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Invariant Manifolds for Singularly Perturbed Linear Functional Differential Equations*†

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The existence of "slow" and "fast" manifolds, and of invariant manifolds approaching the manifold of orbits of the degenerate system, is discussed for singularly perturbed systems of linear retarded functional differential equations (FDE). It is shown that these manifolds exist only in very degenerate situations and, consequently, the geometry of the flow of singularly perturbed ordinary differential equations does not generalize to FDEs.

1. INTRODUCTION

Consider singularly perturbed intial value problems for linear retarded functional differential equations (FDE) of the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 y(t) + A(x_t) + B(y_t) \\ \mu\dot{y}(t) &= C_0 x(t) + D_0 y(t) + C(x_t) + D(y_t) \end{aligned} \qquad (1_\mu)$$

where $t, \mu \in R^+$, $x(t) \in R^m$, $y(t) \in R^n$ $(n > 1)$, the delays lie in the interval $[-r, 0]$ for some fixed $0 < r \leqslant \infty$, x_t, y_t are functions defined on $[-r, 0]$ by $x_t(\theta) = x(t+\theta)$, $y_t(\theta) = y(t+\theta)$, and A, B, C, D are linear operators defined on an appropriate function space. More precisely,

$$A(\phi) = \int_{-r}^{0} a(\theta)\,\phi(\theta)\,d\theta + \sum_{k=1}^{h} A_k \phi(-\omega_k) \qquad (2)$$

and similarly for B, C, D, where a, b, c, d admit exponential bounds $|a(\cdot)|$,

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$|b(\cdot)|$, $|c(\cdot)|$, $|d(\cdot)| \leqslant K_0 e^{\gamma_0 \cdot}$, A_k, B_k, C_k, D_k are real matrices with D_0 having no eigenvalue with zero real part, and the concentrated delays ω_k satisfy $0 < \omega_1 < \omega_2 < \cdots < \omega_h$ for some nonnegative integer h.

When (1_μ) is an ODE, i.e., $A = B = C = D = 0$, the phase space can be decomposed as a direct sum of two invariant manifolds $R^{m+n} = \Sigma_\mu \oplus \Gamma_\mu$, where Σ_μ is the linear span of the generalized eigenspaces of the coefficient matrix which are associated with eigenvalues staying bounded as $\mu \to 0^+$ and Γ_μ is the linear span of the generalized eigenspaces associated with eigenvalues which are unbounded as $\mu \to 0^+$. It is customary to call Σ_μ the "slow" manifold and Γ_μ the "fast" manifold of the system as $\mu \to 0^+$. The manifolds Σ_μ approach the manifold Σ_0 of orbits of the degenerate system (1_0) as $\mu \to 0^+$.

The present paper investigates the existence of "slow" and "fast" manifolds and the existence of invariant manifolds Σ_μ which approach Σ_0 as $\mu \to 0^+$. It will be seen that these manifolds do not, in general, exist for FDEs. Using a change of variables introduced by Chang [2], one may take without loss of generality $B_0 = C_0 = 0$, provided A_0, D_0, A, B, C, D are allowed to depend continuously on μ. Then, necessary and sufficient conditions for the existence of invariant manifolds Σ_μ, approaching Σ_0 as $\mu \to 0^+$, can be given. These conditions are established using ideas introduced, for the particular case where $m = n = 1$, in [5]. In particular, when all the eigenvalues of D_0 have negative real parts, for such manifolds to exist the perturbed equation in system (1_μ) must be an ODE. Thus the geometry of the phase space for singularly perturbed ODEs does not generalize to FDEs.

However, in the particular cases for which there exist invariant linear manifolds approaching the manifold of orbits of the degenerate system, if the coefficient functions appearing in the system are sufficiently smooth those manifolds are "slow," with the solutions with initial data on them "slowly" approaching solutions of the degenerate system in a sense to be made precise in the text.

In the first part of the paper systems with bounded delays are considered, and the case where concentrated delays are not present in the system is discussed separately from the case when they may occur. This is done because it is easier to conduct the discussion on Hilbert spaces defined in terms of square-integrable functions, and, when concentrated delays are present, the manifold of orbits of the degenerate system is not closed in such a space. Systems involving concentrated delays are discussed in phase spaces of continuous functions.

The last two sections are dedicated to systems with unbounded delays. The necessary and sufficient condition for existence of the invariant manifolds Σ_μ approaching Σ_0, as $\mu \to 0^+$, mentioned above for systems with bounded delays, is also sufficient in the case of unbounded delays, but it is no longer necessary. Actually, when this condition does not hold infinitely many families of manifolds Σ_μ, depending continuously on μ for $\mu > 0$ small, may exist in the phase spaces considered.

2. Partial Decoupling

By a linear change of coordinates, system (1_μ) can be transformed into a system where the coupling is done only through delayed values of the variables. This fact can be stated in the following form.

LEMMA 1. *There exist* $\mu_0 > 0$, *and matrices* $R = R(\mu)$, $S = S(\mu)$ *depending continuously on* μ *for* $0 \leqslant \mu \leqslant \mu_0$, *satisfying* $R(0) = D_0^{-1}C_0$ *and* $S(0) = -B_0 S_0^{-1}$, *such that the change of variables*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I_m, & -\mu S \\ -R, & I_n + \mu RS \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{3}$$

where R, S *are solutions as* $\mu \to 0^+$ *of the system*

$$\begin{aligned} D_0 R - \mu RA_0 + \mu RB_0 R - C_0 &= 0 \\ \mu[A_0 - B_0 R]S - S[D_0 + \mu RB_0] - B_0 &= 0 \end{aligned} \tag{4_μ}$$

transforms the system (1_μ) *into a system of the form*

$$\begin{aligned} \dot{v}(t) &= (A_0 - B_0 R(\mu))\, v(t) + \cdots \\ \mu \dot{w}(t) &= (D_0 + \mu R(\mu) B_0)\, w(t) + \cdots \end{aligned}$$

where the dots stand for the contribution of the delayed values of v *and* w.

Proof. The given change of coordinates was introduced by Chang in [2] for decoupling linear ordinary differential equations. We are interested in changes of variables, depending continuously on μ. Such a transformation of variables exists provided there exist solutions of (4_μ) depending continuously on μ in a neighborhood of the origin. The solutions of (4_μ) are the zeros of the function

$$H(R, S, \mu) = \begin{bmatrix} [D_0 R - C_0 - \mu(RA_0 - RB_0 R)]^T \\ -SD_0 - B_0 - \mu(SRB_0 - A_0 S + B_0 RS) \end{bmatrix}.$$

Clearly $H(R(0), S(0), 0) = 0$ for $R(0) = D_0^{-1} C_0$, $S(0) = -B_0 D_0^{-1}$. An application of the Implicit Function Theorem will then finish the proof.

This shows that, without loss of generality, we may assume $B_0 = C_0 = 0$ in (1_μ), provided we allow A_0, D_0, A, B, C, D to depend continuously on $\mu \geqslant 0$

in some neighborhood of the origin. To avoid overburdening the notation we omit the dependence of these elements on μ and use the same symbols as before. It follows directly from Lemma 1 that the new matrix $D_0 = D_0(\mu)$ equals the original D_0 at $\mu = 0$ and therefore has no eigenvalues with zero real part, and has the same number of eigenvalues with positive or negative real parts as D_0 at $\mu = 0$ does.

3. Slow and Fast Manifolds in Systems with Bounded Delays

We consider systems of the form (1_μ), with bounded delays ($r < +\infty$), and take for phase space either[1]

$$X = L^2((-r, 0); R^m) \times L^2((-r, 0); R^n) \times R^m \times R^n$$

or

$$Y = C([-r, 0]; R^m) \times C([-r, 0]; R^n).$$

The solutions of (1_μ) define a C_0-semigroup $T_0(t)$, $t \geqslant 0$, on the phase space, in the usual way (see [3, 4, 8]), whose infinitesimal generators are denoted by $\mathscr{A}_\mu$. The spectrum of $\mathscr{A}_\mu$, denoted by $\sigma(\mathscr{A}_\mu)$, consists only of eigenvalues (see [3, 7, 12]). The function of complex variable Δ_μ is defined as in [10] and is equal to the characteristic function of system (1_μ) with the bottom blocks multiplied by μ.

Definition 2. A one-parameter family $\{M_\mu\}_{\mu \in (0, \mu_0)}$ of submanifolds of the phase space is said to be *fast* under (1_μ) as $\mu \to 0^+$ if each M_μ is invariant under (1_μ) and is the span of linear manifolds lying in generalized eigenspaces of $\mathscr{A}_\mu$ which correspond to eigenvalues satisfying $|\lambda_\mu| \to +\infty$ as $\mu \to 0^+$. The family $\{M_\mu\}$ is said to be *slow* under (1_μ) as $\mu \to 0^+$ if each of the M_μ contains no nonzero elements of such generalized eigenspaces and is invariant under (1_μ).

Fast manifolds do not always exist. A simple example illustrating this is the linear delay equation

$$\mu \dot{y}(t) = -ay(t) + by(t-1), \qquad \text{with} \quad a, b > 0.$$

In fact, each one of the characteristic values λ of this equation stays, as $\mu \to 0^+$, in the same strip $2k\pi \leqslant \operatorname{Im} \lambda \leqslant 2(k+1)\pi$, for some integer k, and approaches a point on the line $\operatorname{Re} \lambda = \ln(b/a)$.

[1] The ambiguity involved in (2) when concentrated delays are present, due to the fact that the elements of X are equivalence classes defined in terms of functions that differ in a set of measure zero, is resolved by considering weak solutions as explained in [11].

A characterization of the systems for which there exists a direct sum decomposition of the phase space in a fast manifold and a slow manifold has not been achieved for the general case of system (1_μ). The following proposition answers this question for some particular cases and summarizes some other facts related to the problem.

PROPOSITION 3. (i) *A sufficient condition for the existence of a decomposition of the phase space in a slow and a fast manifold is* $D = B \equiv 0$ *or* $D = C \equiv 0$. *Under this condition the fast manifold has dimension* n.

If $\operatorname{Re} \sigma(D_0) < 0$ *and either* (1) $n = 1$ *and* $m = 0$ *or* (2) $n = m = 1$, *then the condition is also necessary.*

(ii) *If* $\sigma(\mathscr{A}_\mu)$ *is finite, then*

$$\sigma(\mathscr{A}_\mu) = \sigma(A_0) \cup (1/\mu)\, \sigma(D_0)$$

and there exists a decomposition of the phase space in a slow manifold and a fast manifold. The fast manifold has dimension n.

Proof. (i) Assume $D = B \equiv 0$ or $D = C \equiv 0$. Then

$$\det \Delta_\mu(\lambda) = \det[\lambda I_m - A_0 - A(e^{\lambda \cdot})] \det[\mu\lambda I_n - D_0].$$

Defining Γ_μ to be the linear span of the generalized eigenspaces of $\mathscr{A}_\mu$ associated with the eigenvalues lying in $(1/\mu)\, \sigma(D_0)$, and Σ_μ to be a complementary subspace containing the span of the remaining generalized eigenspaces of $\mathscr{A}_\mu$, we have the phase space decomposed as $\Sigma_\mu \oplus \Gamma_\mu$. The results in [3, 7, 12] guarantee that Γ_μ is a linear manifold invariant under (1_μ), and that there exists a linear manifold Σ_μ, also invariant under (1_μ), such that the phase space is decomposed as $\Sigma_\mu \oplus \Gamma_\mu$. Clearly Γ_μ is fast, Σ_μ is slow, and $\dim \Gamma_\mu = n$.

Assume $\operatorname{Re} \sigma(D_0) < 0$ with $n = 1$ and $m = 0$, then

$$\det \Delta_\mu(\lambda) = \mu\lambda - D_0 - D(e^{\lambda \cdot}).$$

If there exists a fast manifold under (1_μ) as $\mu \to 0^+$, then there exists $\lambda_\mu \in \sigma(\mathscr{A}_\mu)$ such that $|\lambda_\mu| \to +\infty$ as $\mu \to 0^+$. If $\operatorname{Re} \lambda_\mu \to +\infty$ or $\operatorname{Re} \lambda_\mu$ were bounded, we would have

$$\lambda_\mu \in [(1/\mu)\, \sigma(D_0) + o(1/\mu)] \qquad \text{as} \quad \mu \to 0^+$$

which contradicts $\operatorname{Re} \sigma(D_0) < 0$. Consequently, $\operatorname{Re} \lambda_\mu \to -\infty$. Comparing the orders of growth of the terms in the equation $\det \Delta_\mu(\lambda_\mu) = 0$, as $\mu \to 0^+$, we get $\det(\mu\lambda_0 - D_0) = 0$ and, therefore, $\lambda_\mu \in (1/\mu)\, \sigma(D_0)$, which implies $D \equiv 0$.

