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CAYLEY'S FORMULA FOR MULTIDIMENSIONAL TREES *

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This paper concerns extensions of Cayley's enumeration formula to a class of multidimensional tree-like simplicial complexes.

1. Introduction

In 1889 Cayley proved that the number of distinct trees whose vertices are labelled with the integers 1,2, ..., n is given by the formula n^{n-2}. Many authors have considered extensions and refinements of Cayley's formula (see Moon [7]). Included among these results are a number of higher-dimensional analogs. For example, Harary and Palmer [5] defined a *k-tree* to be a simplicial complex constructed in the following way: Start with a $(k-1)$-simplex and add a sequence of new vertices, each suspended over a $(k-1)$-face formed by preceding vertices. Beineke and Pippert [1] proved that the number of k-trees with n labelled vertices is exactly $\binom{n}{k}(kn-k^2+1)^{n-k-2}$, a formula which reduces to Cayley's when $k = 1$. Husimi [6] and Ford and Uhlenbeck [4] considered another class of objects, called *mixed complete star trees* (in the language of [4]). These are constructed inductively from a single vertex, adding new simplices of varying size at each step, with each new simplex connected to exactly one old vertex.

Husimi proved that the number of mixed complete star trees with n vertices and c_i simplices of size i, $i = 1,2, ..., n$, is given by the expression

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$$\frac{(n-1)!}{\Pi_i\,(i!)^{c_i}c_i!}\;n^{(\Sigma_i c_i - 1)}$$

which again reduces to Cayley's formula if $c_2 = n - 1$ and all other c_i's are zero.

In this paper, we derive a common extension of these two formulas, by enumerating a more general class of structures called $(\bar{\alpha},d)$-*trees*. Intuitively, an $(\bar{\alpha},d)$-tree is a simplicial complex formed by successively adding simplices whose size is specified by a multiset $\bar{\alpha}$ of positive integers. Each new simplex intersects an old simplex in a set of size d, and the rest of its vertices are new.

Our main result is the following: the number of $(\bar{\alpha},d)$-trees on n labelled vertices (with n determined by $\bar{\alpha}$ and d) is

$$T_0(\bar{\alpha},d)\,\Delta^{m-2}\,,$$

where m is the number of maximal simplices, Δ is the number of simplices of size d occurring in such a tree, and $T_0(\bar{\alpha},d)$ is the number of $(\bar{\alpha},d)$-trees in which all of the maximal simplices meet in a set of size d.

We give two proofs of our formula: the first is based on one of the many known proofs of Cayley's formula (see [7]), and is the shortest proof we know. The second is obtained by showing that each $(\bar{\alpha},d)$-tree can be canonically associated with an ordinary tree (called its *skeleton*) in such a way that Cayley's formula can be applied directly. In a sense, this explains why "Cayley formulas" exist for $(\bar{\alpha},d)$-trees. It also shows that – in principle, at least – all of the known methods for proving Cayley's formula in the ordinary case can be extended to $(\bar{\alpha},d)$-trees in some form.

We conclude with a discussion of another such method – the Prüfer coding scheme, which permits a purely combinatorial proof of our formula by associating $(\bar{\alpha},d)$-trees with certain sequences of symbols. This section extends work of C. and A. Rényi [9].

2. Main results

Let K be a finite simplicial complex with maximal simplices $A_1, ..., A_m$. If $|A_i| = a_i$, $i = 1, ..., m$, we refer to the multiset $\{a_1, ..., a_m\}$ as the *type of K*. Let d be a positive integer. A *d-vertex* of K is any set of d vertices contained entirely in some A_i (usually called a $(d-1)$-face of K). We say K is *d-connected* if each point is contained in some d-vertex, and for any two d-vertices D_1 and D_2, there exists a sequence $A_1, ..., A_q$ of maximal simplices, such that $D_1 \subseteq A_1$, $D_2 \subseteq A_q$ and $|A_i \cap A_{i+1}| \geqslant d$ for $i = 1, ..., q-1$. We observe that if K has n vertices and is d-connected, then $n \leqslant d + \Sigma_{i=1}^m (a_i - d)$. If $\bar{\alpha} = \{a_1, ..., a_m\}$ is a multiset of integers, and $0 < d < \min \bar{\alpha}$, then an $(\bar{\alpha}, d)$-tree (or *d-tree of type* $\bar{\alpha}$) is a d-connected simplicial complex of type $\bar{\alpha}$ with n vertices, such that $n = d + \Sigma_{i=1}^m (a_i - d)$.

Intuitively, an $(\bar{\alpha}, d)$-tree is a complex formed from a $(d-1)$-simplex by adding successive maximal simplices A_i, where each A_i consists of an old d-vertex together with $a_i - d$ new points. An $(\bar{\alpha}, d)$-tree may be constructed in this manner using any of its d-vertices as a starting point or *root*. If we set $\bar{\alpha} = \{k+1, ..., k+1\}$, then $(\bar{\alpha}, k)$-trees are k-trees in the sense of Harary and Palmer [5]. Mixed complete star trees (Ford and Uhlenbeck [4], Husimi [6]) correspond to $(\bar{\alpha}, d)$-trees with $d = 1$.

