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TRIANGLE-FREE PARTIAL GRAPHS AND EDGE COVERING THEOREMS

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In Section 1 some lower bounds are given for the maximal number of edges of a $(p-1)$-colorable partial graph. Among others we show that a graph on n vertices with m edges has a $(p-1)$-colorable partial graph with at least $mT_{n,p}/\binom{n}{2}$ edges, where $T_{n,p}$ denotes the so called Turán number. These results are used to obtain upper bounds for special edge covering numbers of graphs. In Section 2 we prove the following theorem: If G is a simple graph and μ is the maximal cardinality of a triangle-free edge set of G, then the edges of G can be covered by μ triangles and edges. In Section 3 related questions are examined.

Preliminaries

We will consider only loopless graphs without multiple edges. If $G=(X, E)$ is a graph, then the edge set $F\subset E$ together with the spanned vertices define a *partial graph* of G which will be denoted by the same letter F for simplicity reasons.

K_p stands for a *p-clique* (complete graph on p vertices); K_3 will be called a *triangle*.

The graph G will be called *F-free* if G has no partial graph isomorphic to the graph F.

The *copies* of a given graph F are graphs isomorphic to F.

$\gamma(F)$ denotes the chromatic number of the graph F.

A *k-partition* of the graph $G=(X, E)$ is a partition of X into k classes; a k-partition is said to be *almost equipartite* if each of its classes contains $\lfloor |X|/k \rfloor$ or $\lfloor |X|/k \rfloor+1$ vertices.

1. K_p-free partial graphs

Turán in [1] determined $T_{n,p}$, the maximal number of edges in a K_p-free graph on n vertices. (The value $T_{n,p}$ is about $(p-2)n^2/2(p-1)$.) On the other hand it is known (remarked by Erdös) that every graph with m edges contains a cut of more than $\frac{1}{2}m$ edges. We will prove an extension of this statement giving a sharp lower bound for the number of edges in a maximal K_p-free partial graph of an arbitrary graph.

Suppose that every edge $e \in E$ of the graph G has a real weight $w(e)$ and put

$$W(E') = \sum_{e \in E'} w(e) \quad \text{where } E' \subset E.$$

1.1. Theorem. *For all $p \geq 3$ every graph on n vertices has a $(p-1)$-colorable partial graph with edge set E' satisfying*

$$W(E') \geq W(E) T_{n,p} \Big/ \binom{n}{2}.$$

Proof. The nonedges of G will be considered as weighted edges with weight 0. Then every almost equipartite $(p-1)$-partition P of the vertices of G contains exactly $T_{n,p}$ weighted edges joining vertices of different classes in P. Let t be the number of all almost equipartite $(p-1)$-partitions and let E_i be the set of weighted edges between different classes in the ith partition $(i = 1, 2, \ldots, t)$.

Obviously $S = S(e) = |\{i : e \in E_i\}|$ does not depend on the choice of e, therefore

$$t \cdot T_{n,p} = \sum_{i=1}^{t} |E_i| = \binom{n}{2} \cdot S.$$

On the other hand:

$$\sum_{i=1}^{t} W(E_i \cap E) = \sum_{i=1}^{t} W(E_i) = \sum_{i=1}^{t} \sum_{e \in E_i} w(e)$$

$$= \sum_{\substack{(e,i) \\ e \in E_i}} w(e) = S \sum_{e \in \bigcup_{i=1}^{t} E_i} w(e) = S \sum_{e \in E} w(e) = S \cdot W(E).$$

From this:

$$\left\{\sum_{i=1}^{t} w(E_i \cap E)\right\} \Big/ t = W(E) \cdot S/t = W(E) T_{n,p} \Big/ \binom{n}{2}$$

This implies that for some j $(1 \leq j \leq t)$

$$W(E_j \cap E) \geq W(E) T_{n,p} \Big/ \binom{n}{2} \text{ holds.} \quad \square$$

1.2. Corollary. *Every graph of n vertices and m edges contains a K_p-free partial graph with at least $mT_{n,p}/\binom{n}{2}$ edges.*

Proof. Let $w(e) = 1$ for each $e \in E$. In this case $W(E') = |E'|$ thus Theorem 1.1 gives us a $(p-1)$-colorable partial graph G' with at least $mT_{n,p}/\binom{n}{2}$ edges. G' cannot contain the p-chromatic graph K_p. $\square$

1.3. Corollary. *A graph on n vertices, with m edges contains a bipartite partial graph with at least $m(n+1)/2n$ edges (which is more than $\frac{1}{2}m$).*

