Spherical spline interpolation—basic theory and computational aspects

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Abstract: The purpose of the paper is to adapt to the spherical case the basic theory and the computational method known from surface spline interpolation in Euclidean spaces. Spline functions are defined on the sphere. The solution process is made simple and efficient for numerical computation. In addition, the convergence of the solution obtained by spherical spline interpolation is developed using estimates for Legendre polynomials.

Keywords: Spherical harmonics, spherical splines.

Introduction

Until recently, spherical harmonics constituted the class of functions used more frequently than others to approximate geophysical data or functions. The basic reason was because of their simple mathematical structure. There are, however, several disadvantages in applying spherical harmonics, one of which is their strongly oscillating properties.

In the last years, the method of spherical spline interpolation has been analysed in depth, both in theoretical and computational aspects (see [3,6,15]). Spherical spline functions (s.s.f.) turn out to be natural generalizations to the sphere, of periodic splines on the circle [7,14] and surface splines in Euclidean spaces [9,10,11]; s.s.f. are more adaptable approximating functions than the spherical harmonics; they are ‘smooth’ functions avoiding larger oscillations and severe undulations (cf. the numerical computations in [6,13]). Moreover, in analogy to the interpolation method due to Meinguet [9] the whole solution process can be made surprisingly simple and reasonably efficient for numerical applications.

1. Theoretical background

Throughout this paper we restrict ourselves to the geophysically relevant threedimensional case.

Let Ω denote the unit sphere in Euclidean space \( \mathbb{R}^3 \). As usual, the spherical harmonics \( S_n \) of order \( n \) are defined as the everywhere on \( \Omega \) infinitely differentiable eigenfunctions, corresponding
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to the eigenvalues $\lambda_n = n(n + 1)$ for $n = 0, 1, 2, \ldots$ of the (Beltrami) differential equation

$$\partial_\nu S_n = (\Delta^* + \lambda_n) S_n = 0$$

($\Delta^*$: Beltrami-operator on $\Omega$). Spherical harmonics of different order are orthogonal in the sense of the $L^2$-inner product:

$$\int_{\Omega} S_n(\xi) S_m(\xi) \, d\omega = 0, \quad n \neq m$$

(d$\omega$: surface element). The linear space of all spherical harmonics of order $n$ is of dimension $2n + 1$. In other words, there must be $2n + 1$ linearly independent spherical harmonics $S_{n,1}, \ldots, S_{n,2n+1}$. We assume this system to be orthonormalized in the sense of the $L^2$-inner product. Then, for $\xi, \eta \in \Omega$, the addition theorem gives (cf. [12])

$$\sum_{j=1}^{2n+1} S_{n,j}(\xi) S_{n,j}(\eta) = \frac{2n + 1}{4\pi} P_n(\cos \vartheta_{\xi\eta}),$$

where $P_n$ is the Legendre polynomial of degree $n$ and the angle $\vartheta_{\xi\eta}$ is given by $\xi \eta = \cos \vartheta_{\xi\eta}$. As is well known, the system of all orthonormal bases $\{S_{n,j}\}_{n=0,1,\ldots, j=1,\ldots,2n+1}$ is closed and complete in $L^2(\Omega)$.

Let $\mathcal{H}$ be the vector space of all distributions $F$ (i.e. continuous linear functionals on the space $\mathcal{E}$ of infinitely differentiable functions on $\Omega$, provided with the canonical topology) for which the partial derivative $\partial_{\nu} \cdots \partial_{k} F = \partial_{\nu} \cdots \partial_{k} F$ (in the distributional sense) is square integrable:

$$\mathcal{H} = \{ F \in \mathcal{E}^* | \partial_{\nu} \cdots \partial_{k} F = f \in L^2(\Omega) \}. \quad (4)$$

$\mathcal{H}$ is naturally equipped with the semi-inner product $(\cdot, \cdot)_m$ corresponding to the seminorm

$$|F|_m = \left( \int_{\Omega} |\partial_{\nu} \cdots \partial_{k} F(\eta)|^2 \, d\omega \right)^{1/2}, \quad (5)$$

where $\partial_{\nu} \cdots \partial_{k}$ is to be interpreted in the distributional sense; the kernel of this (Sobolev-like) seminorm $|\cdot|_m$ is the linear space $\mathcal{P} = \mathcal{P}_m$ of dimension $\dim \mathcal{P} = M = (m + 1)^2$ of all spherical harmonics of degree $m$ or less. It should be noted that the norm $|\cdot|_m$ may be physically interpreted (at least for $m = 0$ and under some simplifying linearizations) as the bending energy of a (thin) membrane spanned over the sphere $\Omega$. $F$ denotes the deflection normal to the rest position (supposed of course to be spherical). The following imbedding theorem (cf. [3]) will be one of the key steps in the explanation of spline interpolation.

