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Specialization of Appell's functions to univariate hypergeometric functions

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ABSTRACT

Univariate specializations of Appell's hypergeometric functions F_1, F_2, F_3, F_4 satisfy ordinary Fuchsian equations of order at most 4. In special cases, these differential equations are of order 2 and could be simple (pullback) transformations of Euler's differential equation for the Gauss hypergeometric function. The paper classifies these cases, and presents corresponding relations between univariate specializations of Appell's functions and univariate hypergeometric functions. The computational aspect and interesting identities are discussed.

1. Introduction

Appell's hypergeometric functions $F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$, $F_4(x,y)$ are defined by the following double hypergeometric series:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}(b_1)_n(b_2)_m}{(c)_{n+m}\, n!\, m!} x^n y^m, \tag{1}$$

$$F_2\left(\begin{matrix} a;\ b_1, b_2 \\ c_1, c_2 \end{matrix}\,\middle|\, x, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}(b_1)_n(b_2)_m}{(c_1)_n(c_2)_m\, n!\, m!} x^n y^m, \tag{2}$$

$$F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1)_n(a_2)_m(b_1)_n(b_2)_m}{(c)_{n+m}\, n!\, m!} x^n y^m, \tag{3}$$

$$F_4\left(\begin{matrix} a;\ b \\ c_1, c_2 \end{matrix}\,\middle|\, x, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}(b)_{n+m}}{(c_1)_n(c_2)_m\, n!\, m!} x^n y^m. \tag{4}$$

They are bivariate generalizations of the Gauss hypergeometric series

$$_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(A)_n(B)_n}{(C)_n\, n!} z^n. \tag{5}$$

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In particular, Appell's functions satisfy Fuchsian (that is, linear homogeneous and with regular singularities) systems of partial differential equations, holonomic of rank 3 or 4, which are analogous to Euler's hypergeometric equation for the ${}_2\mathrm{F}_1$ function. Euler's equation for (5) is

$$z(1-z)\frac{d^2y(z)}{dz^2}+\big(C-(A+B+1)z\big)\frac{dy(z)}{dz}-ABy(z)=0. \tag{6}$$

It is a Fuchsian equation with three singularities: $z=0$, $z=1$ and $z=\infty$. Local exponent differences are equal, respectively, to $1-C$, $C-A-B$, $A-B$ at them.

When the two arguments x, y of Appell's functions are algebraically related, the univariate specializations satisfy Fuchsian ordinary differential equations of order at most 4. This paper considers the following questions:

> Which univariate specializations of Appell's functions satisfy a second order Fuchsian ordinary differential equation?
> Which univariate specializations of Appell's functions satisfy Euler's equation (6) for some parameters A, B, C, up to a projective transformation $y(z)\mapsto\theta(z)y(z)$ or a more general pull-back transformation
>
> $$z\mapsto\varphi(x),\quad y(z)\mapsto\theta(x)y\big(\varphi(x)\big), \tag{7}$$
>
> where $\theta(z)$ or $\theta(x)$ is a power function, and $\varphi(x)$ is a rational function?

Other interesting questions are: Which univariate specializations of Appell's functions satisfy a Fuchsian ordinary differential equation of order 3? Up to pull-back transformations, which univariate specializations of Appell's functions satisfy differential equations for generalized hypergeometric functions ${}_3\mathrm{F}_2(z)$ or ${}_4\mathrm{F}_3(z)$?

The paper classifies univariate specializations of Appell's F_2, F_3, F_4 functions satisfying second order ordinary Fuchsian equations, when those ordinary equations follow from the partial differential equations for Appell's functions without reductions due to factorization of the respective differential operators (or reducibility of the monodromy group). In particular, we do not consider isolated solutions of reducible Appell's systems of partial differential equations that are expressible via Gauss hypergeometric functions. Definition 1.1 below specifies the way ordinary differential equations *follow fully* from systems of partial differential equations.

We identify those univariate specialization of Appell's F_2, F_3, F_4 functions that can be expressed in terms of Gauss' hypergeometric function (5). Their ordinary differential equations are pull-back transformations of Euler's equation (6), and *follow fully* from the respective partial differential equations. To deal with univariate specializations onto singularity curves of the respective systems of partial differential equations, we present general ordinary Fuchsian equations for them. In the cases of F_1, F_2, F_3 functions, the specializations onto singularity curves can be expressed in terms of ${}_2\mathrm{F}_1$ or ${}_3\mathrm{F}_2$ hypergeometric functions.

In Appendix A, we briefly review identities between Appell's and Gauss hypergeometric functions existing in literature. Our explicit results that appear more or less novel are the following:

- A case of $F_2(x,2-x)$ function that can be expressed as a ${}_2\mathrm{F}_1$ function with a quadratic argument; see Theorem 2.4. Equivalent identification of $F_3(x,x/(2x-1))$ functions was noticed by Karlsson [9]. These relations apply nicely when the monodromy group of the ${}_2\mathrm{F}_1$ function is dihedral; then the F_2 and F_3 solutions of the same ordinary Fuchsian equation are terminating, and dihedral ${}_2\mathrm{F}_1$ functions can be expressed as elementary functions. We show this application in Section 7.
- A separation of variables case for the F_2 function; see formula (35). It is related to the well-known Bailey case [4] of variable separation for the F_4 function

$$F_4\left(\begin{matrix}a;\,b\\ c,a+b-c+1\end{matrix}\,\middle|\,x(1-y),y(1-x)\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b\\ c\end{matrix}\,\middle|\,x\right){}_2\mathrm{F}_1\left(\begin{matrix}a,b\\ a+b-c+1\end{matrix}\,\middle|\,y\right) \tag{8}$$

  via a known transformation between F_2 and F_4 functions.
- Translation of the above two F_2 cases to relations of F_1 and F_3 functions to Gauss hypergeometric functions.
- A few cases of $F_4(t^2,(1-t)^2)$ functions (that is, specializations to the quadratic singular curve of F_4) expressible as ${}_2\mathrm{F}_1$ or ${}_3\mathrm{F}_2$ functions, or products of two ${}_2\mathrm{F}_1$ solutions of the same Euler's equation (6).

Identities between bivariate and univariate hypergeometric series are usually derived by methods of series manipulation or using integral representations. The method of relating hypergeometric functions as solutions of coinciding differential equations is usually considered tedious and computationally costly. With powerful computer algebra techniques available, the method of identifying differential equations can be worked out rather comprehensively. We discuss general computational techniques in Section 6. Here below and in Section 2 we gradually introduce the algebraic setting, computational methods and some shortcuts.

Consider a holomorphic function $F(x,y)$ on an open set in $\mathbf{C}\times\mathbf{C}$, and a univariate specialization $F(x(t),y(t))$ of it. The full derivatives with respect to t are expressed linearly in terms of the partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial^2F/\partial x^2$, etc. For example,

$$\frac{dF}{dt} = \dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y}, \tag{9}$$

$$\frac{d^2F}{dt^2} = \dot{x}^2\frac{\partial^2 F}{\partial x^2} + 2\dot{x}\dot{y}\frac{\partial^2 F}{\partial x\partial y} + \dot{y}^2\frac{\partial^2 F}{\partial y^2} + \ddot{x}\frac{\partial F}{\partial x} + \ddot{y}\frac{\partial F}{\partial y}. \tag{10}$$

Here $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, $\ddot{x} = d^2x/dt^2$, $\ddot{y} = d^2y/dt^2$. To compute next order full derivatives, one applies the Leibniz rule and lets d/dt act on the partial derivatives by copying the action on F in (9).

We identify partial differential equations with ordinary differential equations in the specialization variable t by identifying their "mixed" forms in the partial derivatives with the coefficients specialized to functions in t. To fix a univariate specialization mapping, consider a holomorphic map Φ from an open subset of $\mathbf{C}$ to $\mathbf{C}\times\mathbf{C}$, mapping $t \mapsto (x(t), y(t))$. If $F(x,y)$ is a holomorphic function on the image of Φ, then $F\circ\Phi$ is the univariate specialization $F(x(t), y(t))$. Geometrically, a univariate specialization is the pullback of $F(x,y)$ with respect to Φ.

Definition 1.1. If E is an ordinary differential equation with respect to t, its *partial differential form* under Φ is the expression where the derivatives with respect to t are replaced by partial derivatives following formulas (9)–(10), etc.

If $\widetilde{E}$ is a partial differential equation with respect to x, y, its *specialized form* under Φ is the expression where the coefficients to the partial derivatives are specialized $x \mapsto x(t)$, $y \mapsto y(t)$.

Let H denote a system of partial differential equations with respect to x, y. The ordinary differential equation E is said to *follow fully* from H under Φ, if its partial differential form under Φ coincides with the specialized form (under Φ) of some partial differential equation following from H by algebraic and partial differentiation operations.

We also say that E is *implied fully* by H, if there is a specialization map Φ such that E follows fully from H under Φ. Here we allow H, E and Φ to have parameters; the parameters of H may specialize (to constants or functions in the parameters of E, Φ) in the specialized forms of its partial differential equations under Φ.

The classical systems of partial differential equations for Appell's functions F_2, F_3, F_4, F_1 are presented, respectively, in formulas (11)–(12), (36)–(37), (43)–(44), (70)–(71) below. Algebraically, linear partial differential equations with polynomial coefficients are identified with partial differential operators in the Weyl algebra $\mathbf{C}[x,y]\langle\partial/\partial x, \partial/\partial y\rangle$. A system of partial differential equations corresponds to a left ideal in the Weyl algebra.

The systems of partial differential equations for F_2, F_3, F_4 functions have rank 4. Correspondingly, Gröbner bases for the mentioned left ideals in the Weyl algebra give linear expressions of all partial derivatives in terms of 4 partial derivatives of low order, say $\partial^2F/\partial x^2$, $\partial F/\partial x$, $\partial F/\partial y$, F. If we take a total degree ordering of partial derivatives, the leading coefficients in the expressions for higher order derivatives vanish only when specialized onto a singularity curve of the differential system.

If a univariate specialization of Appell's F_2, F_3 or F_4 function is not onto a singularity curve of the differential system, the full derivatives can be expressed linearly in terms of the 4 basic partial derivatives, and a linear relation between d^4F/dt^4, d^3F/dt^3, d^2F/dt^2, dF/dt, F gives an ordinary differential equation of order at most 4 for the univariate function. If the univariate specialization is onto a singularity curve, it turns out that we get an ordinary differential equation of order less than 4. All these ordinary differential equations are Fuchsian.

Similarly, the system of partial differential equations for the F_1 function has rank 3, and its univariate specializations satisfy ordinary Fuchsian equations of order at most 3.

We are interested in cases when the ordinary differential equation has order 2. Such a second order differential equation must follow from partial differential equations of order at most 2. For non-singular specializations of F_2, F_3, F_4 functions, linear relations between partial derivatives of order at most 2 are generated by the classical pairs of partial differential equations.

2. Identities with Appell's F_2 function

A system partial differential equations for the $F_2(x,y)$ function is:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - xy\frac{\partial^2 F}{\partial x\partial y} + \big(c_1 - (a+b_1+1)x\big)\frac{\partial F}{\partial x} - b_1 y\frac{\partial F}{\partial y} - ab_1F = 0, \tag{11}$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} - xy\frac{\partial^2 F}{\partial x\partial y} + \big(c_2 - (a+b_2+1)y\big)\frac{\partial F}{\partial y} - b_2 x\frac{\partial F}{\partial x} - ab_2F = 0. \tag{12}$$

This is an equation system of rank 4. Its singular locus on $\mathbf{P}^1\times\mathbf{P}^1$ is the union of the following lines:

$$x=0,\quad x=1,\quad x=\infty,\quad y=0,\quad y=1,\quad y=\infty,\quad x+y=1. \tag{13}$$

First we handle univariate specializations onto the singular lines. We determine that generally these specializations satisfy hypergeometric equations for ${}_2\mathrm{F}_1$ or ${}_3\mathrm{F}_2$ functions. Explicit identification of the following three function pairs is presented after Theorem 2.4. Those identities can be easily derived from power series manipulations as well.

