mathcal{S}$) *is* NP*-hard if and only if* RSAT($\widehat{\mathcal{S}}$) *is* NP*-hard.*

The first statement of Corollary 6 can be used to increase the number of elements of tractable subsets of RCC-8 considerably. With the second statement of Corollary 6 NP-hardness proofs of RSAT can be used to exclude certain relations from being in any tractable subset of RCC-8. The NP-hardness proof of Theorem 3, for example, only contains the relations {PO} and {TPP, TPP^{-1}, NTPP, NTPP^{-1}} when written in RCC-8 relations. So for any subset $\mathcal{S}$ with the two relations contained in its closure $\widehat{\mathcal{S}}$, RSAT($\mathcal{S}$) is NP-hard.

A further NP-hardness proof of RSAT(RCC-8) can be specified to exclude more relations from being in a tractable subset of RCC-8. This proof uses a polynomial transformation from ONE-IN-THREE-3SAT, where R_t = {NTPP} and R_f = {TPP^{-1}}. Three more properties, which can be verified using the composition table (Table 2), are necessary for specifying the clause and the polarity constraints.

Property 7. *Let X, Y, Z be spatial variables, where X{NTPP, TPP^{-1}}Y, Y{NTPP, TPP^{-1}}Z and Z{NTPP, TPP^{-1}}X hold. A refinement of these relations to base relations is consistent only when exactly one of the relations is refined to* {NTPP}.

Property 8. *Let X, Y, Z be spatial variables, where X{NTPP, TPP^{-1}}Z, Y{NTPP, TPP^{-1}}Z and X{PO}Y hold. A refinement of these relations to base relations is consistent only when the relations between (X, Z) and between (Y, Z) are refined to the same base relation.*

Property 9. *Let X, Y, Z be spatial variables, where X{NTPP, TPP^{-1}}Z, Z{NTPP, TPP^{-1}}Y and X{PO}Y hold. A refinement of these relations to base relations is consistent only when the relations between (X, Z) and between (Z, Y) are refined to different base relation.*

Property 7 will be used for the one-in-three condition of the literals in a clause, Properties 8 and 9 are used to propagate a refinement of a constraint to another constraint.

Lemma 10. *Let $\mathcal{S}$ be a subset of* RCC-8 *containing all base relations. If any of the relations* {TPP, NTPP, TPP^{-1}, NTPP^{-1}}, {TPP, TPP^{-1}}, {NTPP, NTPP^{-1}}, {NTPP, TPP^{-1}} *or* {TPP, NTPP^{-1}} *is contained in $\widehat{\mathcal{S}}$, then* RSAT($\mathcal{S}$) *is* NP*-hard.*

Proof. NP-hardness of RSAT({$\mathcal{B}$ ∪ {TPP, NTPP, TPP^{-1}, NTPP^{-1}}}) can be proved by replacing the RCC-5 relations of Theorem 3 with the corresponding RCC-8 relations. NP-hardness of RSAT({$\mathcal{B}$ ∪ {TPP, TPP^{-1}}}) and RSAT({$\mathcal{B}$ ∪ {NTPP, NTPP^{-1}}}) can be proved by replacing {PP} with {TPP} and {PP^{-1}} with {TPP^{-1}} or by replacing {PP} with {NTPP} and {PP^{-1}} with {NTPP^{-1}}, respectively. Then Properties 1 and 2 hold accordingly, so the transformation of Theorem 3 can also be applied using {TPP, TPP^{-1}} or {NTPP, NTPP^{-1}} instead of {PP, PP^{-1}}. A model M for a set of RCC-8-constraints Θ obtained from this revised transformation can be constructed in the same way as specified in Theorem 3.

In order to prove NP-hardness of RSAT({$\mathcal{B}$ ∪ {NTPP, TPP^{-1}}}), Properties 7–9 are required. Then ONE-IN-THREE-3SAT can be polynomially transformed to RSAT with the same transformation steps as specified in Theorem 3. Within these steps {PP} has to be replaced with {NTPP} and {PP^{-1}} with {TPP^{-1}}. The effect of the polarity constraints and the clause constraints can be seen in Figs. 7 and 8. Because of Property 7 exactly one literal must be true in any clause.

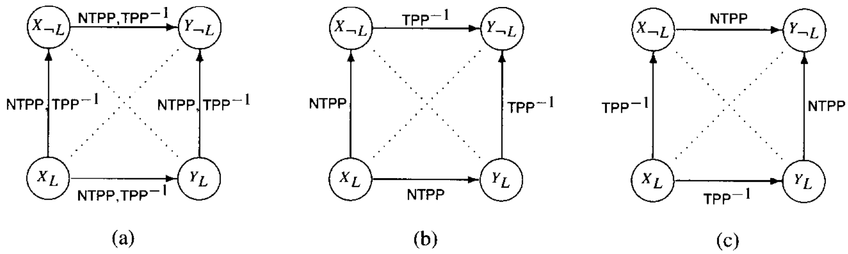

Fig. 7. The polarity constraints (a) for the transformation of ONE-IN-THREE-3SAT assure that positive and negative literals of the same variable have opposite assignments: (b) and (c) are the only possible refinements to base relations.

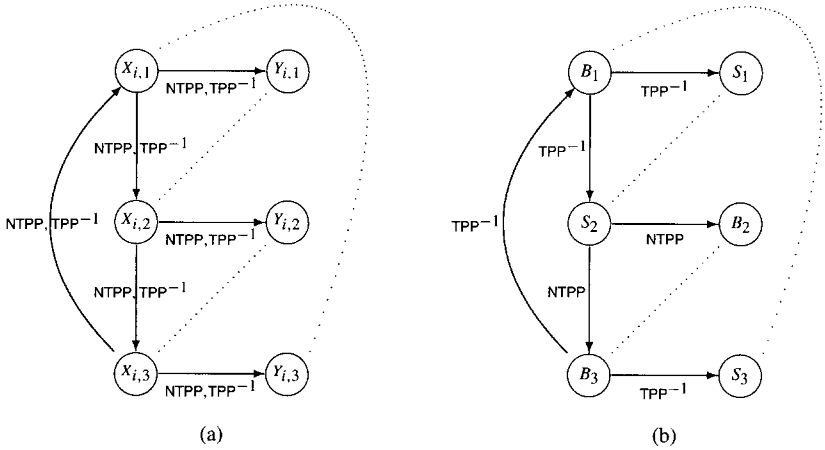

Fig. 8. Transformation of a one-in-three clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ to spatial constraints (a) and a possible refinement of a the clause to base relations (b).

We will now construct a model M for a set of RCC-8-constraints Θ resulting from the transformation of a satisfiable instance of ONE-IN-THREE-3SAT. As in Theorem 3, it can be distinguished between "small" regions S_i and "big" regions B_i where a spatial variable X_i is instantiated with S_i if $X_i\{\text{NTPP}\}Y_i$ holds and is instantiated with B_i if $X_i\{\text{TPP}^{-1}\}Y_i$ holds. If X_i is instantiated with S_i then Y_i is instantiated with B_i and *vice versa*. It can be seen in Fig. 8(b) that (with two exceptions) all small regions partially overlap each other and are part of all big regions which, again, partially overlap each other. In every clause there are two exceptions, namely, that one big region is tangential proper part of another big region and one small region partially overlaps one big region. In Fig. 8(b) this is the case for regions B_1 and B_3 and for regions B_1 and S_3. In order to construct all regions B_i and S_i, we need a region B plus the regions S, s_i, $s_{i,j}$, $s'_{i,j}$, and $b_{i,j}$ for all $i, j = 1, \ldots, n$ (n is the number of spatial variables in Θ) which are all non-tangential proper parts of B. All regions in B are disconnected with all other regions in B except for the following relationships: $s_{i,j}\{\text{EC}\}b_{j,i}$ for all i, j and $s'_{i,j}\{\text{NTPP}\}b_{j,i}$ for all i, j. Let $B'_i := B \setminus \{b_{i,j} \mid j\}$

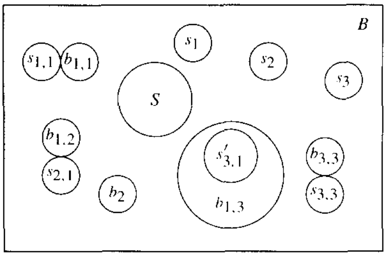

Fig. 9. A model for the six regions in Fig. 8 where $B_1 = B \setminus \{b_{1,1}, b_{1,2}, b_{1,3}, b_{3,3}\}$, $B_2 = B \setminus b_2$, $B_3 = B \setminus b_{3,3}$, $S_1 = S \cup s_1 \cup s_{1,1}$, $S_2 = S \cup s_2 \cup s_{2,1}$, and $S_3 = S \cup s_3 \cup s_{3,3} \cup s'_{3,1}$.

and $S'_i := S \cup s_i$. Then all regions B'_i partially overlap each other, all regions S'_i partially overlap each other and are non-tangential proper part of every region B'_j. B_i and S_i can be obtained from B'_i and S'_i with some modifications (see Fig. 9):

- If $S_i\{\mathsf{TPP}\}B_j$ must hold, then $S'_i := S'_i \cup s_{i,j}$.
- If $S_i\{\mathsf{PO}\}B_j$ must hold, then $S'_i := S'_i \cup s'_{i,j}$.
- If $B_i\{\mathsf{TPP}\}B_j$ must hold, then $B'_i := B'_i \setminus \{b_{j,k} | k\}$.

After these modifications, all relations required by Θ hold for any B'_i and S'_j, so by setting $B_i := B'_i$ and $S_i := S'_i$ for all i we obtain a model of Θ. Thus, an instance of ONE-IN-THREE-3SAT has a solution if and only if the set of RCC-8-constraints obtained by the specified transformation is consistent. The transformation is polynomial, therefore, $\mathsf{RSAT}(\{\mathcal{B} \cup \{\mathsf{NTPP}, \mathsf{TPP}^{-1}\})$ is NP-hard.

If $\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\}$ is contained in $\widehat{\mathcal{S}}$, then $\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}$ is also contained, so $\mathsf{RSAT}(\{\mathcal{B} \cup \{\mathsf{TPP}, \mathsf{NTPP}^{-1}\})$ is also NP-hard. With Corollary 6 the proof is completed. □

By computing the closure of all sets containing the eight base relations together with one additional relation, we obtain the following lemma.

Lemma 11. *$\mathsf{RSAT}(\mathcal{S})$ is NP-hard for any subset $\mathcal{S}$ of RCC-8 containing all base relations together with one of the 72 relations of the following sets:*

$$\mathcal{N}_1 = \{R \mid \{\mathsf{PO}\} \not\subseteq R \text{ and } (\{\mathsf{TPP}, \mathsf{TPP}^{-1}\} \subseteq R \text{ or } \{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\} \subseteq R)\},$$
$$\mathcal{N}_2 = \{R \mid \{\mathsf{PO}\} \not\subseteq R \text{ and } (\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\} \subseteq R \text{ or } \{\mathsf{TPP}^{-1}, \mathsf{NTPP}\} \subseteq R)\}.$$

Proof. The closure of any of the 72 subsets contains one of the five relations of Lemma 10. □

5. Transformation of RSAT to SAT

In the previous section we proved that particular relations cannot be added to the set of RCC-8 base relations without making the consistency problem NP-hard. In order to identify a tractable subset of RCC-8 we have to find out for which set of RCC-8 relations $\mathcal{S}$ the consistency problem $\mathsf{RSAT}(\mathcal{S})$ can be reduced to a tractable decision problem.

We keep on using propositional satisfiability problems and first transform RSAT to the NP-hard propositional satisfiability problem SAT [9], the problem of deciding whether a propositional formula in a conjunctive normal form (CNF) is satisfiable. In the next section we will then determine for which subsets $\mathcal{S}$ of RCC-8 the problem RSAT($\mathcal{S}$) is reduced to tractable fragments of SAT using the transformation developed in this section. In particular we will use HORNSAT, the tractable problem of deciding satisfiability of propositional Horn formulas, i.e., formulas where each clause contains at most one positive literal.

For transforming RSAT to SAT we transform every instance Θ of RSAT to a propositional formula in CNF that is satisfiable if and only if Θ is consistent. We start with analyzing $m(\Theta)$, the modal encoding of Θ, and then show that whenever $m(\Theta)$ is satisfiable, it has a Kripke model of a specific type. This model is then used to transform $m(\Theta)$ to a classical propositional formula.

5.1. Analysis of the modal encoding

In this subsection we analyze the modal encoding of RCC-8 and bring it in a form which is suitable for the further transformation to classical propositional logic. Using the modal encoding of RCC-8 given in Section 3, a set of RCC-8 constraints Θ can be transformed to a modal formula $m(\Theta)$ as follows, where $Reg(\Theta)$ is the set of spatial variables used in Θ:

$$m(\Theta) = \left(\bigwedge_{XRY \in \Theta} m_1(XRY)\right) \wedge \left(\bigwedge_{X \in Reg(\Theta)} m_2(X)\right).$$

m_2 consists of the modal formulas that have to be true for every region $X \in Reg(\Theta)$. It results from the regularity constraint (9) and the non-emptiness constraint $\neg X$. Instead of (9) we use the regularity constraints $\Box(\neg X \to \mathbf{I}\neg X) \wedge \Box(X \to \neg\mathbf{I}\neg\mathbf{I}X)$ which are equivalent to (9) in S4.

$$m_2(X) = \Box(\neg X \to \mathbf{I}\neg X) \wedge \Box(X \to \neg\mathbf{I}\neg\mathbf{I}X) \wedge \neg\Box\neg X.$$

The modal encoding m_1 of a spatial constraint XRY is determined by the base relations B contained in R:

$$m_1(XRY) = \bigvee_{B \subseteq R} m_1(XBY).$$

The modal encoding of spatial constraints containing only base relations results directly from Table 3:

$$\begin{aligned}
m_1(X\{\mathsf{DC}\}Y) &= \Box(\neg(X \wedge Y)),\\
m_1(X\{\mathsf{EC}\}Y) &= \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \wedge \neg\Box(\neg(X \wedge Y)),\\
m_1(X\{\mathsf{PO}\}Y) &= \neg\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \wedge \neg\Box(X \to Y) \wedge \neg\Box(Y \to X),\\
m_1(X\{\mathsf{TPP}\}Y) &= \Box(X \to Y) \wedge \neg\Box(X \to \mathbf{I}Y) \wedge \neg\Box(Y \to X),\\
m_1(X\{\mathsf{TPP}^{-1}\}Y) &= \Box(Y \to X) \wedge \neg\Box(Y \to \mathbf{I}X) \wedge \neg\Box(X \to Y),\\
m_1(X\{\mathsf{NTPP}\}Y) &= \Box(X \to \mathbf{I}Y) \wedge \neg\Box(Y \to X),\\
m_1(X\{\mathsf{NTPP}^{-1}\}Y) &= \Box(Y \to \mathbf{I}X) \wedge \neg\Box(X \to Y),\\
m_1(X\{\mathsf{EQ}\}Y) &= \Box(X \to Y) \wedge \Box(Y \to X).
\end{aligned}$$

As follows from the work by Bennett [3,4], Θ is consistent if and only if $m(\Theta)$ is satisfiable. It is striking that $m(\Theta)$ is composed only of conjunctions and disjunctions of the model and entailment constraints and the regularity constraints without using any other modal operators to combine them. Thus, a classical propositional formula in CNF can be obtained from $m(\Theta)$ by using the following steps:

(i) Consider the model and entailment constraints and the regularity constraints as "propositional atoms".
(ii) Transform $m(\Theta)$, which is a conjunction and disjunction of these "propositional atoms", into CNF.
(iii) Transform the "propositional atoms" into propositional formulas in CNF.

In order to make the following application of these steps more readable, we will introduce some abbreviations for the model and entailment constraints and for the regularity constraints.

Definition 12. Abbreviations for the model constraints: [6]

$$\begin{array}{l|l} \delta_{xy} \equiv \Box(\neg(X \wedge Y)) & \delta\mathrm{i}_{xy} \equiv \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)), \\ \pi_{xy} \equiv \Box(X \to Y) & \pi\mathrm{i}_{xy} \equiv \Box(X \to \mathbf{I}Y), \\ \gamma_{xy} \equiv \Box(Y \to X) & \gamma\mathrm{i}_{xy} \equiv \Box(Y \to \mathbf{I}X). \end{array}$$

Abbreviations for the regularity constraints: [7]

$$\mathrm{CP}_x \equiv \Box(\neg X \to \mathbf{I}\neg X) \qquad \mathrm{RP}_x \equiv \Box(X \to \neg\mathbf{I}\neg\mathbf{I}X).$$

As the entailment constraints are negations of the model constraints, they will be abbreviated as negations of the above abbreviations, e.g., $\neg\delta$ is the abbreviation for the entailment constraint $\neg\Box(\neg(X \wedge Y))$. The entailment constraint $\neg\Box\neg X$ (non-emptiness constraint) can be written as a combination of abbreviations, namely, $\neg\delta_{xy} \vee \neg\pi_{xy}$. When it is obvious which atoms (regions) are used, the abbreviations will be written without indices.

