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Nonnegative-definite and positive-definite solutions to the matrix equation $\mathbf{AXA}^* = \mathbf{B}$ – revisited

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Abstract

The general Hermitian nonnegative-definite solution to the matrix equation $\mathbf{AXA}^* = \mathbf{B}$ is established in a form which can be viewed as a corrected version of that derived by C.G. Khatri and S.K. Mitra (SIAM J. Appl. Math. 31 (1976) 579–585) and an alternative version to that derived by J.K. Baksalary (Linear and Multilinear Algebra 16 (1984) 133–139). The new representation admits an easy way to obtain solutions of minimal and maximal rank, respectively. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $\mathbb{C}_{m\times n}$ denote the set of complex $m \times n$ matrices, and let $\mathbb{C}_m^H$ denote the set of complex Hermitian $m \times m$ matrices. Moreover, let $\mathbb{C}_m^{\geqslant}$ denote the subset of $\mathbb{C}_m^H$ consisting of nonnegative-definite matrices, and let $\mathbb{C}_m^{>}$ denote the subset of $\mathbb{C}_m^{\geqslant}$ consisting of positive-definite matrices. The symbols $\mathbf{A}^*$, $\mathbf{A}^+$, $\mathbf{A}^-$, $\mathscr{R}(\mathbf{A})$, $\mathscr{N}(\mathbf{A})$ and $\mathrm{rk}(\mathbf{A})$ will stand for the conjugate transpose, the Moore–Penrose generalized inverse, any generalized inverse, the range (column space), the null space and the rank, re-

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spectively, of $\mathbf{A} \in \mathbb{C}_{m\times n}$. By $\mathbf{A}^{1/2} \in \mathbb{C}_m^{\geqslant}$ we denote the (unique) Hermitian nonnegative-definite square root of $\mathbf{A} \in \mathbb{C}_m^{\geqslant}$.

We consider the general Hermitian nonnegative-definite solution to the equation

$$\mathbf{AXA}^* = \mathbf{B} \tag{1.1}$$

for given matrices $\mathbf{A} \in \mathbb{C}_{m\times n}$ and $\mathbf{B} \in \mathbb{C}_m^{\geqslant}$. Khatri and Mitra [4, Lemma 2.1] expressed this solution as

$$\mathbf{X} = \mathbf{A}^-\mathbf{B}(\mathbf{A}^-)^* + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\hat{\mathbf{U}}(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*, \tag{1.2}$$

where $\hat{\mathbf{U}}$ is declared to be an arbitrary matrix in $\mathbb{C}_n^{\geqslant}$, while $\mathbf{A}^-$ is declared to be an arbitrary generalized inverse of $\mathbf{A}$. Baksalary [1], however, demonstrated that (1.2) *cannot* be understood in such a way that $\mathbf{A}^-$ is chosen arbitrarily in a first step, and then, in a a second step, every solution is obtained from (1.2) by varying $\hat{\mathbf{U}}$ over $\mathbb{C}_n^{\geqslant}$. As an alternative, Baksalary [1, Theorem 1] proposes a representation

$$\mathbf{X} = \hat{\mathbf{X}}\hat{\mathbf{X}}^*, \quad \hat{\mathbf{X}} = \mathbf{A}^-\mathbf{D} + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}, \tag{1.3}$$

where $\mathbf{A}^-$ is an arbitrary but fixed generalized inverse of $\mathbf{A}$, and $\mathbf{D}$ is an arbitrary but fixed $m \times n$ matrix such that $\mathbf{B} = \mathbf{DD}^*$, whereas $\mathbf{Z}$ is free to vary over $\mathbb{C}_{n\times n}$.