Theorem 2.1. *The following function pairs satisfy the same ordinary differential equations (of order 2 or 3):*

- $F_2\left(\begin{matrix} a;\, b_1, b_2 \\ c_1, c_2 \end{matrix} \,\middle|\, x, 0\right)$ *and* ${}_2\mathrm{F}_1\left(\begin{matrix} a, b_1 \\ c_1 \end{matrix} \,\middle|\, x\right)$;
- $F_2\left(\begin{matrix} a;\, b_1, b_2 \\ c_1, c_2 \end{matrix} \,\middle|\, x, 1\right)$ *and* ${}_3\mathrm{F}_2\left(\begin{matrix} a, b_1, a-c_2+1 \\ c_1, a+b_2-c_2+1 \end{matrix} \,\middle|\, x\right)$;
- $F_2\left(\begin{matrix} a;\, b_1, b_2 \\ c_1, c_2 \end{matrix} \,\middle|\, x, 1-x\right)$ *and* $(1-x)^{-a}{}_3\mathrm{F}_2\left(\begin{matrix} a, c_1-b_1, a-c_2+1 \\ c_1, a+b_2-c_2+1 \end{matrix} \,\middle|\, \frac{x}{x-1}\right)$.

Proof. The first specialization is trivial: after the substitution $y=0$ into the coefficients of (11) we have differentiation with respect to x only.

To handle the other two cases, we use the third order differential equation for the ${}_3\mathrm{F}_2(x)$ function; see formula (100) in Appendix A.

For the $F_2(x,1)$ function, subtract (12) from (11) and set $y=1$ in the coefficients:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} + \big(c_1-(a+b_1-b_2+1)x\big)\frac{\partial F}{\partial x} - (c_2-a+b_1-b_2-1)\frac{\partial F}{\partial y} - a(b_1-b_2)F = 0. \tag{14}$$

Then make the following combination of partial differential equations for the $F_2(x,1)$:

$$x\frac{\partial}{\partial x}[\text{Eq. (14)}] + (a+b_2-c_2+1)[\text{Eq. (11)}] - b_1[\text{Eq. (12)}]. \tag{15}$$

After setting $y=1$ in the coefficients, we get an equation with differentiation by x only. That differential equation is identified as hypergeometric equation (100) with the indicated values of the parameters A, B, C, D, E.

For the $F_2(x,1-x)$ function, add (11) and (12) and set $y=1-x$:

$$\begin{aligned} x(1-x)&\left(\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\partial y} + \frac{\partial^2 F}{\partial y^2}\right) + \big(c_1-(a+b_1+b_2+1)x\big)\frac{\partial F}{\partial x} \\ &+ \big(c_2-a-b_1-b_2-1+(a+b_1+b_2+1)x\big)\frac{\partial F}{\partial y} - a(b_1+b_2)F = 0, \end{aligned} \tag{16}$$

and consider the following combination of partial differential equations for it:

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{a+1}{x-1}\right)[\text{Eq. (16)}] + \frac{a+b_2-c_2+1}{x(1-x)}[\text{Eq. (11)}] + \frac{c_1-b_1}{x(1-x)}[\text{Eq. (12)}]. \tag{17}$$

With $y=1-x$, we recognize the full derivatives

$$\frac{d^3F}{dx^3} = \frac{\partial^3 F}{\partial x^3} - 3\frac{\partial^3 F}{\partial x^2\partial y} + 3\frac{\partial^3 F}{\partial x\partial y^2} - \frac{\partial^3 F}{\partial y^3}, \qquad \frac{d^2F}{dx^2} = \frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\partial y} + \frac{\partial^2 F}{\partial y^2}, \qquad \frac{dF}{dx} = \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \tag{18}$$

in (17), and get an ordinary differential equation of order 3, with the following singularities and local exponents:

$$\begin{aligned} &\text{at } x=0: && 0, && 1-c_1, && c_2-a-b_2, \\ &\text{at } x=1: && 0, && 1-c_2, && c_1-a-b_1, \\ &\text{at } x=\infty: && a, && a+1, && b_1+b_2. \end{aligned} \tag{19}$$

The differential equation for the ${}_3\mathrm{F}_2$ function (times the power factor) has the same data, making the requisite check of the differential equation on the second function worthwhile. □

The cases when ${}_3\mathrm{F}_2$ functions in Theorem 2.1 become ${}_2\mathrm{F}_1$ functions can be found by equating a pair of upper and lower parameters. Note that the case $c_2 = a-b_1+b_2+1$ for the $F_2(x,1)$ function is already evident from (14), while the case $c_1+c_2 = a+b_1+b_2+1$ for the $F_2(x,1-x)$ function is visible in (16). Euler's equation (6) is easily recognizable in those two formulas.

The next two theorems tell that second order Fuchsian equations for univariate F_2 specializations outside the singularity lines, implied fully by (11)–(12), follow from (11)–(12) without differentiation.

Theorem 2.2. *The following univariate specializations of Appell's $F_2(x,y)$ function satisfy ordinary Fuchsian equations of second order (with respect to x or t):*

$$F_2\left(\begin{matrix} a;\, b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, x, 2-x\right), \qquad F_2\left(\begin{matrix} b_1+b_2-\frac{1}{2};\, b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, 1-t^2, 1-\frac{(t+s)^2}{s^2-1}\right). \tag{20}$$

In the second function, s is any constant.

Proof. For the $F_2(x, 2-x)$ function, divide (11) by x and (12) by $-y$, add the two equations, and substitute $y = 2-x$ in the coefficients:

$$(1-x)\left(\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}\right) + \left(\frac{c_1}{x} - a - b_1 - 1 - \frac{b_2 x}{x-2}\right)\frac{\partial F}{\partial x}$$
$$+ \left(\frac{c_2}{x-2} + a + b_2 + 1 - b_1 \frac{2-x}{x}\right)\frac{\partial F}{\partial y} - a\left(\frac{b_1}{x} + \frac{b_2}{x-2}\right)F = 0. \tag{21}$$

If $c_1 = 2b_1$ and $c_2 = 2b_2$, following (19) we recognize the second order equation

$$(1-x)\frac{d^2 F}{dx^2} + \left(\frac{2b_1}{x} - \frac{2b_2}{x-2} - a - b_1 - b_2 - 1\right)\frac{dF}{dx} - a\left(\frac{b_1}{x} + \frac{b_2}{x-2}\right)F = 0. \tag{22}$$

For the second F_2 function, the following differential equation can be checked by substituting (9)–(10) with evaluated $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, then linearly eliminating the derivatives $\partial^2 F/\partial x^2$, $\partial^2 F/\partial y^2$ using equations (11)–(12):

$$\frac{d^2 F}{dt^2} + \left(\frac{4b_1 t}{t^2-1} + \frac{4b_2(t+s)}{t^2+2st+1}\right)\frac{dF}{dt} + \left(b_1 + b_2 - \frac{1}{2}\right)\left(\frac{4b_1}{t^2-1} + \frac{4b_2}{t^2+2st+1}\right)F = 0. \qquad \square \tag{23}$$

Theorem 2.3. *Suppose that a univariate specialization of Appell's $F_2(x, y)$ function satisfies a second order ordinary Fuchsian equation fully implied by the system* (11)–(12)*. Then either the specialization is into singular locus* (13)*, or they are represented by one of the two functions in* (20)*.*

Proof. We can assume that $x(x-1)y(y-1)(x+y-1) \neq 0$. If $F(t) = F_2(x(t), y(t))$ satisfies a second order Fuchsian equation, second order partial derivatives come from the d^2F/dt^2 term following (10). The rank of the differential system (11)–(12) is 4, and there are 2 linearly independent relations between partial derivatives of order at most 2. Therefore Eqs. (11)–(12) linearly generate all partial differential equations of order 2 that algebraically follow from them. (In other words, the corresponding two differential operators in the Weyl algebra $\mathbf{C}[x, y]\langle \partial/\partial x, \partial/\partial y\rangle$ linearly generate order 2 operators of the corresponding left ideal.)

Linear elimination of $\partial^2 F/\partial x^2$, $\partial^2 F/\partial y^2$ from (10) using Eqs. (11)–(12) gives the following coefficient to $\partial^2 F/\partial x \partial y$:

$$\frac{y}{1-x}\dot{x}^2 + 2\dot{x}\dot{y} + \frac{x}{1-y}\dot{y}^2. \tag{24}$$

This expression must be equated to 0. The expression is a quadratic form in the derivatives $\dot{x}$, $\dot{y}$, with the discriminant equal to $4(1-x-y)/(1-x)(1-y)$. We can factorize (24) into linear differential forms if we parametrize the surface $(1-x-y)u^2 = (1-x)(1-y)$. Accordingly, we substitute

$$y = \frac{(1-u^2)(x-1)}{x+u^2-1}. \tag{25}$$

The parameter u is undefined only if $x + y = 1$.

Expression (24) then factorizes as follows:

$$\frac{(2x(x-1)\,du - (u-1)(x-u-1)\,dx)(2x(x-1)\,du - (u+1)(x+u-1)\,dx)}{(x+u^2-1)^3}, \tag{26}$$

where we write du, dx instead of du/dt, dx/dt, because the variable t is irrelevant. (It can be composed with any univariate function without changing our differential relations.) The difference in the two numerator factors is in the sign of u. The differential equation

$$2x(x-1)\,du = (u-1)(x-u-1)\,dx \tag{27}$$

is generally solved by the following algebraic relation between u and x:

$$(u+x-1)^2 + C(x-1)(u-1)^2 = 0, \tag{28}$$

where C is an integration constant. After eliminating u from (25) and (28) we get the algebraic relation between x and y that solves (24):

$$(x+y-2)^2 - 2C\big(y^2 + xy - 2x - 2y + 2\big) + C^2 y^2 = 0. \tag{29}$$

The algebraic curves (28) and (29) can be parametrized as follows:

$$x = 1 - Ct^2, \qquad y = 1 - \frac{C(t+1)^2}{C-1}, \qquad u = \frac{Ct(t+1)}{Ct+1}. \tag{30}$$

The parameter $t = (x+y-2)/2C - y/2$ is undefined if $C = 0$ and $x + y = 2$, which is a special case of (29). We keep in mind the special case $x + y = 2$.

Now we have to check when the elimination of $\partial^2F/\partial x^2$, $\partial^2F/\partial y^2$ from (10) using Eqs. (11)–(12) and parametrization (30) leaves the quotient of the coefficients to $\partial F/\partial x$, $\partial F/\partial y$ equal to $\dot{y}/\dot{x}$, so that we could get rid of the $\partial F/\partial x$, $\partial F/\partial y$ terms by adding a rational multiple of dF/dt as in (9). If $x+y=2$, we are led to compare coefficients to first order partial derivatives in (22). There remains the case (30) with $C\neq 0$. We replace $C\mapsto s^2$, $t\mapsto t/s$ and get the parametrization of x and y as in (20). The derivative d^2F/dx^2 in (10) is reduced by (11)–(12) to the following expression:

$$\left(\frac{2c_1t}{t^2-1}+\frac{4b_2(t+s)}{t^2+2st+1}+\frac{2a+2b_1-2b_2-2c_1+1}{t}\right)2t\frac{\partial F}{\partial x}$$
$$+\left(\frac{4b_1t}{t^2-1}+\frac{2c_2(t+s)}{t^2+2st+1}+\frac{2a-2b_1+2b_2-2c_2+1}{t+s}\right)\frac{2(t+s)}{s^2-1}\frac{\partial F}{\partial y}-\left(\frac{4ab_1}{t^2-1}+\frac{4ab_2}{t^2+2st+1}\right)F.$$

To have the right quotient of coefficients to the first order partial derivatives, we must have $c_1=2b_1$, $c_2=2b_2$ and $2a+1=2b_1+2b_2$. □

Now we indicate univariate hypergeometric solutions of the ordinary differential equations implied by Theorem 2.2.

