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The Resolution of the Generic Residual Intersection of a Complete Intersection

WINFRIED BRUNS*

Universität Osnabrück, Abteilung Vechta, Fachbereich 3 Naturwissenschaften/Mathematik, D-2848 Vechta, West Germany

AND

ANDREW R. KUSTIN† AND MATTHEW MILLER†

Mathematics Department, University of South Carolina, Columbia, South Carolina 29208

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INTRODUCTION

The concept of residual intersection, introduced by Artin and Nagata [1] in 1972, is a fruitful generalization of linkage as the following two examples attest. Let I be a strongly Cohen–Macaulay ideal in a Cohen–Macaulay local ring R. If I satisfies the condition (G_∞), then Huneke [6, Proposition 4.3] has proved that the extended Rees algebra $R[It, t^{-1}]$ is defined by an ideal which is obtained from I by way of residual intersection. Since the extended Rees algebra is a deformation of both the Rees algebra $R[It]$ and the associated graded algebra $\oplus I^i/I^{i+1}$, it contains considerable information about the analytic properties of I. (The definitions of residual intersection, strongly Cohen–Macaulay, and (G_∞) may be found in Section 4.)

Huneke [6, Theorem 4.1] has also shown that the ideal J, generated by the maximal order minors of a generic $n \times m$ matrix, is a residual intersection of a generic codimension two Cohen–Macaulay ideal. Since J is rather poorly behaved with respect to linkage [8], it is promising that it is close to a well understood ideal once we weaken the tie from linkage to residual intersection.

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The modern theory of linkage began with Peskine and Szpiro [11] when they produced a resolution of one ideal of a pair of linked ideals in terms of the resolution of the other ideal and a Koszul complex. It would be very desirable to have an analogous result for residual intersections. In other words, if J is the residual intersection of an ideal I with respect to an ideal generated by elements $z_1, ..., z_m$, then we want to express the resolution of J in terms of the resolution of I and the Koszul complex on $z_1, ..., z_m$. In this paper we begin the project: we resolve J in the case that I is a complete intersection and the z_j are generic linear combinations of the generators of I.

The ideals that we resolve belong to a wide class of ideals defined by "determinantal conditions." Let R be a ring, $Y_{1\times n}$ and $X_{n\times m}$ be matrices of indeterminates, $J_{n,m}$ be the ideal $I_1(YX)+I_{\min\{n,m\}}(X)$ in the ring $R_{n,m}=R[Y, X]$, and $A_{n,m}$ be the quotient $R_{n,m}/J_{n,m}$. If $n\leqslant m$, then Huneke and Ulrich [7, Example 3.4] have proved that $J=J_{n,m}$ is the generic residual intersection of the generic complete intersection $I=I_1(Y)$. We give the $R_{n,m}$-free resolution of $A_{n,m}$. The ideals $J_{n,m}$ with $m\leqslant n$ define the singular locus of a projective algebraic variety that is a complete intersection in projective space. These ideals have been resolved by Buchsbaum and Eisenbud [4]. The ideals $J_{n,m}$ for $m=n$ or $n-1$ are particularly interesting. If $m=n$, then $J_{n,m}$ is an almost complete intersection linked generically in a single step to a generic complete intersection of grade n; this is a Northcott ideal. If $m=n-1$, then $J_{n,m}$ is the Gorenstein generic double link from a generic complete intersection of grade n; this is a Herzog ideal. (For up-to-date information about these two types of $J_{n,m}$ see [2].) We note that Pellikaan [9, Theorems 3.1 and 3.2] has produced a resolution of $I_1(Y)/I_1(YX)$ over $R_{n,m}$; his result does not yield a resolution of $A_{n,m}$, but some similar ideas are involved.

There are two other ways to view the ideals $J=J_{n,m}$. In affine space $\mathbf{A}^{nm+n}(R)$, consider the variety V of complexes

$$0 \longrightarrow R^m \xrightarrow{\theta_2} R^n \xrightarrow{\theta_1} R$$

with rank $\theta_2<\min\{n, m\}$. DeConcini and Strickland [5] have shown that J is the ideal of V. Furthermore, if R is a normal domain, then $R[X, Y]/J$ is a normal domain and J is a perfect ideal with grade equal to the maximum of n and m. Finally, $A_{n,m}$ can also be interpreted as $\mathrm{Symm}_T(\mathrm{coker}\ X)$ for $T=R[X]/I_{\min\{n,m\}}(X)$.

The first section is a review of multilinear algebra, followed by definitions of the main constituents of our resolution. Preliminary acyclicity results are given in Section 2, while the resolution itself may be found in Section 3. Section 4 contains a discussion of the generic case, with applications to residual intersections and canonical modules.

1. Multilinear Algebra

For a quick review of multilinear algebra consult [4]. If G is a free module over a ring R, then we let $S(G)$, $\bigwedge(G)$, and $D(G)$ denote the symmetric algebra, exterior algebra, and divided power algebra of G over R, respectively. Each of these algebras (A) is equipped with a co-multiplication $\Delta: A \to A \otimes A$ which is induced by the diagonal map $G \to G \oplus G$. We shall also let Δ denote any of the R-linear maps given by restriction of Δ followed by projection. For example, $\Delta: \bigwedge^3 G \to \bigwedge^1 G \otimes \bigwedge^2 G$ carries $a \wedge b \wedge c$ to $a \otimes b \wedge c - b \otimes a \wedge c + c \otimes a \wedge b$ and $\Delta: D_3 G \to D_1 G \otimes D_2 G$ carries $a^{(3)}$ to $a \otimes a^{(2)}$ and $a^{(2)}b$ to $a \otimes ab + b \otimes a^{(2)}$ for all a, b, and c in F. The co-multiplication Δ satisfies the co-associativity property. For example, the diagram

$$(1.1)\qquad \begin{tikzcd} \bigwedge^{a+b+c} G \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \bigwedge^{a} G \otimes \bigwedge^{b+c} G \arrow[d, "\mathrm{id} \otimes \Delta"] \\ \bigwedge^{a+b} G \otimes \bigwedge^{c} G \arrow[r, "\Delta \otimes \mathrm{id}"] & \bigwedge^{a} G \otimes \bigwedge^{b} G \otimes \bigwedge^{c} G \end{tikzcd}$$

commutes for all non-negative integers a, b, and c.

We shall frequently, but silently, invoke a handful of canonical indentifications. In particular, we always think of a free module as coming equipped with an orientation. In other words, if G is a free R-module of rank n, then we select and fix an isomorphism $\bigwedge^n G \to R$. Having done this, for each $a \geqslant 0$, we obtain a perfect pairing

$$(1.2)\qquad \mu: \bigwedge^{a} G \otimes \bigwedge^{n-a} G \to \bigwedge^{n} G \simeq R,$$

which is induced by multiplication. There is a similar result exploiting the fact that $D(G)$ is the graded dual of $S(G^*)$. Indeed, for each $a \geqslant 0$ there is a perfect pairing

$$(1.3)\qquad D_a G \otimes S_a(G^*) \to R.$$

Our resolution will be obtained as the total complex of a certain bicomplex. The constituent pieces are all remarkably similar, however, being variations on the Koszul complex. We begin with the most elementary version. If $\psi: F \to R$ is a map of free R-modules then the *Koszul complex* $\mathbf{K}(\psi)$ *induced by* ψ is the exterior algebra $\bigwedge F$ together with a differential map $\partial^a: \bigwedge^a F \to \bigwedge^{a-1} F$ given by

$$\bigwedge^{a} F \xrightarrow{\Delta} \bigwedge^{1} F \otimes \bigwedge^{a-1} F \xrightarrow{\psi \otimes \mathrm{id}} R \otimes \bigwedge^{a-1} F = \bigwedge^{a-1} F.$$

By applying (1.1) to the integers 1, $a-1$, and b we see that the diagram

$$\begin{array}{ccc} \bigwedge^{a+b} F & \xrightarrow{\Delta} & \bigwedge^{a} F \otimes \bigwedge^{b} F \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial \otimes \mathrm{id}} \\ \bigwedge^{a+b-1} F & \xrightarrow{\Delta} & \bigwedge^{a-1} F \otimes \bigwedge^{b} F \end{array} \tag{1.4}$$

commutes for all non-negative a and b. More generally, if $\phi: F \to G^*$ is a map of free R-modules, then we let S be the ring $S(G^*)$, $\tilde{F}$ be the free S-module $S \otimes_R F$, and $\tilde{\phi}: \tilde{F} \to S$ be the composition of $\mathrm{id} \otimes \phi$ with multiplication $S \otimes_R G^* \to S$. Note that if $G^* = \sum_{i=1}^{n} RT_i$, then S can be viewed as the polynomial ring $R[T_1, ..., T_n]$. The differential ∂ on the Koszul complex $\mathbf{K}(\tilde{\phi})$ has a natural decomposition corresponding to the decomposition of $\bigwedge^p \tilde{F}$ as $\sum_{i=0}^{\infty} (S_i G^* \otimes_R \bigwedge^p F)$; following [4] we let ∂_k^j denote the component $S_{k-1} G^* \otimes \bigwedge^j F \to S_k G^* \otimes \bigwedge^{j-1} F$. Hence for each integer $r \geqslant 0$, there is a complex of free R-modules (which we shall still call a "Koszul complex")

$$0 \to S_0 G^* \otimes \bigwedge^r F \to S_1 G^* \otimes \bigwedge^{r-1} F \to \cdots \to S_r G^* \otimes \bigwedge^0 F \to 0. \tag{1.5}$$

EXAMPLE 1.6. Suppose rank $F =$ rank $G = n$ and ϕ is an isomorphism. If $\{e_1, ..., e_n\}$ is a basis for F and we let $T_i = \phi(e_i)$, then $T_1, ..., T_n$ is a regular sequence in S. It follows that $\mathbf{K}(\phi)$ is the S-free resolution of $S/(T_1, ..., T_n) \simeq R$. Hence the complex (1.5) is split exact for $r \geqslant 1$.