It is our aim to deliver a further alternative to (1.2), which can be expressed as

$$\mathbf{X} = \mathbf{A}^=\mathbf{B}(\mathbf{A}^=)^* + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{UU}^*(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*, \tag{1.4}$$

where $\mathbf{A}^=$ represents a class of generalized inverses of $\mathbf{A}$ of the form $\mathbf{A}^= = \mathbf{A}^- + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}(\mathbf{B}^{1/2})^-$. Here $\mathbf{A}^-$ and $(\mathbf{B}^{1/2})^-$ are arbitrary but fixed generalized inverses of $\mathbf{A}$ and $\mathbf{B}^{1/2}$, respectively, whereas $\mathbf{Z}$ is free to vary over $n \times m$, and $\mathbf{U}$ is free to vary over $\mathbb{C}_{n\times(n-b)}$ with $b = \mathrm{rk}(\mathbf{B})$. It will be seen that (1.4) delivers an additive decomposition $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$ such that $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathbf{X}_1) + \mathrm{rk}(\mathbf{X}_2)$, $\mathrm{rk}(\mathbf{X}_1) = \mathrm{rk}(\mathbf{B})$, and $\mathbf{AX}_1\mathbf{A}^* = \mathbf{B}$, where $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{C}_n^{\geqslant}$. As a consequence, representations of general Hermitian nonnegative-definite solutions of minimal and maximal rank, respectively, can be derived.

The importance of Eq. (1.1) has recently been emphasized by Dai and Lancaster [2] within the real setting. See also [3] for some additional notes on the general Hermitian solution to $\mathbf{AXA}^* = \mathbf{B}$. Further application for Hermitian nonnegative-definite solutions to equations of the form (1.1) are given by Young et al. [7]. Their basic Theorem 1, however, is proved to be incorrect, see Section 3.

2. Results

Before stating our main result, we consider the equation $\mathbf{AY} = \mathbf{B}$ for given matrices $\mathbf{A}$ and $\mathbf{B}$. Clearly, there exists a matrix $\mathbf{Y}$ satisfying $\mathbf{AY} = \mathbf{B}$ if and only if $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$. For such a matrix $\mathbf{Y}$ we have $\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{AY}) \leqslant \mathrm{rk}(\mathbf{Y})$ so that a solution of minimal rank is a solution $\mathbf{Y}$ satisfying $\mathrm{rk}(\mathbf{Y}) = \mathrm{rk}(\mathbf{B})$. Mitra [5, Lemma 2.2] demonstrated that every matrix $\mathbf{Y}$ with $\mathbf{AY} = \mathbf{B}$ can be written as $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$

such that $\mathrm{rk}(\mathbf{Y}) = \mathrm{rk}(\mathbf{Y}_1) + \mathrm{rk}(\mathbf{Y}_2)$, $\mathrm{rk}(\mathbf{Y}_1) = \mathrm{rk}(\mathbf{B})$ and $\mathbf{A}\mathbf{Y}_1 = \mathbf{B}$. The following lemma gives a general representation for the set of minimal rank solutions.

Lemma 1. *Let $\mathbf{A} \in \mathbb{C}_{m\times n}$ and $\mathbf{B} \in \mathbb{C}_{m\times p}$ such that $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$. Let $\mathbf{A}^-$ and $\mathbf{B}^-$ denote arbitrary but fixed generalized inverses of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then a representation of the general minimal rank solution to $\mathbf{AY} = \mathbf{B}$ is given by*

$$\mathbf{Y} = \mathbf{A}^-\mathbf{B} + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}\mathbf{B}^-\mathbf{B}, \tag{2.1}$$

where $\mathbf{Z}$ is free to vary over $\mathbb{C}_{n\times p}$.

Proof. Let $\mathbf{Y}$ be given as in (2.1) for some $\mathbf{Z} \in \mathbb{C}_{n\times p}$. Then clearly $\mathbf{AY} = \mathbf{B}$ in view of $\mathbf{AA}^-\mathbf{B} = \mathbf{B}$. This implies $\mathscr{R}(\mathbf{B}^*) \subseteq \mathscr{R}(\mathbf{Y}^*)$. In addition, $\mathbf{Y}$ of the form (2.1) satisfies $\mathscr{R}(\mathbf{Y}^*) \subseteq \mathscr{R}(\mathbf{B}^*)$ so that $\mathscr{R}(\mathbf{Y}^*) = \mathscr{R}(\mathbf{B}^*)$ and hence $\mathrm{rk}(\mathbf{Y}) = \mathrm{rk}(\mathbf{B})$.

