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Graded polynomial identities, group actions, and exponential growth of Lie algebras ☆

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ABSTRACT

Consider a finite dimensional Lie algebra L with an action of a finite group G over a field of characteristic 0. We prove the analog of Amitsur's conjecture on asymptotic behavior for codimensions of polynomial G-identities of L. As a consequence, we prove the analog of Amitsur's conjecture for graded codimensions of any finite dimensional Lie algebra graded by a finite Abelian group.

1. Introduction

In the 1980's, a conjecture about the asymptotic behavior of codimensions of ordinary polynomial identities was made by S.A. Amitsur. Amitsur's conjecture was proved in 1999 by A. Giambruno and M.V. Zaicev [1, Theorem 6.5.2] for associative algebras, in 2002 by M.V. Zaicev [2] for finite dimensional Lie algebras, and in 2011 by A. Giambruno, I.P. Shestakov, M.V. Zaicev for finite dimensional Jordan and alternative algebras [3]. In 2011 the author proved its analog for polynomial identities

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of finite dimensional representations of Lie algebras [4]. Alongside with ordinary polynomial identities of algebras, graded polynomial identities [5,6] and G-identities are important too [7,8]. Therefore the question arises whether the conjecture holds for graded and G-codimensions. E. Aljadeff, A. Giambruno, and D. La Mattina proved [9,10] the analog of Amitsur's conjecture for codimensions of graded polynomial identities of associative algebras graded by a finite Abelian group (or, equivalently, for codimensions of G-identities where G is a finite Abelian group).

This article is concerned with graded codimensions (Theorem 1) and G-codimensions (Theorem 2) of Lie algebras.

1.1. Graded polynomial identities and their codimensions

Let G be an Abelian group. Denote by $L(X^{\mathrm{gr}})$ the free G-graded Lie algebra on the countable set $X^{\mathrm{gr}} = \bigcup_{g\in G} X^{(g)}$, $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \dots\}$, over a field F of characteristic 0, i.e. the algebra of Lie polynomials in variables from X^{gr}. The indeterminates from $X^{(g)}$ are said to be homogeneous of degree g. The G-degree of a monomial $[x_{i_1}^{(g_1)}, \dots, x_{i_t}^{(g_t)}] \in L(X^{\mathrm{gr}})$ (all long commutators in the article are left-normed) is defined to be $g_1 g_2 \dots g_t$, as opposed to its total degree, which is defined to be t. Denote by $L(X^{\mathrm{gr}})^{(g)}$ the subspace of the algebra $L(X^{\mathrm{gr}})$ spanned by all the monomials having G-degree g. Notice that $[L(X^{\mathrm{gr}})^{(g)}, L(X^{\mathrm{gr}})^{(h)}] \subseteq L(X^{\mathrm{gr}})^{(gh)}$, for every $g, h \in G$. It follows that

$$L\left(X^{\mathrm{gr}}\right) = \bigoplus_{g\in G} L\left(X^{\mathrm{gr}}\right)^{(g)}$$

is a G-grading. Let $f = f(x_{i_1}^{(g_1)}, \dots, x_{i_t}^{(g_t)}) \in L(X^{\mathrm{gr}})$. We say that f is a *graded polynomial identity* of a G-graded Lie algebra $L = \bigoplus_{g\in G} L^{(g)}$ and write $f \equiv 0$ if $f(a_{i_1}^{(g_1)}, \dots, a_{i_t}^{(g_t)}) = 0$ for all $a_{i_j}^{(g_j)} \in L^{(g_j)}$, $1 \leqslant j \leqslant t$. The set $\mathrm{Id}^{\mathrm{gr}}(L)$ of graded polynomial identities of L is a graded ideal of $L(X^{\mathrm{gr}})$. The case of ordinary polynomial identities is included for the trivial group $G = \{e\}$.

Example 1. Let $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, $\mathfrak{gl}_2(F) = \mathfrak{gl}_2(F)^{(\bar{0})} \oplus \mathfrak{gl}_2(F)^{(\bar{1})}$ where $\mathfrak{gl}_2(F)^{(\bar{0})} = \left(\begin{smallmatrix} F & 0 \\ 0 & F \end{smallmatrix}\right)$ and $\mathfrak{gl}_2(F)^{(\bar{1})} = \left(\begin{smallmatrix} 0 & F \\ F & 0 \end{smallmatrix}\right)$. Then $[x^{(\bar{0})}, y^{(\bar{0})}] \in \mathrm{Id}^{\mathrm{gr}}(\mathfrak{gl}_2(F))$.

Let S_n be the nth symmetric group, $n \in \mathbb{N}$, and

$$V_n^{\mathrm{gr}} := \left\langle \left[x_{\sigma(1)}^{(g_1)}, x_{\sigma(2)}^{(g_2)}, \dots, x_{\sigma(n)}^{(g_n)}\right] \;\middle|\; g_i \in G,\ \sigma \in S_n \right\rangle_F.$$

The non-negative integer $c_n^{\mathrm{gr}}(L) := \dim(\frac{V_n^{\mathrm{gr}}}{V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L)})$ is called the nth *codimension of graded polynomial identities* or the nth *graded codimension* of L.

The analog of Amitsur's conjecture for graded codimensions can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^{\mathrm{gr}}(L) := \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(L)} \in \mathbb{Z}_+$.

Remark. I.B. Volichenko [11] gave an example of an infinite dimensional Lie algebra L with a non-trivial polynomial identity for which the growth of codimensions $c_n(L)$ of ordinary polynomial identities is overexponential. M.V. Zaicev and S.P. Mishchenko [12,13] gave an example of an infinite dimensional Lie PI-algebra L with a non-trivial polynomial identity such that there exists fractional $\mathrm{PIexp}(L) := \lim_{n\to\infty} \sqrt[n]{c_n(L)}$.

Theorem 1. *Let L be a finite dimensional non-nilpotent Lie algebra over a field F of characteristic* 0, *graded by a finite Abelian group G. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Remark. If L is nilpotent, i.e. $[x_1, \dots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then $V_n^{\text{gr}} \subseteq \text{Id}^{\text{gr}}(L)$ and $c_n^{\text{gr}}(L) = 0$ for all $n \geqslant p$.

Theorem 1 will be obtained as a consequence of Theorem 2 in Section 1.3.

1.2. Polynomial G-identities and their codimensions

Analogously, one can consider polynomial G-identities for any group G. We use the exponential notation for the action of a group and its group algebra. We say that a Lie algebra L is a *Lie algebra with G-action* or a *Lie G-algebra* if there is a fixed linear representation $G \to \text{GL}(L)$ such that $[a,b]^g = [a^g, b^g]$ for all $a, b \in L$ and $g \in G$. Denote by $L(X|G)$ the free Lie algebra over F with free formal generators x_j^g, $j \in \mathbb{N}$, $g \in G$. Define $(x_j^g)^h := x_j^{hg}$ for $h \in G$. Let $X := \{x_1, x_2, x_3, \dots\}$ where $x_j := x_j^1$, $1 \in G$. Then $L(X|G)$ becomes the free G-algebra with free generators x_j, $j \in \mathbb{N}$. Let L be a Lie G-algebra over F. A polynomial $f(x_1, \dots, x_n) \in L(X|G)$ is a *G-identity* of L if $f(a_1, \dots, a_n) = 0$ for all $a_i \in L$. The set $\text{Id}^G(L)$ of all G-identities of L is an ideal in $L(X|G)$ invariant under G-action.

Example 2. Consider $\psi \in \text{Aut}(\mathfrak{gl}_2(F))$ defined by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\psi} := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Then $[x + x^{\psi}, y + y^{\psi}] \in \text{Id}^G(\mathfrak{gl}_2(F))$ where $G = \langle \psi \rangle \cong \mathbb{Z}_2$.

Denote by V_n^G the space of all multilinear G-polynomials in $x_1, \dots, x_n$, i.e.

$$V_n^G = \left\langle \left[x_{\sigma(1)}^{g_1}, x_{\sigma(2)}^{g_2}, \dots, x_{\sigma(n)}^{g_n}\right] \,\middle|\, g_i \in G,\ \sigma \in S_n \right\rangle_F.$$

Then the number $c_n^G(L) := \dim(\frac{V_n^G}{V_n^G \cap \text{Id}^G(L)})$ is called the nth *codimension of polynomial G-identities* or the nth *G-codimension* of L.

Remark. As in the case of associative algebras [1, Lemma 10.1.3], we have

$$c_n(L) \leqslant c_n^G(L) \leqslant |G|^n c_n(L).$$

Here $c_n(L) = c_n^{\{e\}}(L)$ are ordinary codimensions.

Also we have the following upper bound:

Lemma 1. *Let L be a finite dimensional Lie algebra with G-action over any field F and let G be any group. Then $c_n^G(L) \leqslant (\dim L)^{n+1}$.*

Proof. Consider G-polynomials as n-linear maps from L to L. Then we have a natural map $V_n^G \to \text{Hom}_F(L^{\otimes n}; L)$ with the kernel $V_n^G \cap \text{Id}^G(L)$ that leads to the embedding

$$\frac{V_n^G}{V_n^G \cap \text{Id}^G(L)} \hookrightarrow \text{Hom}_F\left(L^{\otimes n}; L\right).$$

Thus

$$c_n^G(L) = \dim\left(\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}\right) \leqslant \dim \mathrm{Hom}_F\left(L^{\otimes n}; L\right) = (\dim L)^{n+1}. \qquad \square$$

The analog of Amitsur's conjecture for G-codimensions can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^G(L) := \lim_{n\to\infty} \sqrt[n]{c_n^G(L)} \in \mathbb{Z}_+$.

Theorem 2. *Let L be a finite dimensional non-nilpotent Lie algebra over a field F of characteristic 0. Suppose a finite group G not necessarily Abelian acts on L. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Remark. If L is nilpotent, i.e. $[x_1, \dots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then, by the Jacobi identity, $V_n^G \subseteq \mathrm{Id}^G(L)$ and $c_n^G(L) = 0$ for all $n \geqslant p$.

Remark. The theorem is still true if we allow G to act not only by automorphisms, but by anti-automorphisms too, i.e. if $G = G_0 \cup G_1$ such that $[a,b]^g = [a^g, b^g]$ for all $a, b \in L$, $g \in G_0$ and $[a,b]^g = [b^g, a^g]$ for all $a, b \in L$, $g \in G_1$. Indeed, we can replace G with $\tilde{G} = G_0 \cup (-G_1)$ where $[a,b]^{-g} = -[a,b]^g = -[b^g, a^g] = [a^{-g}, b^{-g}]$ for all $(-g) \in (-G_1)$. Then $\tilde{G}$ acts on L by automorphisms only. Moreover, n-linear functions from L to L that correspond to polynomials from P_n^G and $P_n^{\tilde{G}}$, are the same. Thus

$$c_n^G(L) = \dim\left(\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}\right) = \dim\left(\frac{V_n^{\tilde{G}}}{V_n^{\tilde{G}} \cap \mathrm{Id}^{\tilde{G}}(L)}\right) = c_n^{\tilde{G}}(L)$$

has the desired asymptotics.

Theorem 2 is proved in Sections 4–6.

1.3. Duality between group gradings and group actions

If F is an algebraically closed field of characteristic 0 and G is finite Abelian, there exists a well-known duality between G-gradings and $\widehat{G}$-actions where $\widehat{G} = \mathrm{Hom}(G, F^*) \cong G$. Details of the application of this duality to polynomial identities can be found, e.g., in [1, Chapters 3 and 10].

A character $\psi \in \widehat{G}$ acts on L in the natural way: $(a_g)^\psi = \psi(g) a_g$ for all $g \in G$ and $a_g \in L^{(g)}$. Conversely, if L is a $\widehat{G}$-algebra, then $L^{(g)} = \{a \in L \mid a^\psi = \psi(g)a$ for all $\psi \in \widehat{G}\}$ defines a G-grading on L.

Note that if G is finite Abelian, then $L(X^{\mathrm{gr}})$ is a free $\widehat{G}$-algebra with free generators $y_j = \sum_{g\in G} x_j^{(g)}$. Thus there exists an isomorphism $\varepsilon : L(X|\widehat{G}) \to L(X^{\mathrm{gr}})$ defined by $\varepsilon(x_j) = \sum_{g\in G} x_j^{(g)}$, that preserves $\widehat{G}$-action and G-grading. The isomorphism has the property $\varepsilon((x_j)^{e_g}) = x_j^{(g)}$ where $e_g := \frac{1}{|G|}\sum_\psi (\psi(g))^{-1}\psi$ is one of the minimal idempotents of $F\widehat{G}$ defined above.

Lemma 2. *Let L be a G-graded Lie algebra where G is a finite Abelian group. Consider the corresponding $\widehat{G}$-action on L. Then*

(1) $\varepsilon(\mathrm{Id}^{\widehat{G}}(L)) = \mathrm{Id}^{\mathrm{gr}}(L)$;
(2) $c_n^{\widehat{G}}(L) = c_n^{\mathrm{gr}}(L)$.

Proof. The first assertion is evident. The second assertion follows from the first one and the equality $\varepsilon(V_n^{\widehat{G}}) = V_n^{\text{gr}}$. □

Remark. Note that $\mathbb{Z}_2$-grading in Example 1 corresponds to $\mathbb{Z}_2$-action in Example 2.

Proof of Theorem 1. Codimensions do not change upon an extension of the base field. The proof is analogous to the cases of ordinary codimensions of associative [1, Theorem 4.1.9] and Lie algebras [2, Section 2]. Thus without loss of generality we may assume F to be algebraically closed. In virtue of Lemma 2, Theorem 1 is an immediate consequence of Theorem 2. □

1.4. *Formula for the PI-exponent*

Theorem 2 is formulated for an arbitrary field F of characteristic 0, but without loss of generality we may assume that F is algebraically closed.

Fix a Levi decomposition $L = B \oplus R$ where B is a maximal semisimple subalgebra of L and R is the solvable radical of L. Note that R is invariant under G-action. By [14, Theorem 1, Remark 3], we can choose B invariant under G-action too.

We say that M is an *L-module with G-action* if M is both left L- and FG-module, and $(a \cdot v)^g = a^g \cdot v^g$ for all $a \in L$, $v \in M$ and $g \in G$. There is a natural G-action on $\operatorname{End}_F(M)$ defined by $\psi^g m = (\psi m^{g^{-1}})^g$, $m \in M$, $g \in G$, $\psi \in \operatorname{End}_F(M)$. Note that $L \to \mathfrak{gl}(M)$ is a homomorphism of FG-modules. Such module M is *irreducible* if for any G- and L-invariant subspace $M_1 \subseteq M$ we have either $M_1 = 0$ or $M_1 = M$. Each G-invariant ideal in L can be regarded as a left L-module with G-action under the adjoint representation of L.

Consider G-invariant ideals $I_1, I_2, \dots, I_r$, $J_1, J_2, \dots, J_r$, $r \in \mathbb{Z}_+$, of the algebra L such that $J_k \subseteq I_k$, satisfying the conditions

(1) I_k/J_k is an irreducible L-module with G-action;
(2) for any G-invariant B-submodules T_k such that $I_k = J_k \oplus T_k$, there exist numbers $q_i \geqslant 0$ such that

$$\big[[T_1, \underbrace{L, \dots, L}_{q_1}], [T_2, \underbrace{L, \dots, L}_{q_2}], \dots, [T_r, \underbrace{L, \dots, L}_{q_r}]\big] \neq 0.$$

Let M be an L-module. Denote by $\operatorname{Ann} M$ its annihilator in L. Let

$$d(L) := \max\left(\dim \frac{L}{\operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_r/J_r)}\right)$$

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \dots, I_r$, $J_1, \dots, J_r$ satisfying conditions (1)–(2). We claim that $\operatorname{PIexp}^G(L) = d(L)$ and prove Theorem 2 for $d = d(L)$.

1.5. *Examples*

Now we give several examples.

Example 3. Let L be a finite dimensional G-simple Lie algebra over an algebraically closed field F of characteristic 0 where G is a finite group. Then there exist $C > 0$ and $r \in \mathbb{R}$ such that $Cn^r (\dim L)^n \leqslant c_n^G(L) \leqslant (\dim L)^{n+1}$.