Assume now that $\operatorname{Re}\sigma(D_0) < 0$ with $m = n = 1$. Then

$$\det \Delta_\mu(\lambda) = [\lambda - A_0 - A(e^{\lambda\cdot})][\mu\lambda - D_0 - D(e^{\lambda\cdot})] - B(e^{\lambda\cdot})\,C(e^{\lambda\cdot})$$

and reasoning as above we get $D = B \equiv 0$ or $D = C \equiv 0$.

(ii) It can be proved, as done by Henry in [6] (see also [12]), that $\sigma(\mathscr{A}_\mu)$ is finite if and only if $\det \Delta_\mu$ is a polynomial. It then follows that

$$\det \Delta_\mu(\lambda) = \det(\lambda I_m - A_0)\det(\mu\lambda I_n - D_0).$$

Consequently, $\sigma(\mathscr{A}_\mu) = \sigma(A_0) \cup (1/\mu)\,\sigma(D_0)$, and the desired decomposition of the phase space follows as in the proof of (i).

The second statement contained in the preceding proposition establishes that a sufficient condition for the decomposition of the phase space in a slow manifold and a fast manifold is that the spectrum of the infinitesimal generator be finite. This is, in fact, a very degenerate situation which was discussed by Henry in [6] in connection with the existence of small solutions. Henry showed that $\sigma(\mathscr{A}_\mu)$ is finite if and only if system (1_μ) is equivalent to an ODE, in the sense that there exists a constant matrix K_μ such that any solution $z_\mu = (x_\mu, y_\mu)$ of (1_μ) satisfies $\dot{z}_\mu(t) = K_\mu z_\mu(t)$, for $t \geqslant r(m+n-1)$. In this case, it is possible to obtain more detailed information on the manifolds Σ_μ appearing in the decomposition of the phase space in a slow manifold and a fast manifold. In particular, the Σ_μ can be decomposed as $\Sigma_\mu = P_\mu \oplus Q_\mu$, where P_μ is the span of the generalized eigenspaces of $\mathscr{A}_\mu$ associated with eigenvalues $\lambda_\mu \in \sigma(A_0)$. If $\phi \in Q_\mu$, then $T_\mu(t)\phi = o(e^{\gamma t})$ as $t \to +\infty$, for all $\gamma < \operatorname{Re}\sigma(\mathscr{A}_\mu)$, and the solution of (1_μ) with initial condition ϕ at $t = 0$ vanishes for $t \geqslant r(m+n-1) - \tau$, where τ is the exponential type of $\det \Delta_\mu$, i.e., $\tau = \overline{\lim}_{|\lambda| \to \infty} (1/\lambda) \log|\det \Delta_\mu(\lambda)|$.

EXAMPLE 4. Let us consider the system

$$\mu\dot{y}(t) = \begin{bmatrix} D_{11}, & D_{12} \\ 0, & D_{22} \end{bmatrix} y(t) + \begin{bmatrix} 0, & 1 \\ 0, & 0 \end{bmatrix} y(t-1) \tag{5_μ}$$

where $y = (u, v) \in R^2$, $\mu \in R^+$, $D_{ij} \in R$, $D_{11}, D_{22} \neq 0$, and $D_{11} \neq D_{22}$, with the initial condition

$$(u_0, v_0) = (\phi, \psi) \in Y \tag{6}$$

and $Y = C([-1, 0]; R^2)$.

The solution of the initial value problem (5_μ)–(6) can be easily computed. In fact, we have

$$\mu\dot{u}(t) = D_{11}u(t) + D_{12}v(t) + v(t-1)$$

$$\mu\dot{v}(t) = D_{22}v(t)$$

and therefore the solution of the initial value problem is such that

$$v_\mu(t) = \begin{cases} \psi(t), & t < 0 \\ e^{D_{22}t/\mu}\psi(0), & t \geqslant 0 \end{cases}$$

and

$$u_\mu(t) = \begin{cases} \phi(t), & t < 0 \\ e^{D_{11}t/\mu}\phi(0) + \dfrac{D_{12}}{D_{22}-D_{11}}\left(e^{D_{22}t/\mu} - e^{D_{11}t/\mu}\right)\psi(0) \\ \qquad + \dfrac{1}{\mu}\displaystyle\int_0^t e^{D_{11}(t-s)/\mu}\psi(s-1)\,ds, & 0 \leqslant t < 1 \\ e^{D_{11}t/\mu}\phi(0) + \dfrac{D_{12}}{D_{22}-D_{11}}\left(e^{D_{22}t/\mu} - e^{D_{11}t/\mu}\right)\psi(0) \\ \qquad + \dfrac{1}{\mu}\displaystyle\int_0^1 e^{D_{11}(t-s)/\mu}\psi(s-1)\,ds \\ \qquad + \dfrac{1}{D_{22}-D_{11}}\left(e^{D_{22}(t-1)/\mu} - e^{D_{11}(t-1)/\mu}\right)\psi(0), & t \geqslant 1. \end{cases}$$

Clearly, the spectrum of the infinitesimal generator $\mathscr{A}_\mu$ of the semigroups defined by the solutions of (5_μ) is

$$\sigma(\mathscr{A}_\mu) = \left\{\frac{D_{11}}{\mu}, \frac{D_{22}}{\mu}\right\}.$$

The correspondent eigenspaces are

$$E\left(\frac{D_{11}}{\mu}\right) = \{(\phi, \psi) \in Y\colon \phi(\theta) = e^{D_{11}\theta/\mu}\alpha,\ \psi \equiv 0,\ \alpha \in R\}$$

$$E\left(\frac{D_{22}}{\mu}\right) = \left\{(\phi, \psi) \in Y\colon \phi(\theta) = e^{D_{22}\theta/\mu}\left(\frac{D_{12} + e^{-D_{22}/\mu}}{D_{22}-D_{11}}\right)\beta,\right.$$
$$\left.\psi(\theta) = e^{D_{22}\theta/\mu}\beta, \beta \in R\right\}.$$

Consequently, there exists a maximal fast manifold

$$\Gamma_\mu = \left\{(\phi, \psi) \in Y\colon \phi(\theta) = e^{D_{11}\theta/\mu}\alpha + e^{D_{22}\theta/\mu}\left(\frac{D_{12} + e^{-D_{22}/\mu}}{D_{22}-D_{11}}\right)\beta,\right.$$
$$\left.\psi(\theta) = e^{D_{22}\theta/\mu}\beta,\ \alpha, \beta \in R\right\}.$$

For complementary manifold we may take

$$\Sigma_\mu = \left\{(\phi, \psi) \in Y: \phi(0) = -\frac{1}{\mu}\int_{-1}^{0} e^{-D_{11}(\theta+1)/\mu}\psi(\theta)\,d\theta,\ \psi(0) = 0\right\}.$$

On the other hand, the manifold of the orbits of the degenerate system is

$$\Sigma_0 = \left\{(\phi, \psi) \in Y: \phi(0) = -\frac{1}{D_{11}}\psi(-1),\ \psi(0) = 0\right\}.$$

Clearly, the following facts are true.

(i) $\Sigma_\mu \oplus \Gamma_\mu = Y$.

(ii) Γ_μ is the linear span of the generalized eigenspaces associated with the unbounded (as $\mu \to 0^+$) eigenvalues of $\mathscr{A}_\mu$. Consequently Γ_μ is fast and Σ_μ is slow.

(iii) The solutions of (5_μ) with initial data in Σ_μ vanish for $t > 1$.

(iv) Any solution y_μ satisfies, for $t \geqslant 1$, the ODE

$$\mu \dot{y}(t) = \begin{bmatrix} D_{11}, & D_{12} + e^{-D_{22}/\mu} \\ 0, & D_{22} \end{bmatrix} y(t).$$

4. Invariant Manifolds Approaching the Manifold of Orbits of the Degenerate System. Systems with Bounded Distributed Delays

It is easier to apply the methods of this section in Hilbert spaces. Accordingly, we take for phase space

$$X = L^2((-r, 0); R^m) \times L^2((-r, 0); R^n) \times R^m \times R^n,$$

with $r < +\infty$, and consider systems in the form (1_μ) with

$$A(\phi) = \int_{-r}^{0} a(\theta)\,\phi(\theta)\,d\theta \tag{7}$$

and similarly for B, C, D, where the functions a, b, c, d satisfy the hypotheses given in the Introduction.

The set of orbits of the degenerate system (1_0) is

$$\Sigma_0 = \{(\phi, \psi, \alpha, \beta) \in X: D_0\beta + C(\phi) + D(\psi) = 0\}.$$

It is a closed linear submanifold of X with codimension n.

It is known that, for $\mu > 0$, the solutions of (1_μ) define a C_0-semigroup $T_\mu(t)$, $t \geqslant 0$, on X by

$$T_\mu(t)(\phi, \psi, \alpha, \beta) = (x_t, y_t, x(t), y(t)), \qquad t \geqslant 0 \tag{8}$$

where (x, y) denotes the solution of (1_μ) with initial condition $(\phi, \psi, \alpha, \beta)$ at $t = 0$. It is possible to associate with the degenerate system a neutral system (obtained from (1_0) by differentiation of the second equation) also defining a C_0-semigroup $T_0(t)$, $t \geqslant 0$, on X by (8). The infinitesimal generators of the T_μ are denoted by $\mathscr{A}_\mu$.

THEOREM 5. *A necessary and sufficient condition for the existence of a family of closed linear manifolds Σ_μ which are invariant under (1_μ) and satisfy $\Sigma_\mu \to \Sigma_0$ as $\mu \to 0^+$ is:*

(C) *The ranges of the matrices $c(\theta)$ and $d(\theta)$ are included in the linear span of the generalized eigenspaces associated with the eigenvalues of D_0 which have positive real parts, a.e. in $\theta \in (-r, 0)$.*

Proof. Denote the inner product in X by

$$\langle (g, h, \delta, \gamma), (\phi, \psi, \alpha, \phi) \rangle = \delta \cdot \alpha + \gamma \cdot \beta + \int_{-r}^{0} g \cdot \phi + \int_{-r}^{0} h \cdot \psi$$

where the dots denote the euclidean inner product.

With Σ_0 we can associate the functions $G_0 = D_0^{-1} c$, $H_0 = D_0^{-1} d$, so that the rows of $(G_0, H_0, 0, I_n)$, where 0 denotes the $n \times m$ null matrix, are normal to Σ_0, i.e.,

$$\Sigma_0 = \left\{ (\phi, \psi, \alpha, \beta) \in X \colon \beta + \int_{-r}^{0} G_0(\theta)\, \phi(\theta)\, d\theta + \int_{-r}^{0} H_0(\theta)\, \psi(\theta)\, d\theta = 0 \right\}.$$

For the Σ_μ to be closed linear manifolds, invariant under (1_μ) and satisfying $\Sigma_\mu \to \Sigma_0$ as $\mu \to 0^+$, there must exist matrix-valued functions G_μ, H_μ[2] whose entries belong to $L^2((-r, 0); R)$ and with dimensions $n \times m$ and $n \times n$, respectively, and an $n \times m$ matrix δ_μ, so that the rows of (G, H, δ, I_n) are normal to Σ_μ, i.e.,

$$\Sigma_\mu = \left\{ (\phi, \psi, \alpha, \beta) \in X \colon \delta\alpha + \beta + \int_{-r}^{0} G(\theta)\, \phi(\theta)\, d\theta + \int_{-r}^{0} H(\theta)\, \psi(\theta)\, d\theta = 0 \right\}.$$

[2] To avoid overburdening the notation, the dependence on μ is dropped when there is no danger of confusion.