Theorem 2.4. *The following function pairs satisfy the same second order ordinary differential equations* (*with respect to x or t*):

- $F_2\left(\begin{matrix}a;\, b_1, b_2\\ 2b_1, 2b_2\end{matrix}\,\middle|\, x, 2-x\right)$ *and* $(x-2)^{-a}\,{}_2F_1\left(\begin{matrix}\frac{a}{2}, \frac{a+1}{2}-b_2\\ b_1+\frac12\end{matrix}\,\middle|\,\frac{x^2}{(2-x)^2}\right)$;
- $F_2\left(\begin{matrix}b_1+b_2-\frac12;\, b_1, b_2\\ 2b_1, 2b_2\end{matrix}\,\middle|\, -\frac{4t}{(t-1)^2}, \frac{(1-s)(st^2-1)}{s(t-1)^2}\right)$ *and* $(1-t)^{2b_1+2b_2-1}\,{}_2F_1\left(\begin{matrix}b_1+b_2-\frac12, b_2\\ b_1+\frac12\end{matrix}\,\middle|\, st^2\right)$.

In the last pair, s is any constant.

Proof. The differential equation (22) for the $F_2(x, 2-x)$ function has the following singularities and local exponents:

at $x=0$: $0,\ 1-2b_1$,
at $x=2$: $0,\ 1-2b_2$,
at $x=1$: $0,\ b_1+b_2-a$,
at $x=\infty$: $a,\ b_1+b_2$.

Local exponent differences at $x=1$ and $x=\infty$ are both equal to b_1+b_2-a. There is a chance that (22) is a pull-back of Euler's equation (6) with respect to the covering $z\mapsto x^2/(2-x)^2$, with its local exponent differences possibly the following:

$$1-C=\frac12-b_1,\qquad A-B=\frac12-b_2,\qquad C-A-B=b_1+b_2-a.$$

The shift of the local exponents at $x=\infty$ and $x=2$ by $-a$ would be needed. Therefore we consider Euler's equation (6) with $C=1/2+b_1$, $A=a/2+1/2-b_2$, $B=a/2$, compute its pullback with respect to the covering $z\mapsto x^2/(2-x)^2$ and projective normalization $y(x)\mapsto (x-2)^{-a}y(x)$, and check that the obtained differential equation indeed coincides with (22).

To get the result for the second F_2 function, we first consider differential equation (23) for the function (20) of Theorem 2.2. The differential equation has the following singularities and local exponents:

at $t=1$ and $t=-1$: $0,\ 1-2b_1$,
at the roots of $t^2+2st+1$: $0,\ 1-2b_2$,
at $t=\infty$: $2b_1+2b_2,\ 2b_1+2b_2-1$.

If the point $t=\infty$ is an apparent singularity, Eq. (23) might be a pull-back of Euler's equation (6) with respect to the covering $z\mapsto K(t+1)^2/(t-1)^2$, with the local exponent differences

$$1-C=\frac12-b_1,\qquad A-B=\frac12-b_1,\qquad C-A-B=1-2b_2,$$

and the constant $K=(s+1)/(s-1)$ adjusted so that the other 2 singular points lie above $z=1$. The shift of the local exponents at $t=\infty$ and $t=1$ by $1-2b_1-2b_2$ would be needed.

Indeed, this is a pullback. We conclude that the function in (20) and

$$(1-t)^{1-2b_1-2b_2}\,{}_2F_1\left(\begin{matrix}b_1+b_2-\frac12, b_2\\ b_1+\frac12\end{matrix}\,\middle|\,\frac{(s+1)(t+1)^2}{(s-1)(t-1)^2}\right)\tag{31}$$

satisfy the same second order differential equation. After substituting $t\mapsto (t+1)/(t-1)$ and $s\mapsto (s+1)/(s-1)$ into the two functions, we obtain the last pair of functions of this theorem. □

Recall [6, Section 2.9] that Euler's equation (6) has 24 hypergeometric ${}_2\mathrm{F}_1$ series solutions (representing 6 different functions) in general, as discovered by Kummer. Generalized hypergeometric equation (100) has 6 hypergeometric ${}_3\mathrm{F}_2$ solutions in general. Therefore one can take alternative univariate hypergeometric solutions in Theorems 2.1 and 2.4; the presented ones look most convenient representatives. In Appendix A, we recall similar sets of Appell's solutions, say (102)–(106), to the classical systems like (11)–(12).

Now we discuss explicit relations between the function pairs in Theorems 2.1 and 2.4. In each case of Theorem 2.1, the two functions differ by a constant multiple in a neighborhood of $x=0$ in general, because they satisfy the same Fuchsian differential equation, start with the same local power exponent at $x=0$, there are no other integer local exponents at $x=0$ in general, and both functions are defined by proper power series around $x=0$. In the first case, the two functions are equal to each other, trivially. For the second function pair, we have the following identity if $\mathrm{Re}(c_2-a-b_2)>0$:

$$F_2\left(\begin{matrix} a;\, b_1, b_2 \\ c_1, c_2 \end{matrix} \,\middle|\, x, 1\right) = \frac{\Gamma(c_2)\Gamma(c_2-a-b_2)}{\Gamma(c_2-a)\Gamma(c_2-b_2)}\, {}_3\mathrm{F}_2\left(\begin{matrix} a, b_1, a-c_2+1 \\ c_1, a+b_2-c_2+1 \end{matrix} \,\middle|\, x\right), \tag{32}$$

Here we evaluated the left-hand side at $x=0$ following definition (2) and using Gauss' formula [1, Theorem 2.2.2] for the ${}_2\mathrm{F}_1(1)$ coefficients to the powers of x. The ${}_2\mathrm{F}_1(1)$ series converge precisely under the condition $\mathrm{Re}(c_2-a-b_2)>0$. Similarly, the F_2 and ${}_3\mathrm{F}_2$ functions of the third case of Theorem 2.1 differ by the same Γ-multiple if $\mathrm{Re}(c_2-a-b_2)>0$.

In the first case of Theorem 2.4, the $F_2(x, 2-x)$ series does not converge. Linear relations between $F_2(x, 2-x)$ and ${}_2\mathrm{F}_1$ solutions of the same second order Fuchsian equation are cumbersome in general, since analytic continuation has to be identified scrupulously.

Nevertheless, if the parameters b_1 and b_2 in the $F_2(x, 2-x)$ series are negative integers, then the series becomes a finite sum of hypergeometric terms, while ${}_2\mathrm{F}_1(x)$ functions have a dihedral monodromy group. The dihedral ${}_2\mathrm{F}_1(x)$ functions can be expressed elementarily in terms of nested radical functions, and the terminating F_2 series occur in the most general form of these elementary expressions. We discuss this in Section 7.

In the second case of Theorem 2.4, let us substitute $s\mapsto s^2$, $t\mapsto t/s$. We conclude that

$$F_2\left(\begin{matrix} b_1+b_2-\frac{1}{2};\, b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, -\frac{4st}{(t-s)^2}, -\frac{(s^2-1)(t^2-1)}{(t-s)^2}\right) \tag{33}$$

and

$$(s-t)^{2b_1+2b_2-1}\, {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{matrix} \,\middle|\, s^2\right) {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{matrix} \,\middle|\, t^2\right) \tag{34}$$

satisfy the same second order ordinary differential equation, with respect to t. In this setting, the ${}_2\mathrm{F}_1(s^2)$ expression is just a constant factor. But the symmetry between s and t suggests that equation system (11)–(12) for $F_2\left(\begin{smallmatrix} b_1+b_2-1/2;\, b_1, b_2 \\ 2b_1, 2b_2 \end{smallmatrix} \,\middle|\, x, y\right)$ can be transformed following (33) to a differential system where the new variables s and t are separated. Moreover, the two separated equations with respect to s or t would be simple transformations of Euler's equation (6). In Section 4 we relate this situation by a known transformation to the well-known case (8) of separation of variables for the $F_4(x, y)$ function.

Here is an identity illustrating the last case of Theorem 2.4, obtainable after substitution $s\mapsto (s+1)/(s-1)$ into (33)–(34) and consideration of F_2 and ${}_2\mathrm{F}_1$ solutions of the same differential system around the point $(t, s)=(0, 0)$:

$$\begin{aligned} &F_2\left(\begin{matrix} b_1+b_2-\frac{1}{2};\, b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, \frac{4(1-s^2)t}{(1+s+t-st)^2}, \frac{4(1-t^2)s}{(1+s+t-st)^2}\right) \\ &\quad = \left(\frac{1+s+t-st}{1-s}\right)^{2b_1+2b_2-1} {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ 2b_2 \end{matrix} \,\middle|\, \frac{4s}{s^2-1}\right) {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{matrix} \,\middle|\, t^2\right). \end{aligned} \tag{35}$$

One can check this identity by comparing low degree terms of bivariate Taylor series around $(t, s)=(0, 0)$.

3. Identities with Appell's F_3 function

A system partial differential equations for Appell's F_3 function is:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} + y\frac{\partial^2 F}{\partial x\partial y} + \big(c-(a_1+b_1+1)x\big)\frac{\partial F}{\partial x} - a_1 b_1 F = 0, \tag{36}$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} + x\frac{\partial^2 F}{\partial x\partial y} + \big(c_2-(a_2+b_2+1)y\big)\frac{\partial F}{\partial y} - a_2 b_2 F = 0. \tag{37}$$

This is an equation system of rank 4. The singular locus of this equation on $\mathbf{P}^1\times\mathbf{P}^1$ is the union of the following curves:

$$x=0,\quad x=1,\quad x=\infty,\quad y=0,\quad y=1,\quad y=\infty,\quad xy=x+y. \tag{38}$$

There is a straightforward relation between $F_2(x, y)$ and $F_3(x, y)$ functions, coming from a transformation between differential systems (11)–(12) and (36)–(37). The relation is that the functions

$$F_2\left(\begin{matrix} a;\ b_1, b_2 \\ c_1, c_2 \end{matrix}\,\middle|\, x, y\right), \qquad x^{-b_1} y^{-b_2} F_3\left(\begin{matrix} 1+b_1-c_1, 1+b_2-c_2;\ b_1, b_2 \\ 1+b_1+b_2-a \end{matrix}\,\middle|\, \frac{1}{x}, \frac{1}{y}\right) \tag{39}$$

both satisfy (11)–(12). Therefore we can translate the results of Theorems 2.1 through 2.4 straightforwardly.

Theorem 3.1. *The following function pairs satisfy the same ordinary differential equations* (*or order* 2 *or* 3*, with respect to* x *or* t):

- $F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, 0\right)$ *and* ${}_2\mathrm{F}_1\left(\begin{matrix} a_1, b_1 \\ c \end{matrix}\,\middle|\, x\right)$;
- $F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, 1\right)$ *and* ${}_3\mathrm{F}_2\left(\begin{matrix} a_1, b_1, c-a_2-b_2 \\ c-a_2, c-b_2 \end{matrix}\,\middle|\, x\right)$;
- $F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, \frac{x}{x-1}\right)$ *and* $x^{1-c}(1-x)^{a_2}\,{}_3\mathrm{F}_2\left(\begin{matrix} 1+a_1+a_2-c, 1+b_1+a_2-c, 1-b_2 \\ 1+a_1+a_2+b_1-c, 1+a_2-b_2 \end{matrix}\,\middle|\, 1-x\right)$;
- $F_3\left(\begin{matrix} 1-b_1, 1-b_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, \frac{x}{2x-1}\right)$ *and*
  $(1-x)^{c-1}(1-2x)^{b_2}\,{}_2\mathrm{F}_1\left(\begin{matrix} \frac{1}{2}(b_2-b_1+c), \frac{1}{2}(b_1+b_2+c-1) \\ c \end{matrix}\,\middle|\, 4x(1-x)\right)$;
- $F_3\left(\begin{matrix} 1-b_1, 1-b_2;\ b_1, b_2 \\ 3/2 \end{matrix}\,\middle|\, -\frac{(t-1)^2}{4t}, \frac{s(t-1)^2}{(1-s)(st^2-1)}\right)$ *and* $t^{b_1}(1-t)^{-1}\,{}_2\mathrm{F}_1\left(\begin{matrix} 1-b_2, b_2 \\ b_1+\frac{1}{2} \end{matrix}\,\middle|\, \frac{st^2}{st^2-1}\right)$.