Proof. The upper bound follows from Lemma 1. Consider G-invariant L-modules $I_1 = L$ and $J_1 = 0$. Then I_1/J_1 is an irreducible L-module, $\operatorname{Ann}(I_1/J_1) = 0$ since a G-simple algebra has zero center, and $\dim(L/\operatorname{Ann}(I_1/J_1)) = \dim L$. Thus $d(L) \geqslant \dim L$ and by Theorem 2 we obtain the lower bound. □

Example 4. Let L be a finite dimensional simple G-graded Lie algebra over an algebraically closed field F of characteristic 0 where G is a finite Abelian group. Then there exist $C>0$ and $r\in\mathbb{R}$ such that $Cn^r(\dim L)^n \leqslant c_n^{\mathrm{gr}}(L)\leqslant (\dim L)^{n+1}$.

Proof. This follows from Example 3 and Lemma 2. $\square$

Example 5. Let L be a finite dimensional Lie algebra with G-action over any field F of characteristic 0 such that $\operatorname{PIexp}^G(L)\leqslant 2$ where G is a finite group. Then L is solvable.

Proof. It is sufficient to prove the statement for an algebraically closed field F. (See the remark before Theorem 2.) Consider the G-invariant Levi decomposition $L=B\oplus R$. If $B\neq 0$, there exists a G-simple Lie subalgebra $B_1\subseteq L$, $\dim B_1\geqslant 3$ and $\operatorname{PIexp}^G(L)=d(L)\geqslant 3$ by Example 3. We get a contradiction. Hence $L=R$ is a solvable algebra. $\square$

Analogously, we derive Example 6 from Example 4.

Example 6. Let L be a finite dimensional G-graded Lie algebra over any field F of characteristic 0 such that $\operatorname{PIexp}^{\mathrm{gr}}(L)\leqslant 2$ where G is a finite Abelian group. Then L is solvable.

Example 7. Let $L=B_1\oplus\dots\oplus B_s$ be a finite dimensional semisimple Lie G-algebra over an algebraically closed field F of characteristic 0 where G is a finite group and B_i are G-minimal ideals. Let $d:=\max_{1\leqslant i\leqslant s}\dim B_i$. Then there exist $C_1,C_2>0$ and $r_1,r_2\in\mathbb{R}$ such that $C_1n^{r_1}d^n\leqslant c_n^G(L)\leqslant C_2n^{r_2}d^n$.

Proof. Note that if I is a G-simple ideal of L, then $[I,L]\neq 0$ and hence $[I,B_i]\neq 0$ for some $1\leqslant i\leqslant s$. However $[I,B_i]\subseteq B_i\cap I$ is a G-invariant ideal. Thus $I=B_i$. And if I is a G-invariant ideal of L, then it is semisimple and each of its simple components coincides with one of B_i. Thus if $I\subseteq J$ are G-invariant ideals of L and I/J is irreducible, then $I=B_i\oplus J$ for some $1\leqslant i\leqslant s$ and $\dim(L/\operatorname{Ann}(I/J))=\dim B_i$. Note that if $I_1=B_{i_1}\oplus J_1$ and $I_2=B_{i_2}\oplus J_2$, $i_1\neq i_2$, then $[[B_{i_1},L,\dots,L],[B_{i_2},L,\dots,L]]=0$. Thus $I_1,\dots,I_r$, $J_1,\dots,J_r$ can satisfy conditions (1)–(2) only if $r=1$. Hence $d(L)=\max_{1\leqslant i\leqslant s}\dim B_i$ and the result follows from Theorem 2. $\square$

Example 8. Let $L=B_1\oplus\dots\oplus B_s$ be a finite dimensional semisimple G-graded Lie algebra over an algebraically closed field F of characteristic 0 where G is a finite Abelian group and B_i are minimal graded ideals. Let $d:=\max_{1\leqslant i\leqslant s}\dim B_i$. Then there exist $C_1,C_2>0$ and $r_1,r_2\in\mathbb{R}$ such that $C_1n^{r_1}d^n\leqslant c_n^{\mathrm{gr}}(L)\leqslant C_2n^{r_2}d^n$.

Proof. This follows from Example 7 and Lemma 2. $\square$

Example 9. Let $m\in\mathbb{N}$, $G\subseteq S_m$ and O_i be the orbits of G-action on

$$\{1,2,\dots,m\}=\coprod_{i=1}^{s}O_i.$$

Denote

$$d:=\max_{1\leqslant i\leqslant s}|O_i|.$$

Let L be the Lie algebra over any field F of characteristic 0 with basis $a_1,\dots,a_m$, $b_1,\dots,b_m$, $\dim L=2m$, and multiplication defined by formulas $[a_i,a_j]=[b_i,b_j]=0$ and

$$[a_i,b_j]=\begin{cases} b_j & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Suppose G acts on L as follows: $(a_i)^\sigma = a_{\sigma(i)}$ and $(b_j)^\sigma = b_{\sigma(j)}$ for $\sigma \in G$. Then there exist $C_1, C_2 > 0$ and $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leqslant c_n^G(L) \leqslant C_2 n^{r_2} d^n.$$

In particular, if

$$G = \langle \tau \rangle \cong \mathbb{Z}_m = \mathbb{Z}/(m\mathbb{Z}) = \{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$$

where $\tau = (1\,2\,3\,\dots\,m)$ (a cycle), then

$$C_1 n^{r_1} m^n \leqslant c_n^G(L) \leqslant C_2 n^{r_2} m^n.$$

However, $c_n(L) = n-1$ for all $n \in \mathbb{N}$.

Proof. If $K \supseteq F$ is a larger field, then $K \otimes_F L$ is defined by the same formulas as L. Since $c_n^G(L) = c_n^{G,K}(K \otimes_F L)$ (see the remark before Theorem 2), we may assume F to be algebraically closed.

Let $B_i := \langle b_j \mid j \in O_i \rangle_F$, $1 \leqslant i \leqslant s$. Suppose I is a G-invariant ideal of L. If $b_i \in I$, then $b_{\sigma(i)} = (b_i)^\sigma \in I$ for all $\sigma \in G$. Thus if $i \in O_j$, then $b_k \in I$ for all $k \in O_j$. Let $c := \sum_{i=1}^m (\alpha_i a_i + \beta_i b_i) \in I$ for some $\alpha_i, \beta_i \in F$. Then $\beta_i b_i = [a_i, c] \in I$ for all $1 \leqslant i \leqslant m$ too. Therefore, $I = A_0 \oplus B_{i_1} \oplus \dots \oplus B_{i_k}$ for some $1 \leqslant i_j \leqslant s$ and $A_0 \subseteq \langle a_1, \dots, a_m \rangle_F$.

If $I, J \subseteq L$ are G-invariant ideals, then $J \subseteq J + [L,L] \cap I \subseteq I$ is a G-invariant ideal too. Suppose I/J is irreducible. Then either $[L,L] \cap I \subseteq J$ and $\operatorname{Ann}(I/J) = L$ or $I \subseteq J + [L,L]$ where $[L,L] = \langle b_1, \dots, b_m \rangle_F$. Thus $\operatorname{Ann}(I/J) \neq L$ implies $J = A_0 \oplus B_{i_1} \oplus \dots \oplus B_{i_k}$ and $I = B_\ell \oplus J$ for some $1 \leqslant \ell \leqslant s$. In this case $\dim(L/\operatorname{Ann}(I/J)) = |O_\ell|$.

Note that if $I_1 = B_{i_1} \oplus J_1$ and $I_2 = B_{i_2} \oplus J_2$, then

$$\big[[B_{i_1}, L, \dots, L], [B_{i_2}, L, \dots, L]\big] = 0.$$

Thus $I_1, \dots, I_r$, $J_1, \dots, J_r$ can satisfy conditions (1)–(2) only if $r = 1$. Hence

$$d(L) = \max_{1 \leqslant i \leqslant s} |O_\ell|$$

and by Theorem 2 we obtain the bounds.

Consider the ordinary polynomial identities. Using the Jacobi identity, any monomial in V_n can be rewritten as a linear combination of left-normed commutators $[x_1, x_j, x_{i_3}, \dots, x_{i_n}]$. Since the polynomial identity

$$\big[[x,y],[z,t]\big] \equiv 0$$

holds in L, we may assume that $i_3 < i_4 < \dots < i_n$. Note that $f_j = [x_1, x_j, x_{i_3}, \dots, x_{i_n}]$, $2 \leqslant j \leqslant n$, are linearly independent modulo $\operatorname{Id}(L)$. Indeed, if $\sum_{k=2}^n \alpha_k f_k \equiv 0$, $\alpha_k \in F$, then we substitute $x_j = b_1$ and $x_i = a_1$ for $i \neq j$. Only f_j does not vanish. Hence $\alpha_j = 0$ and $c_n(L) = n-1$. $\square$

Example 10. Let $m \in \mathbb{N}$, $L = \bigoplus_{\bar{k} \in \mathbb{Z}_m} L^{(\bar{k})}$ be the $\mathbb{Z}_m$-graded Lie algebra with $L^{(\bar{k})} = \langle c_{\bar{k}}, d_{\bar{k}} \rangle_F$, $\dim L^{(\bar{k})} = 2$, multiplication $[c_{\bar{\imath}}, c_{\bar{\jmath}}] = [d_{\bar{\imath}}, d_{\bar{\jmath}}] = 0$ and $[c_{\bar{\imath}}, d_{\bar{\jmath}}] = d_{\bar{\imath}+\bar{\jmath}}$ where F is any field of characteristic 0. Then there exist $C_1, C_2 > 0$ and $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} m^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} m^n.$$

Proof. Again, we may assume F to be algebraically closed. Let $\zeta \in F$ be an mth primitive root of 1. Then $\widehat{G} = \{\psi_0, \dots, \psi_{m-1}\}$ for $G = \mathbb{Z}_m$ where $\psi_\ell(\bar{j}) := \zeta^{\ell j}$. We can identify the algebras from Examples 9 and 10 by formulas $c_{\bar{j}} = \sum_{k=1}^{m} \zeta^{-jk} a_k$ and $d_{\bar{j}} = \sum_{k=1}^{m} \zeta^{-jk} b_k$. The $\mathbb{Z}_m$-grading and $\langle\tau\rangle$-action correspond to each other since $(c_{\bar{j}})^{\tau^\ell} = \zeta^{\ell j} c_{\bar{j}} = \psi_\ell(\bar{j}) c_{\bar{j}}$ and $(d_{\bar{j}})^{\tau^\ell} = \zeta^{\ell j} d_{\bar{j}} = \psi_\ell(\bar{j}) d_{\bar{j}}$. By Lemma 2, $c_n^{\mathrm{gr}}(L) = c_n^{\langle\tau\rangle}(L)$ and the bounds follow from Example 9. □

1.6. S_n-cocharacters

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group S_n acts on the space $\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}$ by permuting the variables. Irreducible FS_n-modules are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and their Young diagrams D_λ. The character $\chi_n^G(L)$ of the FS_n-module $\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}$ is called the nth *cocharacter* of polynomial G-identities of L. We can rewrite it as a sum $\chi_n^G(L) = \sum_{\lambda \vdash n} m(L, G, \lambda)\chi(\lambda)$ of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign} \sigma)\sigma$, be the Young symmetrizers corresponding to a Young tableau T_λ. Then $M(\lambda) = FSe_{T_\lambda} \cong FSe^*_{T_\lambda}$ is an irreducible FS_n-module corresponding to the partition $\lambda \vdash n$. We refer the reader to [1,17,18] for an account of S_n-representations and their applications to polynomial identities.

Our proof of Theorem 2 follows the outline of the proof by M.V. Zaicev [2]. However, in many cases we need to apply new ideas.

In Section 2 we discuss modules with G-action over Lie G-algebras, their annihilators and complete reducibility.

In Section 3 we prove that $m(L, G, \lambda)$ is polynomially bounded. In Section 4 we prove that if $m(L, G, \lambda) \neq 0$, then the corresponding Young diagram D_λ has at most d long rows. This implies the upper bound.

In Section 5 we consider faithful irreducible L_0-modules with G-action where L_0 is a reductive Lie G-algebra. For an arbitrary $k \in \mathbb{N}$, we construct an associative G-polynomial that is alternating in $2k$ sets, each consisting of $\dim L_0$ variables. This polynomial is not an identity of the corresponding representation of L_0. In Section 6 we choose reductive algebras and faithful irreducible modules with G-action, and glue the corresponding alternating polynomials. This allows us to find $\lambda \vdash n$ with $m(L, G, \lambda) \neq 0$ such that $\dim M(\lambda)$ has the desired asymptotic behavior and the lower bound is proved.

2. Lie algebras and modules with G-action

We need several auxiliary lemmas. First, the Weyl theorem [15, Theorem 6.3] on complete reducibility of representations can be easily extended to the case of Lie algebras with G-action.

Lemma 3. *Let M be a finite dimensional module with G-action over a Lie G-algebra L_0. Suppose M is a completely reducible L_0-module disregarding the G-action. Then M is completely reducible L_0-module with G-action.*

Corollary. *If M is a finite dimensional module with G-action over a semisimple Lie G-algebra B_0, then M is a completely reducible module with G-action.*

Proof of Lemma 3. Suppose $M_1 \subseteq M$ is a G-invariant L_0-submodule of M. Then it is sufficient to prove that there exists a G-invariant L_0-submodule $M_2 \subseteq M$ such that $M = M_1 \oplus M_2$.

Since M is completely reducible, there exists an L_0-homomorphism $\pi : M \to M_1$ such that $\pi(v) = v$ for all $v \in M_1$. Consider a homomorphism $\tilde{\pi} : M \to M_1$, $\tilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} \pi(v^{g^{-1}})^g$. Then $\tilde{\pi}(v) = v$ for all $v \in M_1$ too and for all $a \in L_0$, $h \in G$ we have

$$\tilde{\pi}(a\cdot v)=\frac{1}{|G|}\sum_{g\in G}\pi\left((a\cdot v)^{g^{-1}}\right)^{g}=\frac{1}{|G|}\sum_{g\in G}\pi\left(a^{g^{-1}}\cdot v^{g^{-1}}\right)^{g}=\frac{1}{|G|}\sum_{g\in G}a\cdot\pi\left(v^{g^{-1}}\right)^{g}=a\cdot\tilde{\pi}(v),$$

$$\tilde{\pi}\left(v^{h}\right)=\frac{1}{|G|}\sum_{g\in G}\pi\left(\left(v^{h}\right)^{g^{-1}}\right)^{g}=\frac{1}{|G|}\sum_{g\in G}\pi\left(v^{(h^{-1}g)^{-1}}\right)^{h(h^{-1}g)}=\frac{1}{|G|}\sum_{g'\in G}\left(\pi\left(v^{g'^{-1}}\right)^{g'}\right)^{h}=\tilde{\pi}(v)^{h}$$

where $g'=h^{-1}g$. Thus we can take $M_2=\ker\tilde{\pi}$. □

Note that $[L,R]\subseteq N$ by [16, Proposition 2.1.7] where N is the nilpotent radical, which is a G-invariant ideal.

Lemma 4. *There exists a G-invariant subspace $S\subseteq R$ such that $R=S\oplus N$ is the direct sum of subspaces and $[B,S]=0$.*

Proof. Note that R is a B-submodule under the adjoint representation of B on L. Applying the corollary of Lemma 3 to $N\subseteq R$, we obtain a G-invariant complementary subspace $S\subseteq R$ such that $[B,S]\subseteq S$. Thus $[B,S]\subseteq S\cap[L,R]\subseteq S\cap N=0$. □

Therefore, $L=B\oplus S\oplus N$ (direct sum of subspaces).

Let M be an L-module and let T be a subspace of L. Denote $\operatorname{Ann}_T M:=(\operatorname{Ann} M)\cap T$. Lemma 5 is a G-invariant analog of [2, Lemma 4].

Lemma 5. *Let $J\subseteq I\subseteq L$ be G-invariant ideals such that I/J is an irreducible L-module with G-action. Then*

(1) $\operatorname{Ann}_B(I/J)$ *and* $\operatorname{Ann}_S(I/J)$ *are G-invariant subspaces of L*;
(2) $\operatorname{Ann}(I/J)=\operatorname{Ann}_B(I/J)\oplus\operatorname{Ann}_S(I/J)\oplus N$.