Then, the solutions of (1_μ) through initial data $(\phi, \psi, \alpha, \beta) \in \Sigma_\mu$ at $t=0$ satisfy for $t \geqslant 0$

$$\delta x(t) + y(t) + \int_{-r}^{0} G(\theta)\, x(t+\theta)\, d\theta + \int_{-r}^{0} H(\theta)\, y(t+\theta)\, d\theta = 0. \tag{9}$$

It is possible to associate with (9) another equation (obtained through differentiation) which together with the first equation of (1_μ) defines the system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A(x_t) + B(y_t) \\ \dot{y}(t) &= -\delta[A_0 x(t) + A(x_t) + B(y_t)] \\ &\quad - \int_{-r}^{0} G(\theta)\, \dot{x}(t+\theta)\, d\theta - \int_{-r}^{0} H(\theta)\, \dot{y}(t+\theta)\, d\theta. \end{aligned} \tag{10}$$

For Σ_μ to be invariant under (1_μ), the solutions of (1_μ) with initial data in $\Sigma_\mu \cap \mathscr{D}(\mathscr{A}_\mu)$ must agree with the solutions of (10) having the same initial data (note that the solutions of (1_μ) with initial data in $\mathscr{D}(\mathscr{A}_\mu)$ are absolutely continuous). The solutions of the neutral equation (10) define a C_0-semigroup $S_{G,H,\delta}(t)$, $t \geqslant 0$ on X, in a similar way as the semigroup T_0 was defined (see [1] and [11]). Its infinitesimal generator, denoted by $\mathscr{B}_{G,H,\delta}$, is explicitly given by

$$\mathscr{D}(\mathscr{B}_{G,H,\delta}) = \mathscr{D}(\mathscr{A}_\mu)$$

$$\mathscr{B}_{G,H,\delta}(\phi, \psi, \alpha, \beta) = \Big(\phi', \psi', A_0\alpha + A(\phi) + B(\psi),$$
$$- \delta[A_0\alpha + A(\phi) + B(\psi)] - \int_{-r}^{0} G\phi' - \int_{-r}^{0} H\psi'\Big)$$

and must agree with $\mathscr{A}_\mu$ on Σ_μ, i.e.,

$$\mathscr{A}_\mu(\phi, \psi, \alpha, \beta) = \mathscr{B}_{G,H,\delta}(\phi, \psi, \alpha, \beta)$$

for $(\phi, \psi, \alpha, \beta) \in \mathscr{D}(\mathscr{A}_\mu) \cap \Sigma_\mu = \mathscr{D}(\mathscr{B}_{G,H,\delta}) \cap \Sigma_\mu$, or

$$\frac{D_0}{\mu}\beta + \int_{-r}^{0} \frac{c}{\mu}\phi + \int_{-r}^{0} \frac{d}{\mu}\psi$$
$$= -\delta\left(A_0\alpha + \int_{-r}^{0} a\phi + \int_{-r}^{0} b\psi\right) - \int_{-r}^{0} G\phi' - \int_{-r}^{0} H\psi'$$

for all $(\phi, \psi, \alpha, \beta) \in \mathscr{D}(\mathscr{A}_\mu) \cap \Sigma_\mu$. But this implies that G and H are absolutely continuous, with the entries of G', H' belonging to L^2. Using integration by parts, one gets

$$\begin{aligned}
&\frac{D_0}{\mu}\beta+\int_{-r}^{0}\frac{c}{\mu}\phi+\int_{-r}^{0}\frac{d}{\mu}\psi \\
&\quad=-\delta\left(A_0\alpha+\int_{-r}^{0}a\phi+\int_{-r}^{0}b\psi\right)-G(0)\,\alpha-H(0)\,\beta \\
&\qquad+G(-r)\,\phi(-r)+H(-r)\,\psi(-r)+\int_{-r}^{0}G'\phi+\int_{-r}^{0}H'\psi. \qquad (11)
\end{aligned}$$

Since $(\phi, \psi, \alpha, \beta)\in\Sigma_\mu$, we get from (9)

$$\beta=-\delta\alpha-\int_{-r}^{0}G\phi-\int_{-r}^{0}H\psi.$$

Substituting β, as given in this equation, into (11), we obtain the relations

$$\begin{aligned}
&G'+\left(\frac{D_0}{\mu}+H(0)\right)G=\delta a+\frac{c}{\mu} \\
&H'+\left(\frac{D_0}{\mu}+H(0)\right)H=\delta b+\frac{d}{\mu} \\
&\delta A_0-\left(\frac{D_0}{\mu}+H(0)\right)\delta+G(0)=0 \\
&G(-r)=H(-r)=0.
\end{aligned}\qquad (12)$$

Denoting $\Lambda_\mu=D_0+\mu H_\mu(0)$, the first two equations in (12) can be solved, with initial conditions $G(-r)=H(-r)=0$, giving

$$\begin{aligned}
G_\mu(\theta)&=\int_{-r}^{\theta}e^{-\Lambda_\mu(\theta-s)/\mu}\left(\delta a(s)+\frac{c(s)}{\mu}\right)ds \\
H_\mu(\theta)&=\int_{-r}^{\theta}e^{-\Lambda_\mu(\theta-s)/\mu}\left(\delta b(s)+\frac{d(s)}{\mu}\right)ds.
\end{aligned}\qquad (13)$$

Since $\Lambda_\mu=D_0+\mu H(0)$, system (12) has a solution if and only if the following equations can be solved for Λ, δ with μ small:

$$\Lambda-D_0=\int_{-r}^{0}e^{\Lambda s/\mu}(\mu\,\delta b(s)+d(s))\,ds \qquad (14)$$

$$\mu\,\delta A_0-\Lambda\delta=-\int_{-r}^{0}e^{\Lambda s/\mu}(\mu\,\delta a(s)+c(s))\,ds. \qquad (15)$$

For the Σ_μ to approach Σ_0 as $\mu \to 0^+$, we must have $\delta_\mu \to 0$, $G_\mu \to G_0$, $H_\mu \to H_0$ as $\mu \to 0^+$. The last two relations imply $\mu H_\mu(0) = o(1)$ as $\mu \to 0^+$ and so $A_\mu = D_0 + o(1)$ as $\mu \to 0^+$. Without loss of generality, we may assume D_0 is in Jordan canonical form

$$D_0 = \begin{bmatrix} D_0^+ & 0 \\ 0 & D_0^- \end{bmatrix} = \left[\begin{array}{ccc|ccc} J_1^+ & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & J_k^+ & & & \\ \hline & & & J_1^- & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & J_j^- \end{array}\right]$$

where the J_k^+ (J_j^-) are Jordan blocks corresponding to eigenvalues with positive (negative) real parts. For Eq. (14) to have a solution $A_\mu = D_0 + o(1)$ as $\mu \to 0^+$, it is necessary that the lower blocks of the function d, in a partition corresponding to the above partition of D_0, vanish. For Eq. (15) to have such a solution with $\delta_\mu \to 0$ as $\mu \to 0^+$, it is necessary that the lower block of c vanish. Consequently, Condition (C) must be satisfied. This proves the necessity of Condition (C) in the statement.

Assume now that Condition (C) holds. We partition A, δ, b, c, and d according to the partition of D_0, as

$$A = \begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix}, \qquad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$b = [b_1, \quad b_2], \qquad c = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}, \qquad d = \begin{bmatrix} d_{11}, & d_{12} \\ 0, & 0 \end{bmatrix}.$$

Take $A_{21} = 0$, $\delta_2 = 0$. Then

$$e^A = \begin{bmatrix} e^{A_{11}}, & x \\ 0, & e^{A_{22}} \end{bmatrix},$$

and therefore Eq. (14) would be equivalent to the system

$$A_{11} - D_0^+ = \int_{-r}^{0} e^{A_{11}s/\mu}(\mu\delta_1 b_1(s) + d_{11}(s))\, ds \tag{16}$$

$$A_{12} = \int_{-r}^{0} e^{A_{11}s/\mu}(\mu\delta_1 b_2(s) + d_{12}(s))\, ds \tag{17}$$

$$A_{22} - D_0 = 0 \tag{18}$$

and Eq. (15) would be equivalent to

$$\mu\delta_1 A_0 - A_{11}\delta_1 = -\int_{-r}^{0} e^{A_{11}s/\mu}(\mu\delta_1 a(s) + c_1(s))\, ds. \tag{19}$$

Let us now consider the function F, defined for $k \times k$ matrices M having all eigenvalues with positive real parts, $k \times m$ matrices σ, and $\mu > 0$ by

$$F(M, \sigma, \mu) = \begin{bmatrix} M - D_0^+ - \int_{-r}^{0} e^{Ms/\mu}(\mu\sigma b_1(s) + d_{11}(s))\, ds \\ \mu\sigma A_0 - M\sigma + \int_{-r}^{0} e^{Ms/\mu}(\mu\sigma a(s) + c_1(s))\, ds \end{bmatrix} \tag{20}$$

and for $\mu = 0$ by

$$F(M, \sigma, 0) = \begin{bmatrix} M - D_0^+ \\ -M\sigma \end{bmatrix}. \tag{21}$$

The function F is continuous on its domain, and we have the Fréchet derivative $\partial F(M, \sigma, \mu)/\partial(M, \sigma)$ satisfying for $\mu > 0$

$$\frac{\partial F}{\partial(M, \sigma)}(M, \sigma, \mu)(L, \lambda)$$
$$= \begin{bmatrix} L - \int_{-r}^{0} e^{Ms/\mu} \frac{Ls}{\mu}(\mu\sigma b_1(s) + d_{11}(s))\, ds, & -\int_{-r}^{0} e^{Ms/\mu}\mu\lambda b_1(s)\, ds \\ -L\sigma + \int_{-r}^{0} e^{Ms/\mu} \frac{Ls}{\mu}(\mu\sigma a(s) + c_1(s))\, ds, & \mu\lambda A_0 - M\lambda + \int_{-r}^{0} e^{Ms/\mu}\mu\lambda a(s)\, ds \end{bmatrix}$$

and for $\mu = 0$

$$\frac{\partial F}{\partial(M, \sigma)}(M, \sigma, 0)(L, \lambda) = \begin{bmatrix} L, & 0 \\ -L\sigma, & -M\lambda \end{bmatrix}.$$

We also have

$$\lim_{\mu \to 0+} \frac{\partial F}{\partial(M, \sigma)}(M, \sigma, \mu)(L, \lambda) = \begin{bmatrix} L, & 0 \\ -L\sigma, & -M\lambda \end{bmatrix}.$$

The Implicit Function Theorem can then be applied to get unique functions $M^*(\mu)$, $\sigma^*(\mu)$, defined for $\mu \geqslant 0$ small and depending continuously on μ, such that

$$F(M^*(\mu), \sigma^*(\mu), \mu) = 0, \qquad M^*(0) = D_0^+, \qquad \sigma^*(0) = 0.$$

This proves that the system consisting of Eqs. (16) and (19) has a unique solution $(A_{11}^*(\mu), \delta_1^*(\mu))$ for μ small, which depends continuously on μ and satisfies

$$A_{11}^*(\mu) = D_0^+ + o(1), \qquad \delta_1^*(\mu) = o(1) \qquad \text{as} \quad \mu \to 0^+.$$

It follows, as a consequence, that system (16)–(19) has a unique solution $(A_{11}(\mu), \delta(\mu))$ with

$$A(\mu) = \begin{bmatrix} A_{11}^*(\mu), & \int_{-r}^{0} e^{A_{11}^*(\mu)s/\mu}(\mu\delta_1^*(\mu)\, b_2(s) + d_{12}(s))\, ds \\ 0, & D_0^- \end{bmatrix} \tag{22}$$

and

$$A(\mu) = D_0 + o(1), \qquad \delta(\mu) = o(1) \qquad \text{as} \quad \mu \to 0^+. \tag{23}$$

This proves that, under Condition (C), the system (14)–(15) has at least one solution $(A(\mu), \delta(\mu))$ satisfying (22) and (23). Consequently, system (12) also has at least one solution $(G_\mu, H_\mu, \delta_\mu)$ for $\mu > 0$ small, which depends continuously on μ and satisfies $G_\mu \to G_0$, $H_\mu \to H_0$, $\delta_\mu \to 0$ as $\mu \to 0^+$. Therefore, there exist closed linear manifolds Σ_μ, invariant under (1_μ), such that $\Sigma_\mu \to \Sigma_0$ as $\mu \to 0^+$. This finishes the proof of the statement.