Proof.

$$T_{n,3}\Big/\binom{n}{2}=\left[\frac{n^2}{4}\right]\Big/\binom{n}{2}\geq\frac{n+1}{2n}. \qquad \square$$

1.4. Corollary. *A graph on n vertices with m edges contains a K_p-free partial graph with more than $m(p-2)/(p-1)$ edges.*

Proof. Suppose $n=(p-1)k+r$ where $0\leq r\leq p-2$ and $n\geq 2$. Then

$$\begin{aligned}T_{n,p}&=\frac{p-2}{2(p-1)}\cdot(n^2-r^2)+\binom{r}{2}\\&=(n^2-n)\frac{p-2}{2(p-1)}+(n-r^2)\cdot\frac{p-2}{2(p-1)}+\binom{r}{2}\\&=\binom{n}{2}\cdot\frac{p-2}{p-1}+\frac{(n-r)(p-2)+r^2-r}{2(p-1)}>\binom{n}{2}\frac{p-2}{p-1}. \qquad \square\end{aligned}$$

In Section 3 we will use the following strengthening of Corollary 1.3.

1.5. Proposition. *A graph $G=(X,E)$ with $2k+1$ edges $(k\geq 2)$ has a bipartite partial graph with at least $k+2$ edges.*

Proof. Corollary 1.3 gives us a bipartite partial graph $G=(X',E')$ with $|E'|=k+1$. If none of the remaining k edges can be added to E', then each of them completes an odd circuit with some edges of G', thus $X=X'$. Since $|E'|=k+1$, $|X'|\geq 2k+1$ would imply that G' has at most two adjacent edges, therefore in this case we could not obtain $k\geq 2$ odd circuits. Consequently $|X|\leq 2k$ and Corollary 1.3 states that G contains a bipartite partial graph with at least

$$\begin{aligned}|E|\cdot\frac{|X|+1}{2|X|}&=\frac{2k+1}{2}\left(1+\frac{1}{|X|}\right)\geq\frac{2k+1}{2}\left(1+\frac{1}{2k}\right)\\&=\frac{(2k+1)^2}{4k}>k+1 \quad \text{edges.} \qquad \square\end{aligned}$$

1.6. Remark. One can see that Theorem 1.1 and Corollary 1.2 are true in a stronger form when $T_{n,p}/\binom{n}{2}$ is replaced by $T_{\gamma(G),p}/\binom{\gamma(G)}{2}$. This fact with the obvious $\binom{\gamma(G)}{2}\leq m$ results in the following sharpening of Corollary 1.3.

A graph of m edges contains a triangle-free partial graph with at least $\frac{1}{2}m+(\sqrt{8m+1}-1)/8$ edges.

2. Edge covering by triangles

For a simple graph $G=(X,E)$ we define the following values:

Δ is the minimum number of cliques with size ≤ 3 which cover E;

μ is the maximum cardinality of a triangle-free edge set of G;

μ_1 is the maximum cardinality of the subsets of E which contain at most one edge of the triangles of G. (We remark that none of the spanned triangle-free subgraphs of G can contain more than μ_1 edges of G.) The following result was motivated by a question of Erdös.

2.1. Theorem. $\mu_1 \leqslant \Delta \leqslant \mu$.

Proof. (a) If $M_1 \subset E$ is a set of cardinality μ_1 containing at most one edge of the triangles of G, then we need at least μ_1 2- and 3-cliques to cover its edges: $\mu_1 \leqslant \Delta$.

(b) Let $M \subset E$ be a maximal traingle-free set ($|M| = \mu$). Consider the edges $e \in E \setminus M$. We define $\Gamma(e) \subset M$ as the set of edges e' for which there exists a triangle of G containing both e and e'. We will prove the existence of an injection $g : E \setminus M \to M$ satisfying the property: $g(e) \in \Gamma(e)$ holds for each $e \in E \setminus M$. In this case the $|E \setminus M|$ triangles containing e and $g(e)$ cover all the edges of G with the exception of at most $|M| - |E \setminus M|$ edges of M (because the $g(e)$'s are different). Obviously a covering of E has been gained with at most $|M|$ edges and triangles which implies $\Delta \leqslant \mu$.

If $|\bigcup_{e \in K} \Gamma(e)| \geqslant |K|$ holds for each $K \subset E \setminus M$ than the well-known König–Hall Theorem [2] gives us the injection g.