In the last pair, s is any constant. The function pairs represent all cases when a univariate specialization of Appell's F_3 *function satisfies a second order ordinary Fuchsian equation fully implied by* (36)–(37).

Proof. Apart from the first trivial case, we employ correspondence (39) in each case of Theorems 2.1 and 2.4. We substitute

$$x \mapsto \frac{1}{x}, \quad y \mapsto \frac{1}{y}, \quad c_1 \mapsto 1+b_1-a_1, \quad c_2 \mapsto 1+b_2-a_2, \quad a \mapsto 1+b_1+b_2-c,$$

get rid of the power factors $x^{b_1} y^{b_2}$, and choose sometimes a more convenient companion from the 24 Kummer's or six ${}_3\mathrm{F}_2$ series of the second function. □

In the first and fourth function pairs, a direct identity between the two functions holds in a neighborhood of $x=0$. In the second pair, the following formula around $x=0$ holds if $\mathrm{Re}(c-a_2-b_2)>0$:

$$F_3\left(\begin{matrix} a_1, a_2;\ b_1, b_2 \\ c \end{matrix}\,\middle|\, x, 1\right) = \frac{\Gamma(c)\Gamma(c-a_2-b_2)}{\Gamma(c-a_2)\Gamma(c-b_2)}\,{}_3\mathrm{F}_2\left(\begin{matrix} a_1, b_1, c-a_2-b_2 \\ c-a_2, c-b_2 \end{matrix}\,\middle|\, x\right). \tag{40}$$

In the third and last cases, relations between Appell's F_3 and ${}_2\mathrm{F}_1$ solutions of the implied ordinary differential equations can be derived following their analytic continuation.

The $F_3(x, x/(x-1))$ and $F_3(x, x/(2x-1))$ functions are considered in [9]. Karlsson expresses $F_3(x, x/(x-1))$ functions with $c = a_1+a_2+b_1+b_2$ or $c = a_1+b_2$ in terms of Gauss hypergeometric functions; these relations can be obtained by simplifying the respective ${}_3\mathrm{F}_2$ function in Theorem 3.1. Karlsson's relation between $F_3(x, x/(2x-1))$ and ${}_2\mathrm{F}_1(4x(1-x))$ functions differs from ours by Euler's transformation [1, (2.2.7)].

Translating the case (33)–(34) of variable separation, we conclude that the bivariate functions

$$F_3\left(\begin{matrix} 1-b_1, 1-b_2;\ b_1, b_2 \\ 3/2 \end{matrix}\,\middle|\, -\frac{(t-s)^2}{4st}, -\frac{(t-s)^2}{(s^2-1)(t^2-1)}\right) \tag{41}$$

and

$$\frac{s^{b_1} t^{b_1} (1-s^2)^{b_2} (1-t^2)^{b_2}}{s-t}\,{}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{matrix}\,\middle|\, s^2\right){}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{matrix}\,\middle|\, t^2\right) \tag{42}$$

satisfy the same system of partial differential equations with respect to s and t.

4. Identities with Appell's F_4 function

A system partial differential equations for Appell's F_4 function is:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + \big(c_1-(a+b+1)x\big)\frac{\partial F}{\partial x} - (a+b+1)y\frac{\partial F}{\partial y} - abF = 0, \tag{43}$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + \big(c_2-(a+b+1)y\big)\frac{\partial F}{\partial y} - (a+b+1)x\frac{\partial F}{\partial x} - abF = 0. \tag{44}$$

The singular locus on $\mathbf{P}^2$ is $x=0$, $y=0$, the line at infinity, and

$$x^2+y^2+1=2xy+2x+2y. \tag{45}$$

The quadratic singular locus can be loosely described by the equation $\sqrt{x}+\sqrt{y}=1$. It can be parametrized as follows:

$$x=t^2, \qquad y=(1-t)^2. \tag{46}$$

First we characterize univariate specializations outside the singularity curves. The general case can be identified as Bailey's case (8) of variable separation.

Theorem 4.1. *Suppose that a univariate specialization of Appell's $F_4(x,y)$ function is not onto singular locus $xy=0$ or* (45)*, and satisfies a second order ordinary differential equation fully implied by* (43)–(44)*. Then the specialization is represented by the general expression*

$$F_4\left(\begin{matrix} a;\ b \\ c,\ a+b-c+1 \end{matrix}\,\middle|\, st,\ (1-s)(1-t)\right), \tag{47}$$

where s is a constant. The general second order equation is satisfied by ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}\,\middle|\, t\right)$.

Proof. The rank of the differential system (43)–(44) is 4, and the partial differential equations of order 2 are linearly generated by (43)–(44). Linear elimination of $\partial^2F/\partial x\partial y$, $\partial^2F/\partial y^2$ from expression (10) for d^2F/dt^2 gives the following coefficient to $\partial^2F/\partial x^2$:

$$\dot{x}^2+\frac{1-x-y}{y}\dot{x}\dot{y}+\frac{x}{y}\dot{y}^2. \tag{48}$$

This expression must be equated to 0. The expression is a quadratic form in the derivatives, with the discriminant equal to $4(x^2-2xy+y^2-2x-2y+1)/y^2$. We can factorize (48) into linear differential forms if we parametrize the surface $w^2=x^2-2xy+y^2-2x-2y+1$. A parametrization is easy if we set $w=y+u$; then

$$y=\frac{(x+u-1)(x-u-1)}{2(x+u+1)}. \tag{49}$$

Expression (48) factorizes as follows:

$$\frac{(x^2+2xu+u^2-2x+2u+1)^2}{2(x+u+1)^3(x+u-1)(x-u-1)}(du+dx)\big(x\,du-(u+1)\,dx\big), \tag{50}$$

where we write du, dx instead of du/dt, dx/dt because the variable t is irrelevant. The non-differential numerator factor defines the singular locus, since its parametrization $x=t^2$, $u=-(t-1)^2$ is translated to (46) by (49). The differential factors give the following two families of solutions:

$$u=C-x, \qquad u=Cx-1. \tag{51}$$

They translate into the following relations between x and y:

$$y=\frac{C-1}{C+1}x+\frac{1-C}{2}, \qquad y=-\frac{1-C}{2}x+\frac{C-1}{C+1}. \tag{52}$$

The two solution families actually coincide, since they are related by the parameter transformation $C\mapsto(3-C)/(1+C)$. After the substitutions $C\mapsto 2s-1$, $x\mapsto st$, the first family is described by $y=(1-s)(1-t)$, giving us (47).

Adding Eq. (43) multiplied by s with Eq. (44) multiplied by $1-s$, and the substitutions $x=st$, $y=(s-1)(t-1)$ gives us

$$t(1-t)\left(s^2\frac{\partial^2F}{\partial x^2}+2s(s-1)\frac{\partial^2F}{\partial x\partial y}+(s-1)^2\frac{\partial^2F}{\partial y^2}\right)+\big(c_1-(a+b+1)t\big)s\frac{\partial F}{\partial x}$$
$$+\big(a+b+1-c_2-(a+b+1)t\big)(s-1)\frac{\partial F}{\partial y}-abF=0.$$

If $c_2=a+b+1-c_1$, we recognize Euler's equation (6) in t. $\square$

For fixed s, the arguments x and y in (47) are linearly related. The linear relations are precisely the tangent lines to the singularity curve (45). If we let both s and t vary, their symmetry indicates Bailey's case [4] of variable separation for differential system (43)–(44) when $c_1+c_2=a+b+1$. The conclusion is that

$$F_4\left(\begin{matrix} a;\ b \\ c,\ a+b-c+1 \end{matrix}\,\middle|\, st,\ (1-s)(1-t)\right) \quad\text{and}\quad {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, s\right){}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, t\right) \tag{53}$$

satisfy the same system of partial differential equations (with respect to s and t). This is illustrated by identity (8).

The $F_2(x,y)$ case of variable separation in (33)–(34) is related to Bailey's case via the following identity, with $c=2b$:

$$F_4\left(\begin{matrix} a;\ b \\ c, a-b+1 \end{matrix}\,\middle|\, x, y^2\right) = (1+y)^{-2a} F_2\left(\begin{matrix} a;\ b, a-b+\frac{1}{2} \\ c, 2a-2b+1 \end{matrix}\,\middle|\, \frac{x}{(1+y)^2}, \frac{4y}{(1+y)^2}\right). \tag{54}$$

This identity is derived by Srivastava in [15]; it is presented in [16, 9.4.(215)]. An equivalent identity is obtained in [3, p. 27], and presented in [16, 9.4.(97)]. The Appell functions in (33) and (47) are generic F_2 and F_4 functions satisfying the quadratic condition of Sasaki–Yoshida [14, Sections 4.3, 5.5]: four linearly independent solutions of their partial differential equations systems are quadratically related.

In [14, Section 5.4], the following identity between F_4 and F_2 functions is implied:

$$F_4\left(\begin{matrix} \frac{a}{2};\ \frac{a+1}{2} \\ b_1+\frac{1}{2}, b_2+\frac{1}{2} \end{matrix}\,\middle|\, x^2, y^2\right) = (1+x+y)^{-a} F_2\left(\begin{matrix} a;\ b_1, b_2 \\ 2b_1, 2b_2 \end{matrix}\,\middle|\, \frac{2x}{x+y+1}, \frac{2y}{x+y+1}\right). \tag{55}$$

Equivalent identities are presented in [5], [16, 9.4.(175)] and [11, Lemma 5.2]. This identity has no application for $F_2(x, 2-x)$ functions.

There remains to consider specialization of the F_4 function to the quadratic singularity locus (45). Due to parametrization (46), we are looking at $F_4(t^2, (t-1)^2)$ functions. Can such a function satisfy a second order Fuchsian equation?

Theorem 4.2. *The univariate functions*

$$F_4\left(\begin{matrix} a;\ b \\ c, a+b-c+\frac{3}{2} \end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \textit{and} \quad {}_2F_1\left(\begin{matrix} 2a, 2b \\ 2c-1 \end{matrix}\,\middle|\, t\right) \tag{56}$$

satisfy the same Fuchsian equation. This is the only case when the differential system (43)–(44) *fully implies a second order Fuchsian equation.*

Proof. Let J be the differential ideal generated by (43)–(44). After substitution (46) into the coefficients, the rank of partial derivatives becomes 3, and there is another independent linear relation between the partial derivatives of order at most 2. Yet, second order partial derivatives are eliminated from the full differential

$$\frac{d^2F}{dt^2} = 4t^2\frac{\partial^2 F}{\partial x^2} + 8t(t-1)\frac{\partial^2 F}{\partial x\partial y} + 4(t-1)^2\frac{\partial^2 F}{\partial y} + 2\frac{\partial F}{\partial x} + 2\frac{\partial F}{\partial y} \tag{57}$$

already by Eqs. (43)–(44). After we substitute (46) into the coefficients in (43)–(44), add the first equation multiplied by $4/(1-t)$ and the second equation multiplied by $4/t$, we get

$$\begin{aligned} &4t^2\frac{\partial^2 F}{\partial x^2} + 8t(t-1)\frac{\partial^2 F}{\partial x\partial y} + 4(t-1)^2\frac{\partial^2 F}{\partial y^2} + 4\frac{(a+b+1)t-c_1}{t-1}\frac{\partial F}{\partial x} \\ &\quad + 4\frac{(a+b+1)t+c_2-a-b-1}{t}\frac{\partial F}{\partial y} + \frac{4ab}{t(t-1)}F = 0. \end{aligned} \tag{58}$$

Identifying (57) and

$$\frac{dF}{dt} = 2t\frac{\partial F}{\partial x} + 2(t-1)\frac{\partial F}{\partial y} \tag{59}$$

we recognize in (58) the expression

$$\frac{d^2F}{dt^2} + \frac{(2a+2b+1)t+1-2c_1}{t(t-1)}\frac{dF}{dt} + \frac{4c_1+4c_2-4a-4b-6}{t}\frac{\partial F}{\partial y} + \frac{4ab}{t(t-1)}F = 0. \tag{60}$$

We have differentiation here with respect to t only if $c_1+c_2 = a+b+\frac{3}{2}$. Euler's equation is recognizable. □

The general ordinary differential equation for $F_4\left(\begin{smallmatrix} a;\ b \\ c_1, c_2 \end{smallmatrix}\,\middle|\, t^2, (t-1)^2\right)$ is computed by Kato [10, (4.1)]. We reproduce the differential equation in Theorem 6.1 here below as demonstration of general computational techniques; see formula (90) in Theorem 6.1. The most important knowledge at this point is that the differential equation is Fuchsian with singularities at $t=0$, $t=1$, $t=\infty$, and the following local exponents:

$$\begin{array}{llll} \text{at } t=0: & 0, & 2-2c_1, & c_2-a-b, \\ \text{at } t=1: & 0, & 2-2c_2, & c_1-a-b, \\ \text{at } t=\infty: & 2a, & 2b, & c_1+c_2-1. \end{array} \tag{61}$$

If some difference of local exponents at the same singular point is equal to 1, then there is a chance that the accessory parameter is right and the Fuchsian equation is projectively equivalent to Appendix A Eq. (100) for the ${}_3F_2$ function. We present these cases in the following theorem.