EXAMPLE 1.7. Suppose rank $F = m \geqslant n =$ rank G and ϕ is a projection, that is, there are bases $\{e_1, ..., e_m\}$ for F and $\{T_1, ..., T_n\}$ for G such that $\phi(e_i) = T_i$ for $1 \leqslant 1 \leqslant n$ and $\phi(e_i) = 0$ for $i \geqslant n+1$. It is well known that the homology $H_j(\mathbf{K}(\tilde{\phi})) = 0$ for $j \geqslant m-n+1$, and it is not difficult to see that for $0 \leqslant j \leqslant m-n$ the homology at $\mathbf{K}_j(\tilde{\phi})$ is concentrated in the summand $S_0 G^* \otimes \bigwedge^j F$. Hence the complex (1.5) is exact except at $S_0 G^* \otimes \bigwedge^r F$, for $0 \leqslant r \leqslant m-n$.

Dualizing the complexes (1.5) and applying the perfect pairings (1.2) and (1.3), we obtain new complexes that involve divided powers of G. It is convenient, however, to give an independent definition.

DEFINITION 1.8. If $\phi: F \to G^*$ is a map of free R-modules, then the Eagon–Northcott map

$$\eta_a^b: D_a G \otimes \bigwedge^b F \to D_{a-1} G \otimes \bigwedge^{b-1} F$$

is the composition

$$D_aG\otimes\bigwedge^b F\xrightarrow{\Delta\otimes\Delta} D_{a-1}G\otimes D_1G\otimes\bigwedge^1 F\otimes\bigwedge^{b-1} F$$

$$\xrightarrow{\phi} D_{a-1}G\otimes G\otimes G^*\otimes\bigwedge^{b-1} F\xrightarrow{\mathrm{ev}} D_{a-1}G\otimes\bigwedge^{b-1} F,$$

where $\mathrm{ev}\colon G\otimes G^*\to R$ is the evaluation map.

Remark 1.8a. It is easy to verify that $\eta_{a-1}^{b-1}\eta_a^b=0$. We shall call a complex formed by these maps an *Eagon–Northcott* complex.

LEMMA 1.9. *If $\phi\colon F\to G^*$ is a map of free R-modules, then the Eagon–Northcott complexes associated to ϕ are dual to the Koszul complexes of* (1.5).

Proof. Let m and n denote rank F and rank G, respectively. The perfect pairings (1.2) and (1.3) give a perfect pairing

$$\left(D_aG\otimes\bigwedge^b F\right)\otimes\left(S_aG^*\otimes\bigwedge^{m-b} F\right)\to R$$

for all a and b. We claim that the diagram

$$\begin{array}{ccc}
D_aG\otimes\bigwedge^b F & \xrightarrow{\eta_a^b} & D_{a-1}G\otimes\bigwedge^{b-1} F\\
\simeq\Big\downarrow\text{canon} & & \simeq\Big\downarrow\text{canon}\\
\left(S_aG^*\otimes\bigwedge^{m-b} F\right)^* & \xrightarrow{(\partial_a^{m-b+1})^*} & \left(S_{a-1}G^*\otimes\bigwedge^{m-b+1} F\right)^*
\end{array}$$

commutes (up to sign). It suffices to show that $\langle\eta z, y\rangle=(-1)^{b+1}\langle z,\partial y\rangle$ for $z\in D_aG\otimes\bigwedge^b F$ and $y\in S_{a-1}G^*\otimes\bigwedge^{m-b+1} F$. If one picks bases for F and G, then this is a straightforward calculation. ∎

Remark 1.9a. If rank $G=1$, then both the Eagon–Northcott complex and its dual reduce to the simple Koszul complex $\mathbf{K}(\phi)$; in this case, the lemma restates the well-known fact that this complex is self-dual.

The following proposition is a reformulation of Example 1.6.

PROPOSITION 1.10. *Let G be a free R-module of rank $n\geqslant 1$. If a and b* *are non-negative integers with* $b \leqslant n-1$, *then the Eagon–Northcott complexes*

$$0 \to D_{n+a}G \otimes \bigwedge^{n} G^* \to \cdots \to D_{a+1}G \otimes \bigwedge^{1} G^* \to D_a G \otimes R \to 0 \tag{1.10a}$$

$$0 \to D_{n-b}G \otimes \bigwedge^{n} G^* \to \cdots \to D_1 G \otimes \bigwedge^{b+1} G^* \to R \otimes \bigwedge^{b} G^* \to 0 \tag{1.10b}$$

induced by the identity map on G^* *are split exact.*

Proof. If $F = G^*$, $\phi = \mathrm{id}$, and $r \geqslant 1$, then Example 1.6 guarantees that complex (1.5) is split exact. It follows that the dual

$$0 \to D_r G \otimes \bigwedge^{n} G \to \cdots \to D_0 G \otimes \bigwedge^{n-r} G$$

is also exact. If $n < r$, then we obtain (1.10a) by letting $a = r - n$. If $1 \leqslant r \leqslant n$, then we obtain (1.10b) by letting $b = n - r$. ▮

The kernels that appear in the above Eagon–Northcott complexes play a central role in our construction.

DEFINITION 1.11. If G is a free R-module of rank $n \geqslant 1$, then for each pair of non-negative integers a and b, let $K_a^b(G)$ be the kernel of the Eagon–Northcott map $\eta_a^b \colon D_a G \otimes \bigwedge^b G^* \to D_{a-1} G \otimes \bigwedge^{b-1} G^*$ which is induced by the identity map $G^* \to G^*$.

PROPOSITION 1.12. *If* G *is a free* R*-module of rank* $n \geqslant 1$, *then*

(a) $K_0^b(G) = \bigwedge^b G^*$ *for all* $b \geqslant 0$, *and in particular* $K_0^n(G) \simeq R$,

(b) $K_a^0(G) = D_a(G)$ *for all* $a \geqslant 0$,

(c) $K_a^{n-1}(G) \simeq D_{a+1}(G)$ *for all* $a \geqslant 0$,

(d) $K_a^b(G) = 0$ *for all* $a \geqslant 1$ *and* $b \geqslant n$,

(e) $K_a^b(G) = \mathrm{im}(\eta_{a+1}^{b+1})$ *for all* a, $b \geqslant 0$, *except* $K_0^n(G) \neq 0 = \mathrm{im}(\eta_1^{n+1})$.

Proof. The assertions all follow immediately from the definition of K_a^b and Proposition 1.10. ▮

Remark 1.12a. Buchsbaum and Eisenbud [4, pp. 260–262] prefer to work directly with the Koszul complex induced by $\mathrm{id} \colon G^* \to G^*$. They define $L_p^q(G^*) = \ker(\partial_{p+1}^{q-1})$, which is equal to $\mathrm{im}(\partial_p^q)$ unless $p + q = 1$.

Observation 1.12b. If G is a free R-module of rank $n \geqslant 1$ and a and b are integers with $a + n - b \neq 0$, then

$$(K_a^b G)^* \simeq L_{a+1}^{n-b}(G^*).$$

Proof. The Eagon–Northcott complex

$$D_{a+2}G \otimes \bigwedge^{b+2} G^* \to D_{a+1}G \otimes \bigwedge^{b+1} G^* \to K_a^b \to 0$$

is exact by Proposition 1.12(e). It follows that the top row of

$$\text{(1.12c)}\qquad \begin{array}{ccccccc} 0 \to (K_a^b G)^* \to & \left(D_{a+1}G \otimes \bigwedge^{b+1} G^*\right)^* & \xrightarrow{\eta^*} & \left(D_{a+2}G \otimes \bigwedge^{b+2} G\right)^* \\ & \downarrow \simeq & & \downarrow \simeq \\ 0 \to L_{a+1}^{n-b}(G^*) \to & S_{a+1}G^* \otimes \bigwedge^{n-b-1} G & \xrightarrow{\partial} & S_{a+2}G^* \otimes \bigwedge^{n-b-2} G \end{array}$$

is exact. The bottom row of (1.12c) is exact by the definition of $L_p^q(G^*)$. The square commutes by Lemma 1.9. ∎

PROPOSITION 1.13. *If G is a free R-module of rank $n \geqslant 1$, and a and b are non-negative integers with $b \leqslant n$, then $K_a^b(G)$ is a free R-module of rank*

$$\text{(1.13a)}\qquad \binom{n+a-1-b}{a}\binom{n+a}{b}.$$

Proof. The result follows from Observation 1.12b and [4, Proposition 2.5(c)]. Note that $\binom{a-1}{a}$ is defined to be 1 if $a=0$ and zero otherwise. ∎

Remark 1.14. Given maps of free modules $\phi: F \to G^*$ and $\psi: G^* \to H^*$ inducing Eagon–Northcott maps η and λ, respectively, it is trivial to verify that $(1 \otimes \lambda)(\eta \otimes 1) = (\eta \otimes 1)(1 \otimes \lambda)$. But if $H^* = G^*$, then both $1 \otimes \lambda$ and $\eta \otimes 1$ act on $\bigwedge^b F \otimes D_a G \otimes \bigwedge^c G^*$, where the middle term plays two different roles, depending on the choice of association. The diagram

$$\begin{array}{ccc} \bigwedge^b F \otimes D_a G \otimes \bigwedge^c G^* & \xrightarrow{1 \otimes \lambda_a^c} & \bigwedge^b F \otimes D_{a-1} G \otimes \bigwedge^{c-1} G^* \\ \downarrow \eta_a^b \otimes 1 & & \downarrow \eta_{a-1}^b \otimes 1 \\ \bigwedge^{b-1} F \otimes D_{a-1} G \otimes \bigwedge^c G^* & \xrightarrow{1 \otimes \lambda_{a-1}^c} & \bigwedge^{b-1} F \otimes D_{a-2} G \otimes \bigwedge^{c-1} G^* \end{array}$$

commutes, as the reader may verify.

2. First Acyclicity Results

One of the most important tools in the theory of resolutions is the acyclicity lemma of Peskine and Szpiro. The version we use may be found in [4, Corollary 4.2].