Remark 6. If the matrix D_0 has eigenvalues with positive real parts, then there are solutions of (1_μ) which do not converge to solutions of the degenerate equation as $\mu \to 0^+$. On the other hand, as shown in [11], if all the eigenvalues of D_0 have negative real parts, then all solutions of (1_μ) converge to solutions of the degenerate system, as $\mu \to 0^+$. In the latter case, it follows from the preceding theorem that closed linear manifolds Σ_μ, approaching Σ_0 as $\mu \to 0^+$, exist if and only if the second equation in (1_μ) is an ODE, i.e., system (1_μ) is equivalent to

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A(x_t) + B(y_t) \\ \mu\dot{y}(t) &= D_0\, y(t). \end{aligned} \tag{24}$$

We conclude that the geometry of the phase space, associated with singularly perturbed linear ODEs, is not preserved when we pass to retarded FDEs, unless the perturbed equation can itself be written as an ODE. This is an interesting fact, since it provides perhaps the first example of a geometric result valid for ODEs which does not hold in the analogous situation for retarded FDEs.

In case the manifolds Σ_μ with the properties described above exist, it is of interest to inquire how they approach Σ_0 as $\mu \to 0^+$. The following results address this question.

LEMMA 7. *If Condition* (C) *holds, then, for* $\mu \geqslant 0$ *small and arbitrary* $\phi, \psi \in L^2$, $\alpha \in R^m$, *the solutions of* (1_μ) *with initial data on the manifold* Σ_μ, *established in Theorem* 5, *and satisfying* $x(0) = \alpha$, $x(\theta) = \phi(\theta)$, *and* $y(\theta) = \psi(\theta)$ *for* $-r < \theta < 0$ *converge uniformly on compact subsets of* $[0, \infty)$ *as* $\mu \to 0^+$ *to the solution of* (1_0) *which satisfies the same conditions.*

Proof. Let (x_μ, y_μ) denote the solution of (1_μ) satisfying the initial conditions given in the statement, and denote $\bar{x}_\mu = x_\mu - x_0$, $\bar{y}_\mu = y_\mu - y_0$. Then, using the notation in the proof of Theorem 5, we know that $\bar{x}_\mu$ and $\bar{y}_\mu$ must satisfy

$$\dot{\bar{x}}_\mu(t) = A_0 \bar{x}_\mu(t) + \int_{-r}^0 a(\theta)\, \bar{x}_\mu(t+\theta)\, d\theta + \int_{-r}^0 b(\theta)\, \bar{y}_\mu(t+\theta)\, d\theta$$

$$\begin{aligned}\bar{y}_\mu(t) = &-\delta_\mu \bar{x}_\mu(t) - \int_{-r}^0 G_\mu(\theta)\, \bar{x}_\mu(t+\theta)\, d\theta - \int_{-r}^0 H_\mu(\theta)\, \bar{y}_\mu(t+\theta)\, d\theta \\ &- \delta_\mu x_0(t) - \int_{-r}^0 [G_\mu(\theta) - G_0(\theta)]\, x_0(t+\theta)\, d\theta \\ &- \int_{-r}^0 [H_\mu(\theta) - H_0(\theta)]\, y_0(t+\theta)\, d\theta\end{aligned}$$

$$\bar{x}_\mu(0) = 0, \qquad \bar{x}_\mu(\theta) = 0, \qquad \bar{y}_\mu(\theta) = 0 \qquad \text{for} \quad -r < \theta < 0.$$

Extending a, b, G_μ, H_μ to be zero outside the interval $[-r, 0]$ we can write the above equation as

$$\dot{\bar{x}}_\mu(t) - A_0 \bar{x}_\mu(t) = \int_0^t a(\theta - t)\, \bar{x}_\mu(\theta)\, d\theta + \int_0^t b(\theta - t)\, \bar{y}_\mu(\theta)\, d\theta$$

$$\begin{aligned}\bar{y}_\mu(t) = &-\delta_\mu \bar{x}_\mu(t) - \int_0^t G_\mu(\theta - t)\, \bar{x}_\mu(\theta)\, d\theta - \int_0^t H_\mu(\theta - t)\, \bar{y}_\mu(\theta)\, d\theta \\ &- \delta_\mu x_0(t) - \int_0^t [G_\mu(\theta) - G_0(\theta)]\, x_0(t+\theta)\, d\theta \\ &- \int_0^t [H_\mu(\theta) - H_0(\theta)]\, y_0(t+\theta)\, d\theta.\end{aligned}$$

Consequently,

$$\bar{x}_\mu(t) = \int_0^t e^{A_0(t-s)} \int_0^s [a(\theta - s)\, \bar{x}_\mu(\theta) + b(\theta - s)\, \bar{y}_\mu(\theta)]\, d\theta\, ds.$$

Fix a compact interval $[0, T]$, $T > 0$, take $t \in [0, T]$, and let $\sigma > 0$ be the maximum absolute value of the real parts of the eigenvalues of A_0. By changing the order of integration above and using upper bounds for the terms involved, we get, for some $K_1 > 0$,

$$|\bar{x}_\mu(t)| < K_1 t e^{\sigma t} \int_0^t [|\bar{x}_\mu(\theta)| + |\bar{y}_\mu(\theta)|] \, d\theta.$$

Recaling from the proof of Theorem 5 that $\delta_\mu \to 0$, $G_\mu \to G_0$, $H_\mu \to H_0$ as $\mu \to 0^+$, we also get for some $K_2 > 0$

$$|\bar{y}_\mu(t)| \leqslant o(1) |\bar{x}_\mu(t)| + K_2 \int_0^t [|\bar{x}_\mu(\theta)| + |\bar{y}_\mu(\theta)|] \, d\theta$$

$$+ o(1) \left[|x_0(t)| + \int_0^t |y_0(\theta)| \, d\theta + \int_0^t |x_0(\theta)| \, d\theta \right]$$

uniformly in $t \in [0, T]$ as $\mu \to 0^+$.

Consider now a function f defined by

$$f(t) = \max\{|\bar{x}_\mu(t)|, |\bar{y}_\mu(t)|\}.$$

Then, for $t \in [0, T]$ we have

$$f(t) \leqslant \max \left\{ K_1 t e^{\sigma t} \int_0^t f(\theta) \, d\theta, \, o(1) K_1 t e^{\sigma t} \int_0^t f(\theta) \, d\theta + K_2 \int_0^t f(\theta) \, d\theta + o(1) \right\}$$

uniformly in $t \in [0, T]$ as $\mu \to 0^+$. Consequently, there exists $K > 0$ such that, for all $t \in [0, T]$ and $\mu > 0$ in some fixed neighborhood of zero, we have

$$f(t) \leqslant o(1) + K \int_0^t f(\theta) \, d\theta$$

uniformly in $t \in [0, T]$ as $\mu \to 0^+$. An application of Gronwall inequality gives $f(t) = o(1)$ uniformly in $t \in [0, T]$ as $\mu \to 0^+$, and therefore also

$$|\bar{x}_\mu(t)| = o(1), \qquad |\bar{y}_\mu(t)| = o(1)$$

uniformly in $t \in [0, T]$ as $\mu \to 0^+$.

Let $H^1((-r, 0); R)$ denote the Sobolev space of real functions defined on the interval $(-r, 0)$, which are square-integrable and have square-integrable first generalized derivative.

LEMMA 8. *If Condition* (C) *holds, the entries of the matrix-valued functions* c, d *are in* $H^1((-r, 0); R)$ *and* c', d' *are continuous at zero, and* δ_μ, G_μ, H_μ *are as in the proofs of Theorems* 5 *and* 6, *then, for* $\mu > 0$ *sufficiently small, the map* $\mu \to (\delta_\mu, G_\mu, H_\mu)$ *is continuously differentiable and the right derivatives of* G_μ, H_μ *at* $\mu = 0$ *vanish.*

Proof. Consider the function F defined in the proof of Theorem 5 by expressions (20)–(21). For $\mu > 0$ small and M a $k \times k$ matrix with (M, σ, μ) in the domain of F (M is nonsingular), we have

$$\frac{\partial F}{\partial \mu}(M, \sigma, \mu) = \begin{bmatrix} \int_{-r}^{0} e^{Ms/\mu} \left(\frac{Ms\sigma}{\mu} b_1(s) + \frac{Ms}{\mu^2} d_{11}(s) - \sigma b_1(s) \right) ds \\ \sigma A_0 - \int_{-r}^{0} e^{Ms/\mu} \left(\frac{Ms\sigma}{\mu} a(s) + \frac{Ms}{\mu^2} c_1(s) - \sigma a(s) \right) ds \end{bmatrix}.$$

Using integration by parts in the terms containing the factor $1/\mu^2$, we get

$$\frac{\partial F}{\partial \mu}(M, \sigma, \mu) = \begin{bmatrix} \int_{-r}^{0} e^{Ms/\mu} \left(\frac{Ms\sigma}{\mu} b_1(s) - \sigma b_1(s) \right) ds \\ \sigma A_0 - \int_{-r}^{0} e^{Ms/\mu} \left(\frac{Ms\sigma}{\mu} a(s) - \sigma a(s) \right) ds \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{\mu} e^{-Mr/\mu} r d_{11}(-r) - \frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} s \dot{d}_{11}(s)\, ds - \frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} d_{11}(s)\, ds \\ -\frac{1}{\mu} e^{-Mr/\mu} r c_1(-r) + \frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} s \dot{c}_1(s)\, ds + \frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} c_1(s)\, ds \end{bmatrix}.$$

Consequently,

$$\lim_{\mu \to 0+} \frac{\partial F}{\partial \mu}(M, \sigma, \mu) = \begin{bmatrix} -M^{-1} d_{11}(0) \\ \sigma A_0 + M^{-1} c_1(0) \end{bmatrix}.$$

On the other hand, the right derivative of $F(M, \sigma, \cdot)$ at zero is

$$\begin{aligned} \frac{\partial F}{\partial \mu}(M, \sigma, 0^+) &= \lim_{\mu \to 0+} \frac{1}{\mu} [F(M, \sigma, \mu) - F(M, \sigma, 0)] \\ &= \lim_{\mu \to 0+} \begin{bmatrix} -\frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} (\mu\sigma b_1(s) + d_{11}(s))\, ds \\ \sigma A_0 + \frac{1}{\mu} \int_{-r}^{0} e^{Ms/\mu} (\mu\sigma a(s) + c_1(s))\, ds \end{bmatrix} \\ &= \begin{bmatrix} -M^{-1} d_{11}(0) \\ \sigma A_0 + M^{-1} c_1(0) \end{bmatrix}. \end{aligned}$$

It follows that the map $\mu \to F(M, \sigma, \mu)$ is continuously differentiable for $\mu > 0$, has bounded right derivative at zero, and the limit of its derivative as $\mu \to 0^+$ is equal to the right derivative at zero. Consequently, there exists a continuously differentiable extension of F defined for μ in a neighborhood of zero. An Implicit Function Theorem argument gives that the functions $M^*(\mu)$, $\sigma^*(\mu)$ established in the proof of Theorem 4 are continuously differentiable for $\mu > 0$ small and have continuous right derivatives at $\mu = 0$.

This implies that the functions Λ_{11}^* and δ_1^*, established in the proof of Theorem 4, are continuously differentiable for $\mu > 0$ small. Taking into account the definition of the functions Λ, δ in terms of Λ_{11}^*, δ_1^* that was given in the proof of Theorem 5, and the definition of G, H by (13), it follows that the map $\mu \to (\delta_\mu, G_\mu, H_\mu)$ is continuously differentiable for $\mu > 0$ small.

Now, taking into account (13) and (22), we get

$$\begin{aligned}
\left.\frac{\partial G_\mu}{\partial \mu}\right|_{\mu=0^+}(\theta) &= \lim_{\mu\to 0^+} \frac{1}{\mu}\,[G_\mu(\theta) - G_0(\theta)] \\
&= \lim_{\mu\to 0^+} \left[\frac{1}{\mu^2}\int_{-r}^{\theta} e^{-\Lambda_\mu(\theta-s)/\mu}(\mu\delta a(s) + c(s))\,ds - \frac{1}{\mu} D_0^{-1}(\mu\delta a(\theta) + c(\theta))\right] \\
&= \lim_{\mu\to 0^+} \begin{bmatrix} \dfrac{1}{\mu^2}\displaystyle\int_{-r}^{\theta} e^{-\Lambda_{11}^*(\theta-s)/\mu}(\mu\delta_1 a(s) + c_1(s))\,ds \\ \qquad -\dfrac{1}{\mu}(D_0^+)^{-1}(\mu\delta_1 a(\theta) + c_1(\theta)) \\ 0 \end{bmatrix} \\
&= 0
\end{aligned}$$

and similarly for H_μ.