Conversely let $\mathbf{Y}$ satisfy $\mathbf{AY} = \mathbf{B}$ and $\mathrm{rk}(\mathbf{Y}) = \mathrm{rk}(\mathbf{B})$. Then $\mathscr{R}(\mathbf{Y}^*) = \mathscr{R}(\mathbf{B}^*)$ so that $\mathbf{Y} = \mathbf{YB}^-\mathbf{B}$. Since $\mathbf{AY} = \mathbf{B}$, there exists $\mathbf{Z}$ such that $\mathbf{Y} = \mathbf{A}^-\mathbf{B} + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z} = \mathbf{A}^-\mathbf{B} + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}\mathbf{B}^-\mathbf{B}$, thus completing the proof. □

It is obvious that $\mathbf{Y} = \mathbf{A}^-\mathbf{B}$ is a minimal rank solution to $\mathbf{AY} = \mathbf{B}$ irrespective of the choice of $\mathbf{A}^-$, provided $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$. On the other hand, Lemma 1 shows that every minimal rank solution can be written as $\mathbf{A}^=\mathbf{B}$ for some generalized inverse $\mathbf{A}^=$ of the form $\mathbf{A}^= = \mathbf{A}^- + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}\mathbf{B}^-$.

Subsequently we consider the equation $\mathbf{AXA}^* = \mathbf{B}$ for given matrices $\mathbf{A} \in \mathbb{C}_{m\times n}$ and $\mathbf{B} \in \mathbb{C}_m^{\geqslant}$. A necessary and sufficient condition for the consistency of this equation is $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$. In that case $\mathrm{rk}(\mathbf{B}) \leqslant \mathrm{rk}(\mathbf{A}) \leqslant n$.

Theorem 1. *Let $\mathbf{A} \in \mathbb{C}_{m\times n}$ and $\mathbf{B} \in \mathbb{C}_m^{\geqslant}$ with $b = \mathrm{rk}(\mathbf{B})$ such that $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$. Let $\mathbf{A}^-$ and $(\mathbf{B}^{1/2})^-$ denote arbitrary but fixed generalized inverses of $\mathbf{A}$ and $\mathbf{B}^{1/2}$, respectively. Then a representation of the general Hermitian nonnegative-definite solution to $\mathbf{AXA}^* = \mathbf{B}$ is given by*

$$\mathbf{X} = \mathbf{A}^=\mathbf{B}(\mathbf{A}^=)^* + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}\mathbf{U}^*(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^* \tag{2.2}$$

with

$$\mathbf{A}^= = \mathbf{A}^- + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{Z}(\mathbf{B}^{1/2})^-, \tag{2.3}$$

where $\mathbf{Z}$ is free to vary over $\mathbb{C}_{n\times m}$, and $\mathbf{U}$ is free to vary over $\mathbb{C}_{n\times(n-b)}$.

Proof. When $\mathbf{X}$ is given by (2.2) with (2.3), then $\mathbf{X} \in \mathbb{C}_n^{\geqslant}$ and $\mathbf{AXA}^* = \mathbf{AA}^-\mathbf{B}(\mathbf{AA}^-)^* = \mathbf{B}$ in view of $\mathbf{AA}^-\mathbf{B} = \mathbf{B}$.