Proof. Since I/J is a module with G-action, $\operatorname{Ann}(I/J)$, $\operatorname{Ann}_B(I/J)$, and $\operatorname{Ann}_S(I/J)$ are G-invariant. Moreover $[N,I]\subseteq J$ since N is a nilpotent ideal and I/J is a composition factor of the adjoint representation. Hence $N\subseteq\operatorname{Ann}(I/J)$. In order to prove the lemma, it is sufficient to show that if $b+s\in\operatorname{Ann}(I/J)$, $b\in B$, $s\in S$, then $b,s\in\operatorname{Ann}(I/J)$. Denote $\varphi: L\to\mathfrak{gl}(I/J)$. Then $\varphi(b)+\varphi(s)=0$ and

$$\left[\varphi(b),\varphi(B)\right]=\left[-\varphi(s),\varphi(B)\right]=0.$$

Hence $\varphi(b)$ belongs to the center of $\varphi(B)$ and $\varphi(b)=\varphi(s)=0$ since $\varphi(B)$ is semisimple. Thus $b,s\in\operatorname{Ann}(I/J)$ and the lemma is proved. □

Lemma 6. *Let $L_0=B_0\oplus R_0$ be a finite dimensional reductive Lie algebra with G-action, B_0 be a maximal semisimple G-subalgebra, and R_0 be the center of L_0. Let M be a finite dimensional irreducible L_0-module with G-action. Then*

(1) *$M=M_1\oplus\cdots\oplus M_q$ for some L_0-submodules M_i, $1\leqslant i\leqslant q$;*
(2) *elements of R_0 act on each M_i by scalar operators*;
(3) *for every $1\leqslant i\leqslant q$ and $g\in G$ there exists such $1\leqslant j\leqslant q$ that $M_i^g=M_j$ and this action of G on the set $\{M_1,\dots,M_q\}$ is transitive.*

Proof. Denote by φ the homomorphism $L_0\to\mathfrak{gl}(M)$. Then φ is a homomorphism of G-representations. We claim that $\varphi(R_0)$ consist of semisimple operators. Let $r_1,\dots,r_t$ be a basis in R_0. Consider the Jordan decomposition $\varphi(r_i)=r_i'+r_i''$ where each r_i' is semisimple, each r_i'' is nilpotent, and both are polynomials of $\varphi(r_i)$ without a constant term [15, Section 4.2]. Since each $\varphi(r_i)$ commutes with all operators $\varphi(a)$, $a\in L_0$, the elements $(r_i'')^g$, $1\leqslant i\leqslant t$, $g\in G$, generate a nilpotent G-invariant associative ideal K in the enveloping algebra $A\subseteq\operatorname{End}_F(M)$ of the Lie algebra $\varphi(L_0)$. Suppose $KM\neq 0$. Then for some $\varkappa \in \mathbb{N}$ we have $K^{\varkappa+1}M = 0$, but $K^{\varkappa}M \neq 0$. Note that $K^{\varkappa}M$ is a nonzero G-invariant L_0-submodule. Thus $K^{\varkappa}M = M$ and $KM = K^{\varkappa+1}M = 0$. Since $K \subseteq \mathrm{End}_F(M)$, we obtain $K = 0$.

Therefore $\varphi(r_i) = r_i'$ are commuting semisimple operators. They have a common basis of eigenvectors. Hence we can choose subspaces M_i, $1 \leqslant i \leqslant q$, $q \in \mathbb{N}$, such that

$$M = M_1 \oplus \cdots \oplus M_q,$$

and each M_i is the intersection of eigenspaces of $\varphi(r_i)$. Note that $[\varphi(r_i), \varphi(x)] = 0$ for all $x \in L_0$. Thus M_i are L_0-submodules and propositions (1) and (2) are proved.

For every M_i we can define a linear function $\alpha_i : R_0 \to F$ such that $\varphi(r)m = \alpha_i(r)m$ for all $r \in R_0$ and $m \in M_i$. Then $M_i = \bigcap_{r\in R_0} \ker(\varphi(r) - \alpha_i(r)\cdot 1)$ and

$$M_i^g = \bigcap_{r\in R_0} \ker\bigl(\varphi\bigl(r^g\bigr) - \alpha_i(r)\cdot 1\bigr) = \bigcap_{\tilde r\in R_0} \ker\bigl(\varphi(\tilde r) - \alpha_i\bigl(\tilde r^{g^{-1}}\bigr)\cdot 1\bigr)$$

where $\tilde r = r^g$. Therefore, M_i^g must coincide with M_j for some $1 \leqslant j \leqslant q$. The module M is irreducible with respect to L_0- and G-action that implies proposition (3). □

Lemma 7. *Let W be a finite dimensional L-module with G-action. Let $\varphi : L \to \mathfrak{gl}(W)$ be the corresponding homomorphism. Denote by A the associative subalgebra of $\mathrm{End}_F(W)$ generated by the operators from $\varphi(L)$ and G. Then $\varphi([L,R]) \subseteq J(A)$ where $J(A)$ is the Jacobson radical of A.*

Proof. Let $W = W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq W_t = \{0\}$ be a composition chain in W of not necessarily G-invariant L-submodules. Then each W_i/W_{i+1} is an irreducible L-module. Denote the corresponding homomorphism by $\varphi_i : L \to \mathfrak{gl}(W_i/W_{i+1})$. Then by E. Cartan's theorem [16, Proposition 1.4.11], $\varphi_i(L)$ is semisimple or the direct sum of a semisimple ideal and the center of $\mathfrak{gl}(W_i/W_{i+1})$. Thus $\varphi_i([L,L])$ is semisimple and $\varphi_i([L,L]\cap R) = 0$. Since $[L,R] \subseteq [L,L]\cap R$, we have $\varphi_i([L,R]) = 0$ and $[L,R]W_i \subseteq W_{i+1}$. Denote by $\rho : G \to \mathrm{GL}(W)$ the homomorphism corresponding to G-action. The associative G-invariant ideal of A generated by $\varphi([L,R])$ is nilpotent since for any $a_i \in \varphi([L,R])$, $b_{ij} \in \varphi(L)$, $g_{ij} \in G$ we have

$$\begin{aligned}
&a_1\bigl(\rho(g_{10})b_{11}\rho(g_{11})\dots\rho(g_{1,s_1-1})b_{1,s_1}\rho(g_{1,s_1})\bigr)a_2\dots\\
&\quad a_{t-1}\bigl(\rho(g_{t-1,0})b_{t-1,1}\rho(g_{t-1,1})\dots\rho(g_{t-1,s_{t-1}-1})b_{t-1,s_{t-1}}\rho(g_{t-1,s_{t-1}})\bigr)a_t\\
&= a_1\bigl(b_{11}^{g_{10}}\dots b_{1,s_1}^{g'_{1,s_1}}\bigr)a_2^{g_2}\dots a_{t-1}^{g_{t-1}}\bigl(b_{t-1,1}^{g'_{t-1,1}}\dots b_{t-1,s_{t-1}}^{g'_{t-1,s_{t-1}}}\bigr)a_t^{g_t}\rho(g_{t+1}) = 0
\end{aligned}$$

where $g_i, g'_{ij} \in G$ are products of g_{ij} obtained using the property $\rho(g)bw = b^g\rho(g)w$ where $g \in G$, $b \in \varphi(L)$, $w \in W$. Thus $\varphi([L,R]) \subseteq J(A)$. □

3. Multiplicities of irreducible characters in $\chi_n^G(L)$

The aim of the section is to prove

Theorem 3. *Let L be a finite dimensional Lie G-algebra over a field F of characteristic 0 where G is a finite group. Then there exist constants $C > 0$, $r \in \mathbb{N}$ such that*

$$\sum_{\lambda\vdash n} m(L,G,\lambda) \leqslant Cn^r$$

for all $n \in \mathbb{N}$.

Remark. Cocharacters do not change upon an extension of the base field F (the proof is completely analogous to [1, Theorem 4.1.9]), so we may assume F to be algebraically closed.

In [19, Theorem 13(b)] A. Berele, using the duality between S_n- and $\mathrm{GL}_m(F)$-cocharacters [20,21], showed that such sequence for an associative algebra with an action of a Hopf algebra is polynomially bounded. One may repeat those steps for Lie G-algebras and prove Theorem 3 in that way. However we provide an alternative proof based only on S_n-characters.

Let $\{e\}$ be the trivial group, $V_n := V_n^{\{e\}}$, $\chi_n(L) := \chi_n^{\{e\}}(L)$, $m(L,\lambda) := m(L,\{e\},\lambda)$, $\mathrm{Id}(L) := \mathrm{Id}^{\{e\}}(L)$. Then, by [22, Theorem 3.1],

$$\sum_{\lambda\vdash n} m(L,\lambda) \leqslant C_3 n^{r_3} \tag{1}$$

for some $C_3 > 0$ and $r_3 \in \mathbb{N}$.

Let $G_1 \subseteq G_2$ be finite groups and W_1, W_2 be FG_1- and FG_2-modules respectively. Then we denote FG_2-module $FG_2 \otimes_{FG_1} W_1$ by $W_1 \uparrow G_2$. Here G_2 acts on the first component. Let $W_2 \downarrow G_1$ be W_2 with G_2-action restricted to G_1. We use analogous notation for the characters.

Denote by $\mathrm{length}(M)$ the number of irreducible components of a module M.

Consider the diagonal embedding $\varphi : S_n \to S_{n|G|}$,

$$\varphi(\sigma) := \left(\begin{array}{cccc|cccc|c} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n & \dots \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) & n+\sigma(1) & n+\sigma(2) & \dots & n+\sigma(n) & \dots \end{array}\right).$$

Then we have

Lemma 8.

$$\sum_{\lambda\vdash n} m(L,G,\lambda) = \mathrm{length}\left(\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}\right) \leqslant \mathrm{length}\left(\left(\frac{V_{n|G|}}{V_{n|G|}\cap \mathrm{Id}(L)}\right)\downarrow \varphi(S_n)\right).$$

Proof. Consider S_n-isomorphism $\pi : (V_{n|G|} \downarrow \varphi(S_n)) \to V_n^G$ defined by $\pi(x_{n(i-1)+t}) = x_t^{g_i}$ where $G = \{g_1, g_2, \dots, g_{|G|}\}$, $1 \leqslant t \leqslant n$. Note that $\pi(V_{n|G|} \cap \mathrm{Id}(L)) \subseteq V_n^G \cap \mathrm{Id}^G(L)$. Thus FS_n-module $\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)}$ is a homomorphic image of FS_n-module $(\frac{V_{n|G|}}{V_{n|G|}\cap\mathrm{Id}(L)}) \downarrow \varphi(S_n)$. $\square$

Hence it is sufficient to prove that $\mathrm{length}((\frac{V_{n|G|}}{V_{n|G|}\cap\mathrm{Id}(L)}) \downarrow \varphi(S_n))$ is polynomially bounded. However, we start with the study of the restriction on the larger subgroup

$$S\{1,\dots,n\} \times S\{n+1,\dots,2n\} \times \dots \times S\big\{n\big(|G|-1\big),\dots,n|G|\big\} \subseteq S_{n|G|}$$

that we denote by $(S_n)^{|G|}$.

This is a particular case of a more general situation. Let $m = m_1 + \dots + m_t$, $m_i \in \mathbb{N}$. Then we have a natural embedding $S_{m_1} \times \dots \times S_{m_t} \hookrightarrow S_m$. Irreducible representations of $S_{m_1} \times \dots \times S_{m_t}$ are isomorphic to $M(\lambda^{(1)})\sharp \cdots \sharp M(\lambda^{(t)})$ where $\lambda^{(i)} \vdash m_i$. Here

$$M\big(\lambda^{(1)}\big)\sharp \cdots \sharp M\big(\lambda^{(t)}\big) \cong M\big(\lambda^{(1)}\big) \otimes \dots \otimes M\big(\lambda^{(t)}\big)$$

as a vector space and S_{m_i} acts on $M(\lambda^{(i)})$. Denote by $\chi(\lambda^{(1)})\sharp \cdots \sharp \chi(\lambda^{(t)})$ the character of $M(\lambda^{(1)})\sharp \cdots \sharp M(\lambda^{(t)})$.

Analogously, $\chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t)})$ is the character of FS_m-module

$$M(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} M(\lambda^{(t)}) := \left(M(\lambda^{(1)}) \sharp \cdots \sharp M(\lambda^{(t)})\right) \uparrow S_m.$$

Note that if $m_1 = \cdots = m_t = k$, one can define the *inner tensor product*, i.e.

$$M(\lambda^{(1)}) \otimes \cdots \otimes M(\lambda^{(t)})$$

with the diagonal S_k-action. The character of this FS_k-module equals $\chi(\lambda^{(1)}) \ldots \chi(\lambda^{(t)})$.

Recall that irreducible characters of any finite group G_0 are orthonormal with respect to the scalar product $(\chi, \psi) = \frac{1}{|G_0|} \sum_{g \in G_0} \chi(g^{-1})\psi(g)$.

Denote by λ^T the transpose partition of $\lambda \vdash n$. Then λ_1^T equals the height of the first column of D_λ.

Lemma 9. *Let $h, t \in \mathbb{N}$. There exist $C_4 > 0$, $r_4 \in \mathbb{N}$ such that for all $\lambda \vdash m$, $\lambda^{(1)} \vdash m_1, \ldots, \lambda^{(t)} \vdash m_t$, where D_λ lie in the strip of height h, i.e. $\lambda_1^T \leqslant h$, and $m_1 + m_2 + \cdots + m_t = m$, we have*

$$\left(\chi(\lambda) \downarrow (S_{m_1} \times \cdots \times S_{m_t}),\ \chi(\lambda^{(1)}) \sharp \cdots \sharp \chi(\lambda^{(t)})\right) = \left(\chi(\lambda),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t)})\right) \leqslant C_4 m^{r_4}.$$

If $\lambda \vdash m$, $\lambda^{(1)} \vdash m_1, \ldots, \lambda^{(t)} \vdash m_t$, $m_1 + m_2 + \cdots + m_t = m$, and

$$\left(\chi(\lambda) \downarrow (S_{m_1} \times \cdots \times S_{m_t}),\ \chi(\lambda^{(1)}) \sharp \cdots \sharp \chi(\lambda^{(t)})\right) = \left(\chi(\lambda),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t)})\right) \neq 0$$

then $(\lambda^{(i)})_1^T \leqslant \lambda_1^T$ for all $1 \leqslant i \leqslant t$ and $\lambda_1^T \leqslant \sum_{i=1}^t (\lambda^{(i)})_1^T$.

Proof. By Frobenius reciprocity,

$$\begin{aligned}\left(\chi(\lambda) \downarrow (S_{m_1} \times \cdots \times S_{m_t}),\ \chi(\lambda^{(1)}) \sharp \cdots \sharp \chi(\lambda^{(t)})\right) &= \left(\chi(\lambda),\ \left(\chi(\lambda^{(1)}) \sharp \cdots \sharp \chi(\lambda^{(t)})\right) \uparrow S_m\right) \\ &= \left(\chi(\lambda),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t)})\right).\end{aligned}$$

Now we prove the lemma by induction on t. The case $t = 1$ is trivial. Suppose $(\chi(\mu),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t-1)}))$ is polynomially bounded for every $\mu \vdash (m_1 + \cdots + m_{t-1})$ with $\mu_1^T \leqslant h$. We have

$$\begin{aligned}&\left(\chi(\lambda),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t)})\right) \\ &\quad = \left(\chi(\lambda),\ \left(\chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t-1)})\right) \widehat{\otimes} \chi(\lambda^{(t)})\right) \\ &\quad = \sum_{\mu \vdash (m_1 + \cdots + m_{t-1})} \left(\chi(\mu),\ \chi(\lambda^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} \chi(\lambda^{(t-1)})\right)\left(\chi(\lambda), \chi(\mu) \widehat{\otimes} \chi(\lambda^{(t)})\right). \qquad (2)\end{aligned}$$

In order to determine the multiplicity of $\chi(\lambda)$ in $\chi(\mu) \widehat{\otimes} \chi(\lambda^{(t)})$, we are using the Littlewood–Richardson rule (see the algorithm in [23, Corollary 2.8.14]). We cannot obtain D_λ if $(\lambda^{(t)})_1^T > \lambda_1^T$ or $\mu_1^T > \lambda_1^T$, or $\lambda_1^T > (\lambda^{(t)})_1^T + \mu_1^T$. Suppose the Young diagram D_λ lies in the strip of height h. Then we may consider only the case $(\lambda^{(t)})_1^T \leqslant h$ and $\mu_1^T \leqslant h$. Each time the number of variants to add the boxes from a row is bounded by m^h. Since $(\lambda^{(t)})_1^T \leqslant h$, the second multiplier in Eq. (2) is bounded by $(m^h)^h = m^{h^2}$. The number of diagrams in the strip of height h is bounded by m^h. Thus the number of terms in Eq. (2) is bounded by m^h. Together with the inductive assumption this yields the lemma. □

Lemma 10. *There exist $C_5 > 0$, $r_5 \in \mathbb{N}$ such that*

$$\operatorname{length}\left(\left(\frac{V_{n|G|}}{V_{n|G|} \cap \operatorname{Id}(L)}\right) \downarrow (S_n)^{|G|}\right) \leqslant C_5 n^{r_5}$$

for all $n \in \mathbb{N}$. Moreover, if $(\lambda^{(i)})_1^T > \dim L$ for some $1 \leqslant i \leqslant |G|$, then $M(\lambda^{(1)})\sharp \cdots \sharp M(\lambda^{(|G|)})$ does not appear in the decomposition.

