On Univalence and *P*-Matrices

C. B. Garcia and W. I. Zangwill

Graduate School of Business
University of Chicago
Chicago, Illinois 60637

Submitted by R. Cottle

ABSTRACT

Suppose F is a differentiable mapping from a rectangle $R\subset E^n$ into E^n. Gale and Nikaido proved that if the Jacobian of F is a P-matrix in R, then F is univalent in R. Their paper has served as the basis of numerous results on univalence. Recently H. Scarf conjectured a significant extension: that the Jacobian of F need not be a P-matrix everywhere in the rectangle R, but merely on its boundary. This paper proves Scarf's conjecture, and to do so utilizes a conceptually different method of proof than that of Gale and Nikaido. The proof is presented in such a way as to demonstrate a suggestion of Scarf that orientation arguments may provide an alternative proof of the Gale-Nikaido theorem.

1. BACKGROUND

Over a dozen years ago Gale and Nikaido proved a result which is by now well known and has spawned numerous other results regarding univalence of mappings. For example, in the area of linear complementarity, coupled with the work of Samelson, Thrall, and Wesler [13], it underpinned the result that the linear complementarity problem has a unique solution if and only if its matrix is a P-matrix. It led to the notion of a P-function [11]. It is used in [9] to show uniqueness of solutions to nonlinear complementarity problems, building on the earlier work of Cottle [3] and Karamardian [6]. Overall, the Gale-Nikaido work provided fundamental results on when a mapping is univalent.

To be more precise, let $F:R\subset E^n\to E^n$, with R a rectangle and E^n the Euclidean n-space, be a differentiable mapping with Jacobian $F'(x)$. For any $K\subset\{1,2,\dots,n\}$, consider the submatrix remaining from F' after deleting the rows i and columns i for $i\in K$. The determinant of that submatrix forms a principal minor of F', and we denote it by $M(x)_K$. If $K=\{1,2,\cdots,n\}$, then

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all elements are deleted, so we define $M(x)_K = 1$. However, if $K = \varnothing$, observe

$$M(x)_{\varnothing} = \det F'(x), \tag{1.1}$$

where $\det F'(x)$ is the determinant of $F'(x)$.

Gale and Nikaido proved that F is univalent (i.e., injective, one-to-one) if F' is a P-matrix. The Jacobian F' is a P-matrix if all principal minors are positive, that is, if for all $x \in R$ and $K \subset \{1,2,\cdots,n\}$,

$$M(x)_K > 0.$$

Recently, H. Scarf conjectured an interesting strengthening of the Gale-Nikaido result [14]. He suggested that univalence does not require all principal minors to be positive. On the interior of the rectangle R, only $\det F'(x) > 0$ is needed. On the boundary only certain principal minors need be positive. In the next section of the paper we prove Scarf's perceptive conjecture.

2. VERIFYING SCARF'S CONJECTURE

To prove the conjecture for $x = (x_i) \in E^n$, let

$$R = \{x | a_i \leqslant x_i \leqslant b_i,\ i = 1,\cdots,n\}$$

for $a_i < b_i$, a_i, b_i finite. With the rectangle defined let

$$I(x) = \{i | x_i = a_i \text{ or } x_i = b_i\}$$

for $x \in R$.

Suppose we have a continuously differentiable $F: R \to E^n$, with Jacobian $F'(x)$ and principal minors of the Jacobian $M(x)_K$.

Scarf conjectured that if $M(x)_K > 0$ for all $K \subset I(x)$, then F is univalent in R. Observe that this property and (1.1) insure that on all of R, $\det F'(x) > 0$. However, if $I(x) = \varnothing$, no proper principal minors need be positive. Only at the vertices of R—that is, where $I(x) = \{1,\ldots,n\}$—must *all* principal minors be positive.