To see that (2.2) with (2.3) provides the general solution, let

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^* = (\mathbf{V}_1 : \mathbf{V}_2)\begin{pmatrix} \mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{pmatrix} = \mathbf{V}_1\mathbf{\Delta}\mathbf{V}_1^*$$

be the spectral decomposition of $\mathbf{B}$ so that $\mathbf{V} \in \mathbb{C}_{m\times m}$ is unitary and $\mathbf{\Delta} \in \mathbb{C}_b^{>}$ is diagonal with $b = \mathrm{rk}(\mathbf{B})$. Since $b \leqslant \mathrm{rk}(\mathbf{A}) \leqslant n$ we can write $\mathbf{B} = \mathbf{DD}^*$, where $\mathbf{D} = (\mathbf{V}_1\mathbf{\Delta}^{1/2} : \mathbf{0}_{m\times(n-b)})$.

Now, let $\mathbf{X} \in \mathbb{C}_n^{\geqslant}$ satisfy $\mathbf{AXA}^* = \mathbf{B}$. Then $\mathbf{AX}^{1/2}(\mathbf{AX}^{1/2})^* = \mathbf{DD}^*$. Since $\mathbf{AX}^{1/2}$ and $\mathbf{D}$ are both $m \times n$ matrices, there exists a unitary $\mathbf{W} \in \mathbb{C}_{n\times n}$ such that $\mathbf{AX}^{1/2} = \mathbf{DW}$, see e.g. [6, Theorem 8.9.3], or equivalently,

$$\mathbf{AH} = (\mathbf{V}_1\mathbf{\Delta}^{1/2} : \mathbf{0}_{m\times(n-b)}),$$

where $\mathbf{H} = \mathbf{X}^{1/2}\mathbf{W}^*$. By appropriately partitioning $\mathbf{H} = (\mathbf{H}_1 : \mathbf{H}_2)$, we can write

$$\mathbf{AH}_1 = \mathbf{V}_1\mathbf{\Delta}^{1/2} \quad \text{and} \quad \mathbf{AH}_2 = \mathbf{0}_{m\times(n-b)}. \tag{2.4}$$

Then there exists a matrix $\mathbf{U} \in \mathbb{C}_{n\times(n-b)}$ such that

$$\mathbf{H}_2 = (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}.$$

Postmultiplying the first equation in (2.4) by $\mathbf{V}_1^*$ (where $\mathbf{V}_1^*\mathbf{V}_1 = \mathbf{I}_b$) yields

$$\mathbf{AY} = \mathbf{B}^{1/2},$$

where $\mathbf{Y} = \mathbf{H}_1\mathbf{V}_1^*$. That $\mathrm{rk}(\mathbf{Y}) = \mathrm{rk}(\mathbf{H}_1) = \mathrm{rk}(\mathbf{B})$ can be seen from $\mathrm{rk}(\mathbf{H}_1) = \mathrm{rk}(\mathbf{H}_1\mathbf{V}_1^*\mathbf{V}_1) \leqslant \mathrm{rk}(\mathbf{Y}) \leqslant \mathrm{rk}(\mathbf{H}_1)$ and $b = \mathrm{rk}(\mathbf{V}_1\mathbf{\Delta}^{1/2}) = \mathrm{rk}(\mathbf{AH}_1) \leqslant \mathrm{rk}(\mathbf{H}_1) \leqslant b$. From Lemma 1, we conclude that $\mathbf{Y} = \mathbf{A}^{=}\mathbf{B}^{1/2}$ for some $\mathbf{Z} \in \mathbb{C}_{n\times m}$, where $\mathbf{A}^{=}$ is given in (2.3). The matrix $\mathbf{X} = \mathbf{HH}^* = \mathbf{H}_1\mathbf{H}_1^* + \mathbf{H}_2\mathbf{H}_2^* = \mathbf{YY}^* + \mathbf{H}_2\mathbf{H}_2^*$ can therefore be written as

$$\mathbf{X} = \mathbf{A}^{=}\mathbf{B}(\mathbf{A}^{=})^* + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{UU}^*(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*,$$

thus concluding the proof. □

It is easily seen that under the assumptions of Theorem 1, a representation of the general Hermitian nonnegative-definite solution to $\mathbf{AXA}^* = \mathbf{B}$ is also

$$\mathbf{X} = \mathbf{A}^{=}\mathbf{B}(\mathbf{A}^{=})^* + (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\hat{\mathbf{U}}(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*$$

with $\mathbf{A}^{=}$ from (2.3), where $\mathbf{Z}$ is free to vary over $\mathbb{C}_{n\times m}$, and $\hat{\mathbf{U}}$ is free to vary over $\mathbb{C}_n^{\geqslant}$. Such a representation might be advantageous when the rank of $\mathbf{B}$ is not known.