**Theorem 1.** The seminormed space $\mathcal{H}$, defined by (4) and (5), is a functional semi-Hilbert subspace of the space $C(\Omega)$ of continuous functions on $\Omega$. Moreover, for $1 \leq j \leq m$, the derivative $\partial_{\nu} \cdots \partial_{j-1} F$ of a function $F \in \mathcal{H}$ is of class $C(\Omega)$. 

Let there be given a $\mathcal{P}$-unisolvent set $\eta_1, \ldots, \eta_M$ of distinct points on the sphere $\Omega$, i.e. a set such that there exists a unique $P \in \mathcal{P}$ satisfying $P(\eta_i) = y_i, \ i = 1, \ldots, M$ for any prescribed real scalars $y_i, \ i = 1, \ldots, M$. By virtue of the $\mathcal{P}$-unisolveness of the set $\eta_1, \ldots, \eta_M$ of interpolation points there exists in $\mathcal{P}$ a unique basis $B_1, \ldots, B_M$ with $B_j(\eta_i) = \delta_{ij}, \ i, j = 1, \ldots, M$, where $\delta_{ij}$ is the Kronecker symbol. For every $F \in \mathcal{H}$, the (uniquely defined) $\mathcal{P}$-interpolant $pF$ of $F$ on the $\mathcal{P}$-unisolvent set
under consideration is given by the 'Lagrange form'

\[ pF = \sum_{i=1}^{M} F(\eta_i) B_i. \]

The mapping \( p : \mathcal{H} \rightarrow \mathcal{H} \) is a continuous linear projector of \( \mathcal{H} \subset C(\Omega) \) onto \( \mathcal{P} \). Hence, \( p \) determines a \( C(\Omega) \)-topological direct sum decomposition of \( \mathcal{H} \), i.e. \( \mathcal{H} = \mathcal{P} \oplus \mathcal{H} \) with

\[ \mathcal{H} = \{ F \in \mathcal{H} \mid F(\eta_i) = 0, \ i = 1, \ldots, M \}. \tag{6} \]

Thus, any \( F \in \mathcal{H} \) can be written uniquely as \( F = pF + \hat{F}, \ \hat{F} \in \mathcal{H}, \mathcal{H} \), as defined by (6) with \( (\cdot, \cdot)_m \) is a Hilbert function space (it is indeed complete like \( \mathcal{H} \) to which it is isometrically isomorphic).

Since \( \mathcal{H} \) is a Hilbert space (of continuous functions on \( \Omega \)), for each fixed \( \xi \in \Omega \), the evaluation linear functional \( \delta_\xi : \hat{F} \rightarrow \hat{F}(\xi) \) on \( \mathcal{H} \) is bounded (so that \( \delta_\xi \) can be regarded as the Dirac measure at the point \( \xi \in \Omega \)); hence, the following representation formula holds in the Hilbert space \( \mathcal{H} \):

\[ \hat{F}(\xi) = (\delta_\xi, \hat{F}) = (\hat{K}_\xi, \hat{F}), \quad \hat{F} \in \mathcal{H}. \tag{7} \]

where \( \hat{K}_\xi \in \mathcal{H} \) denotes the Riesz representor of \( \delta_\xi \) and \( (\cdot, \cdot) \) is the duality bracket between dual topological vector spaces. Therefore, by definition of differentiation for distributions, we may write for all \( \varphi \in \mathcal{E} \)

\[ \varphi(\xi) - p\varphi(\xi) = (\hat{K}_\xi, \varphi - p\varphi)_m = (\delta_\xi, \hat{K}_\xi, \varphi). \]