Lemma 2.1 [10, Lemma 1.8]. *Let R be a noetherian ring and let*

$$\mathbf{F}: 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$

be a complex of finitely generated free R-modules. If $\mathbf{F}_P$ is acyclic for all prime ideals P of R with grade $P < n$, then $\mathbf{F}$ is acyclic.

Many versions of the next result appear in the literature. It is more convenient to prove the version we use than to ask the reader to translate some other formulation into our context.

Proposition 2.2. *Let $\phi: F \to G^*$ be a map of free R-modules with rank $F = m \geqslant n = \text{rank } G$. Assume that grade $I_n(\phi) \geqslant m - n + 1$. If a and b are non-negative integers with $b \geqslant n - 1$, then the Eagon–Northcott complex*

$$0 \to D_{m-b+a} G \otimes \bigwedge^{m} F \to \cdots \to D_{a+1} G \otimes \bigwedge^{b+1} F \to D_a G \otimes \bigwedge^{b} F$$

is acyclic.

Proof. For each $r \geqslant 1$, consider the Koszul complex (1.5)

$$S_0 G^* \otimes \bigwedge^{r} F \to \cdots \to S_r G^* \otimes \bigwedge^{0} F \to 0.$$

By Lemma 1.9 its dual is the Eagon–Northcott complex

$$0 \to D_r G \otimes \bigwedge^{m} F \to \cdots \to D_0 G \otimes \bigwedge^{m-r} F. \tag{2.3}$$

If we localize at a prime of grade less than $m - n + 1$, then by hypothesis some $n \times n$ minor of ϕ becomes invertible, and we may assume that ϕ is a projection. Hence, by Example 1.7, the Koszul complex is split exact, and so is its dual. From the acyclicity lemma one concludes that the truncation of (2.3) consisting of the last $m - n + 1$ maps is exact. In other words the complexes

$$0 \to D_r G \otimes \bigwedge^m F \to \cdots \to D_0 G \otimes \bigwedge^{m-r} F \qquad \text{for} \quad 1 \leqslant r \leqslant m-n+1$$

$$0 \to D_r G \otimes \bigwedge^m F \to \cdots \to D_{r-(m-n+1)} G \otimes \bigwedge^{n-1} F \qquad \text{for} \quad m-n+1 \leqslant r$$

are all acyclic. ∎

We are now ready to define the rows and columns of our double complex. For the remainder of this section we maintain the following setting:

(2.4) F and G are free R-modules with rank $F = m \geqslant n = \operatorname{rank} G$, $\phi: F \to G^*$ and $\psi: G^* \to R$ are R-module homomorphisms, and $K_a^b = K_a^b(G)$.

The columns pose little problem.

LEMMA 2.5. (a) *For each integer* a, *with* $0 \leqslant a \leqslant m-n$, *there is a complex* $(\mathbf{K}_a, \kappa_a)$,

$$0 \to K_a^{n-1} \to \cdots \to K_a^1 \to K_a^0,$$

where $\kappa_a^j: K_a^j \to K_a^{j-1}$ *is induced by the differential* $\mathrm{id} \otimes \partial^j$ *on* $D_a G \otimes \mathbf{K}(\psi)$.

(b) *The homology* $H_j(\mathbf{K}_a)$ *is annihilated by* $I_1(\psi)^{a+1}$ *for* $0 \leqslant j \leqslant n-2$.

Proof. The identity map on G^* induces an "Eagon–Northcott map" of complexes (see Remark 1.14) $\eta: D_a G \otimes \mathbf{K}(\psi) \to (D_a G \otimes \mathbf{K}(\psi))[-1]$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & D_a G \otimes \bigwedge^n G^* \arrow[r] \arrow[d, "\eta"] & \cdots \arrow[r] & D_a G \otimes \bigwedge^1 G^* \arrow[r] \arrow[d, "\eta"] & D_a G \otimes \bigwedge^0 G^* \arrow[d] \\
D_{a-1} G \otimes \bigwedge^n G^* \arrow[r] & D_{a-1} G \otimes \bigwedge^{n-1} G^* \arrow[r] & \cdots \arrow[r] & D_{a-1} G \otimes \bigwedge^0 G^* \arrow[r] & 0.
\end{tikzcd}$$

The kernel complex is $0 \to K_a^n \to K_a^{n-1} \to \cdots \to K_a^0$ by definition, and by Proposition 1.12(e) the image complex is $0 \to K_{a-1}^{n-1} \to \cdots \to K_{a-1}^0 \to 0$. By Proposition 1.12(d), $K_a^n = 0$ for $a \geqslant 1$. To prove (b) we observe that $\mathbf{K}_0$ is obtained by truncating $\bigwedge^n G^*$ from the end of $\mathbf{K}(\psi)$ and inserting 0 in its place. It is well known that the homology of $\mathbf{K}(\psi)$ is annihilated by $I_1(\psi)$, and the remaining assertion is easily proved by induction on a using the long exact sequence of homology associated to the short exact sequence of complexes $\ker \eta \rightarrowtail D_a G \otimes \mathbf{K}(\psi) \twoheadrightarrow \operatorname{im} \eta$. ∎

The rows present significantly greater problems.

LEMMA 2.6. *Fix the notation as in* (2.4). *Let* c *and* d *be non-negative integers. Then there is a complex* $(\mathbf{R}_{cd}, \rho_{cd})$,

$$\mathbf{R}_{cd}\colon 0 \to K^c_{m-d} \otimes \bigwedge^m F \to \cdots \to K^c_1 \otimes \bigwedge^{d+1} F \to K^c_0 \otimes \bigwedge^d F,$$

in which the differential $\rho^j_{cd}\colon K^c_{j-d} \otimes \bigwedge^j F \to K^c_{j-1-d} \otimes \bigwedge^{j-1} F$ *is induced by* $\mathrm{id} \otimes \eta^j_{j-d}\colon \bigwedge^c G^* \otimes (D_{j-d}G \otimes \bigwedge^j F) \to \bigwedge^c G^* \otimes (D_{j-1-d}G \otimes \bigwedge^{j-1} F)$. *Furthermore, if* $d \geq n-1$ *and* $\operatorname{grade} I_n(\phi) \geq m-n+1$, *then* $\mathbf{R}_{cd}$ *is acyclic.*

Proof. To lighten the notation, let η denote any map induced from an Eagon–Northcott map based on $\phi\colon F \to G^*$, and let λ denote any map induced from an Eagon–Northcott map based on $\mathrm{id}\colon G^* \to G^*$. Then there is a map of complexes

$$\begin{tikzcd}
\mathbf{C}_{cd}\colon 0 \arrow[r] \arrow[d, "\lambda"] & D_{m-d}G \otimes \bigwedge^c G^* \otimes \bigwedge^m F \arrow[r, "\eta"] \arrow[d, "\lambda"] & \cdots \arrow[r] & D_1 G \otimes \bigwedge^c G^* \otimes \bigwedge^{d+1} F \arrow[r, "\eta"] \arrow[d, "\lambda"] & \bigwedge^c G^* \otimes \bigwedge^d F \arrow[d] \\
\mathbf{C}_{c-1\,d+1}[-1]\ 0 \arrow[r] & D_{m-d-1}G \otimes \bigwedge^{c-1} G^* \otimes \bigwedge^m F \arrow[r, "\eta"] & \cdots \arrow[r] & D_0 G \otimes \bigwedge^{c-1} G^* \otimes \bigwedge^{d+1} F \arrow[r] & 0
\end{tikzcd}$$

The kernel complex is $(\mathbf{R}_{cd}, \rho)$ by Definition 1.11, and the image complex is $(\mathbf{R}_{c-1\,d+1}, \rho)$ by Proposition 1.12(e); that is, there is a short exact sequence of complexes $0 \to \mathbf{R}_{cd} \to \mathbf{C}_{cd} \to \mathbf{R}_{c-1\,d+1}[-1] \to 0$.

We now assume that $\operatorname{grade} I_n(\phi) \geq m-n+1$, and, by induction on c, we show that $\mathbf{R}_{cd}$ is acyclic for all $c \geq 0$ and $d \geq n-1$. If $c=0$, then $\mathbf{R}_{cd}$ is simply the Eagon–Northcott complex induced by ϕ and consequently is acyclic for all $d \geq n-1$ by Proposition 2.2. Assume now that $c \geq 1$ and $d \geq n-1$ and $\mathbf{R}_{c-1\,d+1}$ is acyclic. The long exact sequence of homology that is associated to the short exact sequence of complexes above yields that $\mathbf{R}_{cd}$ is acyclic, for $\mathbf{C}_{cd}$ is acyclic by Proposition 2.2, and the homology $H_i(\mathbf{R}_{c-1\,d+1}[-1]) = H_{i-1}(\mathbf{R}_{c-1\,d+1})$ is zero for $i \geq 2$ by the induction hypothesis. ∎

Recall that $K^b_0 = \bigwedge^b G^*$. We define $\varepsilon^b\colon K^b_0 \otimes \bigwedge^n F \to \bigwedge^b F$ as the composition

$$\begin{tikzcd}
\bigwedge^b G^* \otimes \bigwedge^n F \arrow[r, "\Delta"] & \bigwedge^b G^* \otimes \bigwedge^{n-b} F \otimes \bigwedge^b F \arrow[r, "\bigwedge^{n-b}\phi"] & \bigwedge^b G^* \otimes \bigwedge^{n-b} G^* \otimes \bigwedge^b F \arrow[d, "\mu"] \\
 & & \bigwedge^b F,
\end{tikzcd}$$

where the final map is the perfect pairing of (1.2).