Proof. Fix a $|G|$-tuple of partitions $(\lambda^{(1)}, \ldots, \lambda^{(|G|)})$, $\lambda^{(i)} \vdash n$. Then the multiplicity of $M(\lambda^{(1)})\sharp \cdots \sharp M(\lambda^{(|G|)})$ in $(\frac{V_{n|G|}}{V_{n|G|} \cap \operatorname{Id}(L)}) \downarrow (S_n)^{|G|}$ equals

$$\begin{aligned}
&\left(\chi\left(\lambda^{(1)}\right)\sharp \cdots \sharp \chi\left(\lambda^{(|G|)}\right),\ \chi_{n|G|}(L) \downarrow (S_n)^{|G|}\right) \\
&\quad = \sum_{\lambda \vdash n|G|} \left(\chi\left(\lambda^{(1)}\right)\sharp \cdots \sharp \chi\left(\lambda^{(|G|)}\right),\ \chi(\lambda) \downarrow (S_n)^{|G|}\right) m(L, \lambda).
\end{aligned} \tag{3}$$

By [22, Lemma 3.4] (or Lemma 14 for $G = \langle e \rangle$), $m(L, \lambda) = 0$ for all $\lambda \vdash n|G|$ with $\lambda_1^T > \dim L$. Thus Lemma 9 implies that for all $M(\lambda^{(1)})\sharp \cdots \sharp M(\lambda^{(|G|)})$ that appear in $(\frac{V_{n|G|}}{V_{n|G|} \cap \operatorname{Id}(L)}) \downarrow (S_n)^{|G|}$, the Young diagrams $D_{\lambda^{(i)}}$ lie in the strip of height $(\dim L)$. Thus the number of different $(\lambda^{(1)}, \ldots, \lambda^{(|G|)})$ that appear in the decomposition of $(\frac{V_{n|G|}}{V_{n|G|} \cap \operatorname{Id}(L)}) \downarrow (S_n)^{|G|}$ is bounded by $n^{(\dim L)|G|}$. Together with Eqs. (1), (3), and Lemma 9, this yields the lemma. □

Lemma 11. *Let $h, k \in \mathbb{N}$. There exist $C_6 > 0$, $r_6 \in \mathbb{N}$ such that for the inner tensor product $M(\lambda) \otimes M(\mu)$ of any FS_n-modules $M(\lambda)$ and $M(\mu)$, $\lambda, \mu \vdash n$, $\lambda_1^T \leqslant h$, $\mu_1^T \leqslant k$, we have*

$$\operatorname{length}_{S_n}\left(M(\lambda) \otimes M(\mu)\right) \leqslant C_6 n^{r_6}$$

and $(\chi(\lambda)\chi(\mu), \chi(\nu)) = 0$ for any $\nu \vdash n$ with $\nu_1^T > hk$.

Proof. Let T_μ be any Young tableau of the shape μ. Denote by IR_{T_μ} the one-dimensional trivial representation of the Young subgroup (i.e. the row stabilizer) R_{T_μ}. Then

$$FS_n a_{T_\mu} \cong IR_{T_\mu} \uparrow S_n$$

(see [24, Section 4.3]). By [25, Theorem 38.5],

$$M(\lambda) \otimes (IR_{T_\mu} \uparrow S_n) \cong \left(\left(M(\lambda) \downarrow R_{T_\mu}\right) \otimes IR_{T_\mu}\right) \uparrow S_n.$$

Thus

$$\begin{aligned}
M(\lambda) \otimes M(\mu) &\cong M(\lambda) \otimes FS_n e^*_{T_\mu} = M(\lambda) \otimes FS_n b_{T_\mu} a_{T_\mu} \subseteq M(\lambda) \otimes FS_n a_{T_\mu} \\
&\cong M(\lambda) \otimes (IR_{T_\mu} \uparrow S_n) \cong \left(\left(M(\lambda) \downarrow R_{T_\mu}\right) \otimes IR_{T_\mu}\right) \uparrow S_n \cong \left(M(\lambda) \downarrow R_{T_\mu}\right) \uparrow S_n.
\end{aligned}$$

Note that $\operatorname{length}(M(\lambda) \downarrow R_{T_\mu})$ is polynomially bounded by Lemma 9 and $M(\lambda) \downarrow R_{T_\mu}$ is a sum of $M(\varkappa^{(1)})\sharp \cdots \sharp M(\varkappa^{(s)})$, $s = \mu_1^T \leqslant k$, $\varkappa^{(i)} \vdash \mu_i$, $(\varkappa^{(i)})_1^T \leqslant h$. Thus $(M(\lambda) \downarrow R_{T_\mu}) \uparrow S_n$ is a sum of $M(\varkappa^{(1)}) \widehat{\otimes} \cdots \widehat{\otimes} M(\varkappa^{(s)})$. Applying Lemma 9 again, we obtain the lemma. □

Lemma 12. *There exist $C_7 > 0$, $r_7 \in \mathbb{N}$ satisfying the following properties. If $(\lambda^{(1)}, \dots, \lambda^{(|G|)})$ is a $|G|$-tuple of partitions $\lambda^{(i)} \vdash n$ where $(\lambda^{(i)})_1^T \leqslant \dim L$ for all $1 \leqslant i \leqslant |G|$, then*

$$\operatorname{length}_{S_n}\left(M\left(\lambda^{(1)}\right) \otimes \cdots \otimes M\left(\lambda^{(|G|)}\right)\right) \leqslant C_7 n^{r_7}.$$

Proof. Note that

$$M\left(\lambda^{(1)}\right) \otimes \cdots \otimes M\left(\lambda^{(t)}\right) = \left(M\left(\lambda^{(1)}\right) \otimes \cdots \otimes M\left(\lambda^{(t-1)}\right)\right) \otimes M\left(\lambda^{(t)}\right).$$

Using induction on t and applying Lemma 11 with $h = (\dim L)^{t-1}$ and $k = \dim L$, we obtain the lemma. $\square$

Proof of Theorem 3. The theorem is an immediate consequence of Lemmas 8, 10, and 12. $\square$

4. Upper bound

Fix a composition chain of G-invariant ideals

$$L = L_0 \supsetneqq L_1 \supsetneqq L_2 \supsetneqq \cdots \supsetneqq N \supsetneqq \cdots \supsetneqq L_{\theta-1} \supsetneqq L_\theta = \{0\}.$$

Let $\operatorname{ht} a := \max_{a \in L_k} k$ for $a \in L$.

Remark. If $d = d(L) = 0$, then $L = \operatorname{Ann}(L_{i-1}/L_i)$ for all $1 \leqslant i \leqslant \theta$ and $[a_1, a_2, \dots, a_n] = 0$ for all $a_i \in L$ and $n \geqslant \theta + 1$. Thus $c_n^G(L) = 0$ for all $n \geqslant \theta + 1$. Therefore we assume $d > 0$.

Let $Y := \{y_{11}, y_{12}, \dots, y_{1j_1}; y_{21}, y_{22}, \dots, y_{2j_2}; \dots; y_{m1}, y_{m2}, \dots, y_{mj_m}\}$, $Y_1, \dots, Y_q$, and $\{z_1, \dots, z_m\}$ be subsets of $\{x_1, x_2, \dots, x_n\}$ such that $Y_i \subseteq Y$, $|Y_i| = d + 1$, $Y_i \cap Y_j = \varnothing$ for $i \neq j$, $Y \cap \{z_1, \dots, z_m\} = \varnothing$, $j_i \geqslant 0$. Denote

$$f_{m,q} := \operatorname{Alt}_1 \dots \operatorname{Alt}_q\left[\left[z_1^{g_1}, y_{11}^{g_{11}}, y_{12}^{g_{12}}, \dots, y_{1j_1}^{g_{1j_1}}\right], \left[z_2^{g_2}, y_{21}^{g_{21}}, y_{22}^{g_{22}}, \dots, y_{2j_2}^{g_{2j_2}}\right], \dots, \left[z_m^{g_m}, y_{m1}^{g_{m1}}, y_{m2}^{g_{m2}}, \dots, y_{mj_m}^{g_{mj_m}}\right]\right]$$

where Alt_i is the operator of alternation on the variables of Y_i, $g_i, g_{ij} \in G$.

Let $\varphi : L(X|G) \to L$ be a G-homomorphism induced by some substitution $\{x_1, x_2, \dots, x_n\} \to L$. We say that φ is *proper* for $f_{m,q}$ if $\varphi(z_1) \in N \cup B \cup S$, $\varphi(z_i) \in N$ for $2 \leqslant i \leqslant m$, and $\varphi(y_{ik}) \in B \cup S$ for $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant j_i$.

Lemma 13. *Let φ be a proper homomorphism for $f_{m,q}$. Then $\varphi(f_{m,q})$ can be rewritten as a sum of $\psi(f_{m+1,q'})$ where ψ is a proper homomorphism for $f_{m+1,q'}$, $q' \geqslant q - (\dim L)m - 2$. ($Y'$, Y'_i, $z'_1, \dots, z'_{m+1}$ may be different for different terms.)*

Proof. Let $\alpha_i := \operatorname{ht} \varphi(z_i)$. We will use induction on $\sum_{i=1}^m \alpha_i$. (The sum will grow.) Note that $\alpha_i \leqslant \theta \leqslant \dim L$. Denote $I_i := L_{\alpha_i}$, $J_i := L_{\alpha_i + 1}$.

First, consider the case when $I_1, \dots, I_m$, $J_1, \dots, J_m$ do not satisfy conditions (1)–(2). In this case we can choose G-invariant B-submodules T_i, $I_i = T_i \oplus J_i$, such that

$$\left[[T_1, \underbrace{L, \dots, L}_{q_1}], [T_2, \underbrace{L, \dots, L}_{q_2}], \dots, [T_m, \underbrace{L, \dots, L}_{q_m}]\right] = 0 \tag{4}$$

for all $q_i \geqslant 0$. Rewrite $\varphi(z_i) = a'_i + a''_i$, $a'_i \in T_i$, $a''_i \in J_i$. Note that $\operatorname{ht} a''_i > \operatorname{ht} \varphi(z_i)$. Since $f_{m,q}$ is multilinear, we can rewrite $\varphi(f_{m,q})$ as a sum of similar terms $\tilde{\varphi}(f_{m,q})$ where $\tilde{\varphi}(z_i)$ equals either a'_i or a''_i. By Eq. (4), the term where all $\tilde{\varphi}(z_i) = a'_i \in T_i$, equals 0. For the other terms $\tilde{\varphi}(f_{m,q})$ we have $\sum_{i=1}^m \operatorname{ht} \tilde{\varphi}(z_i) > \sum_{i=1}^m \operatorname{ht} \varphi(z_i)$.

Thus without lost of generality we may assume that $I_1, \dots, I_m$, $J_1, \dots, J_m$ satisfy conditions (1)–(2). In this case, $\dim(\operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)) \geqslant \dim(L) - d$. In virtue of Lemma 5,

$$\operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m) = B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m) \\ \oplus S \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m) \oplus N.$$

Choose a basis in B that includes a basis of $B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$ and a basis in S that includes the basis of $S \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$. Since $f_{m,q}$ is multilinear, we may assume that only basis elements are substituted for $y_{k\ell}$. Note that $f_{m,q}$ is alternating in Y_i. Hence, if $\varphi(f_{m,q}) \neq 0$, then for every $1 \leqslant i \leqslant q$ there exists $y_{jk} \in Y_i$ such that either

$$\varphi(y_{jk}) \in B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$$

or

$$\varphi(y_{jk}) \in S \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m).$$

Consider the case when $\varphi(y_{kj}) \in B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$ for some y_{kj}. By the corollary from Lemma 3, we can choose G-invariant B-submodules T_k such that $I_k = T_k \oplus J_k$. We may assume that $\varphi(z_k) \in T_k$ since elements of J_k have greater heights. Therefore $[\varphi(z_k^{g_k}), a] \in T_k \cap J_k$ for all $a \in B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$. Hence $[\varphi(z_k^{g_k}), a] = 0$. Moreover, $B \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$ is a G-invariant ideal of B and $[B, S] = 0$. Thus, applying Jacobi's identity several times, we obtain

$$\varphi\left(\left[z_k^{g_k}, y_{k1}^{g_{k1}}, \dots, y_{kj_k}^{g_{kj_k}}\right]\right) = 0.$$

Expanding the alternations, we get $\varphi(f_{m,q}) = 0$.

Consider the case when $\varphi(y_{k\ell}) \in S \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$ for some $y_{k\ell} \in Y_q$. Expand the alternation Alt_q in $f_{m,q}$ and rewrite $f_{m,q}$ as a sum of

$$\tilde{f}_{m,q-1} := \operatorname{Alt}_1 \dots \operatorname{Alt}_{q-1}\left[\left[z_1^{g_1}, y_{11}^{g_{11}}, y_{12}^{g_{12}}, \dots, y_{1j_1}^{g_{1j_1}}\right], \left[z_2^{g_2}, y_{21}^{g_{21}}, y_{22}^{g_{22}}, \dots, y_{2j_2}^{g_{2j_2}}\right], \dots, \\ \left[z_m^{g_m}, y_{m1}^{g_{m1}}, y_{m2}^{g_{m2}}, \dots, y_{mj_m}^{g_{mj_m}}\right]\right].$$

The operator Alt_q may change indices, however we keep the notation $y_{k\ell}$ for the variable with the property $\varphi(y_{k\ell}) \in S \cap \operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_m/J_m)$. Now the alternation does not affect $y_{k\ell}$. Note that

$$\left[z_k^{g_k}, y_{k1}^{g_{k1}}, \dots, y_{k\ell}^{g_{k\ell}}, \dots, y_{kj_k}^{g_{kj_k}}\right] = \left[z_k^{g_k}, y_{k\ell}^{g_{k\ell}}, y_{k1}^{g_{k1}}, \dots, y_{kj_k}^{g_{kj_k}}\right] \\ + \sum_{\beta=1}^{\ell-1}\left[z_k^{g_k}, y_{k1}^{g_{k1}}, \dots, y_{k,\beta-1}^{g_{k,\beta-1}}, \left[y_{k\beta}^{g_{k\beta}}, y_{k\ell}^{g_{k\ell}}\right], y_{k,\beta+1}^{g_{k,\beta+1}}, \dots, y_{k,\ell-1}^{g_{k,\ell-1}}, y_{k,\ell+1}^{g_{k,\ell+1}}, \dots, y_{kj_k}^{g_{kj_k}}\right].$$

In the first term we replace $[z_k^{g_k}, y_{k\ell}^{g_{k\ell}}]$ with z'_k and define $\varphi'(z'_k) := \varphi([z_k^{g_k}, y_{k\ell}^{g_{k\ell}}])$, $\varphi'(x) := \varphi(x)$ for other variables x. Then $\operatorname{ht} \varphi'(z'_k) > \operatorname{ht} \varphi(z_k)$ and we can use the inductive assumption. If $y_{k\beta} \in Y_j$ for some j, then we expand the alternation Alt_j in this term in $\tilde{f}_{m,q-1}$. If $\varphi(y_{k\beta}) \in B$, then the term is zero. If $\varphi(y_{k\beta}) \in S$, then $\varphi([y_{k\beta}^{g_{k\beta}}, y_{k\ell}^{g_{k\ell}}]) \in N$. We replace $[y_{k\beta}^{g_{k\beta}}, y_{k\ell}^{g_{k\ell}}]$ with an additional variable z'_{m+1} and define $\psi(z'_{m+1}) := \varphi([y_{k\beta}^{g_{k\beta}}, y_{k\ell}^{g_{k\ell}}])$, $\psi(x) := \varphi(x)$ for other variables x. Applying Jacobi's identity several times, we obtain the polynomial of the desired form. In each inductive step we reduce q no more than by 1 and the maximal number of inductive steps equals $(\dim L)m$. This finishes the proof. $\square$

Since N is a nilpotent ideal, $N^p = 0$ for some $p \in \mathbb{N}$.