Incidentally, we recall some Kato's results on reducibility of the monodromy groups of differential system (43)–(44) or the ordinary equation for $F_4(t^2,(1-t)^2)$. The monodromy group of differential system (43)–(44) with $c_1, c_2 \notin \mathbf{Z}$ is reducible if and only if one of the numbers

$$a,\ b,\ c_1-a,\ c_1-b,\ c_2-a,\ c_2-b,\ c_1+c_2-a,\ c_1+c_2-b \tag{62}$$

is an integer [10, Section 8]. If no difference of the local exponents in (61) at a singular point is an integer, then Kato's ordinary equation is reducible if and only if one of the numbers in (62) or $c_1+c_2-a-b-\frac{1}{2}$ is an integer [10, Section 14].

Theorem 4.3. *The univariate functions*

$$F_4\left(\begin{matrix} a;\ b \\ c+\frac{1}{2}, \frac{1}{2}\end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \text{and} \quad {}_3F_2\left(\begin{matrix} 2a, 2b, c \\ a+b+\frac{1}{2}, 2c\end{matrix}\,\middle|\, t\right) \tag{63}$$

satisfy the same ordinary Fuchsian equation. That equation is satisfied by

$$(1-t)F_4\left(\begin{matrix} a+\frac{1}{2};\ b+\frac{1}{2} \\ c+\frac{1}{2}, \frac{3}{2}\end{matrix}\,\middle|\, t^2, (1-t)^2\right), \tag{64}$$

$$(1-t)^{-2a}F_4\left(\begin{matrix} a;\ a+\frac{1}{2} \\ c+\frac{1}{2}, 1+a-b\end{matrix}\,\middle|\, \frac{t^2}{(t-1)^2}, \frac{1}{(t-1)^2}\right) \tag{65}$$

as well. This represents all cases with $c_1, c_2 \notin \mathbf{Z}$ when the monodromy of (43)–(44) *is irreducible and Kato's differential equation* (90) *is projectively equivalent to hypergeometric equation* (100).

Proof. Here are all cases when a difference of local exponents is equal to 1:

- $c_2=\frac{1}{2}$. Once we identify the local exponent sets (61) and (101) under this condition, Kato's equation coincides with (100) up to variable notation and a polynomial factor. We get the relation between the functions in (63) after the substitution $c_1 \mapsto c_1+\frac{1}{2}$ (for a better form). Functions (64)–(65) are solutions of the same ordinary equation by Appendix A transformations (107)–(109).
- $c_2=\frac{3}{2}$. To identify the local exponents in (61) and (101), we apply the projective transformation $y(t)\mapsto(1-t)^{-1}y(t)$. This shifts the local exponents at $t=1$ by 1, and the local exponents at $t=\infty$ by -1. We get the coincidence of Kato's and hypergeometric equations unconditionally. A direct conclusion is that the functions

$$F_4\left(\begin{matrix} a;\ b \\ c, \frac{3}{2}\end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \text{and} \quad \frac{1}{1-t}\,{}_3F_2\left(\begin{matrix} 2a-1, 2b-1, c-\frac{1}{2} \\ a+b-\frac{1}{2}, 2c-1\end{matrix}\,\middle|\, t\right)$$

  satisfy the same ordinary Fuchsian equation. After multiplication of both function by $1-t$ and a simple transformation of parameters we get the ${}_3F_2$ function as in (63), and the F_4 function as in (64).
- $c_1=a+b+1$. The local exponents in (61) and (101) can be identified directly. The accessory parameter is right if $c_2=\frac{1}{2}$ or $a=0$ or $b=0$. In the later two cases, the monodromy group of the system (43)–(44) is reducible.
- $c_1=a+b-1$. The local exponents in (61) and (101) can be identified after the projective transformation $y(t)\mapsto(1-t)^{-1}y(t)$. The accessory parameter is right if $c_2=\frac{3}{2}$ or $a=1$ or $b=1$. The later two cases are reducible.
- $c_1+2c_2=a+b+1$. The local exponents in (61) and (101) can be identified after the projective transformation $y(t)\mapsto(1-t)^{1-2c_2}y(t)$. The accessory parameter is right if $c_2=\frac{1}{2}$ or $c_2=a$ or $c_2=b$. The later two cases are reducible.
- $c_1+2c_2=a+b+3$. The local exponents in (61) and (101) can be identified after the projective transformation $y(t)\mapsto(1-t)^{2-2c_2}y(t)$. The accessory parameter is right if $c_2=\frac{3}{2}$ or $c_2=a+1$ or $c_2=b+1$. The later two cases are reducible.
- $b=a+\frac{1}{2}$. To identify the local exponents in (61) and (101), we permute the singular points $t=1$ and $t=\infty$, and shift the local exponents there by $\pm 2a$. Then Kato's and hypergeometric equations coincide unconditionally. A direct conclusion is that the functions

$$F_4\left(\begin{matrix} a, a+\frac{1}{2} \\ c_1, c_2\end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \text{and} \quad (1-t)^{2a}\,{}_3F_2\left(\begin{matrix} 2a, 2a-2c_2+1, c_1-\frac{1}{2} \\ 2a-c_2+1, 2c_1-1\end{matrix}\,\middle|\, \frac{t}{t-1}\right)$$

  satisfy the same ordinary Fuchsian equation. After straightforward transformations we get the ${}_3F_2$ function as in (63), and the F_4 function as in (65).
- $c_1+c_2=2a$. To identify the local exponents in (61) and (101), we permute the singular points $t=1$ and $t=\infty$, and shift the local exponents there by $\pm(2a-1)$. The accessory parameter is right if $a=b+\frac{1}{2}$ or $a=1$ or $c_1=a$. The later two cases are reducible.

- $c_1+c_2=2a+2$. To identify the local exponents in (61) and (101), we permute the singular points $t=1$ and $t=\infty$, and shift the local exponents there by $\pm 2a$. The accessory parameter is right if $b=a+\frac{1}{2}$ or $a=0$ or $c_1=a+1$. The later two cases are reducible.
- The cases $c_1=\frac{1}{2}$, $c_1=\frac{3}{2}$, $c_2=a+b+1$, $c_2=a+b-1$, $2c_1+c_2=a+b+1$, $2c_1+c_2=a+b+3$, $a=b+\frac{1}{2}$, $c_1+c_2=2b$, $c_1+c_2=2b+2$ are symmetric to the above.

Non-reducible cases fall into the cases when c_1 or c_2 is equal to $\frac{1}{2}$ or $\frac{3}{2}$, or $a-b=\pm\frac{1}{2}$. □

Similarly, we may find the cases when Kato's differential equation is a symmetric tensor square of Euler's hypergeometric equation (6). Recall that if y_1, y_2 is a basis of second order linear ordinary differential equation, then y_1^2, y_2^2, y_1y_2 form a basis of the symmetric square differential equation. One case when Kato's differential equation is a symmetric square of (6) follows from Bailey's case of variable separation. If we set $s=t$ in (53), we conclude that the functions

$$F_4\left(\begin{matrix} a;\ b \\ c, a+b-c+1 \end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \text{and} \quad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, t\right)^2 \tag{66}$$

satisfy the same ordinary Fuchsian equation. Appell himself derived this fact explicitly [2]. Here is characterization of other such cases.

Theorem 4.4. *The univariate functions*

$$F_4\left(\begin{matrix} 2c-\frac{1}{2};\ 3c-1 \\ c+\frac{1}{2}, c+\frac{1}{2} \end{matrix}\,\middle|\, t^2, (1-t)^2\right) \quad \textit{and} \quad {}_2F_1\left(\begin{matrix} c, 3c-1 \\ 2c \end{matrix}\,\middle|\, t\right)^2 \tag{67}$$

satisfy the same ordinary Fuchsian equation of order 3. *That equation is satisfied by*

$$t^{1-2c} F_4\left(\begin{matrix} c;\ 2c-\frac{1}{2} \\ \frac{3}{2}-c, c+\frac{1}{2} \end{matrix}\,\middle|\, t^2, (1-t)^2\right), \tag{68}$$

$$t^{1-2c}(1-t)^{1-2c} F_4\left(\begin{matrix} \frac{1}{2};\ c \\ \frac{3}{2}-c, \frac{3}{2}-c \end{matrix}\,\middle|\, t^2, (1-t)^2\right) \tag{69}$$

as well. Together with (66) *this represents all cases when Kato's differential equation* (90) *is projectively equivalent to a symmetric tensor square of Euler's hypergeometric equation* (6).

Proof. A necessary condition on local exponents (61) is that at each singular point one local exponent is the arithmetic mean of the other two. If we pick for the arithmetic mean at $t=0$ the exponent c_2-a-b, or at $t=1$ the exponent c_1-a-b, or at $t=\infty$ the exponent c_1+c_2-1, we arrive at Appell's case (66). Up to symmetries of parameters, there are three different ways to pick up other local exponents as arithmetic means:

- The exponents $2-2c_1$, $2-2c_2$, $2a$ are the arithmetic means at $t=0$, $t=1$, $t=\infty$, respectively. We have 3 linear equations for the 4 parameters, and the general solution is presented in (67). The local exponents of the ${}_2F_1$ function must be equal to the non-arithmetic mean exponents divided by 2. Explicit computation confirms coincidence of ordinary differential equations; in other words, the accessory parameter turns out to be right. Functions (68)–(69) are solutions of the same ordinary equation by Appendix A transformations (107)–(109).
- The exponents $2-2c_1$, 0, $2a$ are the arithmetic means. A general parametrization of the parameters is the same as in (68). Evidently, we get a related case.
- The exponents 0, 0, $2a$ are the arithmetic means. A general parametrization of the parameters is the same as in (69). □

The function pairs in (63) and (67) differ by these constant multiples, respectively:

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-a-b)}{\Gamma(\frac{1}{2}-a)\Gamma(\frac{1}{2}-b)} \quad \text{if } \mathrm{Re}(a+b)<\frac{1}{2}, \qquad \frac{\Gamma(c+\frac{1}{2})\Gamma(2-4c)}{\Gamma(1-c)\Gamma(\frac{3}{2}-2c)} \quad \text{if } \mathrm{Re}(c)<\frac{1}{2}.$$

Note that the three local exponent differences of the ${}_2F_1$ function in (67) are equal.