In addition, we note that Theorem 1 does not only give a representation of the general Hermitian nonnegative-definite solution to $\mathbf{AXA}^* = \mathbf{B}$, but, simultaneously, delivers an additive decomposition $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$ such that $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathbf{X}_1) + \mathrm{rk}(\mathbf{X}_2)$, $\mathrm{rk}(\mathbf{X}_1) = \mathrm{rk}(\mathbf{B})$ and $\mathbf{AX}_1\mathbf{A}^* = \mathbf{B}$, where $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{C}_n^{\geqslant}$. This motivates us to give representations for the general Hermitian nonnegative-definite solutions of minimal and maximal rank, respectively.

Corollary 1. *Under the assumptions of Theorem* 1, *the following three statements hold*:

(i) *If* $\mathbf{X}$ *is represented as in Theorem* 1, *then* $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathbf{B}) + \mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}]$.

(ii) *The minimal rank of a Hermitian nonnegative-definite matrix* $\mathbf{X}$ *satisfying* $\mathbf{AXA}^* = \mathbf{B}$ *is* $\mathrm{rk}(\mathbf{B})$. *A representation for the general solution is given as in Theorem* 1, *where* $\mathbf{Z}$ *is free to vary over* $\mathbb{C}_{n\times m}$ *and* $\mathbf{U} = \mathbf{0}_{n\times(n-b)}$.

(iii) *The maximal rank of a Hermitian nonnegative-definite matrix* $\mathbf{X}$ *satisfying* $\mathbf{AXA}^* = \mathbf{B}$ *is* $n - [\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})]$. *A representation for the general solution* *is given as in Theorem* 1, *where* $\mathbf{Z}$ *is free to vary over* $\mathbb{C}_{n\times m}$ *and* $\mathbf{U}$ *is free to vary over* $\mathbb{C}_{n\times(n-b)}$ *subject to* $\mathrm{rk}(\mathbf{U} : \mathbf{A}^-\mathbf{A}) = n$.