Now it is possible to write $m(\Theta)$ using only conjunctions and disjunctions of abbreviations of Definition 12. We will call this form of writing $m(\Theta)$ the *abbreviated form* of $m(\Theta)$. The *abbreviated form of a relation* R is the modal encoding $m_1(XRY)$ of R written in abbreviated form. The abbreviated form of the eight base relations is the following:

$$m_1(X\{\mathsf{DC}\}Y) \equiv \delta,$$
$$m_1(X\{\mathsf{EC}\}Y) \equiv \neg\delta \wedge \delta\mathrm{i},$$
$$m_1(X\{\mathsf{PO}\}Y) \equiv \neg\pi \wedge \neg\gamma \wedge \neg\delta\mathrm{i},$$

[6] The abbreviations are the first letters of the meaning of the constraints written in Greek symbols. The constraint $\Box(\neg(X \wedge Y))$ means that X and Y are Disconnected, hence *delta* (δ); $\Box(X \to Y)$ means that X is Part of Y, hence *pi* (π); and $\Box(Y \to X)$ means that X Contains Y, hence *gamma* (γ). The "i" in the three abbreviations $\delta\mathrm{i}$, $\pi\mathrm{i}$, and $\gamma\mathrm{i}$ indicates that the interior of a region is involved in the constraints.

[7] The abbreviations indicate the meaning of the two properties, namely, the Closedness Property (CP) and the Regularity Property (RP).

$$\begin{aligned}
m_1(X\{\mathsf{TPP}\}Y) &\equiv \pi \wedge \neg\gamma \wedge \neg\pi\mathrm{i},\\
m_1(X\{\mathsf{TPP}^{-1}\}Y) &\equiv \neg\pi \wedge \gamma \wedge \neg\gamma\mathrm{i},\\
m_1(X\{\mathsf{NTPP}\}Y) &\equiv \neg\gamma \wedge \pi\mathrm{i},\\
m_1(X\{\mathsf{NTPP}^{-1}\}Y) &\equiv \neg\pi \wedge \gamma\mathrm{i},\\
m_1(X\{\mathsf{EQ}\}Y) &\equiv \pi \wedge \gamma.
\end{aligned}$$

The abbreviated form of the other relations can be obtained by disjunctively connecting the abbreviated form of the contained base relations. We can now regard the abbreviations as "propositional atoms" and write the abbreviated form of $m(\Theta)$ in conjunctive normal form.

5.2. Determining a particular Kripke model

In order to transform the modal encoding $m(\Theta)$ of Θ to a classical propositional formula, we must find a finite Kripke frame by which $m(\Theta)$ can be modelled if it is satisfiable. With respect to this Kripke frame, $m(\Theta)$ will then be transformed to a classical propositional formula. Since the transformation must be polynomial in n, the number of spatial variables of Θ, the Kripke frame, i.e., the number of worlds of the frame must be polynomial in n.

Before identifying a particular Kripke frame, we will first have a look at the conditions that must be satisfied if $m(\Theta)$ is satisfiable. $m(\Theta)$ is satisfiable if it is true in a world w of a Kripke model $\mathcal{M} = \langle W, \{R_\Box = W \times W, R_\mathbf{I} \subseteq W \times W\}, \nu\rangle$, where W is a set of worlds, $R_\Box$ is the accessibility relation of the $\Box$-operator, $R_\mathbf{I}$ is the accessibility relation of the $\mathbf{I}$-operator, and ν is a valuation that assigns a truth value to every atom in every world. The truth conditions for $\mathcal{M}, w \Vdash m(\Theta)$ can be specified as a combination of truth conditions of sub-formulas according to the form of $m(\Theta)$: [8]

$$\begin{aligned}
\mathcal{M}, w \Vdash m(\Theta) \text{ iff } & \mathcal{M}, w \Vdash \bigwedge_{XRY\in\Theta} m_1(XRY) \wedge \mathcal{M}, w \Vdash \bigwedge_{X\in Reg(\Theta)} m_2(X)\\
\text{iff } & \bigwedge_{XRY\in\Theta} \mathcal{M}, w \Vdash m_1(XRY) \wedge \bigwedge_{X\in Reg(\Theta)} \mathcal{M}, w \Vdash m_2(X).
\end{aligned}$$

Since $m(\Theta)$ is composed only of conjunctions of model and entailment constraints and regularity constraints as specified in the previous subsection, we can carry on expressing the truth conditions of $m(\Theta)$ by combining the truth conditions of sub-formulas until we have only combinations of formulas of the type $\mathcal{M}, w \Vdash \Box\varphi$ or $\mathcal{M}, w \Vdash \neg\Box\varphi$. These sub-formulas correspond to the different modal and entailment constraints and the regularity constraints and form fifteen structurally different formulas. The truth conditions of these formulas can be obtained by combining the truth values of the single atoms. We start by specifying the truth conditions of the model constraints. [9]

[8] Note that the logical operators used to combine expressions of the form $\mathcal{M}, w \Vdash \varphi$ are not the usual modal operators, but operators in a meta language.

[9] Since $\Box$ is a strong S5-operator and every world is accessible with $R_\Box$ from any other world, the condition *for all u with $wR_\Box u$* can be replaced with $\forall u$.

$$\mathcal{M}, w \Vdash \delta \quad \text{iff} \quad \forall u.\, (\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \nVdash Y) \tag{10}$$

$$\mathcal{M}, w \Vdash \pi \quad \text{iff} \quad \forall u.\, (\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \Vdash Y) \tag{11}$$

$$\mathcal{M}, w \Vdash \gamma \quad \text{iff} \quad \forall u.\, (\mathcal{M}, u \Vdash X \vee \mathcal{M}, u \nVdash Y) \tag{12}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \delta\mathrm{i} \quad & \text{iff} \quad \forall u.(\mathcal{M}, u \nVdash \mathbf{I}X \vee \mathcal{M}, u \nVdash \mathbf{I}Y) \\ & \text{iff} \quad \forall u.\exists v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, v \nVdash X \vee \mathcal{M}, v \nVdash Y) \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \pi\mathrm{i} \quad & \text{iff} \quad \forall u.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \Vdash \mathbf{I}Y) \\ & \text{iff} \quad \forall u.\forall v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, v \Vdash Y) \end{aligned} \tag{14}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \gamma\mathrm{i} \quad & \text{iff} \quad \forall u.(\mathcal{M}, u \Vdash \mathbf{I}X \vee \mathcal{M}, u \nVdash Y) \\ & \text{iff} \quad \forall u.\forall v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, v \Vdash X \vee \mathcal{M}, u \nVdash Y) \end{aligned} \tag{15}$$

Now the truth conditions of the entailment constraints.

$$\mathcal{M}, w \Vdash \neg\delta \quad \text{iff} \quad \exists u.\, (\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \Vdash Y) \tag{16}$$

$$\mathcal{M}, w \Vdash \neg\pi \quad \text{iff} \quad \exists u.\, (\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \nVdash Y) \tag{17}$$

$$\mathcal{M}, w \Vdash \neg\gamma \quad \text{iff} \quad \exists u.\, (\mathcal{M}, u \nVdash X \wedge \mathcal{M}, u \Vdash Y) \tag{18}$$

$$\mathcal{M}, w \Vdash \neg\delta \vee \neg\gamma \quad \text{iff} \quad \exists u.\, (\mathcal{M}, u \Vdash X) \tag{19}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \neg\delta\mathrm{i} \quad & \text{iff} \quad \exists u.(\mathcal{M}, u \Vdash \mathbf{I}X \wedge \mathcal{M}, u \Vdash \mathbf{I}Y) \\ & \text{iff} \quad \exists u.\forall v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, v \Vdash X \wedge \mathcal{M}, v \Vdash Y) \end{aligned} \tag{20}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \neg\pi\mathrm{i} \quad & \text{iff} \quad \exists u.(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \nVdash \mathbf{I}Y) \\ & \text{iff} \quad \exists u.\exists v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, v \nVdash Y) \end{aligned} \tag{21}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \neg\gamma\mathrm{i} \quad & \text{iff} \quad \exists u.(\mathcal{M}, u \nVdash \mathbf{I}X \wedge \mathcal{M}, u \Vdash Y) \\ & \text{iff} \quad \exists u.\exists v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, v \nVdash X \wedge \mathcal{M}, u \Vdash Y) \end{aligned} \tag{22}$$

Finally, the truth conditions of the regularity constraints.

$$\begin{aligned} \mathcal{M}, w \Vdash \mathrm{CP} \quad & \text{iff} \quad \forall u.(\mathcal{M}, u \Vdash X \vee \mathcal{M}, u \Vdash \mathbf{I}\neg X) \\ & \text{iff} \quad \forall u.\forall v\colon\ uR_{\mathbf{I}}v.(\mathcal{M}, u \Vdash X \vee \mathcal{M}, v \nVdash X) \end{aligned} \tag{23}$$

$$\begin{aligned} \mathcal{M}, w \Vdash \mathrm{RP} \quad & \text{iff} \quad \forall u.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \nVdash \mathbf{I}\neg\mathbf{I}X) \\ & \text{iff} \quad \forall u.\exists s\colon\ uR_{\mathbf{I}}s.\forall t\colon\ sR_{\mathbf{I}}t.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, t \Vdash X) \end{aligned} \tag{24}$$

We call this form of writing $m(\Theta)$ the *explicit form* of $m(\Theta)$.

We will construct a polynomial Kripke frame on which a model $\mathcal{M}$ for $m(\Theta)$ can be based if $m(\Theta)$ is satisfiable at all. For this we will successively add only those worlds to the frame that are explicitly required by the formulas (10)–(24). Amongst them are formulas of the type $\forall w.\varphi$ and formulas of the type $\exists w.\varphi$. Only formulas of the type $\exists w.\varphi$ explicitly require a world with the specified properties φ. If a world with these properties is not present, it has to be introduced. Formulas of the type $\forall w.\varphi$ affect the properties of worlds already introduced, but do not require fresh worlds. These properties must be enforced by the model and not by the frame. So worlds have to be introduced only for existential quantifiers. Since there are also formulas of the type $\forall w.\exists v.\phi$ a fresh world must not be introduced for every occurring existential quantifier. In the following we will analyze which existential quantifiers introduce fresh worlds and for which existential quantifiers there already exist worlds with the specified properties. For this we will classify worlds according to the accessibility relation $R_{\mathbf{I}}$ holding between them.

Definition 13. Let $u, v, w \in W$ be worlds of the frame $\mathcal{F}$.
- u is a world of *level* 0 if $vR_{\mathbf{I}}u$ only holds for $v = u$.
- u is a world of *level* $l+1$ if $vR_{\mathbf{I}}u$ holds for a world v of level l and if there is no world $w \neq u$ of a level higher than l with $wR_{\mathbf{I}}u$.
- If w is a world of level 0 and v is an $R_{\mathbf{I}}$-successor of w, then w is called the *introductory world* of v.
- $W_i \subseteq W$ is the set of worlds of level i.

Every occurrence of one of the formulas (16)–(22), which correspond to the six different entailment constraints, shall introduce a fresh world of level 0. Formulas (21) and (22) introduce additional worlds of level 1. Every model shall be designed according to these principles. Instead of analyzing which world requires a fresh successor by the $\forall w \exists v$ condition of formulas (14) and (22), we will in the following define a Kripke frame of a particular structure and prove that whenever $m(\Theta)$ is consistent there is a model based on that frame.

Definition 14. Let Θ be a set of RCC-8-constraints and $n = |Reg(\Theta)|$ be the number of regions in Θ. An RCC-8-*frame* $\mathcal{F}_l = \langle W, \{R_\Box, R_{\mathbf{I}}\}\rangle$ of level l has the following properties (see Fig. 10):

(i) W contains only worlds up to level l, i.e., $W = \bigcup_{i=0}^{l} W_i$.
(ii) For every world $u \in W_i$ there are exactly $2n$ worlds $v \in W_{i+1}$ with $uR_{\mathbf{I}}v$.
(iii) For every world $v \in W_k$ there is exactly one world $u \in W_i$ with $uR_{\mathbf{I}}v$ for every level $0 \leqslant i \leqslant k$ (if $i = k$ then $u = v$).
(iv) For all worlds $u, v, w \in W$: $wR_\Box v$, $wR_{\mathbf{I}}w$, and $wR_{\mathbf{I}}v$, $vR_{\mathbf{I}}u$ implies $wR_{\mathbf{I}}u$.

An RCC-8-*model* of level l is based on an RCC-8-frame of level l. In a *polynomial* RCC-8-*frame/model* of level l the number of worlds is polynomially bounded by n.

Note that item (iv) guarantees that every RCC-8-frame of level l is an S4-frame. In [38] we used RCC-8-frames of level 2. It turns out, however, that RCC-8-frames of level 1 are sufficient for our purposes.

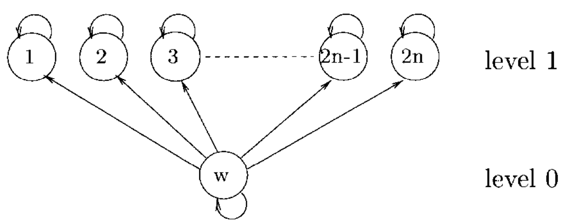

Fig. 10. A world w of level 0 together with its $2n$ $R_{\mathbf{I}}$-successors of an RCC-8-frame of level 1. Worlds are drawn as circles, the arrows indicate the accessibility of worlds with the relation $R_{\mathbf{I}}$.

Lemma 15. *If $m(\Theta)$ is satisfiable, there is a polynomial* RCC-8*-model $\mathcal{M}$ of level* 1 *with* $\mathcal{M}, w \Vdash m(\Theta)$.

Proof. The abbreviated form of every relation contains at most six different negated abbreviations.[10] Since they correspond to entailment constraints, each of these abbreviations introduces a new world of level 0. As there are n regions and therefore n^2 relations, there are at most $6n^2$ worlds of level 0. For each world w and each atom X there might be a sub-formula of $m(\Theta)$ that forces the existence of an $R_{\mathbf{I}}$-successor of w that forces X or that forces $\neg X$. Thus, since there are n different atoms X, a maximal number of $2n$ different $R_{\mathbf{I}}$-successors is sufficient for each world. Therefore, if $m(\Theta)$ is satisfiable, it can be modelled by an RCC-8-model of some level l.

Suppose that $\mathcal{M}' = \langle W', \{R'_{\mathbf{I}}, R'_{\Box}\}, \nu'\rangle$ is such an RCC-8-model that models $m(\Theta)$ and suppose there is no polynomial RCC-8-model of level 1 that models $m(\Theta)$. We will prove by contradiction that a polynomial RCC-8-model of level 1 exists if $m(\Theta)$ is satisfiable. For this we will construct a polynomial RCC-8-model $\mathcal{M} = \langle W, R, \nu\rangle$ of level 1 using $\mathcal{M}'$. Let $W = W'_0 \cup W'_1$, $R_{\mathbf{I}} = \{(u, v) \in R'_{\mathbf{I}} | u, v \in W\}$, $R_{\Box} = W \times W$, and $\nu(w, a) = \nu'(w, a)$ for all $w \in W$ and all propositional atoms a. We now have to find out which of the formulas (10)–(24) hold for $\mathcal{M}'$ but not for $\mathcal{M}$. Trivially, if the formulas (16)–(22), corresponding to the entailment constraints, hold for $\mathcal{M}'$ they also hold for $\mathcal{M}$. The same for the formulas (10)–(12). As the only $R_{\mathbf{I}}$-successor of a world of level 1 is the world itself, formulas (14), (15) and formulas (23) and (24) also hold for $\mathcal{M}$ if they hold for $\mathcal{M}'$. So only formula (13) remains to be checked.