Suppose contrarily that for some $K \subset E \setminus M$ we have $|\bigcup_{e \in K} \Gamma(e)| < |K|$. Put $M' = \bigcup_{e \in K} \Gamma(e)$. Choose a maximal triangle-free edge set M^* in the partial graph defined by $K \cup M'$.

$$|M^*| > \tfrac{1}{2}|K \cup M'| = \tfrac{1}{2}(|K| + |M'|) > |M'|$$

because of Corollary 1.3 and our assumption on $|K|$.

On the other hand M^* and $M \setminus M'$ are disjoint subsets of $M \cup K$, therefore

$$|(M \setminus M') \cup M^*| = |M| - |M'| + |M^*| \geqslant |M|.$$

Because of the maximality of M the set $(M \setminus M') \cup M^*$ contains a triangle of G. M is triangle-free thus for an edge e of this triangle $e \in K$. Consequently all the edges of the triangle are contained by $K \cup \Gamma(e)$, that is M^* contains this triangle— a contradiction. □

Graphs with $\Delta = \mu$ will be characterized in Theorem 3.3.

2.2. Remark. It seems to be true that one can find edge-disjoint covering of E as well by μ edges and triangles. (We have proved it for planar graphs in a more general form.) This result would be a generalization of a theorem of Erdös–Goodman–Pósa [3] which states that the edges of a graph on n vertices can be covered by at most $[\frac{1}{4}n^2]$ pairwise disjoint 2- and 3-cliques.

3. General edge coverings

Like covering by triangles, we propose the problem of edge coverings by copies of a given graph. From here F can be arbitrary (directed or undirected) but fixed graph. For a graph $G=(X,E)$:

Δ_F denotes the minimum number of copies of F and edges covering E;

μ_F is the maximum number of edges in the F-free partial graphs of G. If G is a graph on n vertices and F is bipartite then $\mu_F=o(n^2)$ holds [4]. Because most of the graphs on n vertices have cn^2 edges, one can not expect a good relation between Δ_F and μ_F. Therefore we suppose that the chromatic number of F is not less than 3.

3.1. Theorem. *$\Delta_F \leqslant \mu_F$ holds for every graph F with $\gamma(F)\geqslant 3$.*

To obtain a proof one can amuse himself replacing the word "triangle" by "F" and the number μ by μ_F in the proof of Theorem 2.1.

If F is a ditrangle (a directed circuit of length 3), then from Theorem 3.1 we can get a result analogous to Theorem 2.1.

3.2. Corollary. *If $G=(X,\mathcal{U})$ is a directed graph and μ' is the maximal cardinality of a ditriangle-free arc set of G, then $\mathcal{U}$ can be covered by μ' arcs and ditriangles.*

A question arises in a natural way: is our upper bound for Δ_F the best possible? A slight modification of the proof of Theorem 3.1 gives:

3.3. Theorem. *$\Delta_F=\mu_F$ holds if and only if the graph G is F-free* $(\gamma(F)\geqslant 3)$.

Proof. (a) If $G=(X,E)$ is F-free, then $\mu_F=|E|=\Delta_F$.

(b) Let $M\subset E$ be a maximal F-free partial graph ($|M|=\mu_F$). One supposes that $E\setminus M\neq\emptyset$; then each edge of $E\setminus M$ forms an F with some edges of M (M is maximal F-free).

Let $e_0\in E\setminus M$ and $e_0\in F_0$ where F_0 is a copy of F and $F_0\setminus M=\{e_0\}$. Let e_1 and e_2 be two other edges of F_0.

For $e\in(E\setminus M)\setminus\{e_0\}$ let $\Gamma(e)=\{F'\cap(M\setminus\{e_1,e_2\}): e\in F', F' \text{ is a copy of } F\}$. We will verify the Hall condition between $(E\setminus M)\setminus\{e_0\}$ and $M\setminus\{e_1,e_2\}$ as well as we did it in the proof of Theorem 2.1 between $E\setminus M$ and M.

Suppose contrarily that there exists a set $K\subset(E\setminus M)\setminus\{e_0\}$ with property: $|K|=q+1$ and for the set $M'=\bigcup_{e\in K}\Gamma(e)$, $|M'|=q$ ($q\geqslant 1$ is clear). It follows from Proposition 1.5 that one can choose at least $q+3$ edges from the $2q+3$ ones of $K\cup M'\cup\{e_1,e_2\}$ which form a bipartite consequently F-free partial graph M^*. Put

$$N=((M\setminus\{e_1,e_2\})\setminus M')\cup M^*.$$

Applying the observations that $|M^*|\geqslant q+3$ and $((M\setminus\{e_1,e_2\})\setminus M')\cap M^*=\emptyset$ we obtain:

$$|N|=|M|-|\{e_1,e_2\}|-|M'|+|M^*|\geqslant|M|-2-q+q+3=|M|+1$$

therefore N contains a copy F_1 of F by the maximality of M. Thus there is an edge $e\in F_1\cap K$, consequently $F_1\subset K\cup\Gamma(e)$, which means that $F_1\subset M^*$—contradiction.