Lemma 14. *If $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and $\lambda_{d+1} \geqslant p((\dim L)p + 3)$ or $\lambda_{\dim L+1} > 0$, then $m(L, G, \lambda) = 0$.*

Proof. It is sufficient to prove that $e^*_{T_\lambda} f \in \mathrm{Id}^G(L)$ for every $f \in V_n^G$ and a Young tableau T_λ, $\lambda \vdash n$, with $\lambda_{d+1} \geqslant p((\dim L)p + 3)$ or $\lambda_{\dim L+1} > 0$.

Fix some basis of L that is a union of bases of B, S, and N. Since polynomials are multilinear, it is sufficient to substitute only basis elements. Note that $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ and b_{T_λ} alternates the variables of each column of T_λ. Hence if we make a substitution and $e^*_{T_\lambda} f$ does not vanish, then this implies that different basis elements are substituted for the variables of each column. But if $\lambda_{\dim L+1} > 0$, then the length of the first column is greater than $\dim L$. Therefore, $e^*_{T_\lambda} f \in \mathrm{Id}^G(L)$.

Consider the case $\lambda_{d+1} \geqslant p((\dim L)p + 3)$. Let φ be a substitution of basis elements for the variables $x_1, \dots, x_n$. Then $e^*_{T_\lambda} f$ can be rewritten as a sum of polynomials $f_{m,q}$ where $1 \leqslant m \leqslant p$, $q \geqslant p((\dim L)p + 2)$, and z_i, $2 \leqslant i \leqslant m$, are replaced with elements of N. (For different terms $f_{m,q}$, numbers m and q, variables z_i, y_{ij}, and sets Y_i can be different.) Indeed, we expand symmetrization on all variables and alternation on the variables replaced with elements from N. If we have no variables replaced with elements from N, then we take $m = 1$, rewrite the polynomial f as a sum of long commutators, in each long commutator expand the alternation on the set that includes one of the variables in the inner commutator, and denote that variable by z_1. Suppose we have variables replaced with elements from N. We denote them by z_k. Then, using Jacobi's identity, we can put one of such variables inside a long commutator and group all the variables, replaced with elements from $B \cup S$, around z_k such that each z_k is inside the corresponding long commutator.

Applying Lemma 13 many times, we increase m. The ideal N is nilpotent and $\varphi(f_{p+1,q}) = 0$ for every q and a proper homomorphism φ. Reducing q no more than by $p((\dim L)p + 2)$, we obtain $\varphi(e^*_{T_\lambda} f) = 0$. $\square$

Now we can prove

Theorem 4. *If $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^G(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. In the case $d = 0$, the algebra L is nilpotent.*

Proof. Lemma 14 and [1, Lemmas 6.2.4, 6.2.5] imply

$$\sum_{m(L,G,\lambda)\neq 0} \dim M(\lambda) \leqslant C_8 n^{r_8} d^n$$

for some constants $C_8, r_8 > 0$. Together with Theorem 3 this implies the upper bound. $\square$

5. Alternating polynomials

In this section we prove auxiliary propositions needed to obtain the lower bound.

Lemma 15. *Let $\alpha_1, \alpha_2, \dots, \alpha_q, \beta_1, \dots, \beta_q \in F$, $1 \leqslant k \leqslant q$, $\alpha_i \neq 0$ for $1 \leqslant i < k$, $\alpha_k = 0$, and $\beta_k \neq 0$. Then there exists such $\gamma \in F$ that $\alpha_i + \gamma \beta_i \neq 0$ for all $1 \leqslant i \leqslant k$.*

Proof. It is sufficient to choose $\gamma \notin \{-\frac{\alpha_1}{\beta_1}, \dots, -\frac{\alpha_{k-1}}{\beta_{k-1}}, 0\}$. It is possible to do since F is infinite. $\square$

Let $F\langle X|G\rangle$ be the free associative algebra over F with free formal generators x_j^g, $j \in \mathbb{N}$, $g \in G$. Define $(x_j^g)^h = x_j^{hg}$ for $h \in G$. Then $F\langle X|G\rangle$ becomes the *free associative G-algebra* with free generators $x_j = x_j^1$, $j \in \mathbb{N}$, $1 \in G$. Denote by P_n^G, $n \in \mathbb{N}$, the subspace of associative multilinear G-polynomials in variables $x_1, \dots, x_n$. In other words,

$$P_n^G = \left\{ \sum_{\sigma \in S_n,\, g_1,\dots,g_n \in G} \alpha_{\sigma, g_1,\dots,g_n} x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \dots x_{\sigma(n)}^{g_n} \;\middle|\; \alpha_{\sigma, g_1,\dots,g_n} \in F \right\}.$$

Lemma 16. *Let $L_0 = B_0 \oplus R_0$ be a reductive Lie algebra with G-action, B_0 be a maximal semisimple G-subalgebra, and R_0 be the center of L_0 with a basis $r_1, r_2, \dots, r_t$. Let M be a faithful finite dimensional irreducible L_0-module with G-action. Denote the corresponding representation $L_0 \to \mathfrak{gl}(M)$ by φ. Then there exists such alternating in $x_1, x_2, \dots, x_t$ polynomial $f \in P_t^G$ that $f(\varphi(r_1), \dots, \varphi(r_t))$ is a nondegenerate operator on M.*

Proof. By Lemma 6, $M = M_1 \oplus \dots \oplus M_q$ where M_j are L_0-submodules and r_i acts on each M_j as a scalar operator. Note that it is sufficient to prove that for each j there exists such alternating in $x_1, x_2, \dots, x_t$ polynomial $f_j \in P_t^G$ that $f_j(\varphi(r_1), \dots, \varphi(r_t))$ multiplies each element of M_j by a nonzero scalar. Indeed, in this case Lemma 15 implies the existence of such $f = \gamma_1 f_1 + \dots + \gamma_q f_q$, $\gamma_i \in F$, that $f(\varphi(r_1), \dots, \varphi(r_t))$ acts on each M_i as a nonzero scalar.

Denote by $p_i \in \mathrm{End}_F(M)$ the projection on M_i along $\bigoplus_{k \ne i} M_k$. Fix $1 \leqslant j \leqslant q$. By Lemma 6, proposition (3), we can choose such $g_i \in G$ that $M_i^{g_i} = M_j$, $1 \leqslant i \leqslant q$. Then $p_i^{g_i} = p_j$. Consider $\tilde{f}_j := \sum_{\sigma \in S_q} (\operatorname{sign} \sigma) x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \dots x_{\sigma(q)}^{g_q}$. Note that either $p_{\sigma(1)}^{g_1} p_{\sigma(2)}^{g_2} \dots p_{\sigma(q)}^{g_q} = 0$ or $p_{\sigma(1)}^{g_1} p_{\sigma(2)}^{g_2} \dots p_{\sigma(q)}^{g_q} = p_k$ for some $1 \leqslant k \leqslant s$. Now we prove that $p_{\sigma(1)}^{g_1} p_{\sigma(2)}^{g_2} \dots p_{\sigma(q)}^{g_q} = p_j$ if and only if $\sigma(i) = i$ for all $1 \leqslant i \leqslant q$. Indeed, $p_{\sigma(i)}^{g_i} = p_j$ if and only if $M_{\sigma(i)}^{g_i} = M_j$. Hence $\sigma(i) = i$. This implies that $\tilde{f}_j(p_1, \dots, p_q)$ acts as an identical map on M_j.

We can choose $i_{t+1}, \dots, i_q$ such that $\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t)$, $p_{i_{t+1}}, \dots, p_{i_q}$ form a basis in $\langle p_1, \dots, p_q \rangle_F$. Then $\tilde{f}_j(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t), p_{i_{t+1}}, \dots, p_{i_q})$ acts as a nonzero scalar on M_j. If $t = q$, then we define $f_j = \tilde{f}_j$. Suppose $t < q$. Since the projections commute, we can rewrite

$$\tilde{f}_j\big(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t), p_{i_{t+1}}, \dots, p_{i_q}\big) = \sum_{i=1}^{q} \hat{f}_i\big(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t)\big) p_i$$

where $\hat{f}_i \in P_t^G$ are alternating in $x_1, x_2, \dots, x_t$. Hence $\hat{f}_j(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t))$ acts on M_j as a nonzero scalar operator. We define $f_j := \hat{f}_j$. $\square$

Let L_0 be a Lie algebra with G-action, M be L_0-module with G-action, $\varphi : L_0 \to \mathfrak{gl}(M)$ be the corresponding representation. A polynomial $f(x_1, \dots, x_n) \in F\langle X|G\rangle$ is a *G-identity* of φ if $f(\varphi(a_1), \dots, \varphi(a_n)) = 0$ for all $a_i \in L_0$. The set $\mathrm{Id}^G(\varphi)$ of all G-identities of φ is a two-sided ideal in $F\langle X|G\rangle$ invariant under G-action.

Lemma 17 is an analog of [3, Lemma 1].

Lemma 17. *Let L_0 be a Lie algebra with G-action, M be a faithful finite dimensional irreducible L_0-module with G-action, and $\varphi : L_0 \to \mathfrak{gl}(M)$ be the corresponding representation. Then for some $n \in \mathbb{N}$ there exists a polynomial $f \in P_n^G \setminus \mathrm{Id}^G(\varphi)$ alternating in $\{x_1, \dots, x_\ell\}$ and in $\{y_1, \dots, y_\ell\} \subseteq \{x_{\ell+1}, \dots, x_n\}$ where $\ell = \dim L_0$.*

Proof. Since M is irreducible, by the density theorem, $\operatorname{End}_F(M) \cong M_q(F)$ is generated by operators from G and $\varphi(L_0)$. Here $q := \dim M$. Consider Regev's polynomial

$$\hat{f}(x_1, \dots, x_q; y_1, \dots, y_q) := \sum_{\substack{\sigma \in S_q, \\ \tau \in S_q}} \big(\operatorname{sign}(\sigma\tau)\big) x_{\sigma(1)}\, y_{\tau(1)}\, x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\, y_{\tau(2)} y_{\tau(3)} y_{\tau(4)} \cdots$$
$$x_{\sigma(q^2-2q+2)} \cdots x_{\sigma(q^2)}\, y_{\tau(q^2-2q+2)} \cdots y_{\tau(q^2)}.$$

This is a central polynomial [1, Theorem 5.7.4] for $M_k(F)$, i.e. $\hat{f}$ is not a polynomial identity for $M_q(F)$ and its values belong to the center of $M_q(F)$.

Let $a_1, \dots, a_\ell$ be a basis of L_0. Denote by ρ the representation $G \to \mathrm{GL}(M)$. Note that if we have the product of elements of $\varphi(L_0)$ and $\rho(G)$, we can always move the elements from $\rho(G)$ to the right, using $\rho(g)a = a^g \rho(g)$ for $g \in G$ and $a \in \varphi(L_0)$. Then $\varphi(a_1), \dots, \varphi(a_\ell), (\varphi(a_{i_{11}}) \dots \varphi(a_{i_{1,m_1}}))\rho(g_1), \dots, (\varphi(a_{i_{r,1}}) \dots \varphi(a_{i_{r,m_r}}))\rho(g_r)$, is a basis of $\operatorname{End}_F(M)$ for appropriate $i_{jk} \in \{1, 2, \dots, \ell\}$, $g_j \in G$, since $\operatorname{End}_F(M)$ is generated by operators from G and $\varphi(L_0)$. We replace $x_{\ell+j}$ with $z_{j1} z_{j2} \dots z_{j,m_j} \rho(g_j)$ and $y_{\ell+j}$ with $z'_{j1} z'_{j2} \dots z'_{j,m_j} \rho(g_j)$ in $\hat{f}$ and denote the expression obtained by $\tilde{f}$. Using $\rho(g)a = a^g \rho(g)$ again, we can move all $\rho(g)$, $g \in G$, in $\tilde{f}_q$ to the right and rewrite $\tilde{f}$ as $\sum_{g \in G} f_g\, \rho(g)$ where each $f_g \in P^G_{2\ell + 2\sum_{j=1}^r m_j}$ is an alternating in $x_1, \dots, x_\ell$ and in $y_1, \dots, y_\ell$ polynomial. Note that $\tilde{f}$ becomes a nonzero scalar operator on M under the substitution $x_i = y_i = \varphi(a_i)$ for $1 \leqslant i \leqslant \ell$ and $z_{jk} = z'_{jk} = \varphi(a_{i_{jk}})$ for $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant m_j$. Thus $f_g \notin \mathrm{Id}^G(\varphi)$ for some $g \in G$ and we can take $f = f_g$. $\square$

Let $k\ell \leqslant n$ where $k, \ell, n \in \mathbb{N}$ are some numbers. Denote by $Q^G_{\ell,k,n} \subseteq P^G_n$ the subspace spanned by all polynomials that are alternating in k disjoint subsets of variables $\{x^i_1, \dots, x^i_\ell\} \subseteq \{x_1, x_2, \dots, x_n\}$, $1 \leqslant i \leqslant k$.

Theorem 5 is an analog of [3, Theorem 1].

Theorem 5. *Let $L_0 = B_0 \oplus R_0$ be a reductive Lie algebra with G-action over an algebraically closed field F of characteristic 0, B_0 be a maximal semisimple G-subalgebra, R_0 be the center of L_0, and $\dim L_0 = \ell$. Let M be a faithful finite dimensional irreducible L_0-module with G-action. Denote the corresponding representation $L_0 \to \mathfrak{gl}(M)$ by φ. Then there exists $T \in \mathbb{Z}_+$ such that for any $k \in \mathbb{N}$ there exists $f \in Q^G_{\ell,2k,2k\ell+T} \setminus \mathrm{Id}^G(\varphi)$.*

Proof. Let $f_1 = f_1(x_1, \dots, x_\ell,\ y_1, \dots, y_\ell, z_1, \dots, z_T)$ be the polynomial from Lemma 17 alternating in $x_1, \dots, x_\ell$ and in $y_1, \dots, y_\ell$. Since $f_1 \in Q^G_{\ell,2,2\ell+T} \setminus \mathrm{Id}^G(\varphi)$, we may assume that $k > 1$. Note that

$$f_1^{(1)}(u_1, v_1, x_1, \dots, x_\ell,\ y_1, \dots, y_\ell, z_1, \dots, z_T)$$
$$:= \sum_{i=1}^{\ell} f_1\big(x_1, \dots, \big[u_1, [v_1, x_i]\big], \dots, x_\ell,\ y_1, \dots, y_\ell, z_1, \dots, z_T\big)$$

is alternating in $x_1, \dots, x_\ell$ and in $y_1, \dots, y_\ell$ and

$$f_1^{(1)}(\bar{u}_1, \bar{v}_1, \bar{x}_1, \dots, \bar{x}_\ell,\ \bar{y}_1, \dots, \bar{y}_\ell, \bar{z}_1, \dots, \bar{z}_T)$$
$$= \operatorname{tr}(\operatorname{ad}_{\varphi(L_0)} \bar{u}_1 \operatorname{ad}_{\varphi(L_0)} \bar{v}_1) f_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_\ell,\ \bar{y}_1, \dots, \bar{y}_\ell, \bar{z}_1, \dots, \bar{z}_T)$$

for any substitution of elements from $\varphi(L_0)$ since we may assume $\bar{x}_1, \dots, \bar{x}_\ell$ to be different basis elements. Here $(\operatorname{ad} a)b = [a, b]$.