On the other hand, we have for all \( \varphi \in \mathcal{E} \)

\[ \varphi(\xi) - p\varphi(\xi) = \langle m\delta_\xi - \sum_{i=1}^{M} B_i(\xi)^m \delta_n, \varphi \rangle, \]

where we have set

\[ m\delta_\xi = \delta_\xi - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\xi)S_{n,j}. \]

Thus it follows that \( \hat{K}_\xi \) satisfies the (distributional) partial differential equation

\[ \partial_\xi^2 ... m\hat{K}_\xi = m\delta_\xi - \sum_{i=1}^{M} B_i(\xi)^m \delta_n. \tag{8} \]

in \( \mathcal{H} \subset \mathcal{E} \). The distributional equation

\[ \partial_\xi^2 ... mG_\xi = m\delta_\xi \]

is solvable by the Green function (in enlarged sense) with respect to the operator \( \partial_\xi^2 ... m \) on \( \Omega \):

\[ G_\xi(\eta) = G(\xi, \eta) = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} \sigma_n^2 S_{n,j}(\xi)S_{n,j}(\eta), \quad \eta \in \Omega, \]

where \( \sigma_n \) is given as follows

\[ \sigma_n = [(\lambda_0 - \lambda_n) \cdots (\lambda_m - \lambda_n)]^{-1}, \quad n = m + 1, \ldots \tag{9} \]

(indeed, the solution is unique apart from an additive element of \( \mathcal{P} \)). The theory of Green's functions (in enlarged sense) with respect to (iterated) Beltrami-derivatives on the sphere has
been developed extensively in [2,3,8]. Applying the addition theorem we obtain
\[ G(\xi, \eta) = \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} \sigma_n^2 P_n(\cos \theta_\xi). \]

According to the superposition principle of linear operators, the distribution \( H_\xi \) given by
\[ H_\xi(\eta) = H(\xi, \eta) = G(\xi, \eta) - \sum_{i=1}^{M} B_i(\xi)G(\eta_i, \eta) \]
(10)
satisfies equation (8). For each \( \xi \in \Omega \), \( H_\xi \) is an element of \( \mathcal{H} \) (cf. the considerations given in [3]). Hence, it finally follows that, for each \( \xi \in \Omega \), there exists a unique (continuous) function \( \mathcal{K}_\xi \in \mathcal{H} \) satisfying (8), viz.,
\[ \mathcal{K}_\xi(\eta) = \mathcal{K}(\xi, \eta) = G(\xi, \eta) - \sum_{i=1}^{M} G(\xi, \eta_i)B_i(\eta) \]
(11)
\[ + \sum_{i=1}^{M} B_i(\xi)G(\eta_i, \eta) + \sum_{i=1}^{M} \sum_{j=1}^{M} B_i(\xi)G(\eta_i, \eta_j)B_j(\eta). \]

According to (7), we have for each \( \xi \in \Omega \) and \( \hat{F} \in \mathcal{H} \),
\[ \hat{F}(\xi) = \int_{\Omega} \hat{f}(\eta)\hat{\mathcal{K}}(\xi, \eta) d\omega(\eta). \]
(12)

Theorem 2. The space \( \mathcal{H} \), defined by (6) with norm (5) is a Hilbert function space with the reproducing kernel (11).

2. Spline interpolation

A set \( \eta_1, \ldots, \eta_N \) of points of the unit sphere \( \Omega \) is called an admissible system of order \( m \) if it contains a \( \mathcal{R} \)-unisolvent subset. (In the sequel, the first \( M \) elements \( \eta_1, \ldots, \eta_M \) of each admissible system of order \( m \) are assumed to constitute a \( \mathcal{R} \)-unisolvent system. This is always achievable by reordering).

Definition 1. Given an admissible system \( \eta_1, \ldots, \eta_N \) of order \( m \). Then any function \( S \in \mathcal{H} \) of the form
\[ S(\xi) = \sum_{n=1}^{M} a_n B_n(\xi) + \sum_{n=M+1}^{N} a_n \mathcal{K}(\eta_n, \xi), \quad \xi \in \Omega, \]
(13)
with arbitrarily given (real) coefficients \( a_1, \ldots, a_n \) is called spherical spline function of degree \( m \) in \( \mathcal{H} \) relative to the knots \( \eta_1, \ldots, \eta_N \). (For \( N = M \), \( S \) reduces to the first sum of the right-hand side of (13)).