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COROLLARY 2.7. *For $d=n$ and each $b \geqslant 0$, the map ε^b is an augmentation of the complex $\mathbf{R}_{bn}$.*

Proof. We show that the composition

$$K_1^b \otimes \bigwedge^{n+1} F \xrightarrow{\rho} K_0^b \otimes \bigwedge^{n} F \xrightarrow{\varepsilon^b} \bigwedge^{b} F$$

is zero for $0 \leqslant b \leqslant n$. (For $b=n$ it is even exact since ε^n is an isomorphism and $K_1^n = 0$.) From the definition of ρ there is a commutative diagram

$$\begin{tikzcd}
K_1^b \otimes \bigwedge^{n+1} F \arrow[rr, "\rho"] \arrow[d, "\mathrm{incl}"] & & K_0^b \otimes \bigwedge^{n} F \arrow[d, "="] & \\
\bigwedge^{b} G^* \otimes G \otimes \bigwedge^{n+1} F \arrow[rr, "\mathrm{id}\otimes\eta"] & & \bigwedge^{b} G^* \otimes \bigwedge^{n} F \arrow[r, "\varepsilon^b"] & \bigwedge^{b} F,
\end{tikzcd}$$

where we remind the reader that the Eagon–Northcott map η is the composition

$$G \otimes \bigwedge^{n+1} F \xrightarrow{\varDelta} G \otimes F \otimes \bigwedge^{n} F \xrightarrow{\phi} G \otimes G^* \otimes \bigwedge^{n} F \xrightarrow{\mathrm{ev}} \bigwedge^{n} F.$$

Hence $\varepsilon^b \rho$ is a restriction of the composition $(\mu \circ \bigwedge^{n-b} \phi \circ \varDelta) \circ (\mathrm{ev} \circ \phi \circ \varDelta)$. Use (1.1) and the fact that the diagram

$$\begin{tikzcd}
\bigwedge^{b} G^* \otimes G \otimes \bigwedge^{n+1-b} F \arrow[r, "\varDelta_G"] \arrow[d, "\varDelta"] & \bigwedge^{b-1} G^* \otimes G^* \otimes G \otimes \bigwedge^{n+1-b} F \arrow[r, "\mathrm{ev}"] & \bigwedge^{b-1} G^* \otimes \bigwedge^{n+1-b} F \arrow[d, "\mu \circ \bigwedge^{n+1-b}\phi"] \\
\bigwedge^{b} G^* \otimes G \otimes F \otimes \bigwedge^{n-b} F \arrow[r, "\mathrm{ev}\ \phi"] & \bigwedge^{b} G^* \otimes \bigwedge^{n-b} F \arrow[r, "\mu\circ\bigwedge^{n-b}\phi"] & R
\end{tikzcd}$$

commutes in order to see that $\varepsilon^b \rho$ is also equal to the composition

$$\begin{aligned}
\bigwedge^{b} G^* \otimes G \otimes \bigwedge^{n+1} F &\xrightarrow{\varDelta_G} \bigwedge^{b-1} G^* \otimes G^* \otimes G \otimes \bigwedge^{n+1} F \\
&\xrightarrow{\mathrm{ev}} \bigwedge^{b-1} G^* \otimes \bigwedge^{n+1} F \xrightarrow{\varDelta} \bigwedge^{b-1} G^* \otimes \bigwedge^{n+1-b} F \otimes \bigwedge^{b} F \\
&\xrightarrow{\bigwedge^{n+1-b}\phi} \bigwedge^{b-1} G^* \otimes \bigwedge^{n+1-b} G^* \otimes \bigwedge^{b} F \xrightarrow{\mu} \bigwedge^{b} F.
\end{aligned}$$

Thus $\varepsilon^b \rho = 0$, since $K_1^b \otimes \bigwedge^{n+1} F$ is precisely $\ker(\mathrm{ev} \circ \varDelta_G)$. ∎

Remark 2.8. Henceforth, we drop the second subscript from $\mathbf{R}_{bn}$ and denote this *augmented* complex by $\mathbf{R}_b$ with differential ρ_b^j.

3. The Resolution

Throughout this section R is a commutative noetherian ring, and we retain the setting of (2.4):

(3.1) F and G are free R-modules with rank $F = m \geqslant n = \operatorname{rank} G$, $\phi\colon F \to G^*$ and $\psi\colon G^* \to R$ are R-module homomorphisms, and $K_a^b = K_a^b(G)$.

Our goal is to construct a bicomplex $\mathbf{B} = \mathbf{B}(\psi, \phi)$ of free R-modules. If the ideal $J = I_1(\psi\phi) + I_n(\phi)$ of R has grade at least m, then the total complex $\mathbf{T}$ of $\mathbf{B}$ is a resolution of R/J. Furthermore, if R is graded and the maps ψ and ϕ are homogeneous of positive degree (or if $(R, \mathfrak{m})$ is local, $\phi(F) \subset \mathfrak{m}G^*$, and $\psi(G^*) \subset \mathfrak{m}R$), then $\mathbf{T}$ is a minimal resolution of R/J. Consider the diagram $\mathbf{B}(\psi, \phi)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{m-n}^{n-1} \otimes \bigwedge^{m} F & \longrightarrow & \cdots & \longrightarrow & K_0^{n-1} \otimes \bigwedge^{n} F & \longrightarrow & \bigwedge^{n-1} F \\
 & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & K_{m-n}^{n-2} \otimes \bigwedge^{m} F & \longrightarrow & \cdots & \longrightarrow & K_0^{n-2} \otimes \bigwedge^{n} F & \longrightarrow & \bigwedge^{n-2} F \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & \vdots & & & & \vdots & & \vdots \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_{m-n}^{0} \otimes \bigwedge^{m} F & \longrightarrow & \cdots & \longrightarrow & K_0^{0} \otimes \bigwedge^{n} F & \longrightarrow & \bigwedge^{0} F,
\end{array}$$

where the maps are defined as follows. The column on the right is a truncation of the Koszul complex induced by the composition $\psi \circ \phi\colon F \to R$. The other columns are obtained as $\mathbf{K}_a \otimes \bigwedge^{n+a} F$ (see Lemma 2.5). For each b with $0 \leqslant b \leqslant n-1$, the row

$$0 \longrightarrow K_{m-n}^b \otimes \bigwedge^{m} F \longrightarrow \cdots \longrightarrow K_1^b \otimes \bigwedge^{n+1} F \longrightarrow K_0^b \otimes \bigwedge^{n} F \xrightarrow{\varepsilon^b} \bigwedge^{b} F$$

is the augmented complex $\mathbf{R}_b$ of Corollary 2.7.

Remark 3.1a. It follows from Proposition 1.12 that the top and bottom rows of $\mathbf{B}(\psi, \phi)$ are the Eagon–Northcott complexes

$$0 \longrightarrow D_{m-n+1} G \otimes \bigwedge^{m} F \longrightarrow \cdots \longrightarrow D_1 G \otimes \bigwedge^{n} F \longrightarrow D_0 G \otimes \bigwedge^{n-1} F$$

and

$$0 \longrightarrow D_{m-n} G \otimes \bigwedge^{m} F \longrightarrow \cdots \longrightarrow D_0 G \otimes \bigwedge^{n} F \xrightarrow{\bigwedge^n \phi} R,$$

respectively. The other rows of **B** may be regarded as transitional steps between these two Eagon–Northcott complexes.

PROPOSITION 3.2. *The diagram* $\mathbf{B}(\psi, \phi)$ *is a bicomplex.*

Proof. We must show

(a) that the diagram

$$\begin{array}{ccc} K_a^b \otimes \bigwedge^{n+a} F & \longrightarrow & K_{a-1}^b \otimes \bigwedge^{n+a-1} F \\ \downarrow & & \downarrow \\ K_a^{b-1} \otimes \bigwedge^{n+a} F & \longrightarrow & K_{a-1}^{b-1} \otimes \bigwedge^{n+a-1} F \end{array}$$

commutes for all a and b with $1 \leqslant a \leqslant m-n$, and $1 \leqslant b \leqslant n-1$, and that

(b) the augmentation maps ε^b induce a map of complexes on the two right-hand columns.

Claim (a) is an easy consequence of Remark 1.14. To establish (b) we need to show that the diagram

$$\begin{array}{ccc} \bigwedge^b G^* \otimes \bigwedge^n F & \xrightarrow{\varepsilon^b} & \bigwedge^b F = K_b(\psi\phi) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ \bigwedge^{b-1} G^* \otimes \bigwedge^n F & \xrightarrow{\varepsilon^{b-1}} & \bigwedge^{b-1} F = K_{b-1}(\psi\phi) \end{array}$$

commutes (up to sign). Consider the diagram

$$\begin{array}{ccc} \bigwedge^b G^* \otimes \bigwedge^n F & \xrightarrow{\Delta} & \bigwedge^b G^* \otimes \bigwedge^{n-b} F \otimes \bigwedge^b F \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \partial} \\ \bigwedge^b G^* \otimes \bigwedge^{n+1-b} F \otimes \bigwedge^{b-1} F & \xrightarrow{\partial} & \bigwedge^b G^* \otimes \bigwedge^{n-b} F \otimes \bigwedge^{b-1} F \\ \downarrow{\scriptstyle \bigwedge^{n-b+1}\phi} & & \downarrow{\scriptstyle \bigwedge^{n-b}\phi} \\ \bigwedge^b G^* \otimes \bigwedge^{n+b-1} G^* \otimes \bigwedge^{b-1} F & \xrightarrow{\partial} & \bigwedge^b G^* \otimes \bigwedge^{n-b} G^* \otimes \bigwedge^{b-1} F \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \mu} \\ \bigwedge^{b-1} G^* \otimes \bigwedge^{n+1-b} G^* \otimes \bigwedge^{b-1} F & \xrightarrow{\mu} & \bigwedge^{b-1} F. \end{array}$$

The composition across the top followed by the maps on the right side is equal to $\partial\varepsilon^b$; and the composition down the left side followed by the bottom map is $\varepsilon^{b-1}\partial$. The top square commutes because each map from $\bigwedge^n F$ to $\bigwedge^{n-b} F \otimes \bigwedge^{b-1} F$ is the same as

$$\bigwedge^{n} F \to \bigwedge^{n-1} F \to \bigwedge^{n-b} F \otimes \bigwedge^{b-1} F$$

by (1.4). The middle square commutes because $\phi: F \to G^*$ induces the map $\bigwedge \phi: \mathbf{K}(\psi\phi) \to \mathbf{K}(\phi)$ of Koszul complexes. The bottom square commutes because the Koszul complex $\mathbf{K}(\phi)$ is a differential algebra. ∎

The next lemma is a preliminary calculation. It is used in Lemma 3.4 to establish acyclicity of the rows of $\mathbf{B}(\psi, \phi)$ under the condition grade $I_n(\phi) \geqslant m-n+1$. For integers a and r with $0 \leqslant a, r \leqslant n$, let γ be the "condensation" map given by the composition

$$\gamma: \bigwedge^{a} G^* \otimes \bigwedge^{r} G^* \xrightarrow{\ \Delta\ } \bigwedge^{a} G^* \otimes \bigwedge^{n-a} G^* \otimes \bigwedge^{r-n+a} G^* \xrightarrow{\ \mu \otimes 1\ } \bigwedge^{r-n+a} G^*,$$

and let C_a^r be the kernel of γ.