A two-dimensional example where the property holds is the function $F: [-1,1] \times [-1,1] \to E^2$, defined by

$$F_1(x_1, x_2) = \tfrac{1}{3}x_1^3 + x_1\left(x_2^2 - \tfrac{1}{2}\right) - x_2,$$

$$F_2(x_1, x_2) = x_1 + x_2.$$

Observe that F' is not a P-matrix on the square $[-1,1]\times[-1,1]$, but that $M(x)_K>0$ for all $K\subset I(x)$. Hence, the Scarf conjecture would assert the univalence of F on the square. In fact, as we shall prove in Sec. 3, F is univalent in E^2.

In this paper, we shall prove univalence using a slightly weaker assumption on the principal minors. We say that F has the *S-property* if:

$$M(x)_K \prod_{i\in K} M(x)_i > 0 \tag{2.1}$$

for all $x\in R$ and all $K\subset I(x)$. The S-property permits certain principal minors to be negative.

The proof of the conjecture depends on gently deforming F into a new mapping G which is very similar to F. Then G will be shown one-to-one. Hence F, being almost the same as G, will be also, thereby validating the conjecture. On R, the mapping G will be identically F except near the boundary ∂R. Near ∂R, G will be norm coercive, which means that as x approaches ∂R, $\|G(x)\|\to\infty$, where $\|\cdot\|$ is the Euclidean norm. Also we will insure, letting G' be the Jacobian, that $\det G'(x)>0$ on $\dot{R}$, the interior of R. But now the norm-coerciveness theorem will apply [12, p. 136].

THEOREM 1 (Norm-Coerciveness theorem). *Let $G:\dot{R}\to E^n$ be continuously differentiable and norm coercive, and suppose $\det G'(x)>0$ on $\dot{R}$. Then G is one-to-one and onto (bijective).*

Consequently with G univalent, F will also be univalent.

The actual construction of G is straightforward. Given δ with

$$0<\delta<\min_i\left\{\frac{b_i-a_i}{2}\right\},$$

define a continuously differentiable function of one variable, $D_i:(a_i,b_i)\to E^1$, with derivative D_i' as follows:

$$\begin{array}{lll} \text{If} & a_i+\delta\leqslant x_i\leqslant b_i-\delta, & D_i(x_i)=0. \\ \text{If} & a_i<x_i<a_i+\delta \;\text{ or }\; b_i-\delta<x_i<b_i, & D_i'(x_i)>0. \\ \text{As} & x_i\downarrow a_i \;\text{ or }\; x_i\uparrow b_i, & |D_i(x_i)|\to\infty. \end{array} \tag{2.2}$$

For one example of such a function let

$$D_i(x_i) = -\left(\max\left\{0, \frac{1}{x_i - a_i} - \frac{1}{\delta}\right\}\right)^2 + \left(\max\left\{0, \frac{1}{b_i - x_i} - \frac{1}{\delta}\right\}\right)^2.$$

Then define $G = (G_i)$ as

$$G_i(x) = F_i(x) + D_i(x_i)\operatorname{sgn} M(x)_i, \tag{2.3}$$

where

$$\operatorname{sgn} a = \begin{cases} +1, & a > 0, \\ -1, & a < 0, \end{cases} \quad \text{for} \quad a \neq 0.$$

Clearly $G(x) = F(x)$ except when x is within δ of ∂R, which permits us to define G another way. Note that G is continuously differentiable for small $\delta > 0$, since the S-property insures that $M(x)_i \neq 0$ for all $i \in I(x)$. Let

$$I_\delta(x) = \{i | a_i \leqslant x_i \leqslant a_i + \delta \text{ or } b_i - \delta \leqslant x_i \leqslant b_i\}. \tag{2.4}$$

Then

$$G_i(x) = \begin{cases} F_i(x) + D_i(x_i)\operatorname{sgn} M(x)_i & i \in I_\delta(x) \\ F_i(x) & i \notin I_\delta(x) \end{cases}. \tag{2.5}$$

Further observe that F is bounded, since F is continuous on the compact set R. Therefore, since $|D_i(x_i)| \to \infty$ for some i as x approaches ∂R, G is norm coercive.

With G thus defined we can proceed with the details of the proof and start with a lemma about determinants.