Suppose that formula (13) holds for two atoms X and Y and suppose there is a world $v \in W'_1$ with $\nu'(v, X) = \mathit{true}$ and $\nu'(v, Y) = \mathit{true}$ and an $R_{\mathbf{I}}$-successor of v for which either X or Y is *false*. Then (13) is satisfied in $\mathcal{M}'$ but not in $\mathcal{M}$. In this case $\mathcal{M}$ is not a model for $m(\Theta)$, but it would be a model if the truth value of either $\nu(v, X)$ or $\nu(v, Y)$ can be set to *false* without contradicting any other formula. Therefore it is necessary to find out how an atom can be forced to be *true* in a world of level 1. Formulas (11) and (12) force it only when another atom is already forced to be *true* in the same world. If the atom is forced by one of the formulas (14), (20), (24), or axiom schemata (6) to be *true* in a world then it is also forced to be *true* in all $R_{\mathbf{I}}$-successors of this world.
So, if both X and Y are forced to be *true* in v, they must both be *true* in any $R_{\mathbf{I}}$-successor of v and $\mathcal{M}'$ cannot be a model for $m(\Theta)$, so one of them, say X, is not forced and $\nu(v, X)$ can be set to *false*. We have constructed a polynomial RCC-8-model of level 1 which contradicts our initial assumption that there is no such model. □

In the following we will use the term RCC-8-frame/model to refer to an RCC-8-frame/model of level 1. A spatial interpretation of the RCC-8-models can be found in [37]. In this interpretation, worlds are interpreted as points of the topological space, and the accessibility relation $R_{\mathbf{I}}$ specifies neighborhoods of points.

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[10] This number results from a straightforwardly computed abbreviated form of every relation. It might be decreased by optimizing the abbreviated form with respect to the number of negated abbreviations.

5.3. Transformation to a classical propositional formula

We proved that whenever $m(\Theta)$ is satisfiable, a model for $m(\Theta)$ can be based on a polynomial RCC-8-frame that contains as many worlds of level 0 as entailment constraints are contained in $m(\Theta)$. We will now transform the explicit form of $m(\Theta)$ to a classical propositional formula $p(m(\Theta))$ in CNF such that $p(m(\Theta))$ is satisfiable if and only if $m(\Theta)$ is satisfiable in a polynomial RCC-8-frame $\mathcal{F} = \langle W, \{R_\Box, R_\mathbf{I}\}\rangle$. For this we will introduce a propositional atom for every world $w \in W$ and every region $X \in Reg(\Theta)$ such that the atom is true if and only if $\nu(w, X) = true$. In order to preserve the structure of the RCC-8-frame in propositional logic, two functions have to be defined:

Definition 16. Let $\mathcal{F} = \langle W, \{R_\Box, R_\mathbf{I}\}\rangle$ be an RCC-8-frame.
- $f : W \to W_0$ determines the introductory world of every world,
- $g : W \to \emptyset \cup \{1, \ldots, 2n\}$ provides all worlds of the same introductory world with a specific order, i.e., if the worlds u and v are distinct, have the same level and $f(u) = f(v)$, then $g(u) \neq g(v)$. $g(u) = \emptyset$, if and only if u has level 0.

For every world $w \in W$ and every region $X \in Reg(\Theta)$ we introduce the propositional atom $X_{f(w)}^{g(w)}$. Before transforming $m(\Theta)$ to a classical propositional formula $p(m(\Theta))$ in CNF, we will first transform $m(\Theta)$ in a straightforward manner to a classical propositional formula $q(m(\Theta))$.

Definition 17. The explicit form of $m(\Theta)$ is transformed to a classical propositional formula $q(m(\Theta))$ as follows:
- $\mathcal{M}, w \Vdash X$ is transformed to $X_{f(w)}^{g(w)}$.
- $\mathcal{M}, w \nVdash X$ is transformed to $\neg X_{f(w)}^{g(w)}$.
- The meta operators $\wedge$ and $\vee$ (see Footnote 8) are transformed to the propositional operators $\wedge$ and $\vee$, respectively.

Since all worlds of the RCC-8-frame $\mathcal{F}$ are known, quantifiers can be transformed to propositional operators:
- A universal quantification of particular worlds results in a conjunction over these worlds.
- An existential quantification of a world of level 0 results in a disjunction over all worlds of level 0.
- An existential quantification of an $R_\mathbf{I}$-successor of a world w results in a disjunction over the $2n$ $R_\mathbf{I}$-successor of w.

With the specified transformation, δ, $\neg\delta$ and RP, for example, are transformed to the following formulas:

$$q(\delta) = \bigwedge_{w \in W_0} (\neg X_w \vee \neg Y_w) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (\neg X_w^i \vee \neg Y_w^i),$$

$$q(\neg\delta) = \bigvee_{w \in W_0} (X_w \wedge Y_w),$$

$$q(\mathrm{RP}) = \bigwedge_{w \in W_0} \bigvee_{i=1}^{2n} (\neg X_w \vee X_w^i).$$

Lemma 18. *$q(m(\Theta))$ is satisfiable if and only if $m(\Theta)$ is satisfiable.*

Proof. It follows from Theorem 15 that if $m(\Theta)$ is satisfiable, there is an RCC-8-model $\mathcal{M}$ based on a particular RCC-8-frame $\mathcal{F}$. Since the RCC-8-frame $\mathcal{F}$ is known, $\mathcal{M}$ is determined by its valuation ν which only depends on the model and entailment constraints and on the regularity constraints contained in $m(\Theta)$. The explicit form of $m(\Theta)$ contains all conditions of the valuation ν of $\mathcal{M}$ explicitly, i.e., only conditions of the form $\mathcal{M}, w \Vdash X$ or $\mathcal{M}, w \nVdash X$ are contained, which correspond to $\nu(X, w) = \mathit{true}$ or $\nu(X, w) = \mathit{false}$, respectively. As there is a one-to-one correspondence between $\nu(X, w)$ and the propositional atom $X_{f(w)}^{g(w)}$ of $q(m(\Theta))$ for every world w and every region X, it follows immediately from the transformation specified in Proposition 17 that $q(m(\Theta))$ is satisfiable if and only if $m(\Theta)$ is satisfiable, where $X_{f(w)}^{g(w)} = \mathit{true}$ if and only if $\nu(X, w) = \mathit{true}$. □

It is obvious that because of the transformation of existential quantifiers to disjunctions, $q(m(\Theta))$ is not in CNF. Simply transforming $q(m(\Theta))$ to CNF is, however, not favorable for the further reduction to HORNSAT which is done in the next section. Transforming, for example, $q(\neg\delta)$ to CNF results in non-Horn formulas. Also $q(\mathrm{RP})$ is not a Horn formula because of the disjunction obtained from transforming the existential quantifier. Therefore we have to treat the existential quantifiers differently in order to obtain $p(m(\Theta))$ in CNF.

As stated before, the entailment constraints, i.e., the negated abbreviations, introduce fresh worlds of level 0, so instead of a disjunction over all worlds of level 0, a fresh world of W of level 0 together with it's $2n$ $R_\mathbf{I}$-successors will be explicitly considered in $p(m(\Theta))$. Then $\neg\delta$, for example, is transformed to the Horn formula $p(\neg\delta) = X_w \wedge Y_w$ *(for a fresh world w of level* 0).

Handling existential quantifiers of worlds of level 1 is more difficult. We exploit the fact that for any statement of the form $\exists u.wR_\mathbf{I}u$: $\mathcal{M}, u \Vdash X$ or $\exists u.wR_\mathbf{I}u$: $\mathcal{M}, u \nVdash X$, for all n different spatial variables $X \in \mathit{Reg}(\Theta)$, at most $2n$ $R_\mathbf{I}$-successors are necessary to assure that there exists a world that makes this statement true. Since in the RCC-8-model every world of level 1 has $2n$ $R_\mathbf{I}$-successors, we will reserve one of these $2n$ $R_\mathbf{I}$-successors for every statement of this kind. Therefore we need a function $h : \mathit{Reg}(\Theta) \cup \overline{\mathit{Reg}(\Theta)} \to \{1, \ldots, 2n\}$ that associates every region and the complement of every region with one of the $2n$ $R_\mathbf{I}$-successors. Using this function, the following properties must hold for all $R_\mathbf{I}$-successor:

- If there exists an $R_\mathbf{I}$-successor u of w with $\mathcal{M}, u \Vdash X$, then $X_w^{h(x)} = \mathit{true}$.
- If there exists an $R_\mathbf{I}$-successor u of w with $\mathcal{M}, u \nVdash X$, then $X_w^{h(\neg x)} = \mathit{false}$.
- If the $h(X)$-successor of a world w holds $\neg X$ then every $R_\mathbf{I}$-successor of w holds $\neg X$.
- If the $h(\neg X)$-successor of w holds X then every $R_\mathbf{I}$-successor of w holds X.

To ensure these properties, additional formulas have to be added to $p(m(\Theta))$ for every region X:

$$\bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(X_w^{h(x)} \vee \neg X_w^i\right), \tag{25}$$

$$\bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(\neg X_w^{h(\neg x)} \vee X_w^i\right). \tag{26}$$

Formula (25) ensures the first and third property, formula (26) the second and forth property. With these modifications to $q(m(\Theta))$ we obtain a propositional formula $p(m(\Theta))$ in conjunctive normal form. The transformation is given in the following proposition.

Proposition 19. *$m(\Theta)$ can be transformed to a propositional formula $p(m(\Theta))$ by transforming all model and entailment constraints and the regularity constraints contained in $m(\Theta)$ in the following way:*

$$\begin{aligned}
p(\delta) &= \bigwedge_{w\in W_0} (\neg X_w \vee \neg Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(\neg X_w^i \vee \neg Y_w^i\right),\\
p(\neg\delta) &= X_w \wedge Y_w \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\pi) &= \bigwedge_{w\in W_0} (\neg X_w \vee Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(\neg X_w^i \vee Y_w^i\right),\\
p(\neg\pi) &= X_w \wedge \neg Y_w \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\gamma) &= \bigwedge_{w\in W_0} (X_w \vee \neg Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(X_w^i \vee \neg Y_w^i\right),\\
p(\neg\gamma) &= \neg X_w \wedge Y_w \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\delta\mathrm{i}) &= \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(\neg X_w^i \vee \neg Y_w^i\right),\\
p(\neg\delta\mathrm{i}) &= \bigwedge_{i=1}^{2n}\left(X_w^i \wedge Y_w^i\right) \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\pi\mathrm{i}) &= \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(\neg X_w \vee Y_w^i\right),\\
p(\neg\pi\mathrm{i}) &= X_w \wedge \neg Y_w^{h(\neg y)} \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\gamma\mathrm{i}) &= \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(X_w^i \vee \neg Y_w\right),\\
p(\neg\gamma\mathrm{i}) &= \neg X_w^{h(\neg x)} \wedge Y_w \ (\textit{for a fresh world } w \textit{ of level } 0),\\
p(\mathrm{CP}) &= \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\left(X_w \vee \neg X_w^i\right),
\end{aligned}$$

$$p(\text{RP}) = \bigwedge_{w \in W_0} \left(\neg X_w \vee X_w^{h(x)}\right),$$

$$p(\neg\pi \vee \neg\gamma) = X_w \ (\textit{for a fresh world } w \textit{ of level } 0).$$

Lemma 20. *$p(m(\Theta))$ is satisfiable if and only if $q(m(\Theta))$ is satisfiable.*

Proof. Every entailment constraint introduces a fresh world of level 0 which explicitly fulfills the requirements of the entailment constraint. Thus, whenever one world fulfills the requirements of an entailment constraint it is fulfilled by the world explicitly introduced by the entailment constraint and *vice versa*. If each of the $2n$ R_{I}-successors of a world of level 0 is associated to a different region or the complement of a region and the formulas (25) and (26) are added to $p(m(\Theta))$, then $p(\neg\pi\mathrm{i})$, $p(\neg\gamma\mathrm{i})$, and $p(\text{RP})$ are satisfiable if and only if $q(\neg\pi\mathrm{i})$, $q(\neg\gamma\mathrm{i})$, and $q(\text{RP})$ are satisfiable, respectively. For the model constraints and for the constraint CP, p is equal to q. □

Theorem 21. RSAT(RCC-8) *can be polynomially transformed to* SAT.

Proof. From Lemmas 18 and 20 it follows that $m(\Theta)$ can be transformed to a propositional formula in conjunctive normal form $p(m(\Theta))$ such that $m(\Theta)$ is satisfiable if and only if $p(m(\Theta))$ is satisfiable. Since every world together with every region corresponds to a literal, of which there are at most $12n^4$, and the number of clauses of $p(m(\Theta))$ is polynomial in the number of worlds of the RCC-8-model of $m(\Theta)$ and the size of $m(\Theta)$, the transformation is polynomial. □

As SAT is an NP-complete problem, it follows that RSAT is in NP. Together with Corollary 4 this results in the following theorem.

Theorem 22. RSAT(RCC-8) *is* NP-*complete.*

6. Tractable subsets of RCC-8

In this section we analyze which relations are transformed to propositional Horn formulas using the transformation of RSAT to SAT as specified in the previous section. Since the propositional satisfiability problem for Horn formulas (HORNSAT) is tractable [15], the set of these relations forms a tractable subset of RCC-8. We will then prove that the set of relations identified in this way is maximal with respect to tractability, i.e., no other relation can be added to the set without losing tractability.

6.1. *Identifying a large tractable subset of* RCC-8

In order to identify the relations transformable to Horn formulas, we study the abbreviated form of every relation which consists of a conjunction of disjunctions of abbreviations, i.e., the abbreviated form of relations can be considered as a "propositional formula" in CNF where the abbreviations are the "propositional atoms". We have to find the relations whose abbreviated forms consist only of "clauses" which are transformable to Horn formulas.

Proposition 23. *Applying the transformation p to the model and entailment constraints and to the regularity constraints as specified in Proposition* 19 *leads to Horn formulas. Formulas* (25) *and* (26) *are also Horn formulas.*

With these formulas further Horn formulas can be specified.

Lemma 24. *The following disjunctions of abbreviations are transformed to propositional Horn formulas*:

$$\delta \vee C \ \ \textit{with} \ \ C \in \{\pi, \neg\pi, \gamma, \neg\gamma, \delta\mathrm{i}, \neg\delta\mathrm{i}, \pi\mathrm{i}, \neg\pi\mathrm{i}, \gamma\mathrm{i}, \neg\gamma\mathrm{i}\},$$
$$\delta\mathrm{i} \vee C \ \ \textit{with} \ \ C \in \{\neg\delta, \pi, \neg\pi, \gamma, \neg\gamma, \pi\mathrm{i}, \neg\pi\mathrm{i}, \gamma\mathrm{i}, \neg\gamma\mathrm{i}\}.$$

Proof. The propositional formulas resulting from the model constraints δ and $\delta\mathrm{i}$ are indefinite Horn formulas, i.e., clauses that do not contain any positive literals. Because the propositional formulas resulting from the other constraints are Horn formulas, and the disjunction of an indefinite Horn formula with another Horn formula is again a Horn formula, the lemma holds. □

In our framework some disjunctions of model and entailment constraints are tautologies. These disjunctions of abbreviations can be eliminated from the abbreviated form.

Lemma 25. *The following modal formulas written in abbreviated form are tautologies in* S4 *in the presence of the non-emptiness constraints and the constraints for regular closed regions*:

$$\neg\delta \vee \neg\pi,\ \neg\delta \vee \neg\gamma,\ \neg\delta \vee \delta\mathrm{i},\ \pi \vee \neg\pi\mathrm{i},\ \gamma \vee \neg\gamma\mathrm{i},$$
$$\neg\pi \vee \neg\delta\mathrm{i},\ \neg\gamma \vee \neg\delta\mathrm{i},\ \neg\gamma \vee \neg\pi\mathrm{i} \vee \gamma\mathrm{i},\ \neg\pi \vee \pi\mathrm{i} \vee \neg\gamma\mathrm{i}.$$

Proof. We prove that the above given modal formulas are tautologies by showing that the negation of each of these formulas results in a contradiction. This can be done by using, e.g., tableau based proof procedures for modal logic [13].

$\delta \wedge \pi \equiv \Box(\neg(X \wedge Y)) \wedge \Box(X \to Y)$:
contradicts the non-emptiness constraint $\neg\Box\neg X$.

$\delta \wedge \gamma \equiv \Box(\neg(X \wedge Y)) \wedge \Box(Y \to X)$: analogous.

$\delta \wedge \neg\delta\mathrm{i} \equiv \Box(\neg(X \wedge Y)) \wedge \neg\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$:
results in a contradiction when combined with the S4 axiom schemata $\mathbf{I}X \to X$.