LEMMA 3.3. *Let G be a free R-module of rank n and let a and r be integers with $0 \leqslant a, r \leqslant n$. If $\eta: G \otimes \bigwedge^{r+1} G^* \to \bigwedge^r G^*$ is the Eagon–Northcott map induced by the identity map on G^*, then $\eta \otimes \mathrm{id}: G \otimes \bigwedge^a G^* \otimes \bigwedge^{r+1} G^* \to \bigwedge^a G^* \otimes \bigwedge^r G^*$ induces a surjective map $K_1^a \otimes \bigwedge^{r+1} G^* \to C_a^r$.*

Proof. First note that both $\bigwedge^{r+1} G^*$ and C_a^r are zero for $r = n$ and all a. We may therefore assume that $r \leqslant n-1$, and prove the statement for all a by descending induction on r. The following diagram commutes, and has exact rows and columns. The map $\lambda: G \otimes \bigwedge^a G^* \to \bigwedge^{a-1} G^*$ is the Eagon–Northcott map induced by the identity on G^*:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_1^a \otimes \bigwedge^{r+1} G^* \arrow[r, dashed, "\alpha_a^r"] \arrow[d] & C_a^r \arrow[d] & \\
0 \arrow[r] & K_1^{r+1} \otimes \bigwedge^{a} G^* \arrow[r] \arrow[d, dashed, "\beta_a^{r+1}"] & G \otimes \bigwedge^{a} G^* \otimes \bigwedge^{r+1} G^* \arrow[r, "\eta"] \arrow[d, "\lambda"] & \bigwedge^{a} G^* \otimes \bigwedge^{r} G^* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & C_{a-1}^{r+1} \arrow[r] & \bigwedge^{a-1} G^* \otimes \bigwedge^{r+1} G^* \arrow[r, "\gamma"] \arrow[d] & \bigwedge^{r-n+a} G^* \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We shall actually prove that the induced homomorphisms α_a^r and β_{a-1}^{r+1} are surjective in tandem. Of course they are related, since by the snake lemma $\operatorname{coker}(\alpha_a^r) \simeq \operatorname{coker}(\beta_{a-1}^{r+1})$.

Since $C_{a-1}^n = 0$, the map β_{a-1}^{r+1} is surjective for $r = n-1$, and it follows that α_a^{n-1} is surjective. We assume inductively that

$$(*) \qquad \beta_{a-1}^{r+1} \text{ and } \alpha_a^r \text{ are surjective for all } 0 \leqslant a \leqslant n$$

and demonstrate that the same holds for β_{a-1}^r and α_a^{r-1}. It clearly suffices to prove that β_{a-1}^r is surjective. The key to the argument is that the map $\alpha_{a-1}^r\colon K_1^{a-1} \otimes \bigwedge^{r+1} G^* \to C_{a-1}^r$ is surjective by $(*)$. Consider the following commutative diagram, in which three of the maps are surjections. By Proposition 1.12(e), $\operatorname{im}(\eta) = K_1^r \otimes \bigwedge^a G^*$ and $\operatorname{im}(\lambda) = K_1^{a-1} \otimes \bigwedge^{r+1} G^*$:

$$\begin{CD}
D_2 G \otimes \bigwedge^{r+1} G^* \otimes \bigwedge^a G^* @>{\eta}>> K_1^r \otimes \bigwedge^a G^* \\
@V{\lambda}VV @VV{\beta_{a-1}^r}V \\
K_1^{a-1} \otimes \bigwedge^{r+1} G^* @>{\alpha_{a-1}^r}>> C_{a-1}^r .
\end{CD}$$

It follows that β_{a-1}^r is surjective. ∎

Lemma 3.4. *Retain the notation of* (3.1). *If* grade $I_n(\phi) \geqslant m-n+1$, *then each row of the bicomplex* $\mathbf{B}(\psi, \phi)$ *is acyclic. If, further,* H_b *denotes* $\operatorname{coker}(\varepsilon^b)$, *i.e., the* 0th *homology of the* bth *row, then there is an integer* N, *independent of* b, *such that* $I_n(\phi)^N H_b = 0$.

Proof. Row b, or in other words the augmented complex $\mathbf{R}_b$ given in Corollary 2.7, has length $m-n+1$. By the acyclicity lemma (Lemma 2.1), it suffices to show that $(\mathbf{R}_b)_P$ is exact whenever P is a prime ideal of R with grade $P < m-n+1$. By hypothesis, such a prime cannot contain $I_n(\phi)$. Consequently, it suffices to show that $\mathbf{R}_b$ is acyclic if some $n \times n$ minor of ϕ is invertible. Under the circumstances, we may assume that $F = G^* \oplus F'$ for some F', and $\phi\colon G^* \oplus F' \to G^*$ is the projection map.

From Lemma 2.6 and its proof, we see that it suffices to show that

$$K_1^b \otimes \bigwedge^{n+1} F \xrightarrow{\rho} K_0^b \otimes \bigwedge^n F \xrightarrow{\varepsilon^b} \bigwedge^b F$$

is exact for all b with $0 \leqslant b \leqslant n-1$. For each integer N, write $\bigwedge^N F$ as $\bigoplus_{p+q=N} \bigwedge^p G^* \otimes \bigwedge^q F'$, in order to decompose the above complex as

$$K_1^b \otimes \bigwedge^{n+1} F' \to 0 \to 0$$

plus the direct sum of complexes

$$\sum_{r=0}^{n}\left[K_1^b \otimes \bigwedge^{r+1} G^* \xrightarrow{\rho} \bigwedge^{b} G^* \otimes \bigwedge^{r} G^* \xrightarrow{\gamma} \bigwedge^{r-(n-b)} G^*\right] \otimes \bigwedge^{n-r} F'.$$

Each of these is exact because ρ is induced by η, and hence by Lemma 3.3 the image of this map is precisely $C_b^r = \ker(\gamma)$.

Observe further that each of the maps γ is surjective. This implies that $H_b = \operatorname{coker}(\varepsilon^b)$ has support contained in $\operatorname{Spec} R/I_n(\phi)$. Hence $\operatorname{rad}(\operatorname{ann}(H_b)) \supset \operatorname{rad}(I_n(\phi)) \supset I_n(\phi)$, and the final assertion of the lemma follows. ▌

DEFINITION 3.5. Let $\mathbf{T} = \mathbf{T}(\psi, \phi)$ be the total complex of $\mathbf{B}(\psi, \phi)$. In other words, $\mathbf{T}$ is the complex

$$0 \to P_m \to \cdots \to P_n \to Q_{n-1} \oplus P_{n-1} \to \cdots \to Q_1 \oplus P_1 \to Q_0,$$

where $Q_i = \bigwedge^i F$ and $P_i = \bigoplus K_a^b(G) \otimes \bigwedge^{n+a} F$; the sum is taken over all (a, b) with

(3.5a) $\quad a + b = i - 1, \qquad 0 \leqslant a \leqslant m - n, \qquad$ and $\qquad 0 \leqslant b \leqslant n - 1.$

The maps in $\mathbf{T}$ are taken from $\mathbf{B}$ (there is a standard routine for changing the sign of some of the maps, described in [12, Lemma 11.15], but as this is notationally cumbersome and materially insignificant we suppress the details). If R is a graded ring and ψ and ϕ both are homogeneous maps of degree 1, then each map $P_{i+1} \to P_i$ has degree 1, each map $Q_{i+1} \to Q_i$ has degree 2, and each map $P_{i+1} \to Q_i$ has degree $n - i$. (All maps $Q_{i+1} \to P_i$ are zero.) In this case, $Q_i = R(-2i)^{e(i)}$ and $P_i = R(-(n+i-1))^{f(i)}$, where, by Proposition 2.4,

$$e(i) = \binom{m}{i} \qquad \text{and} \qquad f(i) = \sum \binom{n+a-1-b}{a}\binom{n+a}{b}\binom{m}{n+a},$$

the sum is taken over all a and b as in (3.5a). In particular,

$$P_m = R(-m-n+1)^{f(m)}, \qquad \text{where} \quad f(m) = \binom{m}{n-1}.$$

It is clear that $H_0(\mathbf{T}) = R/J$, where J is the ideal $I_1(\psi\phi) + I_n(\phi)$. Our main result is that $\mathbf{T}$ is a resolution of R/J if the grade of J is at least m. We shall see in Proposition 4.2 that this hypothesis holds in the generic case, and hence the hypothesis on grade $I_n(\phi)$ that appears in the following theorem is actually superfluous.

THEOREM 3.6. *Retain the setting of* (3.1). *If the ideal* $J = I_1(\psi\phi) + I_n(\phi)$ *has grade at least* m, *and* grade $I_n(\phi) \geqslant m - n + 1$, *then* $\mathbf{T}(\psi, \phi)$ *is a resolution of* R/J, *and* J *is perfect of grade exactly* m.

Proof. As usual we shall invoke the acyclicity lemma. The complex $\mathbf{T}$ has length m so it suffices to show that $\mathbf{T}_P$ is acyclic for primes of grade less than m. This can be done directly, but we get the result indirectly by showing that the homology of $\mathbf{T}$ is annihilated by an ideal of grade at least m. We begin by observing that $\operatorname{grade}(I_1(\psi\phi)^N + I_n(\phi)^N) = \operatorname{grade} J \geqslant m$ for any $N > 0$.