LEMMA 1. *Let D_I be an $n \times n$ diagonal matrix with nonzero diagonal entries d_i only in positions $i \in I$. Let P be an arbitrary $n \times n$ matrix. Then,*

$$\det(P + D_I) = \sum_{K \subset I} \left(\prod_{k \in K} d_k\right) M_K,$$

where the M_K's are the principal minors of P.

As this formula may be a bit confusing, here is an example for $n=3$ and $I=\{1,2,3\}$:

$$\det(P+D_I) = \det P + d_1M_1 + d_2M_2 + d_3M_3 + d_1d_2M_{12} + d_1d_3M_{13} + d_2d_3M_{23} + d_1d_2d_3.$$

The proof of the lemma follows easily from the formula for the determinant of the sum of two matrices and by noting that the ith column of D_I is zero except possibly in the ith position.

We may now prove the Scarf conjecture.

THEOREM 2. *Let $F:R\to E^n$ be continuously differentiable on the rectangle R, and suppose the S-property holds. Then F is univalent on R.*

Proof. Assume there exist two distinct points $\bar{x}$ and $\bar{y}$ in R such that $F(\bar{x})=F(\bar{y})$, and seek a contradiction. Without loss of generality, we may assume both points are in $\dot{R}$. (If not, we may always perturb F slightly so that they are, as shown in the Appendix.) Let δ_1 be the distance to the nearest boundary point from either $\bar{x}$ or $\bar{y}$. Thus

$$\delta_1 = \min\{\|x-z\| \mid z\in\partial R,\ x=\bar{x} \text{ or } \bar{y}\}.$$

The S-property holds on any face of R. By continuous differentiability it will also hold near any face. But $I_\delta(x)$ specified which faces of R a point x is near. Thus, via compactness, a $\delta_2>0$ exists such that

$$M(x)_K \prod_{i\in K} M(x)_i > 0 \qquad \text{for all} \quad K\subset I_{\delta_2}(x). \tag{2.6}$$

Let $\delta=\frac{1}{2}\min\{\delta_1,\delta_2\}$, and consider the function G defined in (2.3). Note that $G(\bar{x})=G(\bar{y})$, since from (2.2), $G(x)=F(x)$ for all x at least a distance δ from the boundary. Consequently we need merely prove G satisfies the hypothesis of Theorem 1, as that will force G to be one-to-one thereby providing the contradiction.

We already know G is norm coercive; thus we must verify that $\det G'(x) >0$ on $\dot{R}$. This obviously holds if the distance of x from ∂R is at least δ, for there $F(x)=G(x)$.

Suppose x is within δ of ∂R. Then examining (2.5) we see

$$\det G'(x) = \det\left[F'(x)+D_{I_\delta(x)}\right], \tag{2.7}$$

where $D_{I_\delta(x)}$ is a diagonal matrix with diagonal entries $D_i'(x_i)\operatorname{sgn} M(x)_i$ for those $i \in I_\delta(x)$. From (2.2), $D_i'(x_i) > 0$. Also, since $\delta < \delta_2$, from (2.6) we know that $M(x)_K \prod_{i \in K} M(x)_i > 0$ for all $K \subset I_\delta(x)$. But then applying Lemma 1 we see that all terms in the expansion of (2.7) are positive. Thus $\det G'(x) > 0$ for x within δ of ∂R.

We therefore see that $\det G'(x) > 0$ on $\dot{R}$. Consequently all conditions of the norm-coerciveness theorem hold, so that G is one-to-one. ■

3. THE S-PROPERTY ON E^n

The S-property can be easily extended to the entire space E^n by simple changes in the definitions. Again we are interested in a rectangle $R = \{x = (x_i) | a_i \leq x_i \leq b_i\}$, $a_i < b_i$, a_i, b_i finite, but now we are more concerned about what happens outside of it. Consequently given R, define

$$J_R(x) = \{i | x_i \leq a_i \text{ or } b_i \leq x_i\}.$$

We say that $F: E^n \to E^n$, continuously differentiable, has the S-property on E^n if there exists a bounded rectangle R such that

$$M(x)_K \prod_{i \in K} M(x)_i > 0 \qquad \text{for all} \quad K \subset J_R(x). \tag{3.1}$$

Whereas before we required the S-property to hold on the boundary of R, now we want the property to hold outside of R. Note that the rectangle R can be taken arbitrarily large, although bounded. Thus if $\det F'(x) > 0$ on E^n, we need only concern ourselves about the principal minors very far out.