$\neg\pi \wedge \pi\mathrm{i} \equiv \neg\Box(X \to Y) \wedge \Box(X \to \mathbf{I}Y)$:
results in a contradiction when combined with the constraint that all regions are closed, $\Box(\neg X \to \neg\mathbf{I}X)$.

$\neg\gamma \wedge \gamma\mathrm{i} \equiv \neg\Box(Y \to X) \wedge \Box(Y \to \mathbf{I}X)$: analogous.

$\pi \wedge \delta\mathrm{i} \equiv \Box(X \to Y) \wedge \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$:
the regularity constraint $\Box(X \to \neg\mathbf{I}\neg\mathbf{I}X)$ and the non-emptiness constraint $\neg\Box\neg X$ enforce that there is a world u with $\mathcal{M}, u \Vdash \mathbf{I}X$. Because of the S4 axiom schemata $\mathbf{I}X \to X$, $\mathcal{M}, u \Vdash X$ also holds. π entails $\mathcal{M}, u \Vdash Y$ and $\delta\mathrm{i}$ entails $\mathcal{M}, u \nVdash \mathbf{I}Y$. So there is an $R_\mathbf{I}$-successor v of u with $\mathcal{M}, v \nVdash Y$. Because $\mathcal{M}, u \Vdash \mathbf{I}X$ holds, $\mathcal{M}, v \Vdash X$ also holds. π entails $\mathcal{M}, v \Vdash Y$, which results in a contradiction.

$\gamma \wedge \delta\mathrm{i} \equiv \Box(Y \to X) \wedge \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$: analogous.

$\gamma \wedge \pi\mathrm{i} \wedge \neg\gamma\mathrm{i} \equiv \Box(Y \to X) \wedge \Box(X \to \mathbf{I}Y) \wedge \neg\Box(Y \to \mathbf{I}X)$:
because of $\neg\gamma\mathrm{i}$ there is a world u with $\mathcal{M}, u \nVdash \mathbf{I}X$ and $\mathcal{M}, u \Vdash Y$. Because of γ, $\mathcal{M}, u \Vdash X$ holds and because of $\pi\mathrm{i}$, $\mathcal{M}, u \Vdash \mathbf{I}Y$ holds. Since $\mathcal{M}, u \nVdash \mathbf{I}X$ holds, there is an $R_\mathbf{I}$-successor v of u with $\mathcal{M}, v \nVdash X$. So γ results in $\mathcal{M}, v \nVdash Y$ and $\mathcal{M}, u \Vdash \mathbf{I}Y$ results in $\mathcal{M}, v \Vdash Y$, which is a contradiction.

$\pi \wedge \gamma\mathrm{i} \wedge \neg\pi\mathrm{i} \equiv \Box(X \to Y) \wedge \Box(Y \to \mathbf{I}X) \wedge \neg\Box(X \to \mathbf{I}Y)$: analogous. □

All relations with an abbreviated form using only abbreviations or disjunctions of abbreviations transformable to Horn formulas can be transformed to Horn formulas. In this way 64 relations can be transformed to Horn formulas. They are listed in Table B.1 together with their abbreviated form. We call the subset of RCC-8 containing these relations $\mathcal{H}_8$.

Theorem 26. RSAT($\mathcal{H}_8$) *can be polynomially reduced to* HORNSAT.

Proof. Every constraint using a relation of $\mathcal{H}_8$ is transformable to a Horn formula. So every set Θ of $\mathcal{H}_8$-constraints can be written as a conjunction of the Horn formulas of their elements, which is also a Horn formula. □

Thus, RSAT($\mathcal{H}_8$) $\in$ P and because of Corollary 6 the closure of $\mathcal{H}_8$ is also in P.

Corollary 27. RSAT($\widehat{\mathcal{H}}_8$) $\in$ P.

Apart from the relations of $\mathcal{N}_1$ and $\mathcal{N}_2$, which cannot be included in any tractable subset of RCC-8 that contains all base relations (see Lemma 11), the only relations not contained in $\widehat{\mathcal{H}}_8$ are those that contain EQ and NTPP but not TPP, and the same for the converse relations.

Theorem 28. $\widehat{\mathcal{H}}_8$ *contains the following* 148 RCC-8 *relations*:

$$\widehat{\mathcal{H}}_8 = \text{RCC-8} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3),$$

where

$$\mathcal{N}_3 = \{R \mid \{\mathsf{EQ}\} \subseteq R \textit{ and } ((\{\mathsf{NTPP}\} \subseteq R, \{\mathsf{TPP}\} \not\subseteq R) \textit{ or } (\{\mathsf{NTPP}^{-1}\} \subseteq R, \{\mathsf{TPP}^{-1}\} \not\subseteq R))\}$$

and $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *were defined in Lemma* 11.

6.2. Maximality of $\widehat{\mathcal{H}}_8$ with respect to tractability

Our goal is to find maximal tractable subsets of RCC-8 since these subsets mark the boundary between tractability and NP-hardness, i.e., any subset of one of these sets is tractable, any superset is NP-hard. For proving that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8 we have to show that no relation of $\mathcal{N}_3$ can be added to $\widehat{\mathcal{H}}_8$ without making RSAT intractable. By computing the closure of $\widehat{\mathcal{H}}_8$ with each relation of $\mathcal{N}_3$ we get the following lemma.

Lemma 29. *The closure of every subset of* RCC-8 *containing* $\widehat{\mathcal{H}}_8$ *and one relation of* $\mathcal{N}_3$ *contains the relation* {EQ, NTPP}.

Therefore it is sufficient to prove NP-hardness of RSAT($\widehat{\mathcal{H}}_8 \cup$ {EQ, NTPP}) for showing that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

Theorem 30. RSAT($\widehat{\mathcal{H}}_8 \cup$ {EQ, NTPP}) *is* NP-*complete.*

Proof. Transformation of 3SAT to RSAT($\widehat{\mathcal{H}}_8 \cup$ {EQ, NTPP}).[11] Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of variables and $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses of an arbitrary instance of 3SAT with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$, where $l_{i,j}$ are literals over variables of $\mathcal{V}$. We will construct a set of spatial constraints Θ using only relations of $\widehat{\mathcal{H}}_8 \cup$ {EQ, NTPP}, such that Θ is satisfiable if and only if $\mathcal{C}$ is a positive instance of 3SAT, using the following three transformation steps:

(1) For each variable $v_L \in \mathcal{V}$ the spatial variables X_L, Y_L, $X_{\neg L}$, and $Y_{\neg L}$ are introduced by adding the spatial constraints $X_L\{\mathsf{EQ},\mathsf{NTPP}\}Y_L$ and $X_{\neg L}\{\mathsf{EQ},\mathsf{NTPP}\}Y_{\neg L}$ to Θ. Additionally, the following polarity constraints are added to Θ (see Fig. 11):

$$X_L\{\mathsf{EC},\mathsf{NTPP}\}X_{\neg L},\quad Y_L\{\mathsf{TPP}\}Y_{\neg L},$$
$$X_L\{\mathsf{TPP},\mathsf{NTPP}\}Y_{\neg L},\quad Y_L\{\mathsf{EC},\mathsf{TPP}\}X_{\neg L}.$$

(2) For each literal occurrence $l_{i,j}$ the spatial variables $X_{i,j}$ and $Y_{i,j}$ are introduced by adding the spatial constraint $X_{i,j}\{\mathsf{EQ},\mathsf{NTPP}\}Y_{i,j}$ to Θ. Depending on whether the literal occurrence is positive or negative different polarity constraints have to be added to Θ.

(a) $l_{i,j} \equiv v_L$:

$$X_{i,j}\{\mathsf{EC},\mathsf{NTPP}\}X_{\neg L},\quad Y_{i,j}\{\mathsf{TPP}\}Y_{\neg L},$$
$$X_{i,j}\{\mathsf{TPP},\mathsf{NTPP}\}Y_{\neg L},\quad Y_{i,j}\{\mathsf{EC},\mathsf{TPP}\}X_{\neg L}.$$

(b) $l_{i,j} \equiv \neg v_L$:

$$X_{i,j}\{\mathsf{EC},\mathsf{NTPP}\}X_L,\quad Y_{i,j}\{\mathsf{TPP}\}Y_L,$$
$$X_{i,j}\{\mathsf{TPP},\mathsf{NTPP}\}Y_L,\quad Y_{i,j}\{\mathsf{EC},\mathsf{TPP}\}X_L.$$

[11] The structure of this proof parallels the proof of Theorem 3 in Section 4. Here, however, $R_t =$ NTPP and $R_f =$ EQ. The comments on "polarity constraints" and "clause constraints" given in Section 4 hold accordingly.

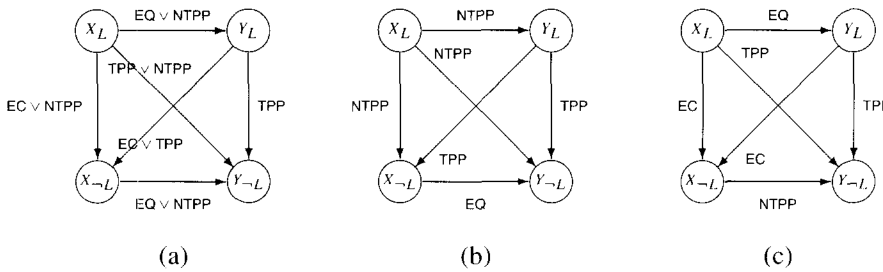

Fig. 11. The polarity constraints (a) for the transformation of 3SAT assure that positive and negative literals of the same variable have opposite assignments: (b) and (c) are the only possible refinements of the relations to base relations.

(3) For each clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ the following clause constraints are added to Θ:

$$Y_{i,1}\{\mathsf{NTPP}^{-1}\}X_{i,2}, \quad Y_{i,2}\{\mathsf{NTPP}^{-1}\}X_{i,3}, \quad Y_{i,3}\{\mathsf{NTPP}^{-1}\}X_{i,1}.$$

With this transformation for every variable as well as for every literal occurrence two spatial variables X and Y (with the appropriate indices) are introduced. When a literal occurrence or a variable is assigned *true*, the corresponding spatial variables hold the relation $X\{\mathsf{NTPP}\}Y$. When a literal occurrence or a variable is assigned *false*, the corresponding spatial variables hold the relation $X\{\mathsf{EQ}\}Y$.

Transformation step (1) introduces the spatial variables corresponding to the positive and the negative literal of each variable. The polarity constraints assure that both literals have complementary assignments (see Fig. 11). Transformation step (2) introduces spatial variables for every literal occurrence. Again, the polarity constraints assure correct assignments. Finally, transformation step (3) makes sure that at least one literal occurrence of every clause is *true*. If all literal occurrences of a clause are *false*, the corresponding spatial variables hold the relation {EQ}. Then there is a path starting at $X_{i,1}$, passing $X_{i,2}$ and $X_{i,3}$, and ending at $X_{i,1}$ where $\{\mathsf{NTPP}^{-1}\}$ and {EQ} are the only occurring relations, which is an inconsistent situation. All other combinations are possible. We now have to show that an instance of 3SAT has a solution if and only if the set of spatial constraints Θ obtained by the given transformation is consistent.

(RSAT ⇒ 3SAT): Suppose that the set of spatial constraints Θ obtained by transformation from a given instance Σ of 3SAT is consistent, and suppose that θ is a consistent instantiation of Θ. Then an assignment σ that satisfies Σ can be obtained in the following way: For every variable $v_L \in V$, if $\theta(X_L)\{\mathsf{NTPP}\}\theta(Y_L)$ holds, then $\sigma(v_L)$ is *true*, otherwise $\sigma(v_L)$ is *false*.

(3SAT ⇒ RSAT): Suppose that Σ is a positive instance of 3SAT and suppose that σ is an assignment that satisfies Σ. Then the set of spatial constraints Θ obtained by the transformation from Σ with respect to σ is consistent. We will show this by constructing a spatial configuration that holds all relations of Θ.

First we will point out some properties of Θ. For every literal and every literal occurrence that is assigned *false*, the corresponding spatial variables hold the relation EQ and can therefore be treated as a single spatial variable. Fig. 12(a) shows the three spatial variables corresponding to a variable P with $\sigma(P) = \mathit{true}$ (placed inside the dashed box)

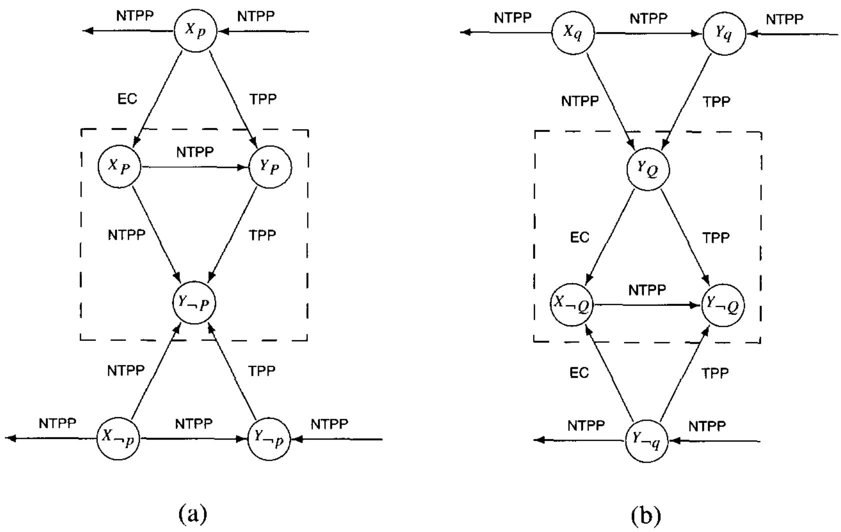

Fig. 12. Spatial variables corresponding to the variables P and Q and to their literal occurrences.

and spatial variables corresponding to a positive and a negative literal occurrence of P. Fig. 12(b) shows the same for a variable Q where $\sigma(Q) = \mathit{false}$. For each variable there might be different corresponding literal occurrences, but all hold the same constraints. The NTPP relations in Fig. 12 pointing to or from nowhere indicate the clause constraints required by transformation step (3). Note that for each variable $V \in \mathcal{V}$ the spatial variable $Y_{\neg V}$ contains all other spatial variables corresponding to V and all spatial variables corresponding to the literal occurrences of V.

For constructing a spatial configuration that holds all constraints of Θ we start with m different regions C_i, one for each clause $c_i \in C$, such that $C_i\{\mathsf{DC}\}C_j$ holds for every $i \neq j$. All regions corresponding to the literal occurrences of a clause c_i are placed within the region C_i. The three regions corresponding to each variable consist of different pieces where a piece is contained in C_i if a literal occurrence of the variable is contained in c_i. All regions within region C_i can be arranged consistently in a way that all required relations hold. Since the regions corresponding to the literal occurrences hold only the relations {TPP}, {NTPP} or {EC} with the pieces of the regions corresponding to the variables, they hold the same relations with the compound regions. All required relations hold for these regions, so we have a consistent model for Θ.

The specified transformation takes linear time in the number of clauses, so it has been proven that RSAT($\widehat{\mathcal{H}}_8 \cup$ {EQ, NTPP}) is NP-hard. Because of Corollary 22 it is also NP-complete. □

Lemma 29 together with Theorem 30 results in the following theorem.

Theorem 31. *$\widehat{\mathcal{H}}_8$ is a maximal tractable subset of* RCC-8.

As it has not been proven that adding relations of $\mathcal{N}_3$ to the base relations results in intractability of RSAT, there might be other maximal tractable subsets of RCC-8 containing all base relations.

As $\widehat{\mathcal{H}}_8$ is a tractable subset of RCC-8, the intersection of RCC-5 and $\widehat{\mathcal{H}}_8$ is also tractable. We will call this subset $\widehat{\mathcal{H}}_5$.

Proposition 32. *$\widehat{\mathcal{H}}_5$ contains all relations of* RCC-5 *except for the relations* $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *and* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$.

The following lemma can easily be obtained by computing the closure of the employed sets.

Lemma 33. *The closure of the set of all* RCC-5 *base relations together with one of the relations* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *or* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *contains* $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$.

Theorem 34. *$\widehat{\mathcal{H}}_5$ is the only maximal tractable subset of* RCC-5 *containing all base relations.*

Proof. $\widehat{\mathcal{H}}_5$ is by definition a tractable subset of RCC-5. To prove NP-hardness of RCC-5 the relations $\{\mathsf{PO}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$ and $\{*\}$ were used in Lemma 3. With Lemma 33 it follows that $\widehat{\mathcal{H}}_5$ contains all relations except for those making a set containing all base relations NP-complete. $\square$

7. Applicability of path-consistency

In the previous section we proved that $\widehat{\mathcal{H}}_8$ is a tractable subset of RCC-8. Thus, RSAT($\widehat{\mathcal{H}}_8$) can be decided in polynomial time. So far this can be done by first transforming a set of $\widehat{\mathcal{H}}_8$-constraints to a propositional Horn formula and then solving the resulting Horn formula in time linear in the number of literals [11]. Because the number of literals is of the order n^4, this way of solving RSAT does not appear to be very efficient.