The space \( \mathcal{S} \equiv \mathcal{S}_m(\eta_1, \ldots, \eta_N) \) of all spline functions \( S \) of degree \( m \) is a \( N \)-dimensional linear subspace of \( \mathcal{H} \) containing the class \( \mathcal{P} \).
Theorem 3. Any \( S \in \mathcal{H} \) admits a unique representation
\[
S = P + Q
\]
with
\[
P(\xi) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} C_{n,j} S_{n,j}(\xi), \quad \xi \in \Omega
\] (14)
and
\[
Q(\xi) = \sum_{i=1}^{N} d_i G(\eta_i, \xi), \quad \xi \in \Omega,
\]
where the coefficients \( d_1, \ldots, d_N \) are real scalars satisfying the linear equations
\[
\sum_{i=1}^{N} d_i S_{n,j}(\eta_i) = 0, \quad n = 0, \ldots, m, \quad j = 1, \ldots, 2n+1.
\] (15)

The proof follows easily from the explicit definition (13). Thus it is obvious that the spherical spline functions developed here may be considered as 'natural' generalizations of the trigonometric splines due to Schoenberg [14].

Remark. The operator \( \partial_{0 \cdots m} \), in fact, produces splines of degree \( m \)
\[
S(\xi) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} S_{n,j}(\xi)
\]
\[
+ \sum_{i=1}^{N} d_i \sum_{n-m+1}^{\infty} \sum_{j=1}^{2n+1} \left[ (\lambda_0 - \lambda_n) \cdots (\lambda_m - \lambda_n) \right]^{-2} S_{n,j}(\eta_i) S_{n,j}(\xi),
\]
whereas the operator \( \partial_{0}^{m+1} \) (cf. [3,15]) gives splines of degree 0
\[
S(\xi) = c_{0,1} S_{0,1}(\xi) + \sum_{i=1}^{N} d_i \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \left[ (\lambda_0 - \lambda_n)^{m+1} \right]^{-2} S_{n,j}(\eta_i) S_{n,j}(\xi).
\]

For a more detailed discussion of splines corresponding to various differential operators the reader is referred to the considerations given by the author in [3].

Suppose now that there are given \( N \) prescribed data points \((\eta_1, y_1), \ldots, (\eta_N, y_N)\) corresponding to an admissible system \( \eta_1, \ldots, \eta_N \) of order \( m \). Consider the problem of determining the 'smoothest' function in the set
\[
J_N = \{ F \in \mathcal{H} \mid F(\eta_i) = y_i, \ i = 1, \ldots, N \}
\] (18)
of all \( \mathcal{H} \) interpolants to the data, where by 'smoothest' we mean that the seminorm \( | \cdot |_m \) is minimized in \( \mathcal{H} \).
Lemma 1. If $F \in J_N$ and $S \in \mathcal{S}$, then
\[
\int_\Omega \partial_0 \ldots \partial_m F(\eta) \partial_0 \ldots \partial_m S(\eta) \, d\omega = \sum_{k=M+1}^{N} a_k \left[ y_k - \sum_{n=1}^{M} y_n B_n(\eta_k) \right].
\]

Lemma 2. There exists a unique $S \in \mathcal{S} \cap J_N$. Denote this spline briefly by $S_N$.

Proof. Any spline $S \in \mathcal{S}$ of the form (13) contains a total of $N$ coefficients $a_1, \ldots, a_N$. Substitution with $\eta = \eta_i$, $i = 1, \ldots, N$, gives $N$ linear equations in these coefficients
\[
\sum_{n=1}^{M} a_n B_n(\eta_i) + \sum_{n=M+1}^{N} a_n \hat{K}(\eta_n, \eta_i) = y_i, \quad i = 1, \ldots, N.
\]
These equations are equivalent to
\[
\sum_{n=M+1}^{N} a_n \hat{K}(\eta_n, \eta_i) = y_i - \sum_{n=1}^{M} y_n B_n(\eta_i), \quad i = M+1, \ldots, N
\]
whose coefficient matrix is invertible (it is indeed symmetric and positive definite as Gram matrix of a sequence of linearly independent elements of $\mathcal{X}$). \[\]

The solution therefore can be obtained simply by standard algorithms, based on the idea of Cholesky factorization. Moreover, as explained in [9], the whole solution process can be described as a recursive algorithm, the data relative to the various interpolation points being exploited in the sequence.