To prove univalence of F we again employ the norm coerciveness theorem and Lemma 1, but utilize somewhat different modifications of F. In particular, boundedness of F is of concern since we are on E^n, but the following lemma applies.

LEMMA 3. *Let $F = (F_i): E^n \to E^n$ be continuously differentiable, and define a mapping $H = (H_i)$ by*

$$H_i = \frac{F_i}{\left[1 + (F_i)^2\right]^{1/2}}.$$

Then F is univalent on E^n if and only if H is. F satisfies the S-property on E^n if and only if H does. Furthermore, H is bounded.

Proof. Clearly H is bounded and H is one-to-one if and only if F is one-to-one.

As for the S-property, let $\nabla H_i(x)$ be a row of the Jacobian matrix for H, and similarly let ∇F_i be a row of the Jacobian of F. Then

$$\nabla H_i = \frac{1}{\left[1+(F_i)^2\right]^{3/2}} \nabla F_i.$$

Each row of H' is precisely a positive multiple of a row of F'. Therefore a principal minor of H' is a positive multiple of the corresponding principal minor of F'. ∎

To prove univalence under the S-property on E^n, we may without loss of generality assume F bounded. If it is not, by Lemma 3 we can transform it into a bounded function with the same S-property holding.

With F bounded let us now create a function G on which we can apply the norm coerciveness theorem. For this define $B_i(x_i): E^1 \to E^1$ with continuous derivative $B_i'(x_i)$ by

$$B_i(x_i) = 0 \qquad \text{if} \quad c_i \leqslant x_i \leqslant d_i,$$

$$B_i'(x_i) > 0 \qquad \text{if} \quad x_i < c_i \ \text{ or } \ x_i > d_i, \tag{3.2}$$

and

$$|B_i(x_i)| \to \infty \qquad \text{if} \quad |x_i| \to \infty.$$

One example would be

$$B_i(x_i) = (\max\{0, x_i - d_i\})^2 - (\max\{0, c_i - x_i\})^2.$$

Now define $G = (G_i)$ by

$$G_i(x) = F_i(x) + B_i(x_i)\operatorname{sgn} M(x)_i. \tag{3.3}$$

As F is bounded but $|B_i(x_i)| \to \infty$ as $|x_i| \to \infty$, we see that $\|G(x)\| \to \infty$ as $\|x\| \to \infty$. Thus G is norm coercive. Also, defining a rectangle

$$T = \{x | c_i \leqslant x_i \leqslant d_i\}, \tag{3.4}$$

note that $G(x) = F(x)$ for $x \in T$.

The proof of the univalence is now at hand.

THEOREM 3. *Let $F: E^n \to E^n$ be continuously differentiable, and let F satisfy the S-property on E^n for some bounded rectangle R. Then F is univalent.*

Proof. Assume that $F(\bar{x}) = F(\bar{y})$ for some $\bar{x}, \bar{y} \in E^n$. Also, without loss of generality, let F be bounded. Define the rectangle T in (3.4) so large that both $\bar{x}$ and $\bar{y}$ are in T, and so that $R \subset T$; that is,

$$c_i \leqslant a_i \quad \text{and} \quad d_i \geqslant b_i. \tag{3.5}$$

Taking the G of (3.3), $F = G$ on T, so that $G(\bar{x}) = G(\bar{y})$. Moreover, G is norm coercive. Thus we need merely prove that on E^n, $\det G'(x) > 0$. But

$$\det G'(x) = \det\left[F'(x) + B_{J_T(x)}\right], \tag{3.6}$$

where $B_{J_T(x)}$ is a diagonal matrix with components $B_i'(x_i)\operatorname{sgn} M(x)_i$ in the diagonal position $i \in J_T(x)$ and $J_T(x) = \{i \mid x_i \leqslant c_i \text{ or } x_i \geqslant d_i\}$.