THEOREM 9. *If the hypotheses in Lemma 8 hold, then, for $\mu > 0$ sufficiently small, the flow of (1_μ) on the manifolds Σ_μ, established in Theorem 5, is "slow" in the sense that solutions of (1_μ) with initial data in Σ_μ converge uniformly on compact sets to solutions of the degenerate equation (1_0) with initial data in Σ_0, as $\mu \to 0^+$, and their time derivatives remain bounded as $\mu \to 0^+$. More precisely, if $(\phi, \psi, \alpha, \beta) \in \Sigma_\mu$ with ϕ, ψ, α fixed, then*

$$\beta = \beta_\mu = -\delta_\mu \alpha - \int_{-r}^{0} H_\mu \phi - \int_{-r}^{0} G_\mu \psi$$

and the solution of (1_μ) with initial condition $(\phi, \psi, \alpha, \beta)$ converges uniformly on compact sets to the solution of (1_0) with initial condition $(x_0, y_0, x(0)) = (\phi, \psi, \alpha)$, with $\beta_\mu \to \beta_0$, $d\beta_\mu/d\mu \to 0$ as $\mu \to 0^+$, and $|\dot{x}_\mu(t; \phi, \psi, \alpha, \beta_\mu)|$, $|\dot{y}_\mu(t; \phi, \psi, \alpha, \beta_\mu)|$ bounded on compact intervals as $\mu \to 0^+$.

Proof. After the two preceding lemmas, it only remains to prove the boundedness, as $\mu \to 0^+$, of the time derivatives of the solutions $x_\mu(t) = x_\mu(t; \phi, \psi, \alpha, \beta_\mu)$, $y_\mu(t) = y_\mu(t; \phi, \psi, \alpha, \beta_\mu)$ for $t \in [0, T]$, with $T > 0$ arbitrarily fixed.

Since the orbit of (x_μ, y_μ) lies on Σ_μ, using the notation in the proof of Theorem 5 we get from (1_μ)

$$\dot{x}_\mu(t) = A_0 x_\mu(t) + \int_{-r}^{0} a(\theta)\, x_\mu(t+\theta)\, d\theta + \int_{-r}^{0} b(\theta)\, y_\mu(t+\theta)\, d\theta$$

$$\begin{aligned}\dot{y}_\mu(t) &= D_0 y_\mu(t) + \int_{-r}^{0} c(\theta)\, x_\mu(t+\theta)\, d\theta + \int_{-r}^{0} d(\theta)\, y_\mu(t+\theta)\, d\theta \\ &= -D_0 \delta_\mu x_\mu(t) + \int_{-r}^{0} [c(\theta) - D_0 G_\mu(\theta)]\, x_\mu(t+\theta)\, d\theta \\ &\quad + \int_{-r}^{0} [d(\theta) - D_0 H_\mu(\theta)]\, y_\mu(t+\theta)\, d\theta.\end{aligned}$$

From Lemma 7, it follows that $x_\mu(t)$, $y_\mu(t)$ are bounded for $t \in [0, T]$ and $\mu > 0$ in some neighborhood of zero. Consequently, there exists a $K > 0$ such that, for $\mu > 0$ in some neighborhood of zero, we have

$$|\dot{x}_\mu(t)| \leqslant K$$

$$|\dot{y}_\mu(t)| \leqslant K \left[\left| \frac{\delta_\mu}{\mu} \right| + \int_{-r}^{0} \left| \frac{c(\theta) - D_0 G_\mu(\theta)}{\mu} \right| d\theta + \int_{-r}^{0} \left| \frac{d(\theta) - D_0 H_\mu(\theta)}{\mu} \right| d\theta \right].$$

The first inequality gives the desired boundedness for $\dot{x}_\mu(t)$, $t \in [0, T]$, as $\mu \to 0^+$. The differentiability of the function $\mu \mapsto (\delta_\mu, G_\mu, H_\mu)$, established in Lemma 8, and the second inequality above give the boundedness of $\dot{y}_\mu(t)$, $t \in [0, T]$, as $\mu \to 0^+$.

Remark 10. (i) The proofs of Theorem 9 and Lemma 8 still work if, instead of assuming $c, d \in H^1$ and c', d' continuous at zero, we only assume these conditions for the blocks c_1, d_{11} in the block representations of c, d relative to canonical coordinates for D_0, as introduced in the proof of Theorem 5.

(ii) In the case that all the eigenvalues of D_0 have negative real parts, it was established in Theorem 5 that a necessary and sufficient condition for the existence of invariant manifolds Σ_μ, approaching Σ_0 as $\mu \to 0^+$, is $C = D = 0$. In this case Proposition 3(i) can be applied to get the existence of an invariant manifold Γ_μ which is fast under (1_μ) as $\mu \to 0^+$, such that $X = \Sigma_\mu \oplus \Gamma_\mu$ and $\dim \Gamma_\mu = n$. The argument in the proof of that proposition implies that the manifolds Σ_μ are uniquely determined for μ in a neighborhood of zero. This takes care of the indeterminacy in the construction of Σ_μ in the proof of Theorem 5.

(iii) Under the assumptions in Lemma 8, the manifold Σ_0 acts as kind of a "center manifold" for (1_μ), as $\mu \to 0^+$, in the sense that the solutions of (1_0) on Σ_0 determine the behavior of the solutions of (1_μ) in the limit $\mu \to 0^+$.

5. Invariant Manifolds Approaching the Manifold of Orbits of the Degenerate System. Systems with Concentrated Delays

In order to describe the geometry of the phase space in analogy with the situation for ODEs, the set of orbits of the degenerate system, Σ_0, must be a closed linear submanifold of the phase space. If there are no concentrated delays present, this will be the case when we take for phase space

$$X = L^2((-r, 0); R^m) \times L^2((-r, 0); R^n) \times R^m \times R^n,$$

as done before. However, when (1_μ) involves concentrated delays, the set of orbits of the degenerate system is no longer a closed linear submanifold of X. To discuss the geometry in analogy with the case of ODEs, it is therefore necessary to use a different phase space. We choose to consider, in this case, a Banach space of continuous functions. More precisely, we then take for phase space

$$Y = C([-r, 0]; R^m) \times C([-r, 0]; R^n)$$

with the supremum norm and $r < +\infty$. Clearly, systems without concentrated delays can also be considered in the phase space Y and, therefore, the remarks in this section also apply to them.

We consider systems of the form (1_μ) with

$$A(\phi) = \int_{-r}^{0} d[a(\theta)]\, \phi(\theta)$$

and similarly for B, C, D, where the integrals are taken in the Riemann–Stieltjes sense and a, b, c, d are functions of bounded variation on $[-r, 0]$, vanishing at $-r$, and with variations over the intervals $[s, 0]$ converging to zero as $s \to 0^-$.[3]

The set of orbits of the degenerate system is, in this case,

$$\Sigma_0 = \{(\phi, \psi) \in Y\colon D_0\phi(0) + C(\phi) + D(\psi) = 0\}.$$

It is a closed linear submanifold of Y with codimension n.

[3] The present setting allows for the occurrence of infinitely many concentrated delays.

It is known (see [3]) that, for $\mu > 0$, the solutions of (1_μ) define a C_0-semigroup $T_\mu(t)$, $t \geqslant 0$, on Y by

$$T_\mu(t)(\phi, \psi) = (x_t(\phi, \psi), y_t(\phi, \psi))$$

where $(x(\phi, \psi), y(\phi, \psi))$ denotes the solution of (1_μ) with initial condition (ϕ, ψ) at $t = 0$, and, for $\mu = 0$, the solutions of the neutral system obtained from (1_0) by replacing the second equation by the one obtained from it through differentiation also define a C_0-semigroup on Y given in terms of the solutions as in the case of $\mu > 0$. The infinitesimal generators of the T_μ, denoted by $\mathscr{A}_\mu$, are given by

$$\mathscr{D}(\mathscr{A}_\mu) = \{(\phi, \psi) \in Y\colon \phi, \psi \text{ are continuous differentiable and}$$
$$\dot{\phi}(0) = A_0\phi(0) + A(\phi) + B(\psi), \mu\dot{\psi}(0) = D_0\psi(0) + C(\phi) + D(\psi)\}$$

and

$$\mathscr{A}_\mu(\phi, \psi) = (\phi', \psi').$$

THEOREM 11.[4] *A necessary and sufficient condition for the existence of a family of closed linear manifolds Σ_μ which are invariant under (1_μ) and approach Σ_0 as $\mu \to 0^+$ is:*

(C) *The ranges of the matrices $c(\theta)$ and $d(\theta)$ are included in the linear span of the generalized eigenspaces associated with the eigenvalues of D_0 which have positive real parts, a.e. in $\theta \in (-r, 0)$.*

Proof. This result can be proved in a way similar to Theorem 5, working with the dual space of Y.

Theorem 7 is valid in the present context. Theorem 9 also holds, provided the functions c, d are assumed to be somewhat smoother (e.g., twice continuously differentiable) than in the hypothesis of that theorem.

EXAMPLE 12. Let us consider again the system of Example 4. Taking into account the facts established there, it is possible to conclude the following:

(i) Γ_μ does not converge in Y as $\mu \to 0^+$.

(ii) Σ_μ converges if and only if $D_{11} > 0$ and then

$$\Sigma_\mu \to \Sigma_0, \qquad \text{as} \quad \mu \to 0^+.$$

[4] This theorem has the same wording as Theorem 5. However, the two statements refer to completely different objects. In particular, the differences between the functions c, d in the two settings should be noticed.

(iii) If $D_{11} > 0$ or $D_{22} > 0$, then the solutions of the perturbed system do not converge to solutions of the degenerate problem for all initial data. Consequently, when we have convergence of solutions for all initial data, the manifolds Σ_μ, Γ_μ do not converge as $\mu \to 0^+$.

(iv) In Theorem 11, a condition (Condition (C)) was given for the existence of invariant linear manifolds Σ_μ, approaching Σ_0 as $\mu \to 0^+$. Applied to the present example, the condition amounts to $D_{11} > 0$, in agreement with the conclusion in (ii). Furthermore, the proof of that theorem would give

$$\Sigma_\mu = \left\{ (\phi, \psi) \in Y \colon (\phi(0), \psi(0)) = -\int_{-r}^{0} d[H_\mu(\theta)] \begin{bmatrix} \phi(\theta) \\ \psi(\theta) \end{bmatrix} \right\}$$

with

$$H_\mu(\theta) = \frac{1}{\mu} \int_{-1}^{\theta} e^{-A_\mu(\theta - s)/\mu} U(s+1)\, ds$$

where U is the Heaviside function which equals 1 for $s > 0$ and vanishes elsewhere, and A_μ satisfies

$$A - D_0 + \frac{1}{\mu} A \int_{-1}^{0} e^{As/\mu} U(s+1)\, ds = 1.$$

The solution of this equation is

$$A_\mu = \begin{bmatrix} D_{11}, & D_{12} + e^{-D_{11}/\mu} \\ 0, & D_{22} \end{bmatrix}$$

and therefore, for $-1 \leqslant \theta \leqslant 0$, we have

$$H_\mu(\theta) = \begin{bmatrix} 0, & 1 - \dfrac{e^{-D_{11}(\theta+1)/\mu}}{D_{11}} \\ 0, & 0 \end{bmatrix}$$

and, consequently,

$$\Sigma_\mu = \left\{ (\phi, \psi) \in Y \colon \phi(0) = -\frac{1}{\mu} \int_{-1}^{0} e^{-D_{11}(\theta+1)/\mu} \psi(\theta)\, d\theta,\ \psi(0) = 0 \right\},$$

in agreement with the definition of Σ_μ in Example 4 and with property (ii).

If $X = L^2((-1, 0); R^2) \times R^2$ is taken for phase space in the preceding example, then the above properties can be discussed for representatives of the (equivalence classes) elements of X. All the properties hold in this setting, except for (i) and (iv). The discussion in (iv) has no meaning in this context, since Σ_0 is not a closed linear submanifold of X, and property (i) should be replaced by (i)′ Γ_μ converges if and only if $D_{11}, D_{22} > 0$, and then

$$\Gamma_\mu \to \{(0, 0, \alpha, \beta) \in X\colon \alpha, \beta \in R\}, \qquad \text{as} \quad \mu \to 0^+.$$

This fact is related to the difference between the convergence properties in spaces of continuous functions and spaces of square-integrable functions.