Lemma 3. If $F \in J_N$, then
\[
\int_\Omega |\partial_0 \ldots \partial_m F(\eta)|^2 \, d\omega = \int_\Omega |\partial_0 \ldots \partial_m S_N(\eta)|^2 \, d\omega + \int_\Omega |\partial_0 \ldots \partial_m (S_N(\eta) - F(\eta))|^2 \, d\omega.
\]

Theorem 4. The interpolation problem
\[
\int_\Omega |\partial_0 \ldots \partial_m S_N(\eta)|^2 \, d\omega = \inf_{F \in J_N} \int_\Omega |\partial_0 \ldots \partial_m F(\eta)|^2 \, d\omega
\]
is well posed in the sense that its solution exists, is unique, and depends continuously on the data $y_1, \ldots, y_N$.

3. Convergence theorem

To every admissible system $\eta_1, \ldots, \eta_N$ of order $m$ and every function $F \in \mathcal{X}$, there exists a unique $S_N = S_N^F$ satisfying $S_N^F(\eta_i) = F(\eta_i)$, $i = 1, \ldots, N$. Furthermore, there is a maximal angle from any point of $\Omega$ to an admissible system $\eta_1, \ldots, \eta_N$ of order $m$:
\[
\Theta_N = \max_{\eta \in \Omega} \left( \min_{1 \leq i \leq N} \vartheta_{\eta \eta_i} \right).
\]
Theorem 5. Let $F$ be a function of class $\mathcal{H}$. Assume that there is given a sequence of admissible systems of order $m$ with $\Theta_N \to 0$ when $N \to \infty$. Then $S_N^f$ converges towards $F$ uniformly on $\Omega$.

Proof. Take a (sufficiently large) $N$. Then, for any given $\eta \in \Omega$, there exists a point $\eta_i$, $1 \leq i \leq N$, with $\theta_{\eta_i} \leq \Theta_N$. It is easy to see that
\[ S_N^f(\eta) - F(\eta) = \hat{S}_N^f(\eta) - \hat{F}(\eta_i) + \hat{F}(\eta_i) - \hat{F}(\eta). \] (24)

Observing $\hat{S}_N^f(\eta_i) = \hat{F}(\eta_i)$, $i = 1, \ldots, N$, we therefore obtain by application of the triangle inequality
\[ |S_N^f(\eta) - F(\eta)| \leq |\hat{S}_N^f(\eta) - \hat{S}_N^f(\eta_i)| + |\hat{F}(\eta_i) - \hat{F}(\eta)|. \] (25)

Now, by Theorem 2, we have
\[ \hat{S}_N^f(\eta) - \hat{S}_N^f(\eta_i) = \int_{\Omega} \partial_{\eta_1} \cdots \partial_{\eta_m} [\hat{K}(\eta, \xi) - \hat{K}(\eta_i, \xi)] \partial_{\eta_1} \cdots \partial_{\eta_m} S_N^f(\xi) \, d\omega(\xi). \] (26)

Cauchy's inequality yields
\[ |\hat{S}_N^f(\eta) - \hat{S}_N^f(\eta_i)| \leq [\kappa(\eta, \eta_i)]^{1/2} \left( \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} S_N^f(\xi)|^2 \, d\omega \right)^{1/2}, \] (27)

where we have used the abbreviation
\[ \kappa(\eta, \eta_i) = \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} [\hat{K}(\eta, \xi) - \hat{K}(\eta_i, \xi)]|^2 \, d\omega(\xi). \] (28)

As interpolating spline, $S_N^f$ is the smoothest $\mathcal{H}$ interpolant. Hence,
\[ \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} S_N^f(\xi)|^2 \, d\omega \leq \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} F(\xi)|^2 \, d\omega. \] (29)

Thus,
\[ |\hat{S}_N^f(\eta) - \hat{S}_N^f(\eta_i)| \leq [\kappa(\eta, \eta_i)]^{1/2} \left( \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} F(\xi)|^2 \, d\omega \right)^{1/2}. \] (30)