Further, by (3.5)

$$J_T(x) \subset J_R(x). \tag{3.7}$$

If we expand (3.6) by Lemma 1, it is again true that all terms are positive.

The mapping G therefore satisfies the conditions of the norm-coerciveness theorem and must be one-to-one, a contradiction. ■

4. DISCUSSION

As mentioned earlier, our goal is to demonstrate that orientation arguments "may provide an alternative proof of the Gale-Nikaido theorem" (see a foreword by H. Scarf in [7, p. 10]). This objective is achieved via the norm-coerciveness theorem, which, though it is often verified by other means, can also be demonstrated via orientation arguments. Such arguments lead to constructive proofs commonly found in the theory of computing fixed points. To be more precise, given $G: \dot{R} \to E^n$, we consider the homotopy equation $H: R \times [0,1] \to E^n$ defined by

$$H(x,t) = G(x) - \left[tG(x^0) + (1-t)y\right] = 0 \tag{4.1}$$

for an arbitrary $x^0 \in \dot{R}$. Then, if G is norm coercive and twice continuously differentiable, and $\det G'(x) > 0$ for all $x \in \dot{R}$, by using orientation arguments one could show that, under a regularity condition on the mapping H, the solution to (4.1) is a path $(x(\theta), t(\theta))$, $(x(1), t(1)) = (x^0, 1)$, which is monotonic in t. This path may be constructively generated by the complementary pivot scheme described in [5]. Because of the norm-coercivity of G, all values of t must be taken in the path $(x(\theta), t(\theta))$. One is therefore assured of finding the unique solution $(x(0), t(0)) = (x^*, 0)$ of $G(x) = y$.

Recall that G is a deformation of the given map F which satisfies the S-property. In fact, we have a sequence of functions $\{G^{\delta_i}\}$ defined by (2.5) for a sequence $\{\delta_i\}$ with $\delta_i \to 0$. For each δ_i, the equation (4.1) will furnish an x^{δ_i} satisfying

$$G(x^{\delta_i}) = y.$$

If $F(x) = y$ has a unique solution x^*, then the sequence $\{x^{\delta_i}\}$ will converge to x^* as $\delta_i \to 0$. Otherwise, if $F(x) = y$ has no solution, then the sequence will approach the boundary of the rectangle R as $\delta_i \to 0$.

If $F: E^n_+ \to E^n$ is a continuously differentiable function such that $F'(x)$ has positive principal minors for all $x \geqslant 0$, then finding a solution to $F(x) = 0$ can be done by way of a nonlinear complementarity algorithm. Since $F'(x)$ is a P-matrix for all $x \geqslant 0$, from [11] we have

$$\max_i (x_i - y_i)[F_i(x) - F_i(y)] > 0$$

for all $x \geqslant 0$, $y \geqslant 0$ (i.e., F is a P-function), so that there can be at most one solution to the complementarity problem

$$x \geqslant 0, \qquad F(x) \geqslant 0, \qquad x^tF(x) = 0, \tag{4.2}$$

where t denotes matric transposition. Hence, if $F(x) = 0$ has a solution, it will be the unique solution to (4.2). Otherwise the solution to (4.2), if it exists, will be such that $F(x) \neq 0$. [Of course nonlinear complementarity algorithms may require further restrictions on F to generate an approximate solution to (4.2).]

Finally, it should be noted that Theorem 3 may be used for proving the existence and uniqueness of solutions to (4.2). We associate with F the mapping $G: E^n \to E^n$ defined by

$$G(z) = z^- - F(z^+), \qquad z \in E^n,$$

where z^+, z^- are such that

$$z = z^+ - z^-, \qquad z^+ \geqslant 0, \qquad z^- \geqslant 0,$$

$$z_i^+ z_i^- = 0 \qquad \text{for all } i.$$

For continuously differentiable F, G is continuous. Clearly, we can make G continuously differentiable by smoothing out the function G at the points where $z_i = 0$ for some i. It is immediate that

$$G(z) = 0 \quad \text{iff} \quad x = z^+ \text{ solves } (4.2).$$

Thus, if G is a continuously differentiable function for which Theorem 3 holds, then (4.2) will have a unique solution. (See [9] for related discussions on this subject.)