6. Systems with Unbounded Delays

In the case the system does not involve concentrated delays, we may take for phase space

$$X_{\gamma_0} = L^2((-\infty, 0); R^m) \times L^2((-\infty, 0); R^n) \times R^m \times R^n$$

where the L^2 spaces are defined relative to a measure ν, with Radon–Nikodym derivative relative to Lebesgue measure equal to $e^{\gamma_0 \cdot}$. The space X_{γ_0} is a Hilbert space with inner product

$$\langle(\phi, \psi, \alpha, \beta), (\eta, \zeta, \gamma, \delta)\rangle$$
$$= \alpha \cdot \gamma + \beta \cdot \delta + \int_{-\infty}^{0} \phi \cdot \eta e^{\gamma_0 \cdot} + \int_{-\infty}^{0} \psi \cdot \zeta e^{\gamma_0 \cdot}.$$

Systems with bounded delays can also be considered in this phase space and the discussion in this section also applies to them.

Theorem 13. *Condition* (C), *given in Theorem* 5, *is sufficient for the existence of a continuous one-parameter family of closed linear manifolds* Σ_μ, *invariant under* (1_μ), *and such that* $\Sigma_\mu \to \Sigma_0$ *as* $\mu \to 0^+$.

Proof. We proceed as in the proof of Theorem 5, and therefore omit the details. In exactly the same way, with

$$\Sigma_\mu = \left\{(\phi, \psi, \alpha, \beta) \in X\colon \delta_\mu \alpha + \beta + \int_{-\infty}^{0} G_\mu \phi e^{\gamma \cdot} + \int_{-\infty}^{0} H_\mu \psi e^{\gamma \cdot} = 0\right\},$$

we get

$$\begin{aligned} G' + \left(\frac{D_0}{\mu} + H(0) + \gamma_0\right) G &= a + \frac{c}{\mu} \\ H' + \left(\frac{D_0}{\mu} + H(0) + \gamma_0\right) H &= b + \frac{d}{\mu} \qquad (25) \\ \delta A_0 - \frac{D_0}{\mu}\delta - H(0)\,\delta + G(0) &= 0 \end{aligned}$$

or, denoting $\Lambda_\mu = D_0 + \mu H_\mu(0) + \mu\gamma_0$,

$$G(\theta) = e^{-\Lambda_\mu \theta/\mu} \left[G(0) - \int_\theta^0 e^{\Lambda_\mu s/\mu} \left(\delta a(s) + \frac{c(s)}{\mu} \right) e^{-\gamma_0 s} \, ds \right]$$
$$H(\theta) = e^{-\Lambda_\mu \theta/\mu} \left[H(0) - \int_\theta^0 e^{\Lambda_\mu s/\mu} \left(\delta b(s) + \frac{d(s)}{\mu} \right) e^{-\gamma_0 s} \, ds \right]. \tag{26}$$

Consequently, the Σ_μ with the desired properties exist if and only if the G_μ, H_μ, δ_μ satisfying (26) are such that $G_\mu, H_\mu \in L^2((-\infty, 0); \nu)$ for $\mu > 0$ small, and $G_\mu \to G_0$, $H_\mu \to H_0$, $\delta_\mu \to 0$ as $\mu \to 0^+$, where

$$G_0(\theta) = D_0^{-1} c(\theta) \, e^{-\gamma_0 \theta} \qquad \text{and} \qquad H_0(\theta) = D_0^{-1} d(\theta) \, e^{-\gamma_0 \theta}.$$

Assume Condition (C) holds. We partition the matrix D_0 as in the proof of Theorem 5, and partition the other matrices in a compatible way. Then, functions G_μ, H_μ, satisfying (26), belong to $L^2((-\infty, 0); \nu)$ if and only if the upper blocks in the partitions of the matrices

$$\left[G_\mu(0) - \int_{-\infty}^0 e^{\Lambda_\mu s/\mu} \left(\delta a(s) + \frac{c(s)}{\mu} \right) e^{-\gamma_0 s} \, ds \right]$$

and (27)

$$\left[H_\mu(0) - \int_{-\infty}^0 e^{\Lambda_\mu s/\mu} \left(\delta b(s) + \frac{d(s)}{\mu} \right) e^{-\gamma_0 s} \, ds \right]$$

vanish. Now, we can proceed as in the proof of Theorem 5 to construct a family of linear manifolds Σ_μ with the required properties.

In the present case, contrary to what happens when the system contains bounded delays and the phase space is defined through L^2 spaces on bounded intervals, Condition (C) is no longer necessary for the existence of the manifolds Σ_μ with the desired properties. However,if Condition (C) is not satisfied, infinitely many one-parameter families of manifolds Σ_μ, depending continuously on μ for μ small, may exist. Rather than stating a general result, we consider the particular case of Eq. (1_μ) with $m = 0$, i.e.,

$$\mu \dot{y}(t) = D_0 y(t) + \int_{-\infty}^0 d(\theta) \, y(t+\theta) \, d\theta, \tag{28_μ}$$

and take for phase space $X_\nu = L^2((-\infty, 0)\colon R^n, \nu) \times R^n$.

PROPOSITION 14. *If condition*

(C_-) *The range of $d(\theta)$ is included in the linear span of the generalized eigenspaces associated with the eigenvalues of D_0 which have negative real parts, a.e. in $\theta \in (-\infty, 0)$*

is satisfied, then there exist infinitely many continuous one-parameter families of closed linear manifolds Σ_μ, *invariant under* (28_μ) *and such that* $\Sigma_\mu \to \Sigma_0$ *as* $\mu \to 0^+$.

Proof. Assume Condition (C_-) holds. Similarly to what was done in the proof of Theorem 13, with

$$\Sigma_\mu = \left\{ (\psi, \beta) \in X_\nu : \beta + \int_{-\infty}^{0} H_\mu \psi e^{\gamma_0 \cdot} = 0 \right\},$$

we get

$$H' + \left(\frac{D_0}{\mu} + H(0) + \gamma_0 \right) H = \frac{d}{\mu} \tag{29}$$

or, denoting $\Lambda_\mu = D_0 + \mu H_\mu(0) + \mu\gamma_0$,

$$H_\mu(\theta) = e^{-\Lambda_\mu \theta/\mu} \left[H_\mu(0) - \frac{1}{\mu} \int_\theta^0 e^{\Lambda_\mu s/\mu} d(s)\, e^{-\gamma_0 s}\, ds \right]. \tag{30}$$

Consequently, the Σ_μ with the desired properties exist if and only if H_μ, satisfying (30), belongs to $L^2((-\infty, 0); R^n, \nu)$ for $\mu > 0$ small and $H_\mu \to H_0$ as $\mu \to 0^+$, where

$$H_0(\theta) = D_0^{-1} d(\theta)\, e^{-\gamma_0 \theta}.$$

We partition D_0 as in the proof of Theorem 4 and partition the other matrices in a compatible way. For H_μ, given by (30), to belong to $L^2((-\infty, 0); R^n, \nu)$, the upper blocks of the partition of the matrix

$$\left[H_\mu(0) - \frac{1}{\mu} \int_{-\infty}^0 e^{\Lambda_\mu s/\mu} d(s)\, e^{-\gamma_0 s}\, ds \right] \tag{31}$$

must vanish. If the partitions of Λ_μ and $H(0)$ are denoted by

$$\Lambda = \begin{bmatrix} \Lambda_{11}, & \Lambda_{12} \\ \Lambda_{21}, & \Lambda_{22} \end{bmatrix}, \qquad H(0) = \begin{bmatrix} H^{11}(0), & H^{12}(0) \\ H^{21}(0), & H^{22}(0) \end{bmatrix},$$

with $\Lambda_{21} = H^{11}(0) = H^{12}(0) = 0$, then the upper blocks of matrix (31) vanish and, because of (29), we have

$$H_\mu^{11}(\theta) = H_\mu^{21}(\theta) = 0$$

$$H_\mu^{21}(\theta) = e^{-(D_0^-/\mu + \gamma_0)\theta} e^{H_\mu^{22}(0)\theta} \left[H_\mu^{21}(0) - \frac{1}{\mu} \int_\theta^0 e^{(D_0^-/\mu + H_\mu^{22}(0))s} d_{21}(s)\, ds \right]$$

$$H_\mu^{22}(\theta) = e^{-(D_0^-/\mu + \gamma_0)\theta} e^{H_\mu^{22}(0)\theta} \left[H_\mu^{22}(0) - \frac{1}{\mu} \int_\theta^0 e^{(D_0^-/\mu + H_\mu^{22}(0))s}\, d_{22}(s)\, ds \right].$$

No matter what $H_\mu^{21}(0)$, $H_\mu^{22}(0)$ are, for $\mu > 0$ small, $H_\mu^{21}, H_\mu^{22} \in L^2((-\infty, 0); \nu)$. If $H_\mu^{21}(0), H_\mu^{22}(0)$ are chosen so that

$$H_\mu(0) \to D_0^{-1} d(0), \qquad \text{as} \quad \mu \to 0^+,$$

we have

$$H_\mu^{21}(\theta) \to (D_0^-)^{-1} \, d_{21}(\theta) \, e^{-\gamma_0 \theta}$$

$$H_\mu^{22}(\theta) \to (D_0^-)^{-1} \, d_{22}(\theta) \, e^{-\gamma_0 \theta}$$

and, consequently, $H_\mu \to H_0$ as $\mu \to 0^+$. Clearly, there are infinitely many choices for $H_\mu^{21}(0), H_\mu^{22}(0)$ such that $H_\mu(0) \to (D_0^-)^{-1} \, d(0)$ as $\mu \to 0^+$. Each one of these corresponds to one family of linear manifolds Σ_μ with the desired properties.

Remark 15. As was pointed out before, the case of interest to have convergence of all the solutions of (28_μ) to solutions of the degenerate equation occurs when all the eigenvalues of D_0 have negative real parts. When this is the case and we are considering Eq. (28_μ) with bounded distributed delays which lie in an interval $[-r, 0]$, $r < +\infty$, the results proved so far imply that:

(i) In $X_\nu = L^2((-\infty, 0); R^n, \nu) \times R^n$ there exist infinitely many one-parameter families of closed linear manifolds Σ_μ, invariant under (28_μ), depending continuously on μ for $\mu > 0$ small, and approaching Σ_0 as $\mu \to 0^+$.

(ii) In $X = L^2((-r, 0); R^n) \times R^n$, a one-parameter family of manifolds Σ_μ with the above properties exists if and only if (28_μ) is an ODE, i.e., $d = 0$.

In the case of the general system (1_μ), the situation is similar.

When considering systems which involve concentrated delays, for the reasons explained in Section 5 above the existence of the manifolds Σ_μ with the desired properties cannot be discussed in the phase space X. However, the results given in this section can be proved taking for phase space

$$Y_{\gamma_0} = C_{\gamma_0}((-\infty, 0]; R^m) \times C_{\gamma_0}((-\infty, 0]; R^n)$$

where the $C_{\gamma_0}(-\infty, 0]$ are Banach spaces defined by

$$C_{\gamma_0}(-\infty, 0] = \{\phi \in C(-\infty, 0] : \lim_{\theta \to -\infty} e^{\gamma_0 \theta} \phi(\theta) = 0\}$$

with the norm

$$\|\phi\| = \sup_{\theta \leqslant 0} |e^{\gamma_0 \theta} \phi(\theta)|,$$

and considering systems of the form (1_μ) with A, B, C, D being bounded linear operators defined on spaces $C_{\gamma_0}(-\infty, 0]$.

EXAMPLE 16. Let us consider the scalar system

$$\mu \dot{y}(t) = D_0 y(t) + y(t-1) \tag{32_μ}$$

with $D_0 < 0$, in the space space $C_{\gamma_0}(-\infty, 0]$.