The previously mentioned path-consistency method with a running time of $O(n^3)$ is much easier to apply than the above described method, but it is not complete in general. In order to apply this simple and popular method also for deciding consistency of RSAT($\widehat{\mathcal{H}}_8$), we have to prove completeness of the path-consistency method for this task.

In this section we first prove that the path-consistency method is sufficient for deciding consistency of RSAT($\mathcal{H}_8$) and based on this that it is also sufficient for deciding consistency of RSAT($\widehat{\mathcal{H}}_8$). This is done by showing that the path-consistency method finds an inconsistency whenever positive unit resolution resolves the empty clause from the corresponding propositional formula. Positive unit resolution (PUR) is a resolution strategy in which in every resolution step at least one of the two resolved clauses is a positive unit clause, i.e., a clause containing a single positive literal. As PUR is refutation-complete for Horn formulas [21], it follows that the path-consistency method decides consistency of RSAT($\mathcal{H}_8$).

7.1. *Applying positive unit resolution to the Horn clauses of* RCC-8

The only way to derive the empty clause using PUR is resolving a positive and a negative unit clause of the same variable. Since the Horn formulas that are used contain only a few different types of clauses, there are only a few ways of deriving unit clauses using PUR. In this subsection we will show how unit clauses can be derived, and how this relates to the structure of the initial set of constraints. In the following we will first point out some important observations made from the transformation of a set of RCC-8 constraints to propositional logic as specified in Proposition 19 in Section 5. For this we need some definitions.

Definition 35.
- R_K denotes the set of all relations of $\mathcal{H}_8$ whose abbreviated form contains the conjunct K (see Table B.1). $R_K(X,Y)$ means that the relation between X and Y is one of R_K.
- $R_{K_1,K_2,\ldots,K_n}$ denotes $R_{K_1} \cup R_{K_2} \cup \cdots \cup R_{K_n}$.
- R_{σ^*} is written instead of $R_{\sigma,\delta\vee\sigma,\delta\mathrm{i}\vee\sigma}$ for any abbreviation σ.
- The clause $\{X_w^*\}$ denotes either one of the clauses $\{X_w\}$ or $\{X_w^i\}$.

Proposition 36. *Let Θ be a set of $\mathcal{H}_8$-constraints and $c(\Theta)$ be the corresponding set of Horn clauses obtained by the transformation specified in Proposition* 19. *The following observations result from the transformation of the abbreviated form of Θ to $c(\Theta)$:*

(i) *For every world w either one or two unit input clauses are contained in $c(\Theta)$ and at least one of these is positive.*
(ii) *If the unit clauses $\{X_w\}$ and $\{Y_w\}$ or $\{X_w^i\}$ and $\{Y_w^i\}$ are input clauses of $c(\Theta)$, then $R_{\neg\delta}(X,Y) \in \Theta$ or $R_{\neg\delta\mathrm{i}}(X,Y) \in \Theta$, respectively.*
(iii) *If the unit clause $\{\neg X_w\}$ or $\{\neg X_w^i\}$ is an input clause of $c(\Theta)$, there must be a spatial variable $Y \in Reg(\Theta)$ with $R_{\neg\gamma}(X,Y) \in \Theta$ or $R_{\neg\gamma\mathrm{i}}(X,Y) \in \Theta$, respectively. In this case $\{Y_w\}$ is also an input clause of $c(\Theta)$.*
(iv) *If none of the unit clauses $\{X_w^*\}$ is an input clause of $c(\Theta)$ and if one of them is derivable from $c(\Theta)$, then there must be a spatial variable $Y \in Reg(\Theta)$ such that $R_{\gamma^*,\gamma\mathrm{i}^*}(X,Y)$ holds and the corresponding unit clause $\{Y_w^*\}$ is present. The clauses corresponding to $R_{\gamma\mathrm{i}^*}$ can only be used to derive the unit clause $\{X_w^i\}$ but not the unit clause $\{X_w\}$.*
(v) *If one of the unit clauses $\{\neg X_w^*\}$ is derivable from $c(\Theta)$, then there is a spatial variable $Y \in Reg(\Theta)$ such that $R_{\delta,\delta\mathrm{i}}(X,Y)$ holds and the corresponding unit clause $\{Y_w^*\}$ is present. The clauses corresponding to $R_{\delta\mathrm{i}}$ can only be used to derive the unit clause $\{\neg X_w^i\}$ but not the unit clause $\{X_w\}$.*
(vi) *If $\{X_w^i\}$ is present, then $\{X_w\}$ can be derived from $c(\mathrm{CP})$.*
(vii) *If no positive unit clause $\{X_w^i\}$ is present for a world w, it can only be derived using a clause introduced by $R_{\gamma\mathrm{i}}(X,Y)$ for some spatial variable Y or using $c(\mathrm{RP})$. In the latter case only the clause $\{X_w^{h(x)}\}$ can be derived.*

The sets of relations used in Proposition 36 contain the following relations (see Table B.1):

$$
\begin{aligned}
R_\gamma &= \{\{\mathsf{TPP}^{-1}\}, \{\mathsf{NTPP}^{-1}\}, \{\mathsf{EQ}\}, \{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}, \{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}\},\\
R_{\delta\vee\gamma} &= \{R \cup \{\mathsf{DC}\} \mid R \in R_\gamma\},\\
R_{\delta\mathrm{i}\vee\gamma} &= \{R \cup \{\mathsf{EC}\} \mid R \in (R_\gamma \cup R_{\delta\vee\gamma})\},\\
R_{\gamma\mathrm{i}} &= \{\{\mathsf{NTPP}^{-1}\}\},\\
R_{\delta\vee\gamma\mathrm{i}} &= \{\{\mathsf{DC}, \mathsf{NTPP}^{-1}\}\},\\
R_{\delta\mathrm{i}\vee\gamma\mathrm{i}} &= \{\{\mathsf{EC}, \mathsf{NTPP}^{-1}\}, \{\mathsf{DC}, \mathsf{EC}, \mathsf{NTPP}^{-1}\}\},\\
R_\delta &= \{\{\mathsf{DC}\}\},\\
R_{\delta\mathrm{i}} &= \{\{\mathsf{DC}\}, \{\mathsf{EC}\}, \{\mathsf{DC}, \mathsf{EC}\}\}.
\end{aligned}
$$

It can be seen, that $R_{\gamma\mathrm{i}} \subseteq R_\gamma$, $R_{\delta\vee\gamma\mathrm{i}} \subseteq R_{\delta\vee\gamma}$ and $R_{\delta\mathrm{i}\vee\gamma\mathrm{i}} \subseteq R_{\delta\mathrm{i}\vee\gamma}$, so, for example, R_{γ^*} can be written instead of $R_{\gamma^*,\gamma\mathrm{i}^*}$.

With PUR, positive unit clauses can only be derived in a very specific way, which is based on the above observations. For this and for the rest of this section, the notion of "chains" will be central:

Definition 37. An R_K*-chain* from X to Y, written as $R^*_K(X, Y)$, is a sequence of constraints $R_K(X, Z)$, $R_K(Z, Z')$, ..., $R_K(Z'', Y)$.

Lemma 38. *Let the clauses* $\{Y^*_w\}$ *and* $\{Z^*_w\}$ *be input clauses of* $c(\Theta)$ *(if there is only one positive input clause for* w *then* $Y \equiv Z$*). A new positive unit clause* $\{X^*_w\}$ *can be derived from* $c(\Theta)$ *only if* Θ *contains an* R_{γ^*}*-chain from* X *to* Y *or an* R_{γ^*}*-chain from* X *to* Z.

Proof. According to Proposition 36(iv), a positive unit clause can only be resolved if there is a spatial variable $Z^1 \in Reg(\Theta)$ such that $R_{\gamma^*}(X, Z^1)$ holds and the corresponding unit clause $\{Z^1_w\}$ is present. If $Z^1 \not\equiv Y$ or $Z^1 \not\equiv Z$ this clause cannot be an input clause so there must be another spatial variable $Z^2 \in Reg(\Theta)$ such that $R_{\gamma^*}(Z^1, Z^2)$ holds and the corresponding unit clause $\{Z^2_w\}$ is present. This goes on until there is a spatial variable Z^n with $Z^n \equiv Y$ or $Z^n \equiv Z$, i.e., there is an R_{γ^*}-chain from X to Y or an R_{γ^*}-chain from X to Z. □

Applying PUR has some side-effects on the possible relations of Θ. In order to demonstrate this side-effect, consider the constraint $X\{\mathsf{DC}, \mathsf{TPP}^{-1}\}Y$. The abbreviated form of the relation $R = \{\mathsf{DC}, \mathsf{TPP}^{-1}\}$ contains the conjunct $\delta \vee \gamma$ (see Table B.1), i.e., $R \in R_{\delta\vee\gamma} \subseteq R_{\gamma^*}$. The clauses corresponding to this conjunct are $\{\neg X^*_u, \neg Y^*_u, X^*_v, \neg Y^*_v\}$ for all $u, v \in W_0$ and are compounded by the clauses corresponding to the abbreviations δ and γ (see Proposition 19). These clauses can be used to derive the positive unit clause $\{X_w\}$ for some w only if the positive unit clauses $\{Y_w\}$, $\{X_u\}$, and $\{Y_u\}$ are present for some u. In this case, i.e., if it is possible to derive $\{X_w\}$ as described above, the clause $\{\neg X_u, \neg Y_u\}$ which corresponds to the abbreviation δ produces the empty clause. Thus, from the initial two possibilities δ or γ the first one becomes inconsistent and, since the relation $\{\mathsf{DC}\} \in R_\delta$, the constraint $X\{\mathsf{TPP}^{-1}\}Y$ must hold. We describe this side-effect by the notion of "refinement by PUR":

Definition 39. A constraint $XRY \in \Theta$ is *refined by PUR* to a constraint $XR'Y$, such that $R' \subset R$, if a clause corresponding to the constraint $XR''Y$, such that $R'' = R \setminus R'$, can be used to produce the empty clause.

In the above example, $X\{\mathsf{DC}, \mathsf{TPP}^{-1}\}Y$ is refined by PUR to $X\{\mathsf{TPP}^{-1}\}Y$.

Lemma 40. *If the positive unit clause $\{X_w^*\}$ is derived using PUR, then every constraint of the required R_{γ^*}-chain will be refined by PUR to a constraint in R_γ.*

Proof. Let Y and Z be two successive regions of the R_{γ^*}-chain, holding either $R_{\delta\vee\gamma}(Y, Z)$ or $R_{\delta\mathrm{i}\vee\gamma}(Y, Z)$. Then the required unit clause $\{Y_w^*\}$ is derived from a clause of the type $\{Y_w^*, \neg Z_w^*, \neg Y_u^*, \neg Z_u^*\}$, which consists of two disjunctively connected parts, the δ or $\delta\mathrm{i}$ part $\{\neg Y_u^*, \neg Z_u^*\}$ and the γ part $\{Y_w^*, \neg Z_w^*\}$. For this resolution the unit clauses $\{Y_u^*\}$ and $\{Z_u^*\}$ are necessary which are inconsistent with the clauses corresponding to $R_\delta(Y, Z)$ and $R_{\delta\mathrm{i}}(Y, Z)$. □

In order to derive a positive unit clause from a particular R_{γ^*}-chain, other positive unit clauses are necessary for which other R_{γ^*}-chains might be required. In order to refer to all R_{γ^*}-chains that are used to derive a particular positive unit clause we introduce the notion of chain structure.

Definition 41. Let XRY with $R \in R_{\delta\vee\gamma,\delta\mathrm{i}\vee\gamma}$ be a constraint of an R_{γ^*}-chain. The *chain structure* of XRY contains all R_{γ^*}-chains used to refine XRY by PUR to a constraint in R_γ.

7.2. Relating positive unit resolution to path-consistency

In this subsection we prove that the path-consistency method is sufficient for deciding consistency of $\mathsf{RSAT}(\mathcal{H}_8)$, by showing that for every set Θ of constraints over $\mathcal{H}_8$ whenever PUR produces the empty clause, the path-consistency method finds an inconsistency in Θ. In order to relate PUR with the path-consistency method, we first show that if a constraint $XRY \in \Theta$ with $R \in R_{\gamma^*}$ is refined by PUR to a constraint in R_γ, then the path-consistency method applied to Θ also refines the constraint to a constraint in R_γ. This is proven by Noetherian induction on chain structures, which is defined on well-founded relations (see e.g. [43]).

Definition 42. A relation $\prec$ on a set M is well-founded, if and only if every non-empty subset of M has a minimal element, i.e.,

$$\forall S \subseteq M.(S \neq \emptyset \rightarrow \exists x \in S.(\neg\exists y \in S.y \prec x)).$$

Theorem 43 (Noetherian Induction). *Let $\prec$ be a well-founded relation on a set M. To prove a property P for all $x \in M$, it suffices to prove:*

$$\forall x \in M.(\forall y \in M.(y \prec x) \rightarrow P(y)) \rightarrow P(x).$$

Proof. Suppose that the set $A \subseteq M$ of elements not satisfying P is not empty. Then A has a minimal element m, i.e., $P(y)$ holds for all $y \prec m$. Then $P(m)$ also holds, which contradicts the assumption. So A must be empty, i.e., P holds for all $x \in M$. $\square$

Before applying Noetherian induction to chain structures, we have to define a relation on chain structures and show that this relation is well-founded.

Definition 44. Let $\prec$ be a relation on chain structures, let S_1 be the chain structure of the constraint $X^1 R_1 Y^1$ and let S_2 be the chain structure of the constraint $X^2 R_2 Y^2$. $S_1 \prec S_2$ holds if and only if the constraint $X^1 R_1 Y^1$ occurs in S_2.

Lemma 45. *$\prec$ is a well-founded relation.*

Proof. By definition, if S is a chain structure where only constraints in R_γ occur, then there is no chain structure S' with $S' \prec S$. Suppose that S is the chain structure of a constraint $XRYY$ and the same constraint is contained in S. If the occurrence of XRY is recursive in S, then S cannot be used to refine XRY to a constraint in R_γ, so S is no chain structure in our sense. If the occurrence of XRY is not recursive in S, then S can be replaced by S', the chain structure belonging to the last occurrence of XRY in S.

There is only a finite number of regions, so every non-empty set of chain structures has at least one minimal element. $\square$

We have to prove that whenever a constraint XRY of an R_{γ^*}-chain is refined by PUR to a constraint in R_γ in the sense of Lemma 40, then the path consistency method also results in R_γ, i.e., the base relations $\{\mathsf{DC}\}$ and $\{\mathsf{EC}\}$ will be excluded from R. For this proof we need the following operations which can be verified using Table 2.

Proposition 46. *Let R be a relation of* RCC-8.

(i) *R_γ is closed under composition.*
(ii) *$R_{\gamma\mathrm{i}} \circ R_\gamma = R_{\gamma\mathrm{i}}$ and $R_\gamma \circ R_{\gamma\mathrm{i}} = R_{\gamma\mathrm{i}}$.*
(iii) *If $\{\mathsf{DC}, \mathsf{EC}\} \cap R = \emptyset$, then $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ R) = \emptyset$ and $\{\mathsf{DC}, \mathsf{EC}\} \cap (R \circ R_\gamma^{\smile}) = \emptyset$.*
(iv) $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ R_\gamma^{\smile}) = \emptyset$.
(v) $\{\mathsf{DC}\} \cap (R_\gamma \circ \{\mathsf{EC}\} \circ R_\gamma^{\smile}) = \emptyset$.
(vi) $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_{\gamma\mathrm{i}} \circ \{\mathsf{EC}\} \circ R_\gamma^{\smile}) = \{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ \{\mathsf{EC}\} \circ R_{\gamma\mathrm{i}}^{\smile}) = \emptyset$.

Lemma 47. *If every constraint of an R_{γ^*}-chain K from X^1 to Y^1 in Θ is refined by PUR to a constraint in R_γ, the path-consistency method applied to Θ results in $R_\gamma(X^1, Y^1)$.*

Proof. Let $P(S) \equiv$ "If a constraint $XRY \in \Theta$, such that $R \in R_{\gamma^*}$, is refined by PUR to a constraint in R_γ using the chain structure S, then the path-consistency method applied to Θ also refines XRY to a constraint in R_γ". We will prove $P(S)$ with Noetherian induction.