Let

$$f_1^{(j)}(u_1,\dots,u_j,v_1,\dots,v_j,x_1,\dots,x_\ell,\ y_1,\dots,y_\ell,z_1,\dots,z_T)$$
$$:=\sum_{i=1}^{\ell} f_1^{(j-1)}\bigl(u_1,\dots,u_{j-1},v_1,\dots,v_{j-1},x_1,\dots,\bigl[u_j,[v_j,x_i]\bigr],\dots,x_\ell,\ y_1,\dots,y_\ell,z_1,\dots,z_T\bigr),$$

$2\leqslant j\leqslant s$, $s=\dim B$. Note that if we substitute an element from $\varphi(R_0)$ for u_i or v_i, then $f_1^{(j)}$ vanish since R_0 is the center of L_0. Again,

$$\begin{aligned} &f_1^{(j)}(\bar u_1,\dots,\bar u_j,\bar v_1,\dots,\bar v_j,\bar x_1,\dots,\bar x_\ell,\ \bar y_1,\dots,\bar y_\ell,\bar z_1,\dots,\bar z_T)\\ &\quad=\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}\bar u_1\operatorname{ad}_{\varphi(L_0)}\bar v_1)\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}\bar u_2\operatorname{ad}_{\varphi(L_0)}\bar v_2)\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}\bar u_j\operatorname{ad}_{\varphi(L_0)}\bar v_j)\\ &\qquad\cdot f_1(\bar x_1,\bar x_2,\dots,\bar x_\ell,\ \bar y_1,\dots,\bar y_\ell,\bar z_1,\dots,\bar z_T). \end{aligned}\tag{5}$$

Let h be the polynomial from Lemma 16. We define

$$\begin{aligned} &f_2(u_1,\dots,u_\ell,v_1,\dots,v_\ell,x_1,\dots,x_\ell,y_1,\dots,y_\ell,z_1,\dots,z_T)\\ &\quad:=\sum_{\sigma,\tau\in S_\ell}\operatorname{sign}(\sigma\tau)f_1^{(s)}(u_{\sigma(1)},\dots,u_{\sigma(s)},v_{\tau(1)},\dots,v_{\tau(s)},x_1,\dots,x_\ell,\ y_1,\dots,y_\ell,z_1,\dots,z_T)\\ &\qquad\cdot h(u_{\sigma(s+1)},\dots,u_{\sigma(\ell)})h(v_{\tau(s+1)},\dots,v_{\tau(\ell)}). \end{aligned}$$

Then $f_2\in Q^G_{\ell,4,4\ell+T}$. Suppose $a_1,\dots,a_s\in\varphi(B_0)$ and $a_{s+1},\dots,a_\ell\in\varphi(R_0)$ form a basis of $\varphi(L_0)$. Consider a substitution $x_i=y_i=u_i=v_i=a_i$, $1\leqslant i\leqslant\ell$. Suppose that the values $z_j=\bar z_j$, $1\leqslant j\leqslant T$, are chosen in such a way that $f_1(a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\neq 0$. We claim that f_2 does not vanish either. Indeed,

$$\begin{aligned} &f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\\ &\quad=\sum_{\sigma,\tau\in S_\ell}\operatorname{sign}(\sigma\tau)f_1^{(s)}(a_{\sigma(1)},\dots,a_{\sigma(s)},a_{\tau(1)},\dots,a_{\tau(s)},a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\\ &\qquad\cdot h(a_{\sigma(s+1)},\dots,a_{\sigma(\ell)})h(a_{\tau(s+1)},\dots,a_{\tau(\ell)})\\ &\quad=\Biggl(\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)f_1^{(s)}(a_{\sigma(1)},\dots,a_{\sigma(s)},a_{\tau(1)},\dots,a_{\tau(s)},a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\Biggr)\\ &\qquad\cdot\Biggl(\sum_{\pi,\omega\in S\{s+1,\dots,\ell\}}\operatorname{sign}(\pi\omega)h(a_{\pi(s+1)},\dots,a_{\pi(\ell)})h(a_{\omega(s+1)},\dots,a_{\omega(\ell)})\Biggr) \end{aligned}$$

since a_j, $s<j\leqslant\ell$, belong to the center of $\varphi(L_0)$ and $f_j^{(s)}$ vanishes if we substitute such a_i for u_i or v_i. Here $S\{s+1,\dots,\ell\}$ is the symmetric group on $\{s+1,\dots,\ell\}$. Note that h is alternating. Using Eq. (5), we obtain

$$\begin{aligned} &f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\\ &\quad=\Biggl(\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_{\sigma(1)}\operatorname{ad}_{\varphi(L_0)}a_{\tau(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_{\sigma(s)}\operatorname{ad}_{\varphi(L_0)}a_{\tau(s)})\Biggr)\\ &\qquad\cdot f_1(a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\bigl((\ell-s)!\bigr)^2\bigl(h(a_{s+1},\dots,a_\ell)\bigr)^2. \end{aligned}$$

Note that

$$\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_{\sigma(1)}\operatorname{ad}_{\varphi(L_0)}a_{\tau(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_{\sigma(s)}\operatorname{ad}_{\varphi(L_0)}a_{\tau(s)})$$
$$=\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_1\operatorname{ad}_{\varphi(L_0)}a_{\tau\sigma^{-1}(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_s\operatorname{ad}_{\varphi(L_0)}a_{\tau\sigma^{-1}(s)})$$
$$\overset{(\tau'=\tau\sigma^{-1})}{=}\sum_{\sigma,\tau'\in S_s}\operatorname{sign}(\tau')\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_1\operatorname{ad}_{\varphi(L_0)}a_{\tau'(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_s\operatorname{ad}_{\varphi(L_0)}a_{\tau'(s)})$$
$$=s!\det\bigl(\operatorname{tr}(\operatorname{ad}_{\varphi(L_0)}a_i\operatorname{ad}_{\varphi(L_0)}a_j)\bigr)_{i,j=1}^{s}=s!\det\bigl(\operatorname{tr}(\operatorname{ad}_{\varphi(B_0)}a_i\operatorname{ad}_{\varphi(B_0)}a_j)\bigr)_{i,j=1}^{s}\neq 0$$

since the Killing form $\operatorname{tr}(\operatorname{ad}x\operatorname{ad}y)$ of the semisimple Lie algebra $\varphi(B_0)$ is nondegenerate. Thus

$$f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\neq 0.$$

Note that if f_1 is alternating in some of $z_1,\dots,z_T$, the polynomial f_2 is alternating in those variables too. Thus if we apply the same procedure to f_2 instead of f_1, we obtain $f_3\in Q^G_{\ell,6,6\ell+T}$. Analogously, we define f_4 using f_3, f_5 using f_4, etc. Eventually, we obtain $f=f_k\in Q^G_{\ell,2k,2k\ell+T}\setminus \operatorname{Id}^G(\varphi)$. □

6. Lower bound

By the definition of $d=d(L)$, there exist G-invariant ideals $I_1,I_2,\dots,I_r$, $J_1,J_2,\dots,J_r$, $r\in\mathbb{Z}_+$, of the algebra L, satisfying conditions (1)–(2), $J_k\subseteq I_k$, such that

$$d=\dim\frac{L}{\operatorname{Ann}(I_1/J_1)\cap\dots\cap\operatorname{Ann}(I_r/J_r)}.$$

We consider the case $d>0$.

Without loss of generality we may assume that

$$\bigcap_{k=1}^{r}\operatorname{Ann}(I_k/J_k)\neq\bigcap_{\substack{k=1,\\k\neq\ell}}^{r}\operatorname{Ann}(I_k/J_k)$$

for all $1\leqslant\ell\leqslant r$. In particular, L has nonzero action on each I_k/J_k.

Our aim is to present a partition $\lambda\vdash n$ with $m(L,G,\lambda)\neq 0$ such that $\dim M(\lambda)$ has the desired asymptotic behavior. We will glue alternating polynomials constructed in Theorem 5 for faithful irreducible modules over reductive algebras. In order to do this, we have to choose the reductive algebras.

Lemma 18. *There exist G-invariant ideals $B_1,\dots,B_r$ in B and G-invariant subspaces $\tilde R_1,\dots,\tilde R_r\subseteq S$ (some of $\tilde R_i$ and B_j may be zero) such that*

(1) $B_1+\dots+B_r=B_1\oplus\dots\oplus B_r$;
(2) $\tilde R_1+\dots+\tilde R_r=\tilde R_1\oplus\dots\oplus\tilde R_r$;
(3) $\sum_{k=1}^r\dim(B_k\oplus\tilde R_k)=d$;
(4) I_k/J_k *is a faithful* $(B_k\oplus\tilde R_k\oplus N)/N$*-module*;
(5) I_k/J_k *is an irreducible* $(\sum_{i=1}^r(B_i\oplus\tilde R_i)\oplus N)/N$*-module with G-action*;
(6) $B_iI_k/J_k=\tilde R_iI_k/J_k=0$ *for* $i>k$.

Proof. Consider $N_\ell := \bigcap_{k=1}^{\ell} \operatorname{Ann}(I_k/J_k)$, $1 \leqslant \ell \leqslant r$, $N_0 = L$. Note that N_ℓ are G-invariant. Since B is semisimple, we can choose such G-invariant ideals B_ℓ that $N_{\ell-1} \cap B = B_\ell \oplus (N_\ell \cap B)$. Also we can choose such G-invariant subspaces $\tilde{R}_\ell$ that $N_{\ell-1} \cap S = \tilde{R}_\ell \oplus (N_\ell \cap S)$. Hence properties (1), (2), (6) hold.

By Lemma 5, $N_k = (N_k \cap B) \oplus (N_k \cap S) \oplus N$. Thus property (4) holds. Furthermore,

$$N_{\ell-1} = B_\ell \oplus (N_\ell \cap B) \oplus \tilde{R}_\ell \oplus (N_\ell \cap S) \oplus N = (B_\ell \oplus \tilde{R}_\ell) \oplus N_\ell$$

(direct sum of subspaces). Hence $L = (\bigoplus_{i=1}^{r} (B_i \oplus \tilde{R}_i)) \oplus N_r$, and properties (3) and (5) hold too. □

Let A be the associative subalgebra in $\operatorname{End}_F(L)$ generated by operators from $\operatorname{ad} L$ and G. Then $J(A)^p = 0$ for some $p \in \mathbb{N}$. Denote by A_2 a subalgebra of $\operatorname{End}_F(L)$ generated by $\operatorname{ad} L$ only. Let $a_{\ell 1}, \dots, a_{\ell, k_\ell}$ be a basis of $\tilde{R}_\ell$.

Lemma 19. *There exist decompositions* $\operatorname{ad} a_{ij} = c_{ij} + d_{ij}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k_i$, *such that* $c_{ij} \in A$ *acts as a diagonalizable operator on* L, $d_{ij} \in J(A)$, *elements* c_{ij} *commute with each other, and* c_{ij} *and* d_{ij} *are polynomials in* $\operatorname{ad} a_{ij}$. *Moreover,* $R_\ell := \langle c_{\ell 1}, \dots, c_{\ell, k_\ell} \rangle_F$ *are* G*-invariant subspaces in* A.

Proof. Consider the solvable G-invariant Lie algebra $(\operatorname{ad} R) + J(A)$. In virtue of the Lie theorem, there exists a basis in L in which all the operators from $(\operatorname{ad} R) + J(A)$ have upper triangular matrices. Denote the corresponding embedding $A \hookrightarrow M_m(F)$ by ψ. Here $m := \dim L$.

Let A_1 be the associative algebra generated by $\operatorname{ad} a_{ij}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k_i$. This algebra is G-invariant since for every fixed i the elements a_{ij}, $1 \leqslant j \leqslant k_i$, form a basis of the G-invariant subspace $\tilde{R}_i$. By the G-invariant Wedderburn–Malcev theorem [14, Theorem 1, Remark 1], $A_1 = \tilde{A}_1 \oplus J(A_1)$ (direct sum of subspaces) where $\tilde{A}_1$ is a G-invariant semisimple subalgebra of A_1. Since $\psi(\operatorname{ad} R) \subseteq \mathfrak{t}_m(F)$, we have $\psi(A_1) \subseteq UT_m(F)$. Here $UT_m(F)$ is the associative algebra of upper triangular matrices $m \times m$. There is a decomposition

$$UT_m(F) = Fe_{11} \oplus Fe_{22} \oplus \cdots \oplus Fe_{mm} \oplus \tilde{N}$$

where

$$\tilde{N} := \langle e_{ij} \mid 1 \leqslant i < j \leqslant m \rangle_F$$

is a nilpotent ideal. Thus there is no subalgebras in A_1 isomorphic to $M_2(F)$ and $\tilde{A}_1 = Fe_1 \oplus \cdots \oplus Fe_t$ for some idempotents $e_i \in A_1$. Denote for every a_{ij} its component in $J(A_1)$ by d_{ij} and its component in $Fe_1 \oplus \cdots \oplus Fe_t$ by c_{ij}. Note that e_i are commuting diagonalizable operators. Thus they have a common basis of eigenvectors in L and c_{ij} are commuting diagonalizable operators too. Moreover

$$\operatorname{ad} a_{ij}^g = c_{ij}^g + d_{ij}^g \in \langle \operatorname{ad} a_{i\ell} \mid 1 \leqslant \ell \leqslant k_i \rangle_F \subseteq \langle c_{i\ell} \mid 1 \leqslant \ell \leqslant k_i \rangle_F \oplus \langle d_{i\ell} \mid 1 \leqslant \ell \leqslant k_i \rangle_F$$

for all $g \in G$. Thus R_i is G-invariant.

We claim that the space $J(A_1) + J(A)$ generates a nilpotent G-invariant ideal I in A. First, $\psi(J(A_1)), \psi(J(A)) \subseteq UT_m(F)$ and consist of nilpotent elements. Thus the corresponding matrices have zero diagonal elements and $\psi(J(A_1)), \psi(J(A)) \subseteq \tilde{N}$. Denote $\tilde{N}_k := \langle e_{ij} \mid i + k \leqslant j \rangle_F \subseteq \tilde{N}$. Then

$$\tilde{N} = \tilde{N}_1 \supsetneqq \tilde{N}_2 \supsetneqq \cdots \supsetneqq \tilde{N}_{m-1} \supsetneqq \tilde{N}_m = \{0\}.$$

Let $\operatorname{ht}_{\tilde{N}} a := k$ if $\psi(a) \in \tilde{N}_k$, $\psi(a) \notin \tilde{N}_{k+1}$.

Recall that $(J(A))^p = 0$. We claim that $I^{m+p} = 0$. Let $\rho : G \to \operatorname{GL}(L)$ be the G-action on L. Using the property

$$\rho(g)a = a^g \rho(g) \tag{6}$$

where $a \in A_2$, $g \in G$, we obtain that the space I^{m+p} is a span of $h_1 j_1 h_2 j_2 \dots j_{m+p} h_{m+p+1} \rho(g)$ where $j_k \in J(A_1) \cup J(A)$, $h_k \in A_2 \cup \{1\}$, $g \in G$. If at least p elements j_k belong to $J(A)$, then the product equals 0. Thus we may assume that at least m elements j_k belong to $J(A_1)$.

Let $j_i \in J(A_1)$, $h_i \in A_2 \cup \{1\}$. We prove by induction on ℓ that $j_1 h_1 j_2 h_2 \dots h_{\ell-1} j_\ell$ can be expressed as a sum of $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\alpha j'_1 j'_2 \dots j'_\beta a$ where $\tilde{j}_i \in J(A_1)$, $j'_i \in J(A)$, $a \in A_2 \cup \{1\}$, and $\alpha + \sum_{i=1}^{\beta} \operatorname{ht}_{\tilde{N}} j'_i \geqslant \ell$. Indeed, suppose that $j_1 h_1 j_2 h_2 \dots h_{\ell-2} j_{\ell-1}$ can be expressed as a sum of $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa a$ where $\tilde{j}_i \in J(A_1)$, $j'_i \in J(A)$, $a \in A_2 \cup \{1\}$, and $\gamma + \sum_{i=1}^{\varkappa} \operatorname{ht}_{\tilde{N}} j'_i \geqslant \ell - 1$. Then $j_1 h_1 j_2 h_2 \dots j_{\ell-1} h_{\ell-1} j_\ell$ is a sum of

$$\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa a h_{\ell-1} j_\ell = \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa [a h_{\ell-1}, j_\ell] + \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa j_\ell (a h_{\ell-1}).$$

Note that, in virtue of the Jacobi identity and Lemma 7, $[a h_{\ell-1}, j_\ell] \in J(A)$. Thus it is sufficient to consider only the second term. However

$$\begin{aligned} \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa j_\ell (a h_{\ell-1}) &= \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j_\ell j'_1 j'_2 \dots j'_\varkappa (a h_{\ell-1}) \\ &\quad + \sum_{i=1}^{\varkappa} \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_{i-1} [j'_i, j_\ell] j'_{i+1} \dots j'_\varkappa (a h_{\ell-1}). \end{aligned}$$

Since $[j'_i, j_\ell] \in J(A)$ and $\operatorname{ht}_{\tilde{N}} [j'_i, j_\ell] \geqslant 1 + \operatorname{ht}_{\tilde{N}} j'_i$, all the terms have the desired form. Therefore,

$$j_1 h_1 j_2 h_2 \dots j_{m-1} h_{m-1} j_m \in \psi^{-1}(\tilde{N}_m) = \{0\},$$

$I^{m+p} = 0$, and

$$J(A) \subseteq J(A_1) + J(A) \subseteq I \subseteq J(A).$$

In particular, $d_{ij} \in J(A_1) \subseteq J(A)$. □

Denote

$$\begin{aligned} \tilde{B} &:= \left(\bigoplus_{i=1}^{r} \operatorname{ad} B_i \right) \oplus \langle c_{ij} \mid 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant k_i \rangle_F, \\ \tilde{B}_0 &:= (\operatorname{ad} B) \oplus \langle c_{ij} \mid 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant k_i \rangle_F \subseteq A. \end{aligned}$$

Lemma 20. *The space L is a completely reducible $\tilde{B}_0$-module with G-action. Moreover, L is a completely reducible $(\operatorname{ad} B_k) \oplus R_k$-module with G-action for any $1 \leqslant k \leqslant r$.*

Proof. By Lemma 3, it is sufficient to show that L is a completely reducible $\tilde{B}_0$-module and a completely reducible $(\operatorname{ad} B_k) \oplus R_k$-module disregarding the G-action. The elements c_{ij} are diagonalizable on L and commute. Therefore, an eigenspace of any c_{ij} is invariant under the action of other $c_{k\ell}$. Using induction, we split $L = \bigoplus_{i=1}^{\alpha} W_i$ where W_i are intersections of eigenspaces of $c_{k\ell}$ and elements $c_{k\ell}$ act as scalar operators on W_i. In virtue of Lemmas 4, 19, and the Jacobi identity, $[c_{ij}, \operatorname{ad} B] = 0$. Thus W_i are B-submodules and L is a completely reducible $\tilde{B}_0$-module and $(\operatorname{ad} B_k) \oplus R_k$-module since B and B_k are semisimple. □

Lemma 21. *There exist complementary subspaces $I_k = \tilde{T}_k \oplus J_k$ such that*

(1) *$\tilde{T}_k$ is a B-submodule and an irreducible $\tilde{B}$-submodule with G-action*;
(2) *$\tilde{T}_k$ is a completely reducible faithful $(\mathrm{ad}\, B_k) \oplus R_k$-module with G-action*;
(3) $\sum_{k=1}^{r} \dim((\mathrm{ad}\, B_k) \oplus R_k) = d$;
(4) *$B_i \tilde{T}_k = R_i \tilde{T}_k = 0$ for $i > k$.*

Proof. By Lemma 20, L is a completely reducible $\tilde{B}_0$-module with G-action. Therefore, for every J_k we can choose a complementary G-invariant $\tilde{B}_0$-submodules $\tilde{T}_k$ in I_k. Then $\tilde{T}_k$ are both B- and $\tilde{B}$-submodules.