5. ADDENDUM

After proving our result we were informed by D. Gale and H. Scarf that A. Mas-Colell has independently and simultaneously verified Scarf's conjecture. Mas-Colell utilized the Poincaré index theorem of differential topology (rather than the more elementary norm-coerciveness approach) to obtain results relating to and including the Scarf conjecture.

The Associate Editor has also informed us of three related papers. Charnes, Raike, and Stutz [1] extend the P-matrix characterizations of Gale and Nikaido. Chua and Lam [2] obtain a slightly different version of the norm-coerciveness theorem that we quoted; their result permits $\det F'$ to be zero at isolated points and positive elsewhere. Kestelman [8] demonstrated that if F is univalent when restricted to ∂R, then F is univalent in R. Scarf's conjecture relates somewhat to Kestelman's result, but specifically provides conditions for the Jacobian matrix in ∂R that guarantee univalence. Kestleman's result, however, is very interesting and may lead to an alternative proof for univalence by detailed examination of F along the boundary ∂R.

6. APPENDIX

Here we show that the two points $\bar{x}$ and $\bar{y}$ selected in Thoerem 2 may be assumed to be in $\dot{R}$ without any loss of generality.

LEMMA. *Let $F: R \to E^n$ be continuously differentiable and satisfy the S-property on the bounded rectangle R. Suppose $F(x^1) = F(x^2)$ for two distinct points x^1 and x^2 in R, not both in $\dot{R}$. Then there exists a continuously differentiable function $\bar{F}$ which satisfies the S-property, where $\bar{F}(y^1) = \bar{F}(y^2)$ for two distinct points y^1 and y^2 in $\dot{R}$.*

Proof. Let x^1 and x^2 be the two distinct points of the hypothesis. We may assume that $x_1^1 \neq x_1^2$. Let y^1 be very close to x^1, and y^2 be very close to x^2, but $y^1, y^2 \in \dot{R}$. (If x^1 or x^2 is already interior, then let $y^1 = x^1$ or $y^2 = x^2$.) Given y^1 and y^2, define

$$\bar{F}_i(x) = F_i(x) - \frac{b_i^1(x_1 - y_1^2)}{y_1^1 - y_1^2} - \frac{b_i^2(x_1 - y_1^1)}{y_1^2 - y_1^1}, \qquad x \in R,$$

where

$$F(y^1) = b^1 \quad \text{and} \quad F(y^2) = b^2. \tag{A.1}$$

Clearly $\bar{F}$ is continuously differentiable, and

$$\frac{\partial \bar{F}_i(x)}{\partial x_j} = \begin{cases} \dfrac{\partial F_i(x)}{\partial x_j} & j \neq 1 \\[2ex] \dfrac{\partial F_i(x)}{\partial x_1} + \dfrac{b_i^1 - b_i^2}{y_1^2 - y_1^1} & j = 1. \end{cases}$$

By hypothesis, $F(x^1) = F(x^2)$. Then from (A.1) by selecting y^1 sufficiently close to x^1, and y^2 sufficiently close to x^2, we can make the term

$$\frac{b_i^1 - b_i^2}{y_1^2 - y_1^1} \tag{A.2}$$

arbitrarily close to zero.

Let $(M^F(x))_K$ and $(M^{\bar{F}}(x))_K$ denote the principal minors of the Jacobians of F and $\bar{F}$ respectively. As F is continuously differentiable and R compact, we may select the term in (A.2) to be so close to zero that if $(M^F(x))_K > 0$ (<0) on a compact subset of R, then $(M^{\bar{F}}(x))_K > 0$ (<0) on that subset also. But then the S-property must hold for $\bar{F}$. Further, there are two distinct points y^1 and y^2 in $\dot{R}$ with $\bar{F}(y^1) = \bar{F}(y^2)$. Also $\bar{F}$ is continuously differentiable. ∎

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