We consider the manifolds

$$\Sigma_\mu^c = \Big\{ \phi \in C_{\gamma_0}(-\infty, 0]: \\ \phi(0) + c \int_{-\infty}^0 e^{-(D_0/\mu + c)}\phi(\theta)\, d\theta - \frac{1}{\mu} \int_{-\infty}^0 e^{-(D_0/\mu + c)}\phi(\theta - 1)\, d\theta = 0 \Big\}.$$

Assume $\phi \in \Sigma_\mu^c$. Then the solution of (32_μ) with initial condition ϕ at $t = 0$ can be written, for $0 < t < 1$, as

$$\begin{aligned} y(t) &= e^{(D_0/\mu)t}\phi(0) + \frac{1}{\mu}\int_0^t e^{(D_0/\mu)(t-s)}\phi(s-1)\, ds \\ &= e^{(D_0/\mu)t} \Big[-c \int_{-\infty}^0 e^{-(D_0/\mu + c)\theta}\phi(\theta)\, d\theta \\ &\quad + \frac{1}{\mu}\int_{-\infty}^0 e^{-(D_0/\mu + c)\theta}\phi(\theta - 1)\, d\theta + \frac{1}{\mu}\int_0^t e^{-(D_0/\mu)\theta}\phi(\theta - 1)\, d\theta \Big]. \end{aligned} \tag{33}$$

We also have

$$\begin{aligned} &\int_{-\infty}^0 e^{-(D_0/\mu + c)\theta} y_t(\theta)\, d\theta \\ &\quad = \int_{-\infty}^{-t} e^{-(D_0/\mu + c)\theta}\phi(t + \theta)\, d\theta + \int_{-t}^0 e^{-(D_0/\mu + c)\theta} y(t + \theta)\, d\theta \end{aligned}$$

and, using (33) with straightforward calculus, we get

$$\begin{aligned} &\int_{-\infty}^0 e^{-(D_0/\mu + c)\theta} y_t(\theta)\, d\theta \\ &\quad = e^{(D_0/\mu)t} \Big[\int_{-\infty}^0 e^{-(D_0/\mu + c)s}\phi(s)\, ds - \frac{1}{c\mu}\int_{-\infty}^0 e^{-(D_0/\mu + c)s}\phi(s-1)\, ds \\ &\qquad - \frac{1}{c\mu}\int_0^t e^{-(D_0/\mu)s}\phi(s-1)\, ds \Big] + \frac{1}{c\mu} e^{(D_0/\mu + c)t}\int_{-\infty}^t e^{-(D_0/\mu + c)s}\phi(s-1)\, ds. \end{aligned}$$

Therefore,

$$y(t)+c\int_{-\infty}^{0} e^{-(D_0/\mu+c)\theta}y_t(\theta)\,d\theta-\frac{1}{\mu}\int_{-\infty}^{0} e^{-(D_0/\mu+c)\theta}y_t(\theta-1)\,d\theta=0.$$

Consequently, $\phi\in\Sigma_\mu^c$ implies $y_t(\phi)\in\Sigma_\mu^c$ for $0<t<1$. This, in turn, implies that Σ_μ^c is invariant under (32_μ), for all $\mu>0$, $c\in R$.

The preceding argument establishes the existence of infinitely many (cardinality of R) closed linear submanifolds of Y_{γ_0}, which are invariant under (32_μ) and have codimension 1. Choosing an arbitrary continuous function $\mu\mapsto c_\mu$, defined for μ small, and such that $c_\mu\to 0$, we get, as $\mu\to 0^+$,

$$c_\mu\int_{-\infty}^{0} e^{-(D_0/\mu+c_\mu)\theta}\phi(\theta)\,d\theta\to 0$$

$$-\frac{1}{\mu}\int_{-\infty}^{0} e^{-(D_0/\mu+c_\mu)\theta}\phi(\theta-1)\,d\theta\to\phi(-1)/D_0.$$

Therefore, $\Sigma_\mu^c\to\Sigma_0$.

Consequently, $\Sigma_\mu\to\Sigma_0$ as $\mu\to 0^+$, for infinitely many continuous one-parameter families of closed linear manifolds Σ_μ, invariant under (32_μ).

7. An Example of a System with Distributed Unbounded Delays

In this section we consider in detail the particular case of Eq. (1_μ) which can be written as[5]

$$\mu\dot{y}(t)=D_0y(t)+\int_{-\infty}^{0} d(\theta)\,y(t+\theta)\,d\theta \qquad (34_\mu)$$

where $y(t)\in R$, $d(\theta)=\sum_{k=1}^{m} d_k e^{\gamma_k\theta}>0$, with $m\in N$, $d_k>0$, $0<\gamma_1<\gamma_2<\cdots<\gamma_m$, and $D_0=-\int_{-\infty}^{0} d$. For simplicity of the presentation it is further assumed that the function of complex variable

$$\Phi(\lambda)=\sum_{k=1}^{m}\frac{d_k/\gamma_k}{\lambda+\gamma_k} \qquad (35)$$

has simple zeros. The general case could, however, be treated in a similar way.

[5] This equation was obtained by linearization of a system occurring in [9] in connection to the rehology of certain polymer filaments.

We take for phase space

$$X_\nu = L^2((-\infty, 0); \nu) \times R$$

where ν is a measure with Radon–Nikodym derivative, relative to Lebesgue measure, equal to $e^{\gamma_0 \cdot}$, with $\gamma_0 < \gamma_1$.

LEMMA 17. *There exists $\mu_0 > 0$ such that the solution of (34_μ), with initial condition $(\phi, \alpha) \in X_\nu$ at $t = 0$, satisfies for $t \geqslant 0$ and $0 < \mu < \mu_0$*

$$y_\mu(t) = \sum_{k=0}^{m} C_k(\mu) \left[\alpha + \frac{1}{\mu} \sum_{i=1}^{m} \frac{d_i}{\gamma_i + \lambda_k(\mu)} \int_{-\infty}^{0} e^{\gamma_i s} \phi(s)\, ds \right] e^{\lambda_k(\mu) t} \tag{36}$$

where the C_k, λ_k are continuous complex-valued functions defined on $[0, \mu_0)$ and having the following properties:

(i) $\lambda_0(\mu) = 0$, $-\beta < \operatorname{Re} \lambda_k(\mu) < 0$, $0 \leqslant \mu < \mu_0$, $k = 1,..., m-1$, *for some fixed $\beta > 0$, and the $\lambda_k(0)$ $(k = 1,..., m-1)$ are the zeros of Φ.*

(ii) λ_m *satisfies the asymptotic relation*

$$\lambda_m(\mu) = -\frac{D_0}{\mu} + o(1/\mu), \qquad as \quad \mu \to 0^+.$$

(iii) *For* $0 < \mu < \mu_0$,

$$C_0(\mu) = \frac{\mu}{\mu + \sum_{i=1}^{m} \frac{d_i}{\gamma_i^2}}$$

$$C_k(\mu) = -\frac{\mu}{\lambda_k(\mu) \sum_{i=1}^{m} \frac{d_i/\gamma_i}{(\gamma_i + \lambda_k(\mu))^2}}, \qquad k = 1,..., m.$$

(iv)

$$\frac{1}{\mu} \sum_{i=1}^{m} \frac{d_i}{(\gamma_i + \lambda_j(\mu))(\gamma_i + \lambda_k(\mu))} = \begin{cases} (1/C_k(\mu)) - 1, & j = k, j, k \neq 0 \\ -1, & j \neq k \\ 1/C_0(\mu), & j = k = 0. \end{cases}$$

Proof. Algebraic manipulations on the characteristic function Δ_μ of (34_μ) show that it can be extended to $\mathbb{C} - \{\gamma_1,..., \gamma_m\}$ as an improper rational function having as zeros $\lambda = 0$ and the roots of the equation

$$\Psi_\mu(\lambda) \stackrel{\text{def}}{=} \frac{1}{\mu} \sum_{k=1}^{m} \frac{d_k/\gamma_k}{\lambda + \gamma_k} = -1. \tag{37}$$

Ψ_μ is a rational function that can be written as

$$\Psi_\mu(\lambda) = \frac{1}{\mu} \frac{\sum_{i=1}^{m-1} c_i \lambda_i}{\prod_{k=1}^{m} (\lambda + \gamma_k)}$$

where the c_i are real constants and $c_{m-1} = -D_0 > 0$. Therefore, Ψ_μ has m distinct first-order poles and only one zero at infinity. Consequently, the root locus of $\Psi_\mu(\lambda) = -1$, when $\mu > 0$ changes, consists of a set of bounded continuous paths and exactly one unbounded path.

Because $\Psi_\mu = \Phi/\mu$, the zeros of Φ are simple by hypothesis, and the root locus of $\Psi_\mu(\lambda) = -1$ has a finite number of branches approaching the zeros of Φ as $\mu \to 0^+$, it follows that there exists $\mu_0 > 0$ such that the equation $\Psi_\mu(\lambda) = -1$ has m distinct simple roots $\lambda_k(\mu)$, $k = 1,..., m$, for each $0 < \mu < \mu_0$ all of them except one, say, $\lambda_m(\mu)$, lying in a fixed bounded region of the complex plane.

The proper rational function Δ_μ^{-1} has, therefore, an expansion in partial fractions of the form

$$\Delta_\mu^{-1}(\lambda) = \sum_{k=0}^{m} \frac{C_k(\mu)}{\lambda - \lambda_k(\mu)} \tag{38}$$

where $\lambda_0 \equiv 0$. By inverse Laplace transformation, we get the fundamental matrix for (34_μ) from which the formula (36), for the general solution, is obtained.

Let $\mathscr{A}_\mu$ denote the infinitesimal generator of the semigroup T_μ defined by the solutions of (34_μ) in a way similar to that described in Section 3 for the general system considered there. We denote by $\rho(\mathscr{A}_\mu)$, $\sigma(\mathscr{A}_\mu)$, and $P_\sigma(\mathscr{A}_\mu)$ the resolvent set, the spectrum, and the point spectrum of $\mathscr{A}_\mu$, respectively, and define for $\delta \in R$

$$\mathbb{C}_\delta = \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > \delta\}.$$

From a result due to Naito [13], we have

$$\mathbb{C}_0 \subset P_\sigma(\mathscr{A}_\mu) \cup \rho(\mathscr{A}_\mu) \subset \bar{\mathbb{C}}_{-\gamma_0/2}. \tag{39}$$

A simple computation shows that, for the system presently considered, $\sigma(\mathscr{A}_\mu) \cap \bar{\mathbb{C}}_0 = \{0\}$. Since the $\lambda_k(\mu)$ with real parts greater than $-\gamma_0/2$ belong to $\sigma(\mathscr{A}_\mu)$, lie in a bounded region of the complex plane for $k = 1,..., m-1$, and converge to zeros of the function Φ as $\mu \to 0^+$, we get property (i) in the statement.

Property (ii) follows from the asymptotic formula

$$\Psi_\mu(\lambda) = \frac{1}{\mu} \frac{\sum_{k=1}^{m} d_k/\gamma_k}{\lambda} + o(1), \qquad \text{as} \quad |\lambda| \to \infty$$

and the fact that $|\lambda_m(\mu)| \to \infty$ as $\mu \to 0^+$.

Property (iii) results from the following consequence of (38):

$$C_k(\mu) = \lim_{\lambda \to \lambda_k(\mu)} \Delta_\mu^{-1}(\lambda)(\lambda - \lambda_k(\mu))$$

$$= \lim_{\lambda \to \lambda_k(\mu)} \frac{\lambda - \lambda_k(\mu)}{\lambda \left(1 + \frac{1}{\mu} \sum_{i=1}^{m} \frac{d_i/\gamma_i}{\lambda + \gamma_i}\right)}.$$

In order to prove property (iv), we apply (iii) and the fact that $\lambda_m(\mu)$ is a root of (37).

PROPOSITION 18. *Let μ_0 be as in Lemma 17. For $0 < \mu < \mu_0$, there exists a direct sum decomposition of the phase space*

$$X_\nu = O \oplus P_0^\mu \oplus P_1^\mu \oplus \cdots \oplus P_m^\mu$$

such that

(i) $\dim P_k^\mu = 1$, $k = 0,..., m$,

(ii) $(\phi, \alpha) \in P_k^\mu - \{(0, 0)\} \Rightarrow \alpha \neq 0$, $k = 0,..., m$,

(iii) $(\phi, \alpha) \in O \Leftrightarrow y_\mu(t; \phi, \alpha) = 0$, $t \geqslant 0$, $0 < \mu < \mu_0$,

(iv) $(\phi, \alpha) \in P_k^\mu \Rightarrow y(t; \phi, \alpha) = \alpha e^{\lambda_k(\mu)t}$, $t \geqslant 0$, $0 < \mu < \mu_0$,

where $y_\mu(\cdot\,; \phi, \alpha)$ is the solution of (34_μ) through the initial datum (ϕ, α) at $t = 0$ and the λ_k are as in Lemma 17.