5. Identities with Appell's F_1 function

A system partial differential equations for Appell's F_1 function is:

$$x(1-x)\frac{\partial^2 F}{\partial x^2} + y(1-x)\frac{\partial^2 F}{\partial x \partial y} + \big(c-(a+b_1+1)x\big)\frac{\partial F}{\partial x} - b_1 y\frac{\partial F}{\partial y} - ab_1 F = 0, \tag{70}$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} + x(1-y)\frac{\partial^2 F}{\partial x \partial y} + \big(c-(a+b_2+1)y\big)\frac{\partial F}{\partial y} - b_2 x\frac{\partial F}{\partial x} - ab_2 F = 0. \tag{71}$$

This is an equation system of rank 3 generally. The equation

$$\left((1-y)\frac{\partial F}{\partial y}-b_2\right)[\text{Eq. (70)}]+\left((x-1)\frac{\partial F}{\partial x}+b_1\right)[\text{Eq. (71)}] \tag{72}$$

can be computed to be $a-c+1$ times

$$(y-x)\frac{\partial^2 F}{\partial x\partial y}+b_2\frac{\partial F}{\partial x}-b_1\frac{\partial F}{\partial y}=0. \tag{73}$$

This equation is of order 2 as well; it holds even for the F_1 functions with $c=a+1$. In the case $c=a+1$, the equation system (70)–(71) is not holonomic; Eq. (73) has to be added to get a finite dimensional space of solutions.

The singular locus of this equation on $\mathbf{P}^1\times\mathbf{P}^1$ is the union of the following lines:

$$x=0,\quad x=1,\quad x=\infty,\quad y=0,\quad y=1,\quad y=\infty,\quad y=x. \tag{74}$$

Univariate specializations to the singular lines are known since [3]:

$$F_1\left(\begin{matrix}a;\ b_1,b_2\\ c\end{matrix}\,\middle|\,x,0\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b_1\\ c\end{matrix}\,\middle|\,x\right), \tag{75}$$

$$F_1\left(\begin{matrix}a;\ b_1,b_2\\ c\end{matrix}\,\middle|\,x,1\right)=\frac{\Gamma(c)\Gamma(c-a-b_2)}{\Gamma(c-a)\Gamma(c-b_2)}\,{}_2\mathrm{F}_1\left(\begin{matrix}a,b_1\\ c-b_2\end{matrix}\,\middle|\,x\right), \tag{76}$$

$$F_1\left(\begin{matrix}a;\ b_1,b_2\\ c\end{matrix}\,\middle|\,x,x\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b_1+b_2\\ c\end{matrix}\,\middle|\,x\right). \tag{77}$$

The second identity holds if $\mathrm{Re}(c-a-b_2)>0$.

Theorem 5.1. *Suppose that a univariate specialization $F_1(x(t),y(t))$ is not onto any of singularity curves in* (74). *Then Eqs.* (70)–(71) *and* (73) *fully imply a second order differential equation for $F_1(x(t),y(t))$ if and only if the functions $x(t)$ and $y(t)$ are related by the non-linear differential equation*

$$\begin{aligned}&\frac{\ddot{x}}{\dot{x}}-\frac{\ddot{y}}{\dot{y}}-(a+1)\left(\frac{\dot{x}}{x-1}-\frac{\dot{y}}{y-1}\right)+c\left(\frac{\dot{x}}{x(x-1)}-\frac{\dot{y}}{y(y-1)}\right)\\ &\quad+\frac{x(x-1)y(y-1)}{(y-x)\dot{x}\dot{y}}\left(\frac{\dot{x}}{x}-\frac{\dot{y}}{y}\right)\left(\frac{\dot{x}}{x(x-1)}-\frac{\dot{y}}{y(y-1)}\right)\left(\frac{b_1\dot{x}}{x-1}+\frac{b_2\dot{y}}{y-1}\right)=0.\end{aligned} \tag{78}$$

Proof. The second order derivative d^2F/dt^2 in (10) can be reduced by Eqs. (70)–(71) and (73) to first order partial derivatives without additional restrictions on $x(t)$, $y(t)$. From equation

$$\frac{\dot{x}^2}{x(1-x)}[\text{Eq. (70)}]+\frac{\dot{y}^2}{y(1-y)}[\text{Eq. (71)}]-\frac{(\dot{x}y-x\dot{y})^2}{xy(y-x)}[\text{Eq. (73)}]$$

we straightforwardly derive

$$\begin{aligned}\frac{d^2F}{dt^2}=&\left(\ddot{x}-\frac{c-(a+b_1+1)x}{x(1-x)}\dot{x}^2+\frac{b_2\dot{x}^2}{x}+\frac{b_2(\dot{x}-\dot{y})^2}{y-x}+\frac{b_2(x+y-1)\dot{y}^2}{y(1-y)}\right)\frac{\partial F}{\partial x}\\ &+\left(\ddot{y}-\frac{c-(a+b_2+1)y}{y(1-y)}\dot{y}^2+\frac{b_1\dot{y}^2}{y}-\frac{b_1(\dot{x}-\dot{y})^2}{y-x}+\frac{b_1(x+y-1)\dot{x}^2}{x(1-x)}\right)\frac{\partial F}{\partial y}+\frac{ab_1\dot{x}^2}{x(1-x)}+\frac{ab_2\dot{y}^2}{y(1-y)}.\end{aligned}$$

We can recognize the derivative dF/dt on the right-hand side precisely when the quotient of the coefficients to $\partial F/\partial x$ and $\partial F/\partial y$ is equal to $\dot{x}/\dot{y}$, according to (9). This leads to Eq. (78). □

Without loss of generality, we can set $\dot{x}=1$, $\ddot{x}=0$ in Eq. (78). Then the non-linear equation can be written in the form $\ddot{y}=g_3(\dot{y})$, where $g_3(\dot{y})$ is a cubic polynomial in $\dot{y}$ with coefficients rational functions in x, y. This is not a canonical form of a non-linear differential equation without movable essential singularities [8, Chapter XIV]. Nevertheless, its Liouville invariants [7, (14)–(15)] are zero, hence the equation is a differentiable transformation $x=\Phi(X,Y)$, $y=\Psi(X,Y)$ of the simplest equation $Y''=0$. However, it is not straightforward to find the transformation.

Using the identity

$$F_1\left(\begin{matrix}a;\ b_1,b_2\\ c\end{matrix}\,\middle|\,x,y\right)=(1-x)^{-b_1}F_3\left(\begin{matrix}c-a,a;\ b_1,b_2\\ c\end{matrix}\,\middle|\,\frac{x}{x-1},y\right) \tag{79}$$

one can translate the two non-singular cases of Theorem 3.1 to a few univariate specializations of the F_1 function relevant to Theorem 5.1. The $F_2(x,2-x)$ or $F_3(x,x/(2x-1))$ case can be translated as follows.

Theorem 5.2. *The following two univariate functions satisfy the same ordinary Fuchsian differential equation of order* 2:

$$F_1\left(\begin{matrix} a;\ 2b, a-b \\ 1+b \end{matrix}\,\middle|\, x, x^2\right) \quad \text{and} \quad (1-x)^{-2a}\,{}_2F_1\left(\begin{matrix} a, \frac{1}{2} \\ 1+b \end{matrix}\,\middle|\, -\frac{4x}{(x-1)^2}\right). \tag{80}$$

The same equation is satisfied by

$$(1-x)^{-a}F_1\left(\begin{matrix} a;\ 1-a, a-b \\ 1+b \end{matrix}\,\middle|\, \frac{x}{x-1}, -x\right), \qquad \left(1-x^2\right)^{-a}F_1\left(\begin{matrix} a;\ 2b, 1-a \\ 1+b \end{matrix}\,\middle|\, \frac{x}{x+1}, \frac{x^2}{x^2-1}\right), \tag{81}$$

$$F_1\left(\begin{matrix} a;\ 2b, a-b \\ 2a \end{matrix}\,\middle|\, 1-x, 1-x^2\right), \qquad x^{-a}F_1\left(\begin{matrix} a;\ a-b, a-b \\ 2a \end{matrix}\,\middle|\, \frac{x-1}{x}, 1-x\right), \tag{82}$$

$$x^{-a}F_1\left(\begin{matrix} a;\ a-b, a-b \\ 1+a-2b \end{matrix}\,\middle|\, \frac{1}{x}, x\right), \qquad (x-1)^{-a}F_1\left(\begin{matrix} a;\ 1-a, a-b \\ 1+a-2b \end{matrix}\,\middle|\, \frac{1}{1-x}, x+1\right), \tag{83}$$

$$x^{-b}(1-x)^{2b-2a}F_1\left(\begin{matrix} 1-2b;\ a-b, a-b \\ 1+a-2b \end{matrix}\,\middle|\, \frac{1}{1-x}, \frac{x}{x-1}\right), \tag{84}$$

$$x^{-b}(1-x)^{1-2a}F_1\left(\begin{matrix} 1-2b;\ a-b, 1-a \\ 1+a-2b \end{matrix}\,\middle|\, x+1, x\right). \tag{85}$$

Proof. The F_3 function in (79) can be directly related to the penultimate case of Theorem 3.1 if $a = 1 - b_2$ and $c = 2 - b_1 - b_2$. The direct conclusion is that the functions

$$F_1\left(\begin{matrix} 1-b_2;\ b_1, b_2 \\ 2-b_1-b_2 \end{matrix}\,\middle|\, x, \frac{x}{x+1}\right) \quad \text{and} \quad \frac{(1+x)^{b_2}}{1-x}\,{}_2F_1\left(\begin{matrix} 1-b_1, \frac{1}{2} \\ 2-b_1-b_2 \end{matrix}\,\middle|\, -\frac{4x}{(x-1)^2}\right) \tag{86}$$

satisfy the same second order ordinary Fuchsian equation. We rename $b_1 \mapsto 1-a$, $b_2 \mapsto a-b$, and adjust the power factor to the ${}_2F_1$ function, so to get the second function in (80). As presented in Appendix A formulas (110)–(121), there are generally 60 F_1 bivariate series giving solutions of the differential system (70)–(71). Investigation of those 60 F_1 series under the given specialization gives the F_1 functions in (80)–(85). □

The F_1 functions in (80)–(85) represent all possible relations between the 2 variables and the parameters among the specialized 60 F_1 series, obtainable from Appendix A formulas (110)–(121) and formally giving univariate solutions of the same ordinary differential equation. In particular, the F_1 function in (86) has the same relation between the arguments and parameters (up to permutation of the two arguments and the b_1, b_2 parameters) as the first function in (81). The four functions in (80)–(81) can be identified in a neighborhood of $x = 0$, while the two functions in (82) can be identified in a neighborhood of $x = 1$. Notice that relation between the two arguments in (84) or (85) is linear.

The variable separation case (41)–(42) for the F_3 function translates into a reducible transformed system (70)–(71). A representative conclusion is that the differential system for

$$F_1\left(\begin{matrix} b+\frac{1}{2};\ b, \frac{1}{2}-b \\ 3/2 \end{matrix}\,\middle|\, \frac{(s-t)^2}{(s+t)^2}, -\frac{(s-t)^2}{(s^2-1)(t^2-1)}\right) \tag{87}$$

has the following elementary solution:

$$\frac{1}{s-t}(s+t)^{2b}\left(1-s^2\right)^{\frac{1}{2}-b}\left(1-t^2\right)^{\frac{1}{2}-b}. \tag{88}$$

If differential condition (78) of Theorem 5.1 is tried for linear solutions $y(x)$, the following cases are found:

- Reducible Appell's functions $F_1\left(\begin{smallmatrix} a;\, b_1, b_2 \\ a+1 \end{smallmatrix}\,\middle|\, x, sx\right)$, $F_1\left(\begin{smallmatrix} a;\, b_1, c-b_1 \\ c \end{smallmatrix}\,\middle|\, 1-x, 1-sx\right)$ and $F_1\left(\begin{smallmatrix} a;\, 0, b_2 \\ c \end{smallmatrix}\,\middle|\, x, s\right)$, where s is a constant.
- The cases (84), (85) with $y = 1-x$, $y = x-1$, respectively, and a symmetric case with $y = x+1$.
- The trivial identity $F_1\left(\begin{smallmatrix} -1;\, b_1, b_2 \\ c \end{smallmatrix}\,\middle|\, x, \frac{c-b_1x}{b_2}\right) = 0$.