Induction hypothesis: $\forall S' \prec S. P(S')$.

Suppose that the constraint XRY is refined by PUR to one of R_γ in order to obtain the clause $\{X_w^*\}$. If R is already in R_γ nothing has to be proven. If R is one of $R_{\delta\vee\gamma,\delta\mathrm{i}\vee\gamma}$, the clause $\{X_w^*, \neg Y_w^*, \neg X_u^*, \neg Y_u^*\}$ is input clause of $c(\Theta)$. Since $R_{\delta,\delta\mathrm{i}}(X, Y)$ can be excluded

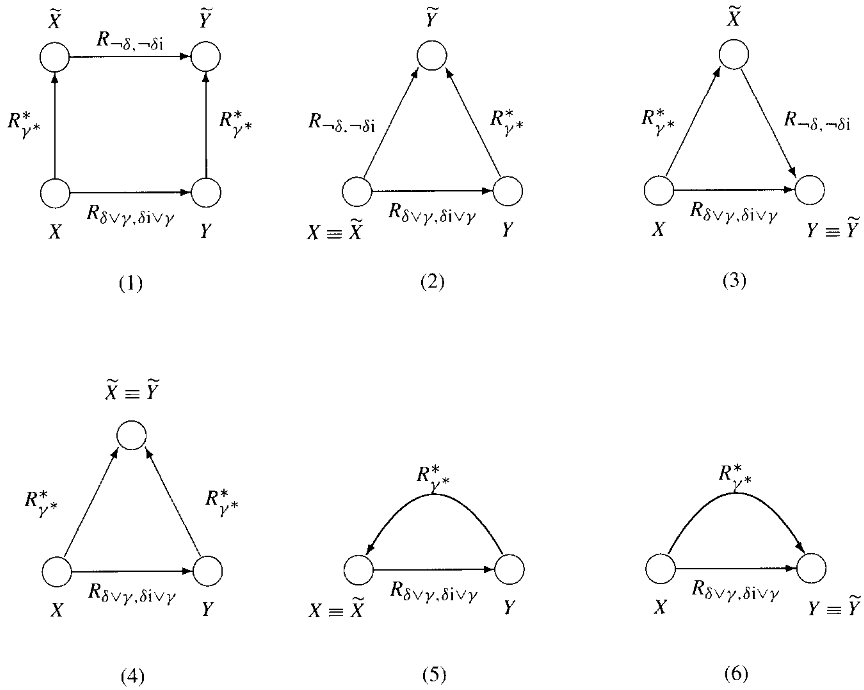

Fig. 13. The six possible cases of the proof to Lemma 47. In the lower row $\widetilde{X}$ is equal to $\widetilde{Y}$.

with PUR, the unit clauses $\{X^*_u\}$ and $\{Y^*_u\}$ for some u must be present. Let the clauses $\{\widetilde{X}^*_u\}$ and $\{\widetilde{Y}^*_u\}$ be the only unit input clauses of u, which can only be introduced by $R_{\neg\delta}(\widetilde{X}, \widetilde{Y})$ or by $R_{\neg\delta\mathrm{i}}(\widetilde{X}, \widetilde{Y})$. (If $\widetilde{X} \equiv \widetilde{Y}$ then the unit clause can be introduced by some other constraint which is not important in this analysis.) Then there are $R_{\gamma*}$-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$ that are part of S. Six different cases, shown in Fig. 13, must be distinguished:

Case 1: A chain structure $S' \prec S$ belongs to every constraint in $R_{\delta\vee\gamma,\delta\mathrm{i}\vee\gamma}$ of the $R_{\gamma*}$-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$. Since $P(S')$ holds by induction hypothesis, and R_γ is closed under composition, $R_\gamma(X, \widetilde{X})$ and $R_\gamma(Y, \widetilde{Y})$ are obtained by the path-consistency method.

(a) If $R \in R_{\delta\vee\gamma}$, then R is refined to one of R_γ by the path-consistency method because of items (iii) and (v) of Proposition 46.

(b) If $R \in R_{\delta\mathrm{i}\vee\gamma}$ then $\{X^i_w\}$ and $\{Y^i_w\}$ must be obtained, so the $R_{\gamma*}$-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$ must each contain at least one relation of $R_{\gamma\mathrm{i}}$ (see Proposition 36(vii)). The two clauses cannot be obtained by $c(\mathrm{CP})$ because in this case they cannot have the same index i. Because of Proposition 46(ii) the path-consistency method results in $R_{\gamma\mathrm{i}}(X, \widetilde{X})$ and $R_{\gamma\mathrm{i}}(Y, \widetilde{Y})$. Because of items (iii) and (vi) of Proposition 46 it results in $R_\gamma(X, Y)$.

The remaining five cases can be handled in the same way as Case 1. All the R_{γ^*}-chains can be reduced to R_γ by applying the induction hypothesis. With the operations of Proposition 46 the path-consistency method reduces R to one of R_γ.

By Noetherian induction we proved that $P(S)$ holds for all chain structures S. Every relation of the R_{γ^*}-chain K can now be reduced to one of R_γ with the path-consistency method. Because R_γ is closed under composition, the path-consistency method results in $R_\gamma(X^1, Y^1)$. □

Using this lemma, we can now prove that the path-consistency method decides RSAT($\mathcal{H}_8$) by showing that whenever the empty clause can be derived, the path-consistency method results in an inconsistency.

Theorem 48. *The path-consistency method decides* RSAT($\mathcal{H}_8$).

Proof. Let Θ be an inconsistent set of $\mathcal{H}_8$-constraints and $c(\Theta)$ the corresponding set of Horn clauses obtained by the transformation specified in Proposition 19. Since Θ is inconsistent, the empty clause can be derived from $c(\Theta)$ using PUR. There are different possibilities for deriving the empty clause.

(i) The empty clause is derived from $\{X_w\}$ and $\{\neg X_w\}$.

(a) $\{\neg X_w\}$ is input clause of $c(\Theta)$. Then, with Proposition 36(iii), $\{Y_w\}$ is also an input clause for some spatial variable Y with $R_{\neg\gamma}(X, Y)$. $\{X_w\}$ is derived from $c(\Theta)$, so there must be an R_{γ^*}-chain from X to Y. Because of Lemma 47, the path-consistency method results in $R_\gamma(X, Y)$, which is a contradiction to $R_{\neg\gamma}(X, Y)$.

(b) $\{X_w\}$ is the only positive input clause of w in $c(\Theta)$. Then $\{\neg X_w\}$ is derived from $c(\Theta)$. According to Proposition 36(v), there must be a spatial variable Y with $R_{\delta,\delta\mathrm{i}}(X, Y)$ and $\{Y_w\}$ must be present. $\{Y_w\}$ is derived using an R_{γ^*}-chain from Y to X, which is reduced to $R_\gamma(Y, X)$ by the path-consistency method. This is a contradiction to $R_{\delta,\delta\mathrm{i}}(X, Y)$.

(c) $\{X_w\}$ is an input clause of $c(\Theta)$ and $\{Z_w\}$ is also an input clause for some spatial variable Z. According to Proposition 36(ii), $R_{\neg\delta}(X, Z)$ must hold. $\{\neg X_w\}$ is derived from $c(\Theta)$, so there must be a spatial variable Y with $R_{\delta,\delta\mathrm{i}}(X, Y)$ (see Proposition 36(v)) and $\{Y_w\}$ must also be derived from $c(\Theta)$. So there is either an R_{γ^*}-chain from Y to X, which is equal to (b), or an R_{γ^*}-chain from Y to Z which is refined to $R_\gamma(Y, Z)$ by the path-consistency method. If $R_\delta(X, Y)$ holds, the path-consistency method results in an inconsistency (see Table 2). If $R_{\delta\mathrm{i}}(X, Y)$ holds, then one relation of the R_{γ^*}-chain from Y to Z must be from $R_{\gamma\mathrm{i}}$, so the path consistency method results in $R_{\gamma\mathrm{i}}(Y, Z)$, which is also inconsistent (see Table 2).

(d) Neither $\{X_w\}$ nor $\{\neg X_w\}$ are input clauses of $c(\Theta)$, so both are derived using PUR. The negative clause can only be derived if there is a spatial variable Y with $R_{\delta,\delta\mathrm{i}}(X, Y)$ (see Proposition 36(v)) and $\{Y_w\}$ can also be derived. As it was shown in the six cases of Lemma 47, $\{X_w\}$ and $\{Y_w\}$ can only be derived from $c(\Theta)$ if the path-consistency method results in an inconsistency with $R_{\delta,\delta\mathrm{i}}(X, Y)$.

(ii) The empty clause is derived from $\{X_w^i\}$ and $\{\neg X_w^i\}$. Only $\{X_w^i\}$ or $\{\neg X_w^{h(\neg x)}\}$ can be input clauses of $c(\Theta)$.
   (a) The empty clause is derived from $\{X_w^{h(\neg x)}\}$ and the input clause $\{\neg X_w^{h(\neg x)}\}$. According to Proposition 36(iii), there is a spatial variable Y with $R_{\neg\gamma\mathrm{i}}(X, Y)$ and $\{Y_w\}$, which is also an input clause. The clause $\{X_w^{h(\neg x)}\}$ is derived using an R_{γ^*}-chain from X to Y. Because of Proposition 36(vii), one of the relations of the chain must be in $R_{\gamma\mathrm{i}}$, so the path consistency method results in $R_{\gamma\mathrm{i}}(X, Y)$ which contradicts $R_{\neg\gamma\mathrm{i}}(X, Y)$.
   (b) $\{X_w^i\}$ is an input clause, then there is a spatial variable Y with $R_{\neg\delta\mathrm{i}}(X, Y)$ (Proposition 36(ii)) and $\{Y_w^i\}$ is also an input clause of $c(\Theta)$. In order to derive $\{\neg X_w^i\}$, there must be a spatial variable Z with $R_{\delta,\delta\mathrm{i}}(X, Z)$ (Proposition 36(v)) and $\{Z_w^i\}$ must be present. $\{Z_w^i\}$ can be derived using an R_{γ^*}-chain from Y to Z. The path-consistency method results in $R_\gamma(Y, Z)$ which is, composed with $R_{\neg\delta\mathrm{i}}(X, Y)$, inconsistent with $R_{\delta,\delta\mathrm{i}}(X, Z)$ (see Table 2).
   (c) Neither $\{X_w^i\}$ nor $\{\neg X_w^i\}$ are input clauses and must be derived from $c(\Theta)$. Then there must be a spatial variable Y with $R_{\delta,\delta\mathrm{i}}(X, Y)$ (Proposition 36(ii)) and $\{Y_w^i\}$ must be present. In order to obtain $\{X_w^i\}$ and $\{Y_w^i\}$ the six cases of Lemma 47 must be considered, where $R_{\delta,\delta\mathrm{i}}(X, Y)$ holds instead of $R_{\gamma^*}(X, Y)$. Since $R_{\delta,\delta\mathrm{i}}$ could be excluded in Lemma 47, it can also be excluded now, so the path-consistency method finds the inconsistency.

Thus, whenever PUR derives the empty clause, an inconsistency is found by the path-consistency method. Since positive unit resolution is refutation-complete for propositional Horn formulas, the path-consistency method is sufficient for deciding consistency of $\mathsf{RSAT}(\mathcal{H}_8)$. $\square$

7.3. Path-consistency for the full set of tractable relations

The path-consistency method can be used to decide $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ when every constraint of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ is transformed to constraints in $\mathcal{H}_8$ according to Theorem 5. As this way of deciding $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ is pretty awkward, we will prove that the path-consistency method can decide $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ directly without any preprocessing. This will be shown in the same way as it was shown for $\mathcal{H}_8$, namely, that the path-consistency method finds an inconsistency whenever PUR derives the empty clause. For this we have to transform constraints in $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ to propositional Horn formulas.

Proposition 49. *Let $\mathcal{S}$ be the set of* RCC-8 *relations transformable to a propositional Horn formula. Then every constraint XSY with $S \in \mathcal{S}$ can be transformed to the propositional Horn formula $p'(S(X, Y))$. If $S \in \mathcal{H}_8$, then $p'(S(X, Y)) = p(m(S(X, Y)))$. Since any relation of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be obtained by composition, converse or intersection of relations of $\mathcal{H}_8$, the propositional Horn formula of XRY where $R \in \widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be obtained using the following construction inductively:*

(i) *If $R = S \circ T$, where $S, T \in \mathcal{S}$ and $R \notin \mathcal{S}$, then introduce a pseudo variable Z that is only related with the spatial variables X and Y, holding XSZ and ZTY. Therefore*

$$p'(XRY) = p'(XSZ) \wedge p'(ZTY).$$

(ii) *If* $R = S \cap T$, *where* $S, T \in \mathcal{S}$ *and* $R \notin \mathcal{S}$, *then*

$$p'(XRY) = p'(XSY) \wedge p'(XTY).$$

(iii) *If* $R = S^{\smile}$, *where* $S \in \mathcal{S}$ *and* $R \notin \mathcal{S}$, *then*

$$p'(XRY) = p'(YSX).$$

If one of the three constructions is possible, add R *to* $\mathcal{S}$.

For proving that the path-consistency method decides $\mathsf{RSAT}(\mathcal{H}_8)$, the R_{γ^*}-chain was the central part as it is the only way to derive a positive unit clause. Some relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be constructed using relations of R_{γ^*}, so these relations can also be used to derive positive unit clauses. As all of these relations must be analyzed separately, we try to keep their number as small as possible. Therefore, we consider only relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ that cannot be constructed without using relations of R_{γ^*}. In Table B.2 we give a list of all relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ and show how they can be functionally constructed from $\mathcal{H}_8$-relations. We have chosen a construction of all relations such that the following analysis is as simple as possible. The set of relations that can be used to derive the empty clause will be denoted by R_e.

Lemma 50. *Additional to* R_{γ^*}, R_e *contains the following relations*:

$$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\} = \{\mathsf{EC}, \mathsf{NTPP}\} \circ \{\mathsf{DC}, \mathsf{EQ}\},$$
$$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{NTPP}^{-1}\} = \{\mathsf{DC}, \mathsf{EQ}\} \circ \{\mathsf{EC}, \mathsf{NTPP}^{-1}\},$$
$$\{\mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\} = \{\mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\} \circ \{\mathsf{EC}, \mathsf{EQ}\},$$
$$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\} = \{\mathsf{EC}, \mathsf{EQ}\} \circ \{\mathsf{DC}, \mathsf{EQ}\},$$

as well as the intersection of these relations with other relations of $\widehat{\mathcal{H}}_8$.

Proof. This can be proven by computing the closure of $\mathcal{H}_8 \setminus R_{\gamma^*}$ under composition and intersection. All relations not contained in this closure can be constructed by intersection of these relations with either the four specified relations or the relations of R_{γ^*}. □

Instead of R_{γ^*}-chains we now consider R_e-chains which contain constraints over R_e. Similar to Definitions 41–44, we define *extended chain structures* on constraints over R_e and a well-founded relation on extended chain structures.

Definition 51.

(i) The *extended chain structure* of XRY with $R \in R_e$ contains all R_e-chains used to refine R to a relation of R_γ.

(ii) Let $\prec_e$ be a relation on extended chain structures, let T_1 be the extended chain structure of the constraint $X^1R_1Y^1$ and T_2 be the extended chain structure of the constraint $X^2R_2Y^2$, where $R_1, R_2 \in R_e$. $T_1 \prec_e T_2$ holds if and only if $X^1R_1Y^1$ occurs in T_2.

Analogous to Lemma 45 it can be proven that $\prec_e$ is well-founded. Similar to Lemma 47 we can relate positive unit resolution and path-consistency also for relations of R_e.

Lemma 52. *If every constraint of an R_e-chain K from X^1 to Y^1 in Θ is refined by PUR to a constraint in R_γ, the path-consistency method applied to Θ results in $R_\gamma(X^1, Y^1)$.*

Proof. $P(T) \equiv$ "If a constraint $XRY \in \Theta$, such that $R \in R_e$, is refined by PUR to a constraint in R_γ using the chain structure T, then the path-consistency method applied to Θ also refines XRY to a constraint in R_γ". We will prove $P(T)$ by Noetherian induction.

Induction hypothesis: $\forall T' \prec T.P(T')$.

Suppose that the constraint XRY, with $R \in R_e$, is refined by PUR to a constraint in R_γ, in order to obtain the clause $\{X_w^*\}$. We have to distinguish different cases for R.