Proof. Let $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$, where $\mathbf{X}_1 = \mathbf{A}^=\mathbf{B}(\mathbf{A}^=)^* \in \mathbb{C}_n^{\geqslant}$ and $\mathbf{X}_2 = (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}\mathbf{U}^*(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^* \in \mathbb{C}_n^{\geqslant}$. If $\mathbf{y}$ is any vector in $\mathscr{R}(\mathbf{X}_1) \cap \mathscr{R}(\mathbf{X}_2)$, then $\mathbf{y} = \mathbf{A}^=\mathbf{B}\mathbf{a} = (\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}\mathbf{b}$ for some vectors $\mathbf{a}$ and $\mathbf{b}$ so that $\mathbf{A}\mathbf{y} = \mathbf{0}$. But since $\mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{A}^=\mathbf{B}\mathbf{a} = \mathbf{B}\mathbf{a}$, we arrive at $\mathbf{y} = \mathbf{0}$. Hence $\mathscr{R}(\mathbf{X}_1) \cap \mathscr{R}(\mathbf{X}_2) = \{\mathbf{0}\}$, and therefore $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathbf{X}_1) + \mathrm{rk}(\mathbf{X}_2)$, cf. [6, Theorem 3.6.1], where $\mathrm{rk}(\mathbf{X}_1) = \mathrm{rk}(\mathbf{A}^=\mathbf{B}) = \mathrm{rk}(\mathbf{B})$ and $\mathrm{rk}(\mathbf{X}_2) = \mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}]$. This shows (i). It is now clear that $b = \mathrm{rk}(\mathbf{B})$ is the minimal rank of $\mathbf{X}$. Moreover, the solution $\mathbf{X}$ of minimal rank is represented by (2.2) with (2.3) where $\mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}] = 0$, i.e., $(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U} = \mathbf{0}$. This is accomplished by choosing $\mathbf{U} = \mathbf{0}$ so that (ii) is shown. To see (iii), note that $\mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})\mathbf{U}] = \mathrm{rk}[\mathbf{U}^*(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*] = \mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*] - \dim\{\mathscr{N}(\mathbf{U}^*) \cap \mathscr{R}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*]\}$, cf. [6, Theorem 3.4.17], where $\mathrm{rk}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*] = n - \mathrm{rk}(\mathbf{A})$. Hence $\mathrm{rk}(\mathbf{X}) = n - [\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})] - \dim\{\mathscr{N}(\mathbf{U}^*) \cap \mathscr{R}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*]\}$ so that the maximal rank of $\mathbf{X}$ is $n - [\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})]$ whenever $\dim\{\mathscr{N}(\mathbf{U}^*) \cap \mathscr{R}[(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})^*]\} = 0$, or equivalently, $\dim[\mathscr{R}(\mathbf{U}) + \mathscr{N}(\mathbf{I}_n - \mathbf{A}^-\mathbf{A})] = n$. Since $\mathscr{N}(\mathbf{I}_n - \mathbf{A}^-\mathbf{A}) = \mathscr{R}(\mathbf{A}^-\mathbf{A})$, the latter is equivalent to $\mathrm{rk}(\mathbf{U} : \mathbf{A}^-\mathbf{A}) = n$. $\square$

Note that part (iii) of Corollary 1 also gives a representation of the general Hermitian positive-definite solution to $\mathbf{A}\mathbf{X}\mathbf{A}^* = \mathbf{B}$, provided such a solution exists, which is the case if $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$ and $\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{A})$, i.e., $\mathscr{R}(\mathbf{B}) = \mathscr{R}(\mathbf{A})$. When $\mathbf{A}\mathbf{X}\mathbf{A}^* = \mathbf{B}$ is consistent, then a sufficient condition for $\mathscr{R}(\mathbf{B}) = \mathscr{R}(\mathbf{A})$ is $\mathbf{B} \in \mathbb{C}_m^{>}$.

If $\mathbf{A}^-$ is chosen as a minimum norm generalized inverse, i.e., $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$ and $(\mathbf{A}^-\mathbf{A})^* = \mathbf{A}^-\mathbf{A}$, then $\mathrm{rk}(\mathbf{U} : \mathbf{A}^-\mathbf{A}) = n$ is equivalent to $\mathrm{rk}(\mathbf{U} : \mathbf{A}^*) = n$.

Let us now reconsider the example given by Baksalary [1], where

$$\mathbf{A} = (1 \quad 1) \quad \text{and} \quad \mathbf{B} = (1).$$

By choosing $\mathbf{A}^- = (1 \quad 0)^*$ and noting $\mathbf{B} = \mathbf{B}^{1/2} = (\mathbf{B}^{1/2})^{-1}$, Theorem 1 yields the general Hermitian nonnegative-definite solution

$$\mathbf{X} = \begin{pmatrix} (1+z)(1+\overline{z}) & -(1+z)\overline{z} \\ -z(1+\overline{z}) & z\overline{z} \end{pmatrix} + \begin{pmatrix} u\overline{u} & -u\overline{u} \\ -u\overline{u} & u\overline{u} \end{pmatrix}, \tag{2.5}$$

where z and u are arbitrary complex numbers. If $u\overline{u} = 0$, then $\mathbf{X}$ has minimal rank, whereas in case $u\overline{u} \neq 0$ the solution $\mathbf{X}$ has maximal rank.