Taking $H'_b = \operatorname{coker}(\varepsilon^b) = H_0(\mathbf{R}_b)$, we obtain a complex

$$\mathbf{H}': 0 \longrightarrow H'_{n-1} \xrightarrow{d''} H'_{n-2} \xrightarrow{d''} \cdots \longrightarrow H'_1 \xrightarrow{d''} H'_0 \longrightarrow 0,$$

where d'' is induced by the differential ∂ on the Koszul complex $\mathbf{K}(\psi\phi) = \bigwedge F$. Since we have assumed that grade $I_n(\phi) \geqslant m - n + 1$, we may apply Lemma 3.4 in order to conclude that each row $\mathbf{R}_b$ of $\mathbf{B}$ is acyclic. It follows that the iterated homology $H''_r(\mathbf{H}')$ is in fact $H_r(\mathbf{T})$; thus $H_r(\mathbf{T}) = 0$ for $r \geqslant n$. Furthermore, we apply Lemma 3.4 again in order to see that each term of $\mathbf{H}'$ is annihilated by $I_n(\phi)^N$ for some sufficiently large N. Hence $I_n(\phi)^N \cdot H_r(\mathbf{T}) = 0$ for $0 \leqslant r \leqslant n - 1$.

The homology of $\mathbf{T}$ can be computed as a spectral sequence limit of iterated homology in two different ways. By reflecting $\mathbf{B}(\psi, \phi)$ across the vertical axis (i.e., the column $\bigwedge F$) one obtains a standard first quadrant spectral sequence induced by the horizontal differential d' of bidegree $(-1, 0)$ and vertical differential d'' of bidegree $(0, -1)$. With the terminology of Rotman [12, p. 317 and Theorems 11.18 and 11.19] in force we obtain

$${}^{II}E^2_{p,q} = H''_p(H'_{q,p}(\mathbf{B})) \overset{p}{\Longrightarrow} H_{p+q}(\mathbf{T}),$$

where

$$H''_p(H'_{q,p}(\mathbf{B})) = \begin{cases} 0 & \text{if } q > 0 \\ H''_p(\mathbf{H}') & \text{if } q = 0. \end{cases}$$

Since all terms off the vertical axis are zero, the sequence collapses and ${}^{II}E^2_{p,q} = {}^{II}E^{\infty}_{p,q}$. Convergence of the spectral sequence means that for each $r \geqslant 0$, the homology $H_r(\mathbf{T})$ has a finite filtration $\{\Phi^j H_r(\mathbf{T})\}$ such that ${}^{II}E^{\infty}_{p,r-p} = \Phi^p H_r(\mathbf{T})/\Phi^{p-1} H_r(\mathbf{T})$. In this case the filtration is trivial and $H_r(\mathbf{T}) = H''_r(\mathbf{H}')$ as claimed above.

The situation is more interesting when we compute column homology first. By Lemma 2.5(b) the homology $H''_{p,q}(\mathbf{B}) = H_q(\mathbf{K}_{p-1}) \otimes \bigwedge^{n+p-1} F$ is annihilated by $I_1(\psi)^p$ for $0 \leqslant q < n - 1$. The homology along the vertical axis is $H''_{0,q}(\mathbf{B}) = H_q(\mathbf{K}(\psi\phi))$ for $0 \leqslant q < n-1$, which is annihilated by $I_1(\psi\phi)$. Taken together, these facts show that $H''_{p,q}(\mathbf{B})$, and hence also the iterated homology $H'_p(H''_{p,q}(\mathbf{B}))$ is annihilated by $I_1(\psi\phi)^N$ for some sufficiently large N, provided that $0 \leqslant q < n-1$. Since we have already determined that $H_r(\mathbf{T}) = 0$ for $r > n-1$, we need only to determine the annihilator of $H'_0(H''_{0,n-1}(\mathbf{B}))$ before wrapping up our spectral sequence argument. Rather than doing this explicitly we resort to an artifice.

Extend $\mathbf{B}$ to $\tilde{\mathbf{B}}$ by adjoining free modules $\bigwedge^n F$ and $K_0^n \otimes \bigwedge^n F$ at positions $(0, n)$ and $(1, n)$, respectively. These modules, and the relevant maps, are found by truncating $\bigwedge F = \mathbf{K}(\psi\phi)$ and $\mathbf{K}_0 \otimes \bigwedge^n F = \mathbf{K}(\psi) \otimes \bigwedge^n F$ one step further back than previously. In other words, we have

$$\begin{tikzcd} \bigwedge^{n} F \arrow[d, "\partial"'] & K_0^n \otimes \bigwedge^{n} F \arrow[l, "\varepsilon^n"'] \arrow[d, "\kappa_0^n"] \\ \bigwedge^{n-1} F & K_0^{n-1} \otimes \bigwedge^{n} F. \arrow[l, "\varepsilon^{n-1}"'] \end{tikzcd}$$

The square commutes (see Lemma 3.2(b)) and moreover ε^n is an isomorphism. If the total complex $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{B}}$ is a resolution of R/J, then so is $\mathbf{T}$ (one simply splits off the extra summands from $\tilde{T}_n$ and $\tilde{T}_{n+1}$); and obviously the first part of our argument needs no alteration since no new row homology has been introduced. But it is apparent that $H''_{0,n-1}(\tilde{\mathbf{B}})$ is annihilated by $I_1(\psi\phi)$, so we now have a convergent spectral sequence

$$'E^2_{p,q} = H'_p(H''_{p,q}(\tilde{\mathbf{B}})) \overset{p}{\Longrightarrow} H_{p+q}(\tilde{\mathbf{T}})$$

in which the terms with $p + q \leqslant n-1$ are annihilated by $I_1(\psi\phi)^N$ for sufficiently large N. This sequence does not collapse, but $\tilde{\mathbf{B}}$ is bounded so it eventually stabilizes and $'E^\infty_{p,q} = {}'E^t_{p,q}$ for some $t \geqslant 2$. In any event $'E^\infty_{p,q}$ is a subquotient of $'E^2_{p,q}$, so it is annihilated by the same ideal. For each $r \leqslant n-1$, $H_r(\tilde{\mathbf{T}})$ has a finite filtration in which the quotients $\Phi^p H_r(\tilde{\mathbf{T}})/\Phi^{p-1} H_r(\tilde{\mathbf{T}})$ are isomorphic to $'E^\infty_{p,r-p}$. Replacing N by a still larger integer, and taking the first part of the argument into account, we conclude that $H_r(\tilde{\mathbf{T}})$ is annihilated by $I_n(\phi)^N + I_1(\psi\phi)^N$.

Finally, since $\mathbf{T}$ is a resolution of R/J, $m \geqslant \operatorname{pd} R/J \geqslant \operatorname{grade} J \geqslant m$. ∎

4. The Generic Case and Applications

It still remains to show that Theorem 3.6 is not vacuous. Following this, we interpret our result in the context of linkage and residual intersection, and use it to compute the canonical module. We establish notation first for the generic case.

DEFINITION 4.1. Let R be a commutative noetherian ring, $Y_{1\times n}$ and $X_{n\times m}$ be matrices of indeterminates, $R_{n,m}$ be the polynomial ring $R[Y, X]$, and $J_{n,m}$ be the ideal $I_1(YX)+I_{\min\{n,m\}}(X)$. Let $A_{n,m}=R_{n,m}/J_{n,m}$.

PROPOSITION 4.2. (a) *The ideal $J_{n,m}$ has grade at least* $\max\{m, n\}$.

(b) *If $m \geqslant n$, then $J_{n,m}$ is perfect of grade m.*

(c) *If $m \geqslant n$, then $J_{n,m}=I_1(YX): I_1(Y)$.*

Proof. (a) Let J denote $J_{n,m}$, $M=\max\{m, n\}$, and $L=\min\{m, n\}$. The claim is trivial to verify if $L=1$. We show that if P is a prime ideal containing J, then depth $S_P \geqslant M$. (Since grade $J=$ depth S_P for some such P, the claim follows.) The inequality certainly holds if all of the indeterminates $\{x_{ij}, y_i\}$ are contained in P.

If y_1 is not an element of P, then there is a $1\times n$ matrix Y' of indeterminates and an isomorphism $\theta: R_{n,m}[y_1^{-1}] \rightarrow R[Y', X, (y_1')^{-1}]$ with $\theta(J)=I_1([1, 0, ..., 0]X)+I_L(X)$. This ideal contains $(x_{11}, ..., x_{1m})+I_L(X')$, where X' is the matrix formed by deleting the top row of X. Hence

$$\text{grade } J_P \geqslant \left.\begin{cases} m & \text{if } \ m\geqslant n \\ m+((n-1)-m+1)=n & \text{if } \ n>m \end{cases}\right\} = M.$$

If x_{11} is not an element of P, then the same type of reasoning shows that grade $J_P \geqslant$ grade $(I_1(YX')+I_L(X'))$, where X' is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & X'' \end{bmatrix}$$

and X'' is an $(n-1)\times(m-1)$ matrix of indeterminates. Thus grade $J_P \geqslant$ grade $((y_1)+I_1([y_2, ..., y_n]X'')+I_{L-1}(X''))$. By induction on L we conclude depth $S_P \geqslant$ grade $J_P \geqslant 1+\max\{n-1, m-1\}=M$.

(b) This is a direct application of Theorem 3.6.

(c) Let $S=R_{n,m}$, $J=J_{n,m}$, and $K=I_1(YX): I_1(Y)$. Cramer's rule shows that $J\subset K$. Consequently, it suffices to show that $J_P=K_P$ for all prime ideals P which are associated to S/J. Fix such a prime ideal P. Since J is perfect, it is grade unmixed; thus P has grade m. Therefore P cannot contain $I_n(X)+I_1(Y)$, which has grade $m+1$. If $I_n(X)\not\subset P$, then J_P and K_P are both equal to S_P. If $I_1(Y)\not\subset P$, then $I_1(Y)_P=S_P$. It follows that K_P is equal to $I_1(YX)_P$, which is contained in J_P. ∎

Remarks. (a) The total complex $\mathbf{T}$ only pertains to $J_{n,m}$ for $n\leqslant m$. The ideals $J_{n,m}$ with $m \leqslant n$ have been completely resolved in [4]. There is no need to estimate their grade here; however, the argument we gave works in either case.