(i) $R \in R_{\gamma^*}$: The six different cases of Lemma 47 can also be applied here. By applying the induction hypothesis, all R_e-chains of the extended chain structure of XRY will be refined to R_γ with the path-consistency method. Therefore R will also be refined to R_γ with the path-consistency method.

(ii) $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Only the second relation of the construction of R is in R_{γ^*}, so $\{X_w^*\}$ cannot be obtained with this relation.

(iii) $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the spatial variables X and Y using the pseudo variable Z. Then clauses of the type $\{\neg X_a^*, \neg Z_a^*, \neg X_b^*, Z_b^*\}$, $\{\neg X_a^*, \neg Z_a^*, X_b^*, \neg Z_b^*\}$, and $\{\neg Z_a^*, \neg Y_a^*, Z_b^*, \neg Y_b^*\}$ for all $a, b \in W_0$ as well as $\{Z_c\}$, $\{Y_c\}$, $\{Z_d\}$ and $\{\neg Y_d\}$ for some $c, d \in W_0$ are input clauses. In order to derive $\{X_w^*\}$, the unit clauses $\{Z_w^*\}$, $\{X_u^*\}$, and $\{Z_u^*\}$ are necessary for some u. There are different possibilities of how $\{Z_u^*\}$ can be derived:

(a) If $\{Z_u\}$ is the only positive input clause of u, then $\{X_u\}$ is derived using $\{Z_v^*\}$ and $\{X_v^*\}$ for some $v \neq u$. Then $\{X_w^*\}$ can also be derived using $\{Z_v^*\}$ and $\{X_v^*\}$, which contradicts the assumption that it is derived using $\{Z_u\}$ and $\{X_u\}$.

(b) If $\{Z_u\}$ and $\{Y_u\}$ are input clauses, $\{X_u\}$ can either be derived as described in (a) which results in a contradiction, or there might be an additional R_e-chain from X to Y, not passing Z. By applying the induction hypothesis, $R_\gamma(X, Y)$ is obtained with the path-consistency method, if it is refined by PUR to the same constraint.

(c) If $\{Z_u^*\}$ is not an input clause, there must be a spatial variable $\widetilde{Z}_u$ that introduces u. If the R_e-chain from Z to $\widetilde{Z}_u$ passes X, then the clauses $\{Z_v^*\}$ and $\{X_v^*\}$ for some $v \neq u$ are necessary. This is again a contradiction to our assumption. If the chain passes Y, then the clauses $\{Y_u^*\}$, $\{Z_v^*\}$ and $\{Y_v^*\}$ for some v are necessary. In order to derive $\{Z_v^*\}$, the clauses $\{Y_v^*\}$, $\{Z_s^*\}$, and $\{Y_s^*\}$ for some $s \neq v$ are necessary. Then $\{Z_u^*\}$ can also be derived using $\{Z_s^*\}$ and $\{Y_s^*\}$, which contradicts the assumption that it is derived using $\{Z_v^*\}$ and $\{Y_v^*\}$.

(iv) $R = \{\mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the spatial variables X and Y using the pseudo variable Z. Then clauses of the type $\{X_a^*, \neg Z_a^*\}$, $\{\neg Z_a^*, \neg Y_a^*, \neg Z_b^*, Y_b^*\}$, and $\{\neg Z_a^*, \neg Y_a^*, Z_b^*, \neg Y_b^*\}$ for all $a, b \in W_0$ as well as $\{X_c\}$, $\{Z_c\}$, $\{Z_d\}$, and $\{Y_d\}$ for some $c, d \in W_0$ are input clauses. In order to derive $\{X_w^*\}$, the unit clause $\{Z_w^*\}$ is necessary. If $\{Z_w\}$ is an input clause, the relation between X and Z is not refined by PUR. If $\{Z_w^*\}$ is not an input clause, the clauses $\{Y_w^*\}$, $\{Z_u^*\}$, and $\{Y_u^*\}$ for some u are necessary. In order to derive $\{Z_u^*\}$ the clauses $\{Y_u^*\}$, $\{Z_v^*\}$ and $\{Y_v^*\}$ for some $v \neq u$ are necessary. Then $\{Z_w^*\}$ can also be derived using $\{Z_v^*\}$ and $\{Y_v^*\}$, which contradicts the assumption that it is derived using $\{Z_u^*\}$ and $\{Y_u^*\}$.

(v) $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the spatial variables X and Y using the pseudo variable Z. Then clauses of the type $\{\neg X_a^*, \neg Z_a^*, \neg X_b^*, Z_b^*\}$, $\{\neg X_a^*, \neg Z_a^*, X_b^*, \neg Z_b^*\}$, $\{\neg Z_a^*, \neg Y_a^*, \neg Z_b^*, Y_b^*\}$, and $\{\neg Z_a^*, \neg Y_a^*, Z_b^*, \neg Y_b^*\}$ for all $a, b \in W_0$, as well as $\{X_c\}$ and $\{Z_c\}$ for some $c \in W_0$ are input clauses. In order to derive $\{X_w^*\}$, the unit clauses $\{Z_w^*\}$, $\{X_u^*\}$, and $\{Z_u^*\}$ are necessary for some u. $\{Z_u^*\}$ can be obtained with an R_e-chain from Z to Z_u passing either X or Y. If it passes X, the clauses $\{X_v^*\}$, and $\{Z_v^*\}$ for some $v \neq u$ are necessary, which is a contradiction. If it passes Y, the clauses $\{Y_u^*\}$, $\{Y_v^*\}$ and $\{Z_v^*\}$ for some $v \neq u$ are necessary. $\{Z_v^*\}$ can be obtained in two ways:
  (a) If $\{Z_v\}$ is an input clause, $\{Y_v\}$ can be derived using the clauses $\{Z_s^*\}$ and $\{Y_s^*\}$ for some s. Then $\{Z_u^*\}$ can also be derived using $\{Z_s^*\}$ and $\{Y_s^*\}$, which contradicts the assumption that it is derived using $\{Z_v^*\}$ and $\{Y_v^*\}$.
  (b) If the chain for obtaining $\{Z_v^*\}$ passes X, the clauses $\{Z_s^*\}$ and $\{X_s^*\}$ for some s are necessary. If it passes Y, the clauses $\{Z_s\}$ and $\{Y_s\}$ for some s are necessary. Both possibilities result in a contradiction to previous assumptions.

(vi) R is constructed by intersection: The intersection of two relations is a refinement of both relations, so if one of the relations is refined to one of R_γ, it also holds for the intersection of the two relations. Some relations are refinements of a relation of R_γ but are not part of R_γ. These relations can be treated as if they were part of R_γ, because the specified properties of Proposition 46 also hold for them.

As R_γ is closed under composition, the proof is completed. □

We are now ready to prove the main theorem of this section, namely, that the path-consistency method is sufficient for deciding $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$.

Theorem 53. *The path-consistency method decides* $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$.

Proof. Let Θ be an inconsistent set of $\widehat{\mathcal{H}}_8$-constraints, $p'(\Theta)$ the equivalent propositional Horn formula as specified in Proposition 49 and $c'(\Theta)$ the corresponding set of Horn clauses. Since Θ is inconsistent, the empty clause can be derived from $c'(\Theta)$ using PUR. Suppose that the empty clause is derived by $\{X_w^*\}$ and $\{\neg X_w^*\}$. The same proof as in Theorem 48 can be applied here, when it is based on Lemma 52 instead of Lemma 47. The only difference is that X might be a pseudo variable. In this case $\{\neg X_w^*\}$ must be input clause, as it cannot be derived by PUR. For this there must either be a spatial variable Y with $R_{\neg\pi}(Y, X)$ or a spatial variable Z with $R_{\neg\gamma,\neg\gamma\mathrm{i}}(X, Z)$. As there is no such spatial variable, X cannot be a pseudo variable, and then the empty clause cannot be derived from $\{X_w^*\}$ and $\{\neg X_w^*\}$. □

This theorem can be easily transferred to RCC-5.

Corollary 54. *The path-consistency method decides* $\mathsf{RSAT}(\widehat{\mathcal{H}}_5)$.

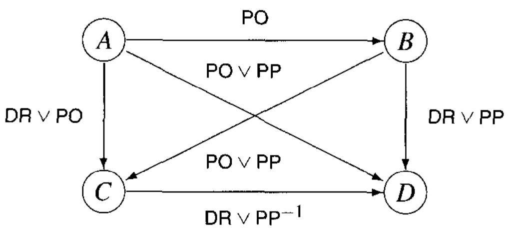

Fig. 14. Path consistent but not minimal constraint graph.

Another interesting question is whether the path-consistency method also decides the minimal-label problem RMIN($\widehat{\mathcal{H}}_8$). As the following theorem shows this is not the case even for the set $\widehat{\mathcal{H}}_5$.

Theorem 55. *The path-consistency method is not sufficient for solving the minimal-label problem* RMIN($\widehat{\mathcal{H}}_5$).

Proof. Fig. 14 shows a constraint graph that is path-consistent but not minimal. The relation between A and D can be refined to PO but not to PP. □

Since RMIN and RSAT are equivalent under polynomial Turing reductions, RMIN($\widehat{\mathcal{H}}_5$) as well as RMIN($\widehat{\mathcal{H}}_8$) are solvable in polynomial time.

8. Applicability of the maximal tractable set $\widehat{\mathcal{H}}_8$

In this section we will discuss some practical advantages of the theoretical results obtained so far. One obvious advantage of the maximal tractable subset $\widehat{\mathcal{H}}_8$ is that the path-consistency method is sufficient for deciding RSAT if it is possible to restrict the relations used in an application to the relations of $\widehat{\mathcal{H}}_8$.

In many applications this is certainly not possible. In spatial configuration tasks or queries to spatial databases, e.g., RCC-8 base relations and negations of the base relations are often used, as in *Find a region which is in A but does not overlap B*. Since the closure of these 16 relations is the whole set of 256 RCC-8 relations, all relations have to be considered in this kind of applications.

As in the case of temporal reasoning where the usage of the maximal tractable subset ORD-HORN has been extensively studied [32], $\widehat{\mathcal{H}}_8$ can also be used to speed up backtracking algorithms for the general NP-complete RSAT problem using all RCC-8 relations. Previously, every spatial constraint had to be refined to a base relation before the path-consistency method could be applied to decide consistency. In the worst case this has to be done for all possible refinements. Supposing that the relations are uniformly distributed, the average branching factor, i.e., the average number of different refinements of a single relation to relations of $\mathcal{B}$ is 4.0.

Using our results it is sufficient to make refinements of all relations to relations of $\widehat{\mathcal{H}}_8$. Except for four relations, each of the 108 relations not contained in $\widehat{\mathcal{H}}_8$ can be

Table 4
Comparison of the average size of the search space

$\lvert Reg(\Theta)\rvert$	$\mathcal{B}$ (4.0)	$\widehat{\mathcal{H}}_8$ (1.4375)
5	10^6	38
7	4.4×10^{12}	2041
10	1.2×10^{27}	1.2×10^7

expressed as a union of two relations of $\widehat{\mathcal{H}}_8$, the four relations can only be expressed as a union of three $\widehat{\mathcal{H}}_8$ relations. This reduces the average branching factor to 1.4375 ($= (148 + 104 \times 2 + 4 \times 3)/256$). Both branching factors are of course worst-case measures because the search space can be considerably reduced when path-consistency is used as a forward checking method [25].

Table 4 shows the average size of the search space for the average branching factors given above. The size of the search space is computed as $b^{(n^2-n)/2}$ where b is the average branching factor, n the number of spatial variables contained in Θ, and $(n^2-n)/2$ the number of different constraints.

In [39] we made an empirical study of reasoning with RCC-8 by randomly generating instances of up to 100 regions and solving them using different strategies. It turned out that those strategies applying $\widehat{\mathcal{H}}_8$ were much more effective in finding a fast solution and solving instances in reasonable time than those strategies applying $\mathcal{B}$ or $\widehat{\mathcal{B}}$. To our surprise, almost all apparently hard instances of the phase-transition region could be solved in a few seconds if the different strategies were run in parallel.

9. Discussion and related work

Other complexity results on qualitative spatial reasoning were obtained by Grigni et al. [19] who worked with Egenhofer's topological relations [12]. These relations are defined using the so-called *9-intersection*, the 9 possible intersections of the interior, the boundary, and the complement of two regions. Apart from the more restricting definition of regions, 8 different relations can be defined which use the same distinctions of regions as RCC-8. Grigni et al. [19] considered two different notions of satisfiability, the purely syntactical notion of *relational consistency* and the semantic notion of *realizability*, which are both different from what we call consistency. Relational consistency means that there is a path-consistent refinement of all relations to base relations, realizability means that there is a model consisting of simply connected planar regions. Both kinds of satisfiability were found to be NP-hard for Egenhofer's eight relations which Grigni et al. [19] called *high resolution case* as well as a subset of five relations called *medium resolution case* which uses different distinctions of regions than RCC-5. [12] Both NP-hardness results are independent from our NP-hardness result. The notion of realizability is much more constraining

[12] In particular PO and EC were joined to a base relation of the medium resolution case instead of DC and EC as it was done in RCC-5.

than our notion of consistency. It is also computationally much harder—realizability for the eight base relations of RCC-8, e.g., is not known to be in NP. "Realizability", i.e., finding one-piece regions, in three and higher dimensional space, however, is equivalent to our notion of consistency [37].

Nebel [31] proved tractability for the set of RCC-8 base relations by transforming the propositional intuitionistic encoding of the base relations given by Bennett [2] to 2CNF formulas. This tractability result, however, is not applicable in our case, since Nebel did not consider the regularity condition. The regularity condition is necessary to rule out certain counterintuitive regions, e.g., regions which only consist of a boundary. Moreover, since we are restricting our analysis to closed regions, the regularity condition is required in order to guarantee inferences according to the RCC-8 composition table. Consider, e.g., a non-regular region B with a spike consisting of a piece of the boundary, like a balloon with a cord, such that only the spike intersects the two regions A and C, where $A\{\mathsf{NTPP}\}C$. In this case, since B intersects A and C but the interior of B does not intersect A and C, B is externally connected to both A and C, which is not consistent with the composition table.

Jonsson and Drakengren [23] made a complete classification of tractability in RCC-5. Apart from $\widehat{\mathcal{H}}_5$, the only maximal tractable subclass of RCC-5 containing all base relations, they discovered three other tractable subclasses not containing all base relations. For all subclasses not contained in one of the four tractable subsets, RSAT is NP-complete. While a complete classification of tractability is certainly worthwhile from a theoretical point of view, the practical usage of subclasses not containing all base relations is limited, as it is not possible to express definite knowledge within a given calculus even if it is available. Furthermore, these subclasses cannot be used to speed up backtracking algorithms as it is not possible to refine every relation to relations of these subclasses. So far we have not been able to either identify other maximal tractable subclasses containing all base relations or prove that $\widehat{\mathcal{H}}_8$ is the only such set.

10. Summary

We analyzed the computational properties of the qualitative spatial calculus RCC-8 and identified the boundary between polynomial and NP-hard fragments. Using a modification of Bennett's encoding of RCC-8 in propositional modal logic, we transformed the RCC-8 consistency problem to a problem in propositional logic and isolated the relations that are representable as Horn clauses. As it turns out, the fragment identified in this way is also a maximal fragment that contains all base relations and is still computationally tractable. Further, we showed that for this fragment path-consistency is sufficient for deciding consistency. As in the case of qualitative temporal reasoning, our result allows to check whether the relations that are used in an application allow for a polynomial reasoning algorithm. Further, if the application requires an expressive power beyond the polynomial fragment, it can be used to speed up backtracking algorithms as in the case of qualitative temporal reasoning [32].

Research on this topic has to be continued, as it is still an open question whether there are other maximal tractable fragments of RCC-8 that also contain all base relations. Among other open problems, the question for a fragment that permits the determination of minimal labels by the path-consistency method seems to be interesting. Further, the determination of the computational properties of more expressive calculi like RCC-23 [2] and the design of efficient algorithms for the case of connected regions with a fixed dimensionality [19] appear to be interesting in an application context. In order to approach the goal of a general qualitative theory of space, it seems to be useful to extend RCC-8 by other aspects of space such as direction and distance. A first step towards this goal has been taken in [16] where qualitative size relations were added to RCC-8.