Note that $(\mathrm{ad}\, a_{ij})w = c_{ij} w$ for all $w \in I_k/J_k$ since I_k/J_k is an irreducible A-module and $J(A)\, I_k/J_k = 0$. Hence, by Lemma 18, I_k/J_k is a faithful $(\mathrm{ad}\, B_k) \oplus R_k$-module, $R_i\, I_k/J_k = 0$ for $i > k$ and the elements c_{ij} are linearly independent. Moreover, by property (5) of Lemma 18, I_k/J_k is an irreducible $(\sum_{i=1}^{r}(B_i \oplus \tilde{R}_i) \oplus N)/N$-module with G-action. However $(\sum_{i=1}^{r}(B_i \oplus \tilde{R}_i) \oplus N)/N$ acts on I_k/J_k by the same operators as $\tilde{B}$. Thus $\tilde{T}_k \cong I_k/J_k$ is an irreducible $\tilde{B}$-module with G-action. Property (1) is proved. By Lemma 20, L is a completely reducible $(\mathrm{ad}\, B_k) \oplus R_k$-module with G-action for any $1 \leqslant k \leqslant r$. Using the isomorphism $\tilde{T}_k \cong I_k/J_k$, we obtain properties (2) and (4) from the remarks above. Property (3) is a consequence of property (3) of Lemma 18. □

Lemma 22. *For all $1 \leqslant k \leqslant r$ we have*

$$\tilde{T}_k = T_{k1} \oplus T_{k2} \oplus \cdots \oplus T_{km}$$

where T_{kj} are faithful irreducible $(\mathrm{ad}\, B_k) \oplus R_k$-submodules with G-action, $m \in \mathbb{N}$, $1 \leqslant j \leqslant m$.

Proof. By Lemma 21, property (2), $\tilde{T}_k = T_{k1} \oplus T_{k2} \oplus \cdots \oplus T_{km}$ for some irreducible $(\mathrm{ad}\, B_k) \oplus R_k$-submodules with G-action. Suppose T_{kj} is not faithful for some $1 \leqslant j \leqslant m$. Hence $bT_{kj} = 0$ for some $b \in (\mathrm{ad}\, B_k) \oplus R_k$, $b \neq 0$. Note that $\tilde{B} = ((\mathrm{ad}\, B_k) \oplus R_k) \oplus \tilde{B}_k$ where

$$\tilde{B}_k := \bigoplus_{i \neq k} (\mathrm{ad}\, B_i) \oplus \bigoplus_{i \neq k} R_i$$

and $[(\mathrm{ad}\, B_k) \oplus R_k, \tilde{B}_k] = 0$. Denote by $\widehat{B}_k$ the associative subalgebra of $\mathrm{End}_F(\tilde{T}_k)$ with 1 generated by operators from $\tilde{B}_k$. Then

$$\left[(\mathrm{ad}\, B_k) \oplus R_k, \widehat{B}_k\right] = 0$$

and $\sum_{a \in \widehat{B}_k} aT_{kj} \supseteq T_{kj}$ is a G-invariant $\tilde{B}$-submodule of $\tilde{T}_k$ since

$$\left(\sum_{a \in \widehat{B}_k} aT_{kj}\right)^g = \sum_{a \in \widehat{B}_k} a^g T_{kj}^g = \sum_{a \in \widehat{B}_k} a^g T_{kj} = \sum_{a' \in \widehat{B}_k} a' T_{kj}$$

for all $g \in G$. Thus $\tilde{T}_k = \sum_{a \in \widehat{B}_k} aT_{kj}$ and

$$b\tilde{T}_k = \sum_{a \in \widehat{B}_k} baT_{kj} = \sum_{a \in \widehat{B}_k} a(bT_{kj}) = 0.$$

We get a contradiction with faithfulness of $\tilde{T}_k$. □

By condition (2) of the definition of d, there exist numbers $q_1, \dots, q_r \in \mathbb{Z}_+$ such that

$$\big[[\tilde{T}_1, \underbrace{L, \dots, L}_{q_1}], [\tilde{T}_2, \underbrace{L, \dots, L}_{q_2}] \dots, [\tilde{T}_r, \underbrace{L, \dots, L}_{q_r}]\big] \neq 0.$$

Choose $n_i \in \mathbb{Z}_+$ with the maximal $\sum_{i=1}^r n_i$ such that

$$\left[\left[\left(\prod_{k=1}^{n_1} j_{1k}\right)\tilde{T}_1, \underbrace{L, \dots, L}_{q_1}\right], \left[\left(\prod_{k=1}^{n_2} j_{2k}\right)\tilde{T}_2, \underbrace{L, \dots, L}_{q_2}\right] \dots, \left[\left(\prod_{k=1}^{n_r} j_{rk}\right)\tilde{T}_r, \underbrace{L, \dots, L}_{q_r}\right]\right] \neq 0$$

for some $j_{ik} \in J(A)$. Let $j_i := \prod_{k=1}^{n_i} j_{ik}$. Then $j_i \in J(A) \cup \{1\}$ and

$$\big[[j_1\tilde{T}_1, \underbrace{L, \dots, L}_{q_1}], [j_2\tilde{T}_2, \underbrace{L, \dots, L}_{q_2}], \dots, [j_r\tilde{T}_r, \underbrace{L, \dots, L}_{q_r}]\big] \neq 0,$$

but

$$\big[[j_1\tilde{T}_1, \underbrace{L, \dots, L}_{q_1}], \dots, \big[j_k(j\tilde{T}_k), \underbrace{L, \dots, L}_{q_k}\big], \dots, [j_r\tilde{T}_r, \underbrace{L, \dots, L}_{q_r}]\big] = 0 \tag{7}$$

for all $j \in J(A)$ and $1 \leqslant k \leqslant r$.

In virtue of Lemma 22, for every k we can choose a faithful irreducible $(\operatorname{ad} B_k) \oplus R_k$-submodule with G-action $T_k \subseteq \tilde{T}_k$ such that

$$\big[[j_1 T_1, \underbrace{L, \dots, L}_{q_1}], [j_2 T_2, \underbrace{L, \dots, L}_{q_2}] \dots, [j_r T_r, \underbrace{L, \dots, L}_{q_r}]\big] \neq 0. \tag{8}$$

Lemma 23. *Let $\psi : \bigoplus_{i=1}^r (B_i \oplus \tilde{R}_i) \to \bigoplus_{i=1}^r ((\operatorname{ad} B_i) \oplus R_i)$ be the linear isomorphism defined by formulas $\psi(b) = \operatorname{ad} b$ for all $b \in B_i$ and $\psi(a_{i\ell}) = c_{i\ell}$, $1 \leqslant \ell \leqslant k_\ell$. Let f_i be multilinear associative G-polynomials, $h_1^{(i)}, \dots, h_{n_i}^{(i)} \in \bigoplus_{i=1}^r B_i \oplus \tilde{R}_i$, $\bar{t}_i \in \tilde{T}_i$, $\bar{u}_{ik} \in L$, be some elements. Then*

$$\begin{aligned}&\big[\big[j_1 f_1\big(\operatorname{ad} h_1^{(1)}, \dots, \operatorname{ad} h_{n_1}^{(1)}\big)\bar{t}_1, \bar{u}_{11}, \dots, \bar{u}_{1q_1}\big], \dots, \big[j_r f_r\big(\operatorname{ad} h_1^{(r)}, \dots, \operatorname{ad} h_{n_r}^{(r)}\big)\bar{t}_r, \bar{u}_{r1}, \dots, \bar{u}_{rq_r}\big]\big] \\ &\quad = \big[\big[j_1 f_1\big(\psi\big(h_1^{(1)}\big), \dots, \psi\big(h_{n_1}^{(1)}\big)\big)\bar{t}_1, \bar{u}_{11}, \dots, \bar{u}_{1q_1}\big], \dots, \big[j_r f_r\big(\psi\big(h_1^{(r)}\big), \dots, \psi\big(h_{n_r}^{(r)}\big)\big)\bar{t}_r, \bar{u}_{r1}, \dots, \bar{u}_{rq_r}\big]\big].\end{aligned}$$

In other words, we can replace $\operatorname{ad} a_{i\ell}$ with $c_{i\ell}$ and the result does not change.

Proof. We rewrite $\operatorname{ad} a_{i\ell} = c_{i\ell} + d_{i\ell} = \psi(a_i) + d_{i\ell}$ and use the multilinearity of f_i. By Eq. (7), terms with $d_{i\ell}$ vanish. □

Denote by $A_3 \subseteq \operatorname{End}_F(L)$ the linear span of products of operators from $\operatorname{ad} L$ and G such that each product contains at least one element from $\operatorname{ad} L$.

Lemma 24. $J(A) \subseteq A_3$.

Proof. Note that A_3 is a G-invariant two-sided ideal of A and $A_3 + \tilde{A}_3 = A$ where $\tilde{A}_3 \subseteq \operatorname{End}_F(L)$ is the associative subalgebra generated by operators from G. Thus $A/A_3 \cong \tilde{A}_3/(\tilde{A}_3 \cap A_3)$ is a semisimple algebra since $\tilde{A}_3$ is a homomorphic image of the semisimple group algebra FG. Thus $J(A) \subseteq A_3$. □

Lemma 25. *If $d \neq 0$, then there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ there exist disjoint subsets $X_1, \ldots, X_{2k} \subseteq \{x_1, \ldots, x_n\}$, $k := [\frac{n-n_0}{2d}]$, $|X_1| = \cdots = |X_{2k}| = d$ and a polynomial $f \in V_n^G \setminus \mathrm{Id}^G(L)$ alternating in the variables of each set X_j.*

Proof. Denote by $\varphi_i : (\mathrm{ad}\, B_i) \oplus R_i \to \mathfrak{gl}(T_i)$ the representation corresponding to the action of $(\mathrm{ad}\, B_i) \oplus R_i$ on T_i. In virtue of Theorem 5, there exist constants $m_i \in \mathbb{Z}_+$ such that for any k there exist multilinear polynomials $f_i \in Q^G_{d_i,2k,2kd_i+m_i} \setminus \mathrm{Id}^G(\varphi_i)$, $d_i := \dim((\mathrm{ad}\, B_i) \oplus R_i)$, alternating in the variables from disjoint sets $X_\ell^{(i)}$, $1 \leqslant \ell \leqslant 2k$, $|X_\ell^{(i)}| = d_i$.

In virtue of (8),

$$\big[[j_1\bar{t}_1, \bar{u}_{11}, \ldots, \bar{u}_{1,q_1}], [j_2\bar{t}_2, \bar{u}_{21}, \ldots, \bar{u}_{2,q_2}], \ldots, [j_r\bar{t}_r, \bar{u}_{r1}, \ldots, \bar{u}_{r,q_r}]\big] \neq 0,$$

for some $\bar{u}_{i\ell} \in L$ and $\bar{t}_i \in T_i$. All $j_i \in J(A) \cup \{1\}$ are polynomials in elements from G and $\mathrm{ad}\, L$. Denote by $\tilde{m}$ the maximal degree of them.

Recall that each T_i is a faithful irreducible $(\mathrm{ad}\, B_i) \oplus R_i$-module with G-action. Therefore by the density theorem, $\mathrm{End}_F(T_i)$ is generated by operators from G and $(\mathrm{ad}\, B_i) \oplus R_i$. Note that $\mathrm{End}_F(T_i) \cong M_{\dim T_i}(F)$. Thus every matrix unit $e^{(i)}_{j\ell} \in M_{\dim T_i}(F)$ can be represented as a polynomial in operators from G and $(\mathrm{ad}\, B_i) \oplus R_i$. Choose such polynomials for all i and all matrix units. Denote by m_0 the maximal degree of those polynomials.

Let $n_0 := r(2m_0 + \tilde{m} + 1) + \sum_{i=1}^r (m_i + q_i)$. Now we choose f_i for $k = [\frac{n-n_0}{2d}]$. Since $f_i \notin \mathrm{Id}^G(\varphi_i)$, there exist $\bar{x}_{i1}, \ldots, \bar{x}_{i,2kd_i+m_i} \in (\mathrm{ad}\, B_i) \oplus R_i$ such that $f_i(\bar{x}_{i1}, \ldots, \bar{x}_{i,2kd_i+m_i}) \neq 0$. Hence

$$e^{(i)}_{\ell_i\ell_i} f_i(\bar{x}_{i1}, \ldots, \bar{x}_{i,2kd_i+m_i}) e^{(i)}_{s_is_i} \neq 0$$

for some matrix units $e^{(i)}_{\ell_i\ell_i}, e^{(i)}_{s_is_i} \in \mathrm{End}_F(T_i)$, $1 \leqslant \ell_i, s_i \leqslant \dim T_i$. Thus

$$\sum_{\ell=1}^{\dim_{T_i}} e^{(i)}_{\ell\ell_i} f_i(\bar{x}_{i1}, \ldots, \bar{x}_{i,2kd_i+m_i}) e^{(i)}_{s_i\ell}$$

is a nonzero scalar operator in $\mathrm{End}_F(T_i)$.