The Hall condition is true and König–Hall Theorem gives now the existence of an injection $g:((E\setminus M)\setminus\{e_0\})\to(M\setminus\{e_1,e_2\})$ satisfying that $g(e)\in\Gamma(e)$ holds for each $e\in(E\setminus M)\setminus\{e_0\}$. This injection guarantees that at most $|M\setminus\{e_1,e_2\}|=\mu_F-2$ copies of F and edges from M cover all edges of G except the edges e_0, e_1, e_2 which are covered by F_0. Consequently $\Delta_F\leqslant\mu_F-1$. □

3.4. Theorem. *Let F be K_p, the p-clique ($p\geqslant 3$ and $|F|=\binom{p}{2}$). If G contains a p-clique, then $\Delta_F\leqslant\mu_F-|F|+2$.*

Proof. For $p=3$ our assertion is given by Theorem 3.3. Let $p\geqslant 4$ and $M\subset E$ be a maximal K_p-free set ($|M|=\mu_F$). Let $e_0\in E\setminus M$ and C_0 be a p-clique of G for which $C_0\setminus M=\{e_0\}$. We will prove the existence of an injection $g:(E\setminus M)\setminus C_0\to M\setminus C_0$ satisfying that for each $e\in(E\setminus M)\setminus C_0$, $g(e)$ and e are contained in a p-clique of G. In this case we have at most $|M\setminus C_0|+1=\mu_F-(|F|-1)+1=\mu_F-|F|+2$ p-cliques and edges which cover E. Supposing that the Hall condition does not hold, one can find the edge sets $K\subset(E\setminus M)\setminus C_0$ with $|K|=q+1$ and $M'\subset M\setminus C_0$ with $|M'|=q$ satisfying that each $e\in M\setminus C_0$ contained in a p-clique meeting K is an element of M'. We observe that

$$q\geqslant p-2. \tag{1}$$

Indeed, if $e_1\in K$ and C_1 is p-clique of G for which $C_1\setminus M=\{e_1\}$, then $C_1\neq C_0$; thus one vertex v_1 spanned by C_1 is not spanned by C_0; since there are at least $p-2$ edges of C_1 starting from v_1, and contained in $M\setminus C_0$, $|C_1\cap(M\setminus C_0)|\geqslant p-2$; finally since M' contains $C_1\cap(M\setminus C_0)$, thus

$$q=|M'|\geqslant|C_1\cap(M\setminus C_0)|\geqslant p-2$$

which gives (1). Let us consider the graph $G'=(X',E')$ defined by the edge set $E'=K\cup M'\cup(C_0\setminus\{e_0\})$ and denote by M^* its maximal K_p-free partial graph. Using Corollary 1.4 we have:

$$|M^*|>|E'|\cdot\frac{p-2}{p-1}=\left(2q+\binom{p}{2}\right)\cdot\frac{p-2}{p-1}. \tag{2}$$

It is easy to see that $((M\setminus C_0)\setminus M')\cup M^*$ is K_p-free, therefore

$$\begin{aligned}|M|&\geqslant|((M\setminus C_0)\setminus M')\cup M^*|\\&=|M|-(|C_0|-1)-|M'|+|M^*|\\&=|M|+|M^*|-q-\binom{p}{2}+1.\end{aligned}$$

From this:

$$|M^*|\leqslant q+\binom{p}{2}-1. \tag{3}$$

Combining (3) and (2) we have

$$q+\frac{p(p-1)}{2}-1>2q\cdot\frac{p-2}{p-1}+\frac{p(p-2)}{2}.$$

For $p \geqslant 4$ the last inequality is equivalent to

$$(p-2)\left(\frac{1}{2}+\frac{1}{p-3}\right)>q. \tag{4}$$

It can be seen immediately that (1) and (4) have only one common integer solution: $p=4$, $q=2$. In this case there is only one vertex of X' which does not belong to the vertex set spanned by C (see the argument after (1)), therefore $|X'|=5$ thus X' can span at most 10 different edges in G. But the fact is that it spans at least

$$|E' \cup \{e_1\}| = 2q+\binom{p}{2}+1=11$$

edges—contradiction. □

References

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