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Appendix A. Basics on modal logics

Propositional modal logics [7,14,22] have the same syntax as standard propositional logic except for an additional unary operator $\Box$. The modal logics we are interested in are the so-called *normal modal logics*, i.e., the family of logics that are obtained by extending the basic normal modal logic K. K contains all tautologies, the axiom schema $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, and is closed under the following inference rules: modus ponens (if $\phi, \phi \to \psi \in \mathsf{K}$ then $\psi \in \mathsf{K}$), uniform substitution (if $\phi \in \mathsf{K}$ and p occurs in ϕ then $\phi[\psi|p] \in \mathsf{K}$ for any ψ), and the rule of necessitation (if $\phi \in \mathsf{K}$ then $\Box\phi \in \mathsf{K}$).

Modal formulas are usually interpreted by means of *Kripke semantics*. A *Kripke model* $\mathcal{M} = \langle W, R, \nu\rangle$ is built upon a *frame* and a *valuation*. A frame $\mathcal{F} = \langle W, R\rangle$ consists of a set of *worlds* W together with an *accessibility relation* $R \subseteq W \times W$. A valuation ν assigns truth values to all the propositional atoms in every world. If a world $v \in W$ is accessible from a world $w \in W$, i.e., $(w, v) \in R$, we say that v is an *R-successor* of w.

The truth of a modal formula ϕ in a world w of a model $\mathcal{M}$, written as $\mathcal{M}, w \Vdash \phi$, is defined inductively on the structure of ϕ:

$\mathcal{M}, w \Vdash a$ for an atom a	iff	$\nu(w, a) = \text{true}$
$\mathcal{M}, w \Vdash \neg\phi$	iff	$\mathcal{M}, w \nVdash \phi$
$\mathcal{M}, w \Vdash \phi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \vee \psi$	iff	$\mathcal{M}, w \Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \to \psi$	iff	$\mathcal{M}, w \nVdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \Box\phi$	iff	for all u with wRu: $\mathcal{M}, u \Vdash \phi$

Note that the modal operator $\Box$ is related to the accessibility relation R (see [41]).

Table 5
Modal axioms and the corresponding constraints on the accessibility relation

Name	Axiom	Constraint on R
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	–
T	$\Box\phi \to \phi$	reflexive
4	$\Box\phi \to \Box\Box\phi$	transitive
5	$\neg\Box\phi \to \Box\neg\Box\phi$	Euclidean
		(wRu and wRv implies uRv)

Other normal modal logics are obtained by extending K with axioms that formalize properties of R. Some well-known examples of modal axioms and corresponding constraints on the accessibility relation are given in Table 5.

Two modal logics, which are of particular interest in this paper, are S4 and S5. S4 is the extension of the modal logic K by **T** and **4**, closed under modus ponens, uniform substitution, and the rule of necessitation. This is equivalent to specifying that the accessibility relation is reflexive and transitive. S5 is a similar extension of K by **T**, **4**, and **5**, which is equivalent to specifying that the accessibility relation is an equivalence relation. The modal operator is named according to the modal logic, so, e.g., the operator of an S5-frame is called S5 operator.

Multi-modal logics contain more than one modal operator $\Box$. Each different $\Box_i$ is associated to a different accessibility relation $R_i \subseteq W \times W$, i.e.,

$$\mathcal{M}, w \Vdash \Box_i\phi \quad \text{iff} \quad \text{for all } u \text{ with } wR_iu\text{: } \mathcal{M}, u \Vdash \phi.$$

Appendix B. Enumeration of the relations of $\widehat{\mathcal{H}}_8$

A complete list of all relations of $\widehat{\mathcal{H}}_8$ is given here. The list is separated into two parts. Table B.1 contains all the relations of $\mathcal{H}_8$ together with their abbreviated form. Table B.2 contains all relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ and their functional construction from relations of $\mathcal{H}_8$. The construction of the relations is chosen according to Lemma 50. $\overline{R}$ specifies the complement of R, i.e., $\overline{R}$ contains all base relations not included in R.

Table B.1
Relations of $\mathcal{H}_8$ and their abbreviated form

Relations	Abbreviations
{DC}	δ
{EC}	$\neg\delta \wedge \delta\mathrm{i}$
{DC, EC}	$\delta\mathrm{i}$
{PO}	$\neg\pi \wedge \neg\gamma \wedge \neg\delta\mathrm{i}$
{DC, PO}	$\neg\pi \wedge \neg\gamma \wedge (\delta \vee \neg\delta\mathrm{i})$

Table B.1 (Continued)

{EC, PO}	$\neg\delta \wedge \neg\pi \wedge \neg\gamma$
{DC, EC, PO}	$\neg\pi \wedge \neg\gamma$
{TPP}	$\pi \wedge \neg\gamma \wedge \neg\pi\mathrm{i}$
{DC, TPP}	$\neg\gamma \wedge \neg\pi\mathrm{i} \wedge (\delta \vee \pi)$
{EC, TPP}	$\neg\delta \wedge \neg\gamma \wedge \neg\pi\mathrm{i} \wedge (\delta\mathrm{i} \vee \pi)$
{DC, EC, TPP}	$\neg\gamma \wedge \neg\pi\mathrm{i} \wedge (\delta\mathrm{i} \vee \pi)$
{PO, TPP}	$\neg\gamma \wedge \neg\delta\mathrm{i} \wedge \neg\pi\mathrm{i}$
{DC, PO, TPP}	$\neg\gamma \wedge \neg\pi\mathrm{i} \wedge (\delta \vee \neg\delta\mathrm{i})$
{EC, PO, TPP}	$\neg\delta \wedge \neg\gamma \wedge \neg\pi\mathrm{i}$
{DC, EC, PO, TPP}	$\neg\gamma \wedge \neg\pi\mathrm{i}$
{NTPP}	$\neg\gamma \wedge \pi\mathrm{i}$
{DC, NTPP}	$\neg\gamma \wedge (\delta \vee \pi\mathrm{i})$
{EC, NTPP}	$\neg\delta \wedge \neg\gamma \wedge (\delta\mathrm{i} \vee \pi\mathrm{i})$
{DC, EC, NTPP}	$\neg\gamma \wedge (\delta\mathrm{i} \vee \pi\mathrm{i})$
{TPP, NTPP}	$\pi \wedge \neg\gamma$
{DC, TPP, NTPP}	$\neg\gamma \wedge (\delta \vee \pi)$
{EC, TPP, NTPP}	$\neg\delta \wedge \neg\gamma \wedge (\delta\mathrm{i} \vee \pi)$
{DC, EC, TPP, NTPP}	$\neg\gamma \wedge (\delta\mathrm{i} \vee \pi)$
{PO, TPP, NTPP}	$\neg\gamma \wedge \neg\delta\mathrm{i}$
{DC, PO, TPP, NTPP}	$\neg\gamma \wedge (\delta \vee \neg\delta\mathrm{i})$
{EC, PO, TPP, NTPP}	$\neg\delta \wedge \neg\gamma$
{DC, EC, PO, TPP, NTPP}	$\neg\gamma$
{TPP^{-1}}	$\neg\pi \wedge \gamma \wedge \neg\gamma\mathrm{i}$
{DC, TPP^{-1}}	$\neg\pi \wedge \neg\gamma\mathrm{i} \wedge (\delta \vee \gamma)$
{EC, TPP^{-1}}	$\neg\delta \wedge \neg\pi \wedge \neg\gamma\mathrm{i} \wedge (\delta\mathrm{i} \vee \gamma)$
{DC, EC, TPP^{-1}}	$\neg\pi \wedge \neg\gamma\mathrm{i} \wedge (\delta\mathrm{i} \vee \gamma)$
{PO, TPP^{-1}}	$\neg\pi \wedge \neg\delta\mathrm{i} \wedge \neg\gamma\mathrm{i}$
{DC, PO, TPP^{-1}}	$\neg\pi \wedge \neg\gamma\mathrm{i} \wedge (\delta \vee \neg\delta\mathrm{i})$
{EC, PO, TPP^{-1}}	$\neg\delta \wedge \neg\pi \wedge \neg\gamma\mathrm{i}$
{DC, EC, PO, TPP^{-1}}	$\neg\pi \wedge \neg\gamma\mathrm{i}$
{NTPP^{-1}}	$\neg\pi \wedge \gamma\mathrm{i}$
{DC, NTPP^{-1}}	$\neg\pi \wedge (\delta \vee \gamma\mathrm{i})$
{EC, NTPP^{-1}}	$\neg\delta \wedge \neg\pi \wedge (\delta\mathrm{i} \vee \gamma\mathrm{i})$
{DC, EC, NTPP^{-1}}	$\neg\pi \wedge (\delta\mathrm{i} \vee \gamma\mathrm{i})$

Table B.1 (Continued)

{TPP^{-1}, NTPP^{-1}}	$\neg\pi \wedge \gamma$
{DC, TPP^{-1}, NTPP^{-1}}	$\neg\pi \wedge (\delta \vee \gamma)$
{EC, TPP^{-1}, NTPP^{-1}}	$\neg\delta \wedge \neg\pi \wedge (\delta\mathrm{i} \vee \gamma)$
{DC, EC, TPP^{-1}, NTPP^{-1}}	$\neg\pi \wedge (\delta\mathrm{i} \vee \gamma)$
{PO, TPP^{-1}, NTPP^{-1}}	$\neg\pi \wedge \neg\delta\mathrm{i}$
{DC, PO, TPP^{-1}, NTPP^{-1}}	$\neg\pi \wedge (\delta \vee \neg\delta\mathrm{i})$
{EC, PO, TPP^{-1}, NTPP^{-1}}	$\neg\delta \wedge \neg\pi$
{DC, EC, PO, TPP^{-1}, NTPP^{-1}}	$\neg\pi$
{EQ}	$\pi \wedge \gamma$
{DC, EQ}	$(\delta \vee \pi) \wedge (\delta \vee \gamma)$
{EC, EQ}	$\neg\delta \wedge (\delta\mathrm{i} \vee \pi) \wedge (\delta\mathrm{i} \vee \gamma)$
{DC, EC, EQ}	$(\delta\mathrm{i} \vee \pi) \wedge (\delta\mathrm{i} \vee \gamma)$
{TPP, NTPP, EQ}	π
{DC, TPP, NTPP, EQ}	$(\delta \vee \pi)$
{EC, TPP, NTPP, EQ}	$\neg\delta \wedge (\delta\mathrm{i} \vee \pi)$
{DC, EC, TPP, NTPP, EQ}	$(\delta\mathrm{i} \vee \pi)$
{TPP^{-1}, NTPP^{-1}, EQ}	γ
{DC, TPP^{-1}, NTPP^{-1}, EQ}	$(\delta \vee \gamma)$
{EC, TPP^{-1}, NTPP^{-1}, EQ}	$\neg\delta \wedge (\delta\mathrm{i} \vee \gamma)$
{DC, EC, TPP^{-1}, NTPP^{-1}, EQ}	$(\delta\mathrm{i} \vee \gamma)$
{PO, TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ}	$\neg\delta\mathrm{i}$
{DC, PO, TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ}	$(\delta \vee \neg\delta\mathrm{i})$
{EC, PO, TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ}	$\neg\delta$

Table B.2
Construction of the relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ from $\mathcal{H}_8$-relations

Relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$	Construction
$\overline{\{\mathrm{EQ}\}}$	{EC} $\circ$ {DC, PO}
{EC, PO, TPP, NTPP, EQ}	{EC, TPP, NTPP, EQ} $\circ$ {TPP, NTPP, EQ}
$\overline{\{\mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}}$	{DC, TPP, NTPP, EQ} $\circ$ {TPP, NTPP, EQ}
$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$	{EC} $\circ$ {EC}
$\overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}^{-1}\}}$	{PO, TPP^{-1}} $\circ$ {TPP}
$\overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}\}}$	{EC, PO, TPP^{-1}} $\circ$ {TPP}
$\overline{\{\mathrm{NTPP}^{-1}\}}$	{EC} $\circ$ {EC, PO}

Table B.2 (Continued)

$\overline{\{\mathrm{NTPP}\}}$	$\{\mathrm{EC}\} \circ \{\mathrm{DC}, \mathrm{EC}\}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}^{-1}\}}$
$\overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}\}}$
$\overline{\{\mathrm{NTPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{NTPP}^{-1}\}}$
$\overline{\{\mathrm{NTPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{NTPP}\}}$
$\overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}\}}$
$\overline{\{\mathrm{EC}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}\}}$
$\overline{\{\mathrm{DC}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}\}}$
$\{\mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{DC}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{EC}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{EC}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}^{-1}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{EC}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}\}}$
$\overline{\{\mathrm{EC}, \mathrm{NTPP}^{-1}\}}$	$\overline{\{\mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{EC}\}}$
$\overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}\}}$	$\overline{\{\mathrm{NTPP}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}\}}$
$\overline{\{\mathrm{EC}, \mathrm{NTPP}\}}$	$\overline{\{\mathrm{NTPP}\}} \cap \overline{\{\mathrm{EC}\}}$
$\overline{\{\mathrm{DC}, \mathrm{NTPP}\}}$	$\overline{\{\mathrm{NTPP}\}} \cap \overline{\{\mathrm{DC}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\} \cap \overline{\{\mathrm{EC}, \mathrm{EQ}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\}$	$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\} \cap \{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}\}$
$\overline{\{\mathrm{EC}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{NTPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{NTPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{EC}, \mathrm{NTPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{NTPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{NTPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{EQ}\}$

Table B.2 (Continued)

$\overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$	$\{\mathrm{EC}, \mathrm{NTPP}\} \circ \{\mathrm{DC}, \mathrm{EQ}\}$
$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}}$	$\{\mathrm{DC}, \mathrm{EQ}\} \circ \{\mathrm{EC}, \mathrm{NTPP}^{-1}\}$
$\overline{\{\mathrm{DC}, \mathrm{TPP}, \mathrm{NTPP}\}}$	$\{\mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\} \circ \{\mathrm{EC}, \mathrm{EQ}\}$
$\overline{\{\mathrm{TPP}, \mathrm{NTPP}\}}$	$\{\mathrm{EC}, \mathrm{EQ}\} \circ \{\mathrm{DC}, \mathrm{EQ}\}$
$\{\mathrm{PO}, \mathrm{NTPP}\}$	$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{NTPP}\}$	$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\}$	$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\}$	$\overline{\{\mathrm{EC}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\}$	$\overline{\{\mathrm{DC}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}, \mathrm{TPP}^{-1}\}$	$\overline{\{\mathrm{NTPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \{\mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}\}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{NTPP}, \mathrm{EQ}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{EQ}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{NTPP}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{TPP}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{DC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{EC}, \mathrm{TPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{EQ}\}}$
$\overline{\{\mathrm{DC}, \mathrm{TPP}, \mathrm{EQ}\}}$	$\overline{\{\mathrm{TPP}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EQ}\}}$
$\{\mathrm{PO}, \mathrm{EQ}\}$	$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{TPP}\}}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{TPP}, \mathrm{NTPP}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{TPP}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}\} \cap \overline{\{\mathrm{TPP}, \mathrm{NTPP}\}}$
$\{\mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \{\mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}$
$\{\mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{TPP}\}}$

Table B.2 (Continued)

$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \overline{\{\mathrm{TPP}, \mathrm{NTPP}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{TPP}\}}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\}}$
$\{\mathrm{DC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \{\mathrm{DC}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}$
$\{\mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\{\mathrm{DC}, \mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\} \cap \{\mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}$
$\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{NTPP}, \mathrm{NTPP}^{-1}\}} \cap \overline{\{\mathrm{TPP}, \mathrm{NTPP}\}}$
$\{\mathrm{PO}, \mathrm{TPP}^{-1}, \mathrm{NTPP}^{-1}, \mathrm{EQ}\}$	$\overline{\{\mathrm{DC}, \mathrm{NTPP}, \mathrm{TPP}\}} \cap \overline{\{\mathrm{DC}, \mathrm{EC}, \mathrm{NTPP}\}}$
$\overline{\{\mathrm{EC}, \mathrm{TPP}, \mathrm{NTPP}\}}$	$\overline{\{\mathrm{TPP}, \mathrm{NTPP}\}} \cap \overline{\{\mathrm{EC}, \mathrm{NTPP}\}}$
$\{\mathrm{PO}, \mathrm{NTPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{NTPP}^{-1}\}$
$\{\mathrm{DC}, \mathrm{PO}, \mathrm{NTPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{EC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}}$
$\{\mathrm{EC}, \mathrm{PO}, \mathrm{NTPP}, \mathrm{NTPP}^{-1}\}$	$\overline{\{\mathrm{TPP}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}} \cap \overline{\{\mathrm{DC}, \mathrm{TPP}^{-1}, \mathrm{EQ}\}}$

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