Hence

$$\left[\left[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(\bar{x}_{11}, \ldots, \bar{x}_{1,2kd_1+m_1}) e^{(1)}_{s_1\ell}\right)\bar{t}_1, \bar{u}_{11}, \ldots, \bar{u}_{1q_1}\right], \ldots, \right.$$
$$\left.\left[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(\bar{x}_{r1}, \ldots, \bar{x}_{r,2kd_r+m_r}) e^{(r)}_{s_r\ell}\right)\bar{t}_r, \bar{u}_{r1}, \ldots, \bar{u}_{rq_r}\right]\right] \neq 0.$$

Denote $X_\ell := \bigcup_{i=1}^r X_\ell^{(i)}$. Let Alt_ℓ be the operator of alternation in the variables from X_ℓ. Consider

$$\tilde{f}(x_{11}, \ldots, x_{1,2kd_1+m_1}, \ldots, x_{r1}, \ldots, x_{r,2kd_r+m_r})$$
$$:= \mathrm{Alt}_1 \mathrm{Alt}_2 \ldots \mathrm{Alt}_{2k}\left[\left[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(x_{11}, \ldots, x_{1,2kd_1+m_1}) e^{(1)}_{s_1\ell}\right)\bar{t}_1, \bar{u}_{11}, \ldots, \bar{u}_{1q_1}\right], \ldots, \right.$$
$$\left.\left[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(x_{r1}, \ldots, x_{r,2kd_r+m_r}) e^{(r)}_{s_r\ell}\right)\bar{t}_r, \bar{u}_{r1}, \ldots, \bar{u}_{rq_r}\right]\right].$$

Then

$$\tilde{f}(\bar{x}_{11},\dots,\bar{x}_{1,2kd_1+m_1},\dots,\bar{x}_{r1},\dots,\bar{x}_{r,2kd_r+m_r})$$
$$= (d_1!)^{2k}\dots(d_r!)^{2k}\left[\left[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(\bar{x}_{11},\dots,\bar{x}_{1,2kd_1+m_1}) e^{(1)}_{s_1\ell}\right)\bar{t}_1,\bar{u}_{11},\dots,\bar{u}_{1q_1}\right],\dots,\right.$$
$$\left.\left[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(\bar{x}_{r1},\dots,\bar{x}_{r,2kd_r+m_r}) e^{(r)}_{s_r\ell}\right)\bar{t}_r,\bar{u}_{r1},\dots,\bar{u}_{rq_r}\right]\right] \neq 0,$$

since f_i are alternating in each $X^{(i)}_\ell$ and, by Lemma 21, $((\operatorname{ad} B_i)\oplus R_i)\tilde{T}_\ell = 0$ for $i > \ell$. Now we rewrite $e^{(i)}_{\ell j}$ as polynomials in elements of $(\operatorname{ad} B_i)\oplus R_i$ and G. Using linearity of $\tilde{f}$ in $e^{(i)}_{\ell j}$, we can replace $e^{(i)}_{\ell j}$ with the products of elements from $(\operatorname{ad} B_i)\oplus R_i$ and G, and the expression will not vanish for some choice of the products. Using Eq. (6), we can move all $\rho(g)$ to the right. By Lemma 23, we can replace all elements from $(\operatorname{ad} B_i)\oplus R_i$ with elements from $B_i\oplus\tilde{R}_i$ and the expression will be still nonzero. Denote by $\psi:\bigoplus_{i=1}^r(B_i\oplus\tilde{R}_i)\to\bigoplus_{i=1}^r((\operatorname{ad} B_i)\oplus R_i)$ the corresponding linear isomorphism. Now we rewrite j_i as polynomials in elements $\operatorname{ad} L$ and G. Since $\tilde{f}$ is linear in j_i, we can replace j_i with one of the monomials, i.e. with the product of elements from $\operatorname{ad} L$ and G. Using Eq. (6), we again move all $\rho(g)$ to the right. Then we replace the elements from $\operatorname{ad} L$ with new variables, and

$$\hat{f} := \operatorname{Alt}_1\operatorname{Alt}_2\dots\operatorname{Alt}_{2k}\big[\big[[y_{11},[y_{12},\dots[y_{1\alpha_1},[z_{11},[z_{12},\dots,[z_{1\beta_1},$$
$$\big(f_1(\operatorname{ad} x_{11},\dots,\operatorname{ad} x_{1,2kd_1+m_1})\big)^{g_1}\big[w_{11},\big[w_{12},\dots,\big[w_{1\gamma_1},t_1^{h_1}\big]\dots\big],u_{11},\dots,u_{1q_1}\big],\dots,$$
$$\big[[y_{r1},[y_{r2},\dots,[y_{r\alpha_r},[z_{r1},[z_{r2},\dots,[z_{r\beta_r},$$
$$\big(f_r(\operatorname{ad} x_{r1},\dots,\operatorname{ad} x_{r,2kd_r+m_r})\big)^{g_r}\big[w_{r1},\big[w_{r2},\dots,\big[w_{r\gamma_r},t_r^{h_r}\big]\dots\big],u_{r1},\dots,u_{rq_r}\big]\big]$$

for some $0\leqslant\alpha_i\leqslant\tilde{m}$, $0\leqslant\beta_i,\gamma_i\leqslant m_0$, $g_i,h_i\in G$, $\bar{y}_{i\ell},\bar{z}_{i\ell},\bar{w}_{i\ell}\in L$ does not vanish under the substitution $t_i=\bar{t}_i$, $u_{i\ell}=\bar{u}_{i\ell}$, $x_{i\ell}=\psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell}=\bar{y}_{i\ell}$, $z_{i\ell}=\bar{z}_{i\ell}$, $w_{i\ell}=\bar{w}_{i\ell}$.

Note that $\hat{f}\in V^G_{\tilde{n}}$, $\tilde{n}:=2kd+r+\sum_{i=1}^r(m_i+q_i+\alpha_i+\beta_i+\gamma_i)\leqslant n$. If $n=\tilde{n}$, then we take $f:=\hat{f}$. Suppose $n>\tilde{n}$. Let $b\in(\operatorname{ad} B_1)\oplus R_1$, $b\neq 0$. Then $e^{(1)}_{jj}be^{(1)}_{\ell\ell}\neq 0$ for some $1\leqslant j,\ell\leqslant\dim T_1$ and $(\sum_{s=1}^{\dim T_1}(e^{(1)}_{sj}be^{(1)}_{\ell s}))^{n-\tilde{n}}\bar{t}_1=\mu\bar{t}_1$, $\mu\in F\backslash\{0\}$. Hence $\hat{f}$ does not vanish under the substitution $t_1=(\sum_{s=1}^{\dim T_1}(e^{(1)}_{sj}be^{(1)}_{\ell s}))^{n-\tilde{n}}\bar{t}_1$; $t_i=\bar{t}_i$ for $2\leqslant i\leqslant r$; $u_{i\ell}=\bar{u}_{i\ell}$, $x_{i\ell}=\psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell}=\bar{y}_{i\ell}$, $z_{i\ell}=\bar{z}_{i\ell}$, $w_{i\ell}=\bar{w}_{i\ell}$.

By Lemma 24,

$$b\in J(A)\oplus\operatorname{ad}(B_1\oplus\tilde{R}_1)\subseteq A_3$$

and using Eq. (6) we can rewrite $(\sum_{s=1}^{\dim T_1}(e^{(1)}_{sj}be^{(1)}_{\ell s}))^{n-\tilde{n}}\bar{t}_1$ as a sum of elements $[\bar{v}_1,[\bar{v}_2,[\dots,[\bar{v}_q,\bar{t}_1^g]\dots]$, $q\geqslant n-\tilde{n}$, $\bar{v}_i\in L$, $g\in G$. Hence $\hat{f}$ does not vanish under a substitution $t_1=[\bar{v}_1,[\bar{v}_2,[\dots,[\bar{v}_q,\bar{t}_1^g]\dots]$ for some $q\geqslant n-\tilde{n}$, $\bar{v}_i\in L$, $g\in G$; $t_i=\bar{t}_i$ for $2\leqslant i\leqslant r$; $u_{i\ell}=\bar{u}_{i\ell}$, $x_{i\ell}=\psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell}=\bar{y}_{i\ell}$, $z_{i\ell}=\bar{z}_{i\ell}$, $w_{i\ell}=\bar{w}_{i\ell}$. Therefore,

$$f := \operatorname{Alt}_1\operatorname{Alt}_2\dots\operatorname{Alt}_{2k}\big[\big[[y_{11},[y_{12},\dots[y_{1\alpha_1},[z_{11},[z_{12},\dots,[z_{1\beta_1},$$
$$\big(f_1(\operatorname{ad} x_{11},\dots,\operatorname{ad} x_{1,2kd_1+m_1})\big)^{g_1}[w_{11},[w_{12},\dots,[w_{1\gamma_1},$$
$$\big[v_1^{h_1},\big[v_2^{h_1},\big[\dots,\big[v_{n-\tilde{n}}^{h_1},t_1^{h_1}\big]\dots\big]\dots\big],u_{11},\dots,u_{1q_1}\big],$$

$$[[y_{21},[y_{22},\dots[y_{2\alpha_2},[z_{21},[z_{22},\dots,[z_{2\beta_2},$$
$$\big(f_2(\operatorname{ad}x_{21},\dots,\operatorname{ad}x_{2,2kd_2+m_2})\big)^{g_2}\big[w_{21},\big[w_{22},\dots,\big[w_{2\gamma_2},t_2^{h_2}\big]\dots\big],u_{21},\dots,u_{2q_2}\big],$$
$$\dots,[[y_{r1},[y_{r2},\dots,[y_{r\alpha_r},[z_{r1},[z_{r2},\dots,[z_{r\beta_r},$$
$$\big(f_r(\operatorname{ad}x_{r1},\dots,\operatorname{ad}x_{r,2kd_r+m_r})\big)^{g_r}\big[w_{r1},\big[w_{r2},\dots,\big[w_{r\gamma_r},t_r^{h_r}\big]\dots\big],u_{r1},\dots,u_{rq_r}\big]\big]$$

does not vanish under the substitution $v_\ell=\bar v_\ell$, $1\leqslant \ell\leqslant n-\tilde n$, $t_1=[\bar v_{n-\tilde n+1},[\bar v_{n-\tilde n+2},[\dots,[\bar v_q,\bar t_1^g]\dots]$; $t_i=\bar t_i$ for $2\leqslant i\leqslant r$; $u_{i\ell}=\bar u_{i\ell}$, $x_{i\ell}=\psi^{-1}(\bar x_{i\ell})$, $y_{i\ell}=\bar y_{i\ell}$, $z_{i\ell}=\bar z_{i\ell}$, $w_{i\ell}=\bar w_{i\ell}$. Note that $f\in V_n^G$ and satisfies all the conditions of the lemma. □

Lemma 26. *Let k, n_0 be the numbers from Lemma 25. Then for every $n\geqslant n_0$ there exists a partition $\lambda=(\lambda_1,\dots,\lambda_s)\vdash n$, $\lambda_i\geqslant 2k-C$ for every $1\leqslant i\leqslant d$, with $m(L,G,\lambda)\neq 0$. Here $C:=p((\dim L)p+3)((\dim L)-d)$ where $p\in\mathbb{N}$ is such number that $N^p=0$.*

Proof. Consider the polynomial f from Lemma 25. It is sufficient to prove that $e^*_{T_\lambda}f\notin \operatorname{Id}^G(L)$ for some tableau T_λ of the desired shape λ. It is known that $FS_n=\bigoplus_{\lambda,T_\lambda}FS_ne^*_{T_\lambda}$ where the summation runs over the set of all standard tableax T_λ, $\lambda\vdash n$. Thus $FS_nf=\sum_{\lambda,T_\lambda}FS_ne^*_{T_\lambda}f\not\subseteq \operatorname{Id}^G(L)$ and $e^*_{T_\lambda}f\notin\operatorname{Id}^G(L)$ for some $\lambda\vdash n$. We claim that λ is of the desired shape. It is sufficient to prove that $\lambda_d\geqslant 2k-C$, since $\lambda_i\geqslant\lambda_d$ for every $1\leqslant i\leqslant d$. Each row of T_λ includes numbers of no more than one variable from each X_i, since $e^*_{T_\lambda}=b_{T_\lambda}a_{T_\lambda}$ and a_{T_λ} is symmetrizing the variables of each row. Thus $\sum_{i=1}^{d-1}\lambda_i\leqslant 2k(d-1)+(n-2kd)=n-2k$. In virtue of Lemma 14, $\sum_{i=1}^{d}\lambda_i\geqslant n-C$. Therefore $\lambda_d\geqslant 2k-C$. □

Proof of Theorem 1. The Young diagram D_λ from Lemma 26 contains the rectangular subdiagram D_μ, $\mu=(\underbrace{2k-C,\dots,2k-C}_{d})$. The branching rule for S_n implies that if we consider the restriction of S_n-action on $M(\lambda)$ to S_{n-1}, then $M(\lambda)$ becomes the direct sum of all non-isomorphic FS_{n-1}-modules $M(\nu)$, $\nu\vdash(n-1)$, where each D_ν is obtained from D_λ by deleting one box. In particular, $\dim M(\nu)\leqslant\dim M(\lambda)$. Applying the rule $(n-d(2k-C))$ times, we obtain $\dim M(\mu)\leqslant\dim M(\lambda)$. By the hook formula,

$$\dim M(\mu)=\frac{(d(2k-C))!}{\prod_{i,j}h_{ij}}$$

where h_{ij} is the length of the hook with edge in (i,j). By Stirling formula,

$$c_n^G(L)\geqslant\dim M(\lambda)\geqslant\dim M(\mu)\geqslant\frac{(d(2k-C))!}{((2k-C+d)!)^d}$$
$$\sim\frac{\sqrt{2\pi d(2k-C)}\,(\frac{d(2k-C)}{e})^{d(2k-C)}}{(\sqrt{2\pi(2k-C+d)}\,(\frac{2k-C+d}{e})^{2k-C+d})^d}\sim C_9k^{r_9}d^{2kd}$$

for some constants $C_9>0$, $r_9\in\mathbb{Q}$, as $k\to\infty$. Since $k=[\frac{n-n_0}{2d}]$, this gives the lower bound. The upper bound has been proved in Theorem 4. □

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References

[1] A. Giambruno, M.V. Zaicev, Polynomial Identities and Asymptotic Methods, Math. Surveys Monogr., vol. 122, American Mathematical Society, Providence, RI, 2005.
[2] M.V. Zaitsev, Integrality of exponents of growth of identities of finite-dimensional Lie algebras, Izv. Math. 66 (2002) 463–487.
[3] A. Giambruno, I.P. Shestakov, M.V. Zaicev, Finite-dimensional non-associative algebras and codimension growth, Adv. Appl. Math. 47 (2011) 125–139.
[4] A.S. Gordienko, Codimensions of polynomial identities of representations of Lie algebras, Proc. Amer. Math. Soc., in press.
[5] Yu.A. Bahturin, M.V. Zaicev, Identities of graded algebras, J. Algebra 205 (1998) 1–12.
[6] Yu.A. Bahturin, M.V. Zaicev, Identities of graded algebras and codimension growth, Trans. Amer. Math. Soc. 356 (10) (2004) 3939–3950.
[7] Yu.A. Bakhturin, M.V. Zaĭtsev, S.K. Sehgal, G-identities of non-associative algebras, Sb. Math. 190 (11) (1999) 1559–1570.
[8] Yu.A. Bahturin, A. Giambruno, M.V. Zaicev, G-identities on associative algebras, Proc. Amer. Math. Soc. 127 (1) (1999) 63–69.
[9] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011) 83–100.
[10] A. Giambruno, D. La Mattina, Graded polynomial identities and codimensions: computing the exponential growth, Adv. Math. 225 (2010) 859–881.
[11] I.B. Volichenko, Varieties of Lie algebras with identity $[[X_1, X_2, X_3], [X_4, X_5, X_6]] = 0$ over a field of characteristic zero, Sibirsk. Mat. Zh. 25 (3) (1984) 40–54 (in Russian).
[12] M.V. Zaicev, S.P. Mishchenko, An example of a variety of Lie algebras with a fractional exponent, J. Math. Sci. (N. Y.) 93 (6) (1999) 977–982.
[13] S.P. Mishchenko, A.B. Verevkin, M.V. Zaitsev, A sufficient condition for coincidence of lower and upper exponents of the variety of linear algebras, Moscow Univ. Math. Bull. 66 (2) (2011) 86–89.
[14] E.J. Taft, Invariant Wedderburn factors, Illinois J. Math. 1 (1957) 565–573.
[15] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York, 1978.
[16] M. Goto, F. Grosshans, Semisimple Lie Algebras, Marcel Dekker, New York, Basel, 1978.
[17] Yu.A. Bakhturin, Identical Relations in Lie Algebras, VNU Science Press, Utrecht, 1987.
[18] V.S. Drensky, Free Algebras and PI-Algebras: Graduate Course in Algebra, Springer-Verlag, Singapore, 2000.
[19] A. Berele, Cocharacter sequences for algebras with Hopf algebra actions, J. Algebra 185 (1996) 869–885.
[20] V.S. Drensky, Representations of the symmetric group and varieties of linear algebras, Math. USSR Sb. 43 (1) (1982) 85–101.
[21] A. Berele, Homogeneous polynomial identities, Israel J. Math. 42 (3) (1982) 258–272.
[22] A. Giambruno, A. Regev, M.V. Zaicev, Simple and semisimple Lie algebras and codimension growth, Trans. Amer. Math. Soc. 352 (4) (2000) 1935–1946.
[23] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison–Wesley, Massachusetts, 1981.
[24] W. Fulton, J. Harris, Representation Theory: A First Course, Springer-Verlag, New York, 1991.
[25] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Pure Appl. Math., vol. XI, John Wiley & Sons, New York, London, Sydney, 1962.