(b) DeConcini and Strickland [5] use the technique of algebras with straightening law to show that if R is a field, then $J_{n,m}$ is a perfect ideal of grade equal to $\max\{n, m\}$. One can use the basis for $A_{n,m}$ found in [5] in order to prove that if R is the ring of integers $\mathbf{Z}$, then $J_{n,m}$ is a perfect ideal of grade $\max\{n, m\}$, and $A_{n,m}$ is faithfully flat over $\mathbf{Z}$. Now if R is an arbitrary commutative noetherian ring, the theory of generic perfection enables us to conclude that $J_{n,m}$ is perfect of grade exactly equal to $\max\{n, m\}$. Our proof of Proposition 4.2 uses the standard technique for estimating the grade of a "determinantal-like" ideal and does not depend on the results of DeConcini and Strickland.

(c) Our proof of (c) is extracted from [7], although it is not possible to quote a precise result from that paper.

Having established the generic case, it follows by the principle of generic perfection that we only need to know that the grade of J is large enough in order to conclude exactness of $\mathbf{T}(\psi, \phi)$.

Corollary 4.3. *Retain the setting of* (3.1). *If the ideal* $J = I_1(\psi\phi) + I_n(\phi)$ *has grade at least* m, *then* $\mathbf{T}(\psi, \phi)$ *is a resolution of* R/J, *and* J *is perfect of grade exactly* m.

Next we interpret the resolution $\mathbf{T}$ and the annihilator condition (c) above in the context of residual intersection. The definitions we use are taken from Huneke and Ulrich [7].

Definition 4.4. Let R be a Cohen–Macaulay local ring, $A \subset I$ be ideals of R with $A \neq I$, and let J be the ideal $A : I$. If $\mathrm{ht}(J) \geqslant m \geqslant \mathrm{ht}(I)$, and $m \geqslant \mu(A)$, then J is called an *m-residual intersection* of I (with respect to A). If furthermore, $I_P = A_P$ for all prime ideals P of R which contain I and have height at most m, then J is a *geometric m-residual intersection* of I.

The notion of residual intersection works best when the ideal I is strongly Cohen–Macaulay and satisfies a mild condition on the local number of generators.

Definition 4.5. An ideal I is *strongly Cohen–Macaulay* (SCM) if the Koszul homology modules for any set of generators of I are Cohen–Macaulay. An ideal I satisfies the condition (G_m) if $\mu(I_P) \leqslant \mathrm{ht}(P)$ for all prime ideals P with $\mathrm{ht}(P) \leqslant m-1$. An ideal satisfies the condition (G_∞), if it satisfies (G_m) for all m.

Example 4.6. If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring and $m \geqslant n$, then $J_{n,m}$ is a geometric m-residual intersection of the ideal $I = I_1(Y)$ in the local ring $(R_{n,m})_{(m, Y, X)}$. In this case I is clearly (SCM) and (G_∞), $A = I_1(YX)$, and $J_{n,m} = A : I$ by Proposition 4.2(c). We call $J_{n,m}$ a generic m-residual intersection of the generic complete intersection I.

The following result, due to Huneke and Ulrich, gives conditions under which a residual intersection can be deformed. We use it to prove Theorem 4.8, which is the main theorem of this section: the resolution of an arbitrary geometric residual intersection of a complete intersection is given by the total complex **T**. The proof of the theorem involves deforming the residual intersection to be a generic residual intersection, applying Proposition 4.2 and Theorem 3.6 obtain the desired structure and resolution for the generic case, and finally specializing back to the original setting.

LEMMA 4.7. *Let S be a Cohen–Macaulay local ring,* **f** *be a regular S-sequence, R be the ring $S/(\mathbf{f})$, and $\theta: S \to R$ be the natural map. Suppose that I', J', and A' are ideals of S with J' the geometric m-residual intersection of I' with respect to A'; and I, J, and A are ideals of R with J the geometric m-residual intersection of I with respect to A. Suppose further that* **f** *is a regular sequence on S/I', and that I' is strongly Cohen–Macaulay and satisfies the condition (G_m). If $\theta(I') = I$ and $\theta(A') = A$, then $\theta(J') = J$.*

Proof. This is a collation of [7, Proposition 4.2(ii) and Theorem 1.5]. ∎

THEOREM 4.8. *Let (R, m) be a Cohen–Macaulay local ring and let I be a complete intersection ideal in R. If J is a geometric residual intersection of I, then the total complex* **T** *of Section 3 is a resolution of R/J by free R-modules.*

Proof. Let $y_1, ..., y_n$ be a regular sequence that generates I. Suppose that J is a geometric m-residual intersection of I with respect to an ideal $A = (z_1, ..., z_m)$. Let **y** and **z** be the matrices $[y_1, ..., y_n]$ and $[z_1, ..., z_m]$, respectively. Since $A \subset I$, there is an $n \times m$ matrix **x** with entries from R with $\mathbf{z} = \mathbf{yx}$. View **y** as a map from R^n to R and **x** as a map from R^m to R^n. We show that the total complex **T** of $\mathbf{B}(\mathbf{y}, \mathbf{x})$ is a resolution of R/J. The ideal J is an m-residual intersection of I, so by definition J is equal to $A : I$ and has grade at least m. It is apparent that $I_1(\mathbf{yx}) + I_n(\mathbf{x}) \subset J$. Once we show equality, then the proof is finished by appealing to Corollary 4.3.

Let $Y = [Y_1, ..., Y_n]$ and $X = (X_{ij})_{n \times m}$ be matrices of indeterminates. Let **f** be the sequence of elements $Y_1 - y_1, ..., Y_n - y_n, X_{11} - x_{11}, ..., X_{nm} - x_{nm}$ in the ring $R[X, Y]$, M be the maximal ideal generated by m and **f**, and S be the Cohen–Macaulay local ring $R[X, Y]_M$. Let $I' = I_1(Y)$, $A' = I_1(YX)$, and $J' = I_1(YX) + I_n(X)$ in S. It is clear that **f** is a regular sequence on S, and that R is isomorphic to $S/(\mathbf{f})$. Let $\theta: S \to R$ be the canonical map. Observe that $\theta(I')=I$, $\theta(A')=A$, and $\theta(J')=I_1(\mathbf{yx})+I_n(\mathbf{x})$. Furthermore, $A':I'=J'$ by Proposition 4.2(c) and localization at M. Since $\mathrm{ht}(I_n(X)+I')=m+1$, it is clear that $I'_P=A'_P$ for all prime ideals P of height at most m which contain I'. Thus J' is the geometric m-residual intersection of I' with respect to A'. Finally, we consider the ideal I'. Since $\mathrm{ht}(I')=n=\mathrm{ht}(I)$, we see that $\mathbf{f}$ is a regular sequence on S/I'. Obviously the complete intersection I' is strongly Cohen–Macaulay and satisfies the condition (G_∞). Hence by Lemma 4.7, $I_1(\mathbf{yx})+I_n(\mathbf{x})=\theta(J')=A:I=J$. ∎

EXAMPLE 4.9. Let R be a Gorenstein local ring; I be a grade n complete intersection ideal in R; $z_1, ..., z_n$ be a regular R-sequence in I; and J be the n-residual intersection of I with respect to $(z_1, ..., z_n)$. In this case J is linked to I by way of $(z_1, ..., z_n)$. There are two ways to obtain the R-resolution of R/J. One could use the bicomplex $\mathbf{B}$ of Section 3, or one could use the technique of linkage. The point of this example is to show that these two methods are equivalent.

Consider first the linkage interpretation. Let $\mathbf{z}$ be the matrix $[z_1, ..., z_n]$, and let $\mathbf{x}_{n\times n}$ and $\mathbf{y}_{1\times n}$ be matrices with entries from R with $I_1(\mathbf{y})=I$ and $\mathbf{z}=\mathbf{yx}$. View $\mathbf{x}$ as a map $F\to G^*$ and $\mathbf{y}$ as a map $G^*\to R$. In effect this means choosing bases for F and G^*. If $\{e_1, ..., e_n\}$ is the basis for F then we shall use the obvious orientation, namely $\bigwedge^n F=Re_1\wedge e_2\wedge\cdots\wedge e_n$. The Koszul complexes $\mathbf{K}(\mathbf{z})$ and $\mathbf{K}(\mathbf{y})$ resolve $R/(\mathbf{z})$ and R/I, respectively. Let $\alpha:\mathbf{K}(\mathbf{z})\to\mathbf{K}(\mathbf{y})$ be the complex map given by $\alpha_1=\mathbf{x}$ and $\alpha_i=\bigwedge^i(\mathbf{x})$. The first theorem of linkage [11, Proposition 2.6] states that the mapping cone of $\alpha^*:(\mathbf{K}(\mathbf{y}))^*\to(\mathbf{K}(\mathbf{z}))^*$ is a resolution of R/J. In particular, J is equal to $I_1(\mathbf{z})+I_n(\mathbf{x})$.

On the other hand, the grade of J is at least n by hypothesis. Since we now know that $J=I_1(\mathbf{z})+I_n(\mathbf{x})$, by Corollary 4.3 the total complex $\mathbf{T}(\mathbf{y},\mathbf{x})$ of the bicomplex $\mathbf{B}(\mathbf{y},\mathbf{x})$—or equivalently, and more conveniently, the total complex $\tilde{\mathbf{T}}$ of the extended bicomplex $\tilde{\mathbf{B}}$ given in the proof of Theorem 3.6—is a resolution of R/J. In this case the bicomplex $\tilde{\mathbf{B}}$ consists of two columns $\varepsilon:\mathbf{K}_0\otimes\bigwedge^n F\to\mathbf{K}(\mathbf{z})$. Since $\mathbf{K}_0=\mathbf{K}(\mathbf{y})$, and $\bigwedge^n F\simeq R$, it is not difficult to use the self-duality of the Koszul complex (Remark 1.9a) in order to see that the bicomplex $\tilde{\mathbf{B}}$ is isomorphic to the complex map α^*.