6. Computational aspects

Identities between bivariate and univariate hypergeometric series are usually derived by methods of series manipulation, or using integral representations. The method of relating differential equations is usually considered as tedious and computationally costly. In [16, p. 314] Srivastava and Karlsson characterize this method as follows:

> This method (though simple in theory) is rather laborious in practice, and is not very useful for discovering new transformations.

They immediately mention that Appell himself [2] showed that the functions in (66) satisfy the same differential equation of order 3.

Here we demonstrate the basic computational routine to translate linear differential equations in partial derivatives to ordinary differential equations for univariate specialization of their solutions. In our terminology, we seek to recognize *partial differential forms* of ordinary equations within *specialized forms* of given systems of partial differential equations.

In the specialization setting $x \mapsto x(t)$, $y \mapsto y(t)$ of Definition 1.1, let J denote the left ideal of the Weyl algebra $\mathbf{C}[x,y]\langle \partial/\partial x, \partial/\partial y\rangle$ determined by a given system of partial differential equations. We work in the $\mathbf{C}(t)\langle d/dt\rangle$ left module M that is linearly generated over $\mathbf{C}(t)$ by the full derivatives 1, d/dt, d^2/dt^2, etc., and the partial derivatives $\partial/\partial x$, $\partial/\partial y$, $\partial^2/\partial x^2$, $\partial^2/\partial x\partial y$, etc. We let d/dt act from the left following the Leibniz rule, with the action on the partial derivatives defined by the identity

$$\frac{d}{dt} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}, \tag{89}$$

compatible with (9). Let P be the submodule of M generated by $d/dt - \dot{x}\partial/\partial x - \dot{y}\partial/\partial y$. The elements of P give expressions of the full derivatives d/dt, d^2/dt^2, etc., in terms of partial derivatives, like the series of formulas starting with (9)–(10).

Let $\widetilde{J}$ denote the submodule of M generated by special forms of the elements in J. Let $\widetilde{P}$ denote the union of $\widetilde{J}$ and P, as a submodule of M. We propose to use Gröbner basis computations for $\widetilde{P}$ with respect to an ordering of full and partial derivatives suitable for elimination of the partial derivatives.

Here we briefly demonstrate this strategy by computing Kato's differential equation [10, (4.1)] for the general $F_4(t^2,(1-t)^2)$ function. We used this equation in Section 4, and presented its singularities and local exponents in (61).

Theorem 6.1. *The univariate function* $F_4\left(\begin{smallmatrix} a;\, b \\ c_1, c_2\end{smallmatrix}\,\middle|\, t^2,(t-1)^2\right)$ *satisfies the differential equation*

$$\begin{aligned} &\frac{d^3F}{dt^3} + \left(\frac{a+b+2c_1-c_2+1}{t} + \frac{a+b-c_1+2c_2+1}{t-1}\right)\frac{d^2F}{dt^2} \\ &\quad + \left(\frac{(2c_1-1)(a+b-c_2+1)}{t^2} + 2\frac{a+b-c_1-c_2+1+2ab+2c_1c_2}{t(t-1)}\right. \\ &\quad \left. + \frac{(2c_2-1)(a+b-c_1+1)}{(t-1)^2}\right)\frac{dF}{dt} + \frac{2ab(2(c_1+c_2-1)t-2c_1+1)}{t^2(t-1)^2}F = 0. \end{aligned} \tag{90}$$

Proof. In the course of the proof of Theorem 4.2 we basically derived Eq. (60) as an element of $\widetilde{P}$. We differentiate (60) with respect to t, keeping in mind that d/dt acts on the partial derivative $\partial F/\partial y$ following (89), or more specifically, following (59). We get the partial derivatives $\partial^2F/\partial x\partial y$ and $\partial^2F/\partial y^2$ in the computation; we can eliminate them using original equations (43)–(44), but then first order derivatives $\partial F/\partial x$, $\partial F/\partial y$ occur. For linear elimination of these derivatives, we use (59) and Eq. (60) itself. Eq. (90) is obtained as

$$\begin{aligned} &\frac{d}{dt}[\text{Eq. (60)}] + \frac{2(2c_1+2c_2-2a-2b-3)}{(t-1)^2}\left([\text{Eq. (43)}] + \frac{1-t^2}{t^2}[\text{Eq. (44)}]\right) \\ &\quad + \frac{(c_1-a-b-1)(2c_1+2c_2-2a-2b-3)}{t(t-1)^2}[\text{Eq. (59)}] + \frac{(c_1+c_2+1)t+c_2-a-b-2}{t(t-1)}[\text{Eq. (60)}], \end{aligned}$$

where the second term is added to eliminate second order partial derivatives, the next term—to eliminate $\partial F/\partial x$, and the last term—to eliminate $\partial F/\partial y$. □

Kato himself [10] derived equation (90) via a conversion to Pfaffian systems. The specialization transformation of Pfaffian systems is particularly elegant, but Kato's method requires conversion of a matrix differential equation to an ordinary differential equation. The specialization problem for partial differential equations can be considered as a restriction problem for D-modules [13]. However, available implementations of algorithms for D-modules worked very slowly on our examples. Border bases [12] rather than Gröbner bases look promisingly suitable for our strategy. Towards the end of Section 1 we mentioned the simplest computational view of looking for a linear relation between full derivatives (in the specialized variable) expressed in terms of a basis of partial derivatives.

7. Application: Dihedral hypergeometric functions

In the first case of Theorem 2.4, we have a general function of the form ${}_2F_1\left(\begin{smallmatrix} a,b \\ c\end{smallmatrix}\,\middle|\, t^2\right)$. The corresponding differential equation is a general pull-back transformation of Euler's equation (6) with respect to a degree 2 covering ramified above two (of its three) singular points. The pull-backed equation has 4 singularities in general; it has 3 (or less) singularities when a local exponent difference below a ramification point is equal to $\frac{1}{2}$. Theorem 2.4 says the general pull-back equation has solutions expressible as an $F_2(x,y)$ function with $x+y=2$. As we mentioned, the F_2 series does not converge unless it terminates.

An interesting case is when the parameters b_1, b_2 in the first case of Theorem 2.4 are zero or negative integers, say $b_1=-k$, $b_2=-\ell$. The immediate conclusion is that the functions

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}+\ell\\ \frac12-k\end{matrix}\,\middle|\,z\right)\quad\text{and}\quad (1+\sqrt{z})^{-a}F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,\frac{2\sqrt{z}}{1+\sqrt{z}},\frac{2}{1+\sqrt{z}}\right) \tag{91}$$

and satisfy the same second order ordinary equation, with respect to z. The monodromy group of the ${}_2\mathrm{F}_1(z)$ function is a dihedral group; the local exponent differences at $z=0$ and $z=\infty$ are the half-integers $k+\frac12$, $\ell+\frac12$, respectively. Functions with a dihedral monodromy group can be expressed as elementary functions, since after a quadratic pull-back transformation (ramified above $z=0$, $z=\infty$, in our case) the monodromy group of the pull-backed equation is a cyclic group, and hypergeometric solutions become simple power or logarithmic functions. An example of such an elementary expression is

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}\\ \frac12\end{matrix}\,\middle|\,z\right)=\frac{(1-\sqrt{z})^{-a}+(1+\sqrt{z})^{-a}}{2}. \tag{92}$$

Relatedly,

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac{a+1}{2},\ \frac{a+2}{2}\\ \frac32\end{matrix}\,\middle|\,z\right)=\frac{(1-\sqrt{z})^{-a}-(1+\sqrt{z})^{-a}}{2a\sqrt{z}}\quad (a\neq 0), \tag{93}$$

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac12,\ 1\\ \frac32\end{matrix}\,\middle|\,z\right)=\frac{\log(1+\sqrt{z})-\log(1-\sqrt{z})}{2\sqrt{z}}, \tag{94}$$

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}\\ a+1\end{matrix}\,\middle|\,z\right)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-a}. \tag{95}$$

The ${}_2\mathrm{F}_1$ function in (91) is a contiguous family of Gauss hypergeometric equations with the same monodromy group.

The relation between the two functions in (91) allows us to generalize formula (92). The generalization is

$$\begin{aligned} \frac{(\frac{a+1}{2})_\ell}{(\frac12)_\ell}\,{}_2\mathrm{F}_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}+\ell\\ \frac12-k\end{matrix}\,\middle|\,z\right)&=\frac{(1+\sqrt{z})^{-a}}{2}F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,\frac{2\sqrt{z}}{1+\sqrt{z}},\frac{2}{1+\sqrt{z}}\right)\\ &\quad+\frac{(1-\sqrt{z})^{-a}}{2}F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,\frac{2\sqrt{z}}{\sqrt{z}-1},\frac{2}{1-\sqrt{z}}\right). \end{aligned} \tag{96}$$

The F_2 series on the right-hand side are finite sums with $(k+1)(\ell+1)$ terms. If $(\frac{a+1}{2})_\ell\neq 0$, this is an expression of the ${}_2\mathrm{F}_1$ function as a linear combination of two explicit solutions of the same differential equation. The identity can be proved by using the symmetry with respect to the conjugation of $\sqrt{z}$ and checking the value of both sides at $z=0$, which leads to evaluation of ${}_2\mathrm{F}_1\left(\begin{smallmatrix}a,-\ell\\ -2\ell\end{smallmatrix}\,\middle|\,2\right)$ obtainable by Zeilberger's algorithm. See [17] for details.

Similarly, a generalization of (93) is

$$\begin{aligned} &\frac{(\frac{a+1}{2})_k(\frac{a}{2})_{k+\ell+1}}{(\frac12)_k(\frac12)_{k+1}(\frac12)_\ell}(-1)^k z^{k+\frac12}\,{}_2\mathrm{F}_1\left(\begin{matrix}\frac{a+1}{2}+k,\ \frac{a}{2}+k+\ell+1\\ \frac32+k\end{matrix}\,\middle|\,z\right)\\ &\quad=\frac{(1+\sqrt{z})^{-a}}{2}F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,\frac{2\sqrt{z}}{1+\sqrt{z}},\frac{2}{1+\sqrt{z}}\right)-\frac{(1-\sqrt{z})^{-a}}{2}F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,\frac{2\sqrt{z}}{\sqrt{z}-1},\frac{2}{1-\sqrt{z}}\right). \end{aligned} \tag{97}$$

To show this identity, one can verify that both sides satisfy the same recurrence relations with respect to k and ℓ, and check the identity for a few values (k,ℓ).

Recalling the relation between $F_3(x,x/(2x-1))$ and ${}_2\mathrm{F}_1$ functions in Theorem 3.1, Karlsson's identity [16, 9.4.(90)] gives the following generalization of (95):

$$ {}_2\mathrm{F}_1\left(\begin{matrix}\frac{a-\ell}{2},\ \frac{a+\ell+1}{2}\\ a+k+1\end{matrix}\,\middle|\,z\right)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-a-k}(1-z)^{k/2}F_3\left(\begin{matrix}k+1,\ell+1;\ -k,-\ell\\ a+k+1\end{matrix}\,\middle|\,\frac{\sqrt{1-z}-1}{2\sqrt{1-z}},\frac{1-\sqrt{1-z}}{2}\right). \tag{98}$$

Note that the F_3 sum is finite for any integers k, ℓ, since it is invariant under the substitutions $k\mapsto -k-1$ and $\ell\mapsto-\ell-1$. The F_2 sums in (96) and (97) can be written as F_3 sums, or vice versa in (98), by reversing the double summation in both directions:

$$ F_2\left(\begin{matrix}a;\ -k,-\ell\\ -2k,-2\ell\end{matrix}\,\middle|\,x,y\right)=\frac{k!\,\ell!\,(a)_{k+\ell}}{(2k)!(2l)!}x^k y^\ell F_3\left(\begin{matrix}k+1,\ell+1;\ -k,-\ell\\ 1-a-k-\ell\end{matrix}\,\middle|\,\frac1x,\frac1y\right). \tag{99}$$

These explicit expressions are analyzed more thoroughly in [17].

Appendix A

Here we recall a couple of relevant facts about ordinary Fuchsian equations, and review related explicit results on Appell' functions existing in literature.