3. Extensions

It is easily seen that the above results extend naturally to systems of four equations

$$\mathbf{A}_i\mathbf{X}\mathbf{A}_j^* = \mathbf{B}_i\mathbf{B}_j^*, \quad i, j = 1, 2, \tag{3.1}$$

for given matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}_{m\times n}$ and $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_{m\times k}$. A representation for the general common Hermitian nonnegative-definite solution to (3.1), and the condition for its existence, can be given in terms of

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_1\mathbf{B}_1^* & \mathbf{B}_1\mathbf{B}_2^* \\ \mathbf{B}_2\mathbf{B}_1^* & \mathbf{B}_2\mathbf{B}_2^* \end{pmatrix},$$

by applying Theorem 1 and noting that (3.1) can be written as $\mathbf{AXA}^* = \mathbf{B}$, where the matrix $\mathbf{B}$ is Hermitian nonnegative-definite.

For the choice $k = m$, Theorem 1 in [7] claims $\mathbf{X} = \tilde{\mathbf{X}}\tilde{\mathbf{X}}^*$ as a representation for the general common Hermitian nonnegative-definite solution (provided its existence) to the two equations

$$\mathbf{A}_i\mathbf{X}\mathbf{A}_i^* = \mathbf{B}_i\mathbf{B}_i^*, \quad i = 1, 2, \tag{3.2}$$

where

$$\begin{aligned}\tilde{\mathbf{X}} = {} & \mathbf{A}_1^+\mathbf{B}_1 + (\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)]^+(\mathbf{B}_2 - \mathbf{A}_2\mathbf{A}_1^+\mathbf{B}_1) \\ & +(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)(\mathbf{I}_n - [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)]^+[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)])\mathbf{Z},\end{aligned} \tag{3.3}$$

and $\mathbf{Z}$ being free to vary over $\mathbb{C}_{n\times n}$. However, this cannot be true. For a counterexample, disproving the asserted general representation, let

$$\mathbf{A}_1 = (1 \quad 1), \quad \mathbf{A}_2 = (0 \quad 1) \quad \text{and} \quad \mathbf{B}_1 = (1 \quad 0) = \mathbf{B}_2.$$

It is easily computed that

$$(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)(\mathbf{I}_n - [\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)]^+[\mathbf{A}_2(\mathbf{I}_n - \mathbf{A}_1^+\mathbf{A}_1)]) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and that $\tilde{\mathbf{X}}$ from (3.3) is uniquely given as

$$\tilde{\mathbf{X}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now,

$$\mathbf{X} = \tilde{\mathbf{X}}\tilde{\mathbf{X}}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

cannot be the general common Hermitian nonnegative-definite solution to (3.2), since every matrix of the form (2.5) with $z\overline{z} + u\overline{u} = 1$ is also a solution. Take for example

$$\mathbf{X} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

Nonetheless,

$$\mathbf{X} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is the only common Hermitian nonnegative-definite solution to the four equations (3.1).

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References

[1] J.K. Baksalary, Nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$, Linear and Multilinear Algebra 16 (1984) 133–139.
[2] H. Dai, P. Lancaster, Linear matrix equations from an inverse problem of vibration theory, Linear Algebra Appl. 246 (1996) 31–47.
[3] J. Groß, A note on the general Hermitian solution to $AXA^* = B$, Bull. Malay. Math. Soc. 21 (1998) 57–62.
[4] C.G. Khatri, S.K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math. 31 (1976) 579–585.
[5] S.K. Mitra, Fixed rank solutions of linear matrix equations, Sankhyā Ser. A 35 (1972) 387–392.
[6] A.R. Rao, P. Bhimasankaram, Linear Algebra, Tata McGraw-Hill, New Delhi, 1992.
[7] D.M. Young, J.W. Seaman Jr., L.M. Meaux, Independence distribution preserving covariance structures for the multivariate model, J. Multivariate Anal. 68 (1999) 165–175.