In the situation of linkage there is an identification of the canonical module $\omega_{R/J}=\mathrm{Ext}^n_R(R/J,R)$ with $I/(\mathbf{z})$. If the linkage is geometric, then in turn $I/(\mathbf{z})\simeq(I+J)/J$. Huneke and Ulrich [7] have generalized this result for residual intersections of ideals (such as complete intersections) that are in the even linkage class of ideals satisfying (SCM) and (G_∞). We shall prove a rather general and formal result that identifies the Ext-module, from which the result of [7] can be recovered whenever the two sets of hypotheses coincide.

THEOREM 4.10. *Let R be a commutative noetherian ring; $\mathbf{y}_{1\times n}$ and $\mathbf{x}_{n\times m}$ be matrices of elements of R with $n\leqslant m$; and let I, J, and A denote the ideals $I_1(\mathbf{y})$, $I_1(\mathbf{yx})+I_n(\mathbf{x})$, and $I_1(\mathbf{yx})$, respectively. Assume that grade $I\geqslant n$ and grade $J\geqslant m$. Then*

$$\operatorname{Ext}_R^m(R/J, R)=S_{m-n+1}(I/A).$$

Proof. From Corollary 4.3, we know that J is perfect and that the total complex $\mathbf{T}$ of $\mathbf{B}$ is a resolution of R/J. It follows that the dual $\mathbf{T}^*=\operatorname{Hom}_R(\mathbf{T}, R)$ of $\mathbf{T}$ is a resolution of $\operatorname{Ext}_R^m(R/J, R)$. The argument consists of two distinct parts. First, we find a presentation of I/A by free R-modules in order to express the symmetric module $S_r(I/A)$ in terms of the data $\mathbf{x}$ and $\mathbf{y}$. Then, we form the dual $\mathbf{B}^*$ of $\mathbf{B}$ in order to read off $H_0(\mathbf{T}^*)$.

Let $\phi: F\to G^*$ and $\psi: G^*\to R$ be represented by the matrices $\mathbf{x}$ and $\mathbf{y}$, respectively. Observe that the diagram

$$F\oplus \bigwedge^2 G^* \xrightarrow{[\phi,\,\partial]} G^* \xrightarrow{\psi} (I/A) \longrightarrow 0 \tag{4.11}$$

is a presentation of I/A, where ∂ is the Koszul complex map induced by ψ. (Indeed, if $\gamma\in G^*$ with $\psi(\gamma)\in A$, then $\psi(\gamma)=\psi(\phi f)$ for some $f\in F$; hence $\gamma-\phi(f)$ is in the kernel of $\psi: G^*\to I$. On the other hand, I is a complete intersection generated by the entries of the matrix $\mathbf{y}$, which represents ψ. So the kernel of ψ is the image of ∂.) It follows that

$$S(I/A)=R[T_1, ..., T_n] \bigg/ \Big(y_i T_j - y_j T_i, \sum_i x_{ij} T_i\Big).$$

In particular, the homogeneous summand $S_{m-n+1}(I/A)$ of $S(I/A)$ is equal to

$$S_{m-n+1}(G^*)/(\mathrm{Im}_1+\mathrm{Im}_2),$$

where Im_1 is the image of the composition

$$S_{m-n}G^*\otimes \bigwedge^2 G^* \xrightarrow{1\otimes\partial} S_{m-n}G^*\otimes G^* \xrightarrow{\mu} S_{m-n+1}G^*,$$

∂ is the Koszul map, and μ is (as always) the multiplication map; and where Im_2 is the image of the composition

$$S_{m-n}G^*\otimes F \xrightarrow{1\otimes\phi} S_{m-n}G^*\otimes G^* \xrightarrow{\mu} S_{m-n+1}G^*.$$

We now turn to the second part of the argument. If $m=n$, then it is clear that

$$F \oplus \bigwedge^2 G^* \xrightarrow{[\phi, \partial]} G^* \longrightarrow \operatorname{Ext}^m_R(R/J, R) \longrightarrow 0$$

is a presentation of $\operatorname{Ext}^m_R(R/J, R)$. (Use Example 4.9 if necessary.) Henceforth, we take $n<m$. We have observed that $\mathbf{T}^*$ is a resolution of $\operatorname{Ext}^m_R(R/J, R)$. But $\mathbf{T}^* = (\operatorname{Tot}(\mathbf{B}))^* = \operatorname{Tot}(\mathbf{B}^*)$; so

$$\operatorname{Ext}^m_R(R/J, R) = \left(K^{n-1}_{m-n} \otimes \bigwedge^m F\right)^* / (I_1 + I_2),$$

where

$$I_1 = \operatorname{im}(\kappa^*) = \operatorname{im}\left[\left(K^{n-2}_{m-n} \otimes \bigwedge^m F\right)^* \to \left(K^{n-1}_{m-n} \otimes \bigwedge^m F\right)^*\right]$$

with κ induced by ∂ (see Lemma 2.5), and

$$I_2 = \operatorname{im}(\rho^*) = \operatorname{im}\left[\left(K^{n-1}_{m-n-1} \otimes \bigwedge^{m-1} F\right)^* \to \left(K^{n-1}_{m-n} \otimes \bigwedge^m F\right)^*\right]$$

with ρ induced by the Eagon–Northcott map η (see Lemma 2.6). Our goal now is to identify $(K^{n-1}_{m-n} \otimes \bigwedge^m F)^*$ with $S_{m-n+1}(G^*)$ and to prove that $I_1 = \operatorname{Im}_1$ and $I_2 = \operatorname{Im}_2$ under this identification. By Remark 1.9a and Observation 1.12b, there is a commutative diagram

$$\begin{array}{ccc}
\left(K^{n-2}_{m-n} \otimes \bigwedge^m F\right)^* & \xrightarrow{\kappa^*} & \left(K^{n-1}_{m-n} \otimes \bigwedge^m F\right)^* \\
\downarrow \simeq & & \downarrow \simeq \\
\ker\left[S_{m-n+1} G^* \otimes \bigwedge^1 G^* \xrightarrow{\delta} S_{m-n+2} G^* \otimes R\right] & \xrightarrow{1 \otimes \partial} & S_{m-n+1} G^*
\end{array}$$

in which δ is the Koszul map induced by id_{G^*}, and ∂ is induced by ψ. It follows from Example 1.6 that I_1 can be identified with the image of the composition

$$S_{m-n} G^* \otimes \bigwedge^2 G^* \xrightarrow{\Delta} S_{m-n} G^* \otimes G^* \otimes G^* \xrightarrow{\mu \otimes 1} S_{m-n+1} G^* \otimes G^* \xrightarrow{1 \otimes \psi} S_{m-n+1} G^*$$

and this is evidently the same as Im_1. Use Remark 3.1a and Lemma 1.9 in order to see that I_2 is the image of

$$S_{m-n} G^* \otimes F \xrightarrow{\partial} S_{m-n+1} G^*,$$

where here ∂ is induced by $\phi\colon F \to G^*$. It is clear that ∂ is given by

$$S_{m-n}G^* \otimes F \xrightarrow{1 \otimes \phi} S_{m-n}G^* \otimes G^* \xrightarrow{\mu} S_{m-n+1}G^*;$$

so $I_2 = \mathrm{Im}_2$, as claimed. ∎

The next result is a special case of both Theorem 4.10 and [7, Theorems 2.3 and 5.3].

COROLLARY 4.12. *Let R be a Gorenstein local ring and let I be a complete intersection of height n in R. If J is a geometric m-residual intersection of I, then the canonical module of R/J is isomorphic to $S_{m-n+1}((I+J)/J)$.*

Proof. Since J is a *geometric* residual intersection of I, we may use Theorem 4.8 in order to see that Theorem 4.10 applies. The result follows because by [6, Theorem 3.1], $I \cap J = A$ in the case of geometric residual intersection. ∎

We also recover a result of Bruns. In the setting of Theorem 4.10 we assume that $R = S[\mathbf{x}, \mathbf{y}]$, where $\mathbf{x}$ and $\mathbf{y}$ are matrices of indeterminates over a Gorenstein normal domain S. DeConcini and Strickland have proved that R/J is also a normal domain. Bruns [3, Theorem 3.1] has proved that the class group of R/J is cyclic, torsion-free, and generated by the class of the prime ideal $K = (I_{n-1}(\mathbf{x}') + J)/J$, where $\mathbf{x}'$ is the submatrix of $\mathbf{x}$ consisting of the first $n-1$ columns. He [3, Theorem 4.1] also proves

COROLLARY 4.13. *Given the notation of the above paragraph, then*

$$[\omega_{R/J}] = (m-n+1)[K]$$

in the class group of R/J.

Proof. In the generic situation it follows from [7, Theorem 3.3] that $I \cap J = A$ and that the symmetric power $S_{m-n+1}((I+J)/J)$ is isomorphic to the ordinary power $(I^{m-n+1} + J)/J$. Consequently $\mathrm{Ext}_R^m(R/J, R)$ is isomorphic to

$$S_{m-n+1}(I/A) \simeq S_{m-n+1}((I+J)/J) \simeq (I^{m-n+1} + J)/J.$$

It remains to show that the ideals K and $(I+J)/J$ of R/J are isomorphic. Let $\Delta_i = (-1)^{i+1}\det(\mathbf{x}'_i)$, where $\mathbf{x}'_i$ is the matrix $\mathbf{x}'$ with row i deleted, and let $\theta\colon G^* \to K$ be given by $[\Delta_1, ..., \Delta_n]$. It is easy to see that

$$F \oplus \bigwedge^2 G^* \xrightarrow{[\phi, \partial]} G^* \xrightarrow{\theta} K \longrightarrow 0$$

is a complex. It follows from the presentation of I/A in (4.11) that there is a surjective map from the ideal $(I+J)/J$ of R/J to the non-zero ideal K. This map must be an isomorphism since R/J is a domain. ∎

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