The third order hypergeometric differential equation for ${}_3\mathrm{F}_2\left({A,B,C \atop D,E}\,\middle|\,x\right)$ can be written

$$x^2(1-x)\frac{d^3y(x)}{dx^3}+x\big(D+E+1-(A+B+C+3)x\big)\frac{d^2y(x)}{dx^2} + \big(DE-(AB+AC+BC+A+B+C+1)x\big)\frac{dy(x)}{dx}-ABC\,y(x)=0. \tag{100}$$

The singularities and local exponents are:

$$\begin{array}{llll} \text{at } x=0: & 0, & 1-D, & 1-E, \\ \text{at } x=1: & 0, & 1, & D+E-A-B-C, \\ \text{at } x=\infty: & A, & B, & C. \end{array} \tag{101}$$

Recall that a third order Fuchsian equation with 3 singularities is determined by the local exponents and one *accessory parameter*. Say, one can add a scalar multiple of $y(x)/(x-1)$ to the left-hand side of Eq. (100) without changing the local exponents in (101).

Solutions of differential systems (11)–(12), (36)–(37), (43)–(44), (70)–(71) in terms of bivariate hypergeometric series were considered by many authors ([6, pp. 222–242], [16, pp. 291–305]), starting from Le Vavasseur, Appell, Kampé de Fériet, Borngässer, Erdélyi, Olsson.

In general, each Appell's $F_2(x,y)$ function can be represented by four F_2 series at the origin $(0,0)$:

$$F_2\left({a;\ b_1,b_2 \atop c_1,c_2}\,\middle|\,x,y\right)=(1-x)^{-a}F_2\left({a;\ c_1-b_1,b_2 \atop c_1,c_2}\,\middle|\,\frac{x}{x-1},\frac{y}{1-x}\right) \tag{102}$$

$$=(1-y)^{-a}F_2\left({a;\ b_1,c_2-b_2 \atop c_1,c_2}\,\middle|\,\frac{x}{1-y},\frac{y}{y-1}\right) \tag{103}$$

$$=(1-x-y)^{-a}F_2\left({a;\ c_1-b_1,c_2-b_2 \atop c_1,c_2}\,\middle|\,\frac{x}{x+y-1},\frac{y}{x+y-1}\right). \tag{104}$$

Note that the parameters of the $F_2(x,2-x)$ function in (20) are invariant under these transformations; this invariance is discussed in [14, Section 10]. Besides, in general there are four distinct $F_2(x,y)$ functions that are local solutions of (11)–(12) at the origin. The other 3 functions are represented by the series

$$x^{1-c_1}F_2\left({1+a-c_1;\ 1+b_1-c_1,b_2 \atop 2-c_1,c_2}\,\middle|\,x,y\right),\qquad y^{1-c_2}F_2\left({1+a-c_2;\ b_1,1+b_2-c_2 \atop c_1,2-c_2}\,\middle|\,x,y\right), \tag{105}$$

$$x^{1-c_1}y^{1-c_2}F_2\left({2+a-c_1-c_2;\ 1+b_1-c_1,1+b_2-c_2 \atop 2-c_1,2-c_2}\,\middle|\,x,y\right). \tag{106}$$

The F_2 system (11)–(12) has Horn's H_2 series local solutions at $(x,y)=(0,\infty)$ and $(x,y)=(\infty,0)$, and four distinct F_3 local solutions at $(x,y)=(\infty,\infty)$. The latter are obtainable by applying relation (39) to the four series in (102)–(104). Application of relation (39) to (105)–(106) does not give different F_3 series, but only shows invariance of (3) under the permutations $a_1\leftrightarrow b_1$ and $a_2\leftrightarrow b_2$ of upper parameters. Hypergeometric solutions of the F_3 system (36)–(37) are described similarly.

The system (43)–(44) has four generally different F_4 local solutions at $(x,y)=(0,0)$. Besides (4) we have

$$x^{1-c_1}F_4\left({1+a-c_1;\ 1+b-c_1 \atop 2-c_1,c_2}\,\middle|\,x,y\right),\qquad y^{1-c_2}F_4\left({1+a-c_2;\ 1+b-c_2 \atop c_1,2-c_2}\,\middle|\,x,y\right), \tag{107}$$

$$x^{1-c_1}y^{1-c_2}F_4\left({2+a-c_1-c_2;\ 2+b-c_1-c_2 \atop 2-c_1,2-c_2}\,\middle|\,x,y\right). \tag{108}$$

Similar sets of four local solutions exist at $(x,y)=(0,\infty)$ and $(\infty,0)$. A connection formula between (4) and the following two functions is presented in [3, p. 26] and [16, 9.4.(69)], for example:

$$y^{-a}F_4\left({a;\ 1+a-c_2 \atop c_1,1+a-b}\,\middle|\,\frac{x}{y},\frac{1}{y}\right),\qquad y^{-b}F_4\left({1+b-c_2,b \atop c_1,1+b-a}\,\middle|\,\frac{x}{y},\frac{1}{y}\right). \tag{109}$$

The full set of F_1 solutions to the system (70)–(71) consists of 60 F_1 series in general. First of all, the F_1 solutions are identified in sextets:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix} \,\middle|\, x, y\right) = (1-x)^{-b_1}(1-y)^{-b_2} F_1\left(\begin{matrix} c-a;\ b_1, b_2 \\ c \end{matrix} \,\middle|\, \frac{x}{x-1}, \frac{y}{y-1}\right) \tag{110}$$

$$= (1-x)^{-a} F_1\left(\begin{matrix} a;\ c-b_1-b_2, b_2 \\ c \end{matrix} \,\middle|\, \frac{x}{x-1}, \frac{x-y}{x-1}\right) \tag{111}$$

$$= (1-y)^{-a} F_1\left(\begin{matrix} a;\ b_1, c-b_1-b_2 \\ c \end{matrix} \,\middle|\, \frac{x-y}{1-y}, \frac{y}{y-1}\right) \tag{112}$$

$$= (1-x)^{c-a-b_1}(1-y)^{-b_2} F_1\left(\begin{matrix} c-a;\ c-b_1-b_2, b_2 \\ c \end{matrix} \,\middle|\, x, \frac{x-y}{1-y}\right) \tag{113}$$

$$= (1-x)^{-b_1}(1-y)^{c-a-b_2} F_1\left(\begin{matrix} c-a;\ b_1, c-b_1-b_2 \\ c \end{matrix} \,\middle|\, \frac{x-y}{x-1}, y\right). \tag{114}$$

Consequently, there are 10 different F_1 solutions of (70)–(71) in general. The other 9 are represented by the following series:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ 1+a+b_1+b_2-c \end{matrix} \,\middle|\, 1-x, 1-y\right), \qquad x^{-b_1} y^{-b_2} F_1\left(\begin{matrix} 1+b_1+b_2-c;\ b_1, b_2 \\ 1+b_1+b_2-a \end{matrix} \,\middle|\, \frac{1}{x}, \frac{1}{y}\right), \tag{115}$$

$$x^{-a} F_1\left(\begin{matrix} a;\ 1+a-c, b_2 \\ 1+a-b_1 \end{matrix} \,\middle|\, \frac{1}{x}, \frac{y}{x}\right), \qquad y^{-a} F_1\left(\begin{matrix} a;\ b_1, 1+a-c \\ 1+a-b_2 \end{matrix} \,\middle|\, \frac{x}{y}, \frac{1}{y}\right), \tag{116}$$

$$(1-x)^{-b_1}(1-y)^{c-a-b_2} F_1\left(\begin{matrix} c-a;\ b_1, c-b_1-b_2 \\ c-a-b_2+1 \end{matrix} \,\middle|\, \frac{1-y}{1-x}, 1-y\right), \tag{117}$$

$$(1-x)^{c-a-b_1}(1-y)^{-b_2} F_1\left(\begin{matrix} c-a;\ c-b_1-b_2, b_2 \\ c-a-b_1+1 \end{matrix} \,\middle|\, 1-x, \frac{1-x}{1-y}\right), \tag{118}$$

$$x^{-b_1} y^{b_1-c+1} F_1\left(\begin{matrix} 1+b_1+b_2-c;\ b_1, 1+a-c \\ 2+b_1-c \end{matrix} \,\middle|\, \frac{y}{x}, y\right), \tag{119}$$

$$x^{b_2-c+1} y^{-b_2} F_1\left(\begin{matrix} 1+b_1+b_2-c;\ 1+a-c, b_2 \\ 2+b_2-c \end{matrix} \,\middle|\, x, \frac{x}{y}\right), \tag{120}$$

$$x^{b_1+b_2-c}(x-y)^{1-b_1-b_2}(1-x)^{c-a-1} F_1\left(\begin{matrix} 1-b_1;\ 1+a-c, c-b_1-b_2 \\ 2-b_1-b_2 \end{matrix} \,\middle|\, \frac{x-y}{x-1}, \frac{x-y}{x}\right). \tag{121}$$

Besides, Appel's F_1 functions can be expressed as F_3 series following formula (79). Appell's F_2 series realize F_1 functions as well; the following identity hold in the neighborhood of the point $(x/y, y) = (1, 0)$ of the blow-up at $(x, y) = (0, 0)$:

$$F_1\left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix} \,\middle|\, x, y\right) = (y/x)^{b_1} F_2\left(\begin{matrix} b_1+b_2;\ b_1, a \\ b_1+b_2, c \end{matrix} \,\middle|\, \frac{x-y}{x}, y\right). \tag{122}$$

In the remainder of Appendix A, we summarize identities in [16, Section 9.4] that relate different Appell's functions to each other or to Gauss hypergeometric functions. The expressions for Appell's functions with reducible (up to possibly a quadratic transformation) monodromy groups are formulas [16, 9.4, (87), (88), (98), (99), (108)–(114)]; several of these formulas are attributed to Bailey or to [3]. It looks convenient to substitute $x \mapsto x/(x-1)$, $y \mapsto y/(y-1)$ in Bailey's formulas [16, 9.4, (110)–(113)]. We mentioned special relations (54)–(55) between F_2 and F_4 functions; another similar relation is [16, 9.4, (216)]. The following 5 formulas are [16, 9.4, (190)–(192), (149), (179)]; they give a taste of what can be expected from an exhaustive research of univariate specializations of Appell's functions to ${}_4F_3$ or ${}_3F_2$ functions. The first three formulas are attributed to Bailey, and (126) is due to Burchnall:

$$F_2\left(\begin{matrix} a;\ b, b \\ c, c \end{matrix} \,\middle|\, x, -x\right) = {}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, b, c-b \\ c, \frac{c}{2}, \frac{c+1}{2} \end{matrix} \,\middle|\, x^2\right), \tag{123}$$

$$F_2\left(\begin{matrix} a;\ b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, x, -x\right) = {}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{b_1+b_2}{2}, \frac{b_1+b_2+1}{2} \\ b_1+\frac{1}{2}, b_2+\frac{1}{2}, b_1+b_2 \end{matrix} \,\middle|\, x^2\right), \tag{124}$$

$$F_3\left(\begin{matrix} a, a;\ b, b \\ c \end{matrix} \,\middle|\, x, -x\right) = {}_4F_3\left(\begin{matrix} a, b, \frac{a+b}{2}, \frac{a+b+1}{2} \\ a+b, \frac{c}{2}, \frac{c+1}{2} \end{matrix} \,\middle|\, x^2\right), \tag{125}$$

$$F_4\left(\begin{matrix} a;\ b \\ c_1, c_2 \end{matrix} \,\middle|\, x, x\right) = {}_4F_3\left(\begin{matrix} a, b, \frac{c_1+c_2}{2}, \frac{c_1+c_2-1}{2} \\ c_1, c_2, c_1+c_2-1 \end{matrix} \,\middle|\, 4x\right), \tag{126}$$

$$F_4\left(\begin{matrix} a;\ b \\ c, c \end{matrix} \,\middle|\, x, -x\right) = {}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2} \\ c, \frac{c}{2}, \frac{c+1}{2} \end{matrix} \,\middle|\, -4x^2\right). \tag{127}$$

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