On the other hand,
\[ |\hat{F}(\eta) - \hat{F}(\eta_i)| \leq [\kappa(\eta, \eta_i)]^{1/2} \left( \int_{\Omega} |\partial_{\eta_1} \cdots \partial_{\eta_m} F(\xi)|^2 \, d\omega \right)^{1/2}. \] (31)

Elementary calculations give
\[ \kappa(\eta, \eta_i) = G(\eta, \eta) - 2G(\eta, \eta_i) + G(\eta_i, \eta_i) + \sum_{k=1}^M \sigma_k(\eta, \eta_i) \left[ B_k(\eta) - B_k(\eta_i) \right], \] (32)

where, for $k = 1, \ldots, M$, we have used the abbreviation
\[ \sigma_k(\eta, \eta_i) = -2\left[ G(\eta, \eta_k) - G(\eta_i, \eta_k) \right] + \sum_{j=1}^M G(\eta_j, \eta_k) \left[ B_j(\eta) - B_j(\eta_i) \right]. \] (33)

For estimating the expression (32) we need some results about Legendre polynomials. It is well
known that
\[ |P_n(\cos \theta)| \leq 1, \quad \theta \in [0, \pi]. \] (34)
Furthermore, for each \( t \in [-1, 1] \), we have
\[ 1 - P_n(t) = \int_t^1 P_n'(s) \, ds \leq P_n'(1)(1 - t). \] (35)

Now,
\[ P_n'(1) = \frac{1}{2} (n(n + 1)). \] (36)
Therefore, by setting \( t = \cos \theta, \theta \in [0, \pi] \), we find
\[ 1 - P_n(\cos \theta) \leq n(n + 1) \sin^2 \frac{1}{2} \theta \leq \frac{1}{4} n(n + 1) \theta^2 \leq \frac{1}{2} (n \theta)^2. \] (37)
Consequently, the Legendre polynomial satisfies the estimate
\[ 0 \leq 1 - P_n(\cos \theta) \leq A(n\theta)^{2\tau}, \quad \theta \in [0, \pi], \] (38)
provided that \( \tau \in [0, 1] \). From (34) we obtain for all \( \xi, \eta \in \Omega \)
\[ |G(\xi, \eta)| \leq \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} \sigma_n^2. \] (39)
An easy calculation yields for all \( \xi, \eta \in \Omega \)
\[ G(\eta, \eta) - 2G(\xi, \eta) + G(\xi, \xi) = 2 \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} \sigma_n^2 \left[ 1 - P_n(\cos \theta_{\xi}) \right]. \] (40)
Hence, from (38), we get
\[ |G(\xi, \eta) - 2G(\xi, \xi) + G(\eta, \eta)| \leq 2A\Theta_{\xi}^{2\tau} \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} n^{2\tau} \sigma_n^2 \] (41)
(eespecially, for \( m = 0, \) assume \( \tau < 1 \)). As continuous function on \( \Omega \), each \( B_k \in \mathcal{P}, k = 1, \ldots, M, \) is bounded on \( \Omega \). Thus, for \( k = 1, \ldots, M \), we are able to prove that there exists a positive constant \( B \) (independent of \( k \) and \( \eta, \eta \)) such that
\[ |\sigma_k(\eta, \eta)| \leq B. \] (42)
Let \( \mu_k(\Theta) \) denote the modulus of continuity of \( B_k \) on \( \Omega \):
\[ \mu_k(\Theta) = \max_{\xi, \eta \in \Omega, \xi \in \Theta} |B_k(\xi) - B_k(\eta)|. \] (43)
Then, in connection with (41) and (42), we are able to find, for suitably large \( N \), positive constants \( C, D \) such that
\[ |\kappa(\eta, \eta)| \leq C \cdot \Theta^{2\tau} + D \sum_{k=1}^{M} \mu_k(\Theta_k). \] (44)
Combining (25), (30), (31) and (44) we finally obtain
\[ |S_N^F(\eta) - F(\eta)| \leq 2C\Theta_{\xi}^{2\tau} + D \sum_{k=1}^{M} \mu_k(\Theta_k)\left( \int_{\Omega} |\partial_0 \cdots \partial_m F(\xi)|^2 d\omega \right)^{1/2}. \] (45)
Hence, on passing to the limit \( N \to \infty \), \( S_N^F \) converges to \( F \) uniformly on \( \Omega \). \( \square \)
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