Furthermore, the subspaces O, $O \oplus P_k^\mu$ are all invariant under the semigroup T_μ defined by the solutions of (34_μ) as $T_\mu(t)(\phi, \alpha) = ((y_\mu)_t(\phi, \alpha),$ $y_\mu(t; \phi, \alpha))$.

Proof. Let $G: X_\nu \to R^{m+1}$ be such that

$$G(\phi, \alpha) = \begin{bmatrix} \alpha \\ \int_{-\infty}^{0} e^{\gamma_1 s} \phi(s)\, ds \\ \vdots \\ \int_{-\infty}^{0} e^{\gamma_m s} \phi(s)\, ds \end{bmatrix}$$

and

$$O = \{(\phi, \alpha) \in X_\nu : G(\phi, \alpha) = 0\}. \tag{40}$$

The set O is a closed linear submanifold of X_ν. Let Π_0 be the projection operator projecting orthogonally X_ν upon O. We have

$$\|(\phi, \alpha) - \Pi_0(\phi, \alpha)\|^2 = \min_{(\psi, \beta) \in O} \|(\phi, \alpha) - (\psi, \beta)\|^2$$

$$= \min_{(\psi, \beta) \in O} \{\|(\phi, \alpha)\|^2 + \|(\psi, \beta)\|^2 - 2\langle(\phi, \alpha), (\psi, \beta)\rangle\}.$$

Therefore $\Pi_0(\phi, \alpha)$ is the unique minimum, over the set O, of the functional given by

$$V_{(\phi,\alpha)}(\psi, \beta) = \langle(\psi, \beta), (\psi, \beta)\rangle - 2\langle(\phi, \alpha), (\psi, \beta)\rangle.$$

It is easy to show that $V_{(\phi,\alpha)}$ and G are C^1, and that G maps X_r onto R^{m+1}. Therefore, if (ϕ_0, α_0) is a minimizer of V_ϕ over O, the Lagrange Multiplier Theorem implies the existence of $z^* \in (R^{m+1})^*$ such that

$$DV_{(\phi,\alpha)}(\phi_0, \alpha_0) + z^* DG(\phi_0, \alpha_0) = 0.$$

Therefore, for some $K_i(\phi, \alpha) \in R$ $(i = 0,..., m)$, we get

$$\Pi_0(\phi, \alpha) = \left(\phi - \sum_{i=1}^{m} K_i(\phi, \alpha)\, e^{(\gamma_i - \gamma_0)\cdot}, \alpha - K_0(\phi, \alpha)\right) \tag{41}$$

with the $K_i(\phi, \alpha)$ such that $G(\Pi_0(\phi, \alpha)) = 0$, i.e.,

$$\begin{aligned} K_0(\phi, \alpha) &= \alpha_0 \\ \sum_{i=1}^{m} \frac{K_i(\phi, \alpha)}{\gamma_i + \gamma_j - \gamma_0} &= \int_{-\infty}^{0} e^{\gamma_i s} \phi(s)\, ds. \qquad j = 1,..., m. \end{aligned} \tag{42}$$

Since the above minimization problem corresponds to an orthogonal projection in Hilbert space, it has a unique solution. Therefore the matrix

$$R \stackrel{\text{def}}{=} \left[\frac{1}{\gamma_i + \gamma_j - \gamma_0}\right]_{i,j=1}^{m,m} \tag{43}$$

is nonsingular. Because G maps X_v onto R^{m+1}, it follows from (41) and (42) that

$$O^\perp = \left\{(\phi, \alpha) \in X_v : \phi(s) = \sum_{i=1}^{m} K_i e^{(\gamma_i - \gamma_0)s},\ s < 0,\ K_i \in R\right\}. \tag{44}$$

Clearly, $\dim O^\perp = m + 1$.

For $y(t) = \alpha e^{\lambda_k(\mu)t}$, $t \geq 0$, to be a solution of (34_μ) with initial condition (ϕ, α) at $t = 0$ it is necessary and sufficient that, for $t \geq 0$,

$$\begin{aligned} \mu\alpha\lambda_k(\mu)\, e^{\lambda_k(\mu)t} = &- \left(\sum_{i=1}^{m} \frac{d_i/\gamma_i}{\gamma_i + \lambda_k(\mu)}\right) \alpha\lambda_k(\mu)\, e^{\lambda_k(\mu)t} \\ &+ \sum_{i=1}^{m} d_i e^{-\gamma_i t} \left[\int_{-\infty}^{0} e^{\gamma_i s}\phi(s)\, ds - \frac{\alpha}{\gamma_i + \lambda_k(\mu)}\right]. \end{aligned}$$

This equality is only possible for the $\lambda_k(\mu)$ $(k = 0,..., m)$ appearing in Lemma 17. We have that $\lambda_0(\mu) = 0$ and the $\lambda_k(\mu)$ $(k = 1,..., m)$ are the roots of Eq. (37) for $0 < \mu < \mu_0$, with μ_0 also as in Lemma 17. Consequently, the set of initial data (ϕ, α) for which $\alpha e^{\lambda_k(\mu)t}$, $t > 0$, is a nontrivial ($\alpha \neq 0$ for $\phi \neq 0$) solution of Eq. (34_μ) is

$$P_k^\mu = \left\{(\phi, \alpha) \in O^\perp\colon \alpha = (\gamma_i + \lambda_k(\mu)) \int_{-\infty}^{0} e^{\gamma_i s}\phi(s)\, ds,\ i = 1,\dots, m\right\} \tag{45}$$

for $k = 0,\dots, m$, $0 < \mu < \mu_0$.

We now introduce the matrices

$$S_k(\mu) = \left[\frac{1}{\gamma_1 + \lambda_k(\mu)}, \frac{1}{\gamma_2 + \lambda_k(\mu)}, \dots, \frac{1}{\gamma_m + \lambda_k(\mu)}\right], \qquad k = 0,\dots, m \tag{46}$$

$$S(\mu) = \begin{bmatrix} S_0(\mu) \\ S_1(\mu) \\ \vdots \\ S_m(\mu) \end{bmatrix}, \qquad U(\mu) = \left[\begin{array}{c|c} 1 & \\ 1 & \\ \vdots & S(\mu) R^{-1} \\ 1 & \end{array}\right] \tag{47}$$

$$E(s) = \begin{bmatrix} e^{(\gamma_1 - \gamma_0)s} \\ \vdots \\ e^{(\gamma_m - \gamma_0)s} \end{bmatrix}. \tag{48}$$

Then, for $k = 0,\dots, m$ and $0 < \mu < \mu_0$, it follows from (44) and (45) that, for some row matrix $W = [w_1, w_2, \dots, w_m]$, we have (ϕ, α) belonging to P_k^μ if and only if

$$\phi(s) = WE(s), \qquad s < 0$$

$$\alpha S_k(\mu) = WR$$

or

$$\phi(s) = \alpha S_k(\mu) R^{-1} E(s), \qquad s < 0$$

for some $\alpha \in R$. Thus

$$P_k(\mu) = \{(\phi, \alpha) \in X_\nu\colon \phi(s) = \alpha S_k(\mu) R^{-1} E(s),\ s < 0\}. \tag{49}$$

For the decomposition $X_\nu = O \oplus P_0^\mu \oplus \cdots \oplus P_m^\mu$ to hold, it is necessary and sufficient that the following system have exactly one solution for each $1 \times m$ matrix W and $\alpha \in R$:

$$\sum_{j=0}^{m} \xi_j = \alpha$$

$$\sum_{j=0}^{m} \xi_j S_j(\mu) R^{-1} E(s) = WE(s).$$

With $\Xi = [\xi_1, ..., \xi_m]$, this system can be written as

$$\Xi U(\mu) = [\alpha, W]. \tag{50}$$

Now, Lemma 17 implies

$$\left[\begin{array}{c|c} 1 & \\ \cdot & \\ \cdot & \left[\dfrac{1}{\gamma_j + \lambda_k(\mu)}\right]_{k,j=0,1}^{m,m} \\ \cdot & \\ 1 & \end{array}\right] \left[\begin{array}{c} 0, \quad 1, \quad ..., \quad 1 \\ \hline \left[\dfrac{C_i(\mu)\, d_j}{\mu(\gamma_j + \lambda_i(\mu))}\right]_{j,i=1,0}^{m,m} \end{array}\right] = I_{m+1}. \tag{51}$$

The first matrix in this product is equal to

$$U(\mu)\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right].$$

Therefore, $U(\mu)$ is nonsingular and, consequently, system (50) has a unique solution Ξ, establishing, as a consequence, the desired decomposition of X_μ.

Directly from (49) it follows that $\dim P_k^\mu = 1$ and that $\alpha \neq 0$ for all $(\varphi, \alpha) \in P_k^\mu - \{(0, 0)\}$, proving (i) and (ii) in the statement.

Now, from Eq. (36) in Lemma 17 we get that $y_\mu(t; \phi, \alpha) = 0$ for $t \geqslant 0$ if and only if

$$\left[\begin{array}{cc} 1 & \\ \cdot & \\ \cdot & \left[\dfrac{1}{\gamma_j + \lambda_k(\mu)}\right]_{k,j=0,1}^{m,m} \\ \cdot & \\ 1 & \end{array}\right] \left[\begin{array}{c} \alpha \\ \dfrac{d_1}{\mu} e^{\gamma_1 s}\phi(s)\, ds \\ \vdots \\ \dfrac{d_m}{\mu} e^{\gamma_m s}\phi(s)\, ds \end{array}\right] = 0.$$

From (51), the first matrix in the preceding expression is nonsingular. Consequently, the second matrix must vanish, i.e.,

$$y_\mu(t; \phi, \alpha) = 0, \qquad t \geqslant 0 \Leftrightarrow G(\phi, \alpha) = 0 \Leftrightarrow (\phi, \alpha) \in O.$$

This proves (iii) in the statement.

Property (iv) is a consequence of the discussion preceding the definition of the P_k^μ in (45).

The invariance of the subspaces O and $O \oplus P_k^\mu$, under the semigroup T_μ, is a direct consequence of properties (iii)–(iv) and the definition of T_μ.

COROLLARY 19. *The manifold*

$$\Sigma_\mu = O \oplus P_0^\mu \oplus \cdots \oplus P_{m-1}^\mu,$$

defined for $0 < \mu < \mu_0$, *is a closed linear submanifold of* X_ν, *has codimension* 1, *is invariant under* (34_μ), *and approaches the manifold* Σ_0 *of orbits of the degenerate equation, as* $\mu \to 0^+$.

Proof. Everything in the statement is a direct consequence of the preceding proposition, except the convergence of Σ_μ to Σ_0.

It is easy to see from (49), (46), and Lemma 17(i) that, for $k = 0,..., m-1$, $P_k^\mu \to P_k^0$ as $\mu \to 0^+$, where

$$P_k^0 = \{(\phi, \alpha) \in X_\nu\colon \phi(s) = \alpha S_k(0) R^{-1} E(S), s < 0\}.$$

On the other hand, for $k = 0, 1,..., m-1$, we have

$$D_0 \alpha + \int_{-\infty}^0 d(\theta)[\alpha S_k(0) R^{-1} E(\theta)]\, d\theta = \alpha \left[-\sum_{i=1}^m \frac{d_i}{\gamma_i} + \sum_{i=1}^m \frac{d_i}{\gamma_i + \lambda_k(0)} \right]. \tag{52}$$

The $\lambda_k(\mu)$ $(k = 0,..., m-1)$ are the zeros of the function Δ_μ which remain bounded as $\mu \to 0^+$, and this function satisfies

$$\Delta_\mu(\lambda) = \mu\lambda - \left(-\sum_{i=1}^m \frac{d_i}{\gamma_i} + \sum_{i=1}^m \frac{d_i}{\gamma_i + \lambda} \right).$$

Therefore, the right-hand side of Eq. (52) vanishes. Noting that

$$\Sigma_0 = \left\{ (\phi, \alpha) \in X_\nu\colon D_0 \alpha + \int_{-\infty}^0 d(\theta)\, \phi(\theta)\, d\theta = 0 \right\},$$

we conclude that $P_k^\mu \subset \Sigma_0$ for $k = 0,..., m-1$. Since the P_k^0 $(k = 0,..., m-1)$ are one-dimensional linear subspaces generated by independent elements of X_ν, and $O \subset \Sigma_\mu$, it follows that $\Sigma_\mu \to \Sigma_0$, as $\mu \to 0^+$.

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