It is clear from the definition that the number n of vertices of an $(\bar{\alpha}, d)$-tree is determined by the type $\bar{\alpha}$ and the integer d. If we denote by Δ the number of d-vertices of an $(\bar{\alpha}, d)$-tree, then Δ also depends only on $\bar{\alpha}$ and d. In fact, $\Delta = 1 + \Sigma_{i=1}^m (\binom{a_i}{d} - 1)$, as can be seen by counting the number of new d-vertices which are added along with each new maximal simplex.

Theorem 2.1 (Cayley's formula for $(\bar{\alpha}, d)$-trees). *Let $T(\bar{\alpha}, d)$ denote the number of $(\bar{\alpha}, d)$-trees with n labelled vertices ($n = d + \Sigma_{i=1}^m (a_i - d)$). Let $T_0(\bar{\alpha}, d)$ be the number of $(\bar{\alpha}, d)$-trees in which all of the maximal simplices meet in a single d-vertex.* Then $T(\bar{\alpha}, d) = T_0(\bar{\alpha}, d)\Delta^{m-2}$.*

For ordinary trees this reduces to Cayley's formula since in this case $m = n - 1$, $\Delta = n$ and $T_0 = n$. If $d = k$ and $\bar{\alpha} = \{k+1, ..., k+1\}$, then $m = n - k$, $\Delta = 1 + k(n-k)$, $T_0 = \binom{n}{k}$, and we have $T(\bar{\alpha}, d) = \binom{n}{k}(kn - k^2 + 1)^{n-k-2}$ which is the formula for k-trees discovered by Beineke and Pippert [1]. For $(\bar{\alpha}, 1)$-trees with $\bar{\alpha}$ given by $1^{c_1} 2^{c_2} ... k^{c_k}$ we have

$$m = \sum_i c_i, \quad \Delta = n, \quad T_0 = n \frac{(n-1)!}{\Pi_i (i!)^{c_i} c_i!},$$

* If $\bar{\alpha} = 1^{c_1} 2^{c_2} ... k^{c_k}$ (in the usual notation for partitions), then $T_0(\bar{\alpha}, d) = \binom{n}{d}(n-d)!/(\Pi(i!)^{c_i} c_i!)$.

and our result reduces to

$$\frac{(n-1)!}{\Pi_i (i!)^{c_i} c_i!} n^{\Sigma_i c_i - 1}.$$

This is Husimi's formula for the number of mixed complete star trees with block sizes specified by $\bar{\alpha}$.

First Proof. This is based on the idea of Clarke's proof [3] of Cayley's formula and makes use of the following simple lemma, easily proved by induction.

Lemma 2.2. *Let $f(k)$, $k = 1, \ldots, m$, be a function which satisfies $(m-k) f(k) = k(D-1) f(k+1)$ for some integer D and $k = 1, \ldots, m-1$. Then*

$$f(k) = f(m) \binom{m-1}{m-k} (D-1)^{m-k} \quad \textit{and}$$

$$\sum_{k=1}^{m} f(k) = f(m) D^{m-1}.$$

To apply this lemma we let D_0 be a specific d-subset of vertices, and let $f(k)$, $k = 1, \ldots, m$, denote the number of $(\bar{\alpha}, d)$-trees with labelled vertices for which D_0 is a d-vertex of *degree* k (that is, D_0 is contained in exactly k maximal simplices). Next, we compute the number of pairs (A,B) of $(\bar{\alpha}, d)$-trees such that D_0 has degree k in A, degree $k+1$ in B, and A is obtained from B by removing one of the $k+1$ "branches" of the tree attached to D_0 and reattaching it at a d-vertex D_i different from D_0 (to insure that reattachment is a well-defined operation let us specify a linear ordering of all the vertex labels, and require that the correspondence between vertices of D_0 and D_i be order-preserving).

To begin with, there are $f(k)$ ways of choosing A. For each such choice there are $m-k$ simplices *not* attached to D_0, each of which can be removed and attached to D_0 to form an $(\bar{\alpha}, d)$-tree B which stands in the required relation to A. Hence there are $(m-k) f(k)$ related pairs (A,B). On the other hand, there are $f(k+1)$ ways of first choosing B. For each choice we can obtain an A by removing one of the $k+1$ branches attached to D_0 and reattaching it elsewhere. There are $\Delta - 1$ choices of a d-vertex D_j ($j \neq 0$) for reattachment, and for each such choice there are k different ways to choose a branch to remove from D_0 and reattach to D_j. So for each B there are $k(\Delta - 1)$ choices of A, hence $k(\Delta - 1)f(k + 1)$ also counts the number of pairs (A,B). Thus we have the recursion $(m - k)f(k) = k(\Delta - 1)f(k + 1)$ for $k = 1, ..., m - 1$. By the lemma the number of $(\bar{\alpha},d)$-trees rooted at D_0 is given by

$$R_0(\bar{\alpha},d) = \sum_{k=1}^{m} f(k) = f(m)\,\Delta^{m-1} .$$

Multiplying $R_0(\bar{\alpha},d)$ by $\binom{n}{d}$ for the ways of selecting D_0, and dividing by Δ to eliminate the root, yields

$$T(\alpha,d) = \binom{n}{d} f(m)\,\Delta^{m-2} = T_0(\bar{\alpha}, d)\,\Delta^{m-2} .$$

This proof contains (as in the case of ordinary trees) the following

Corollary 2.3. *The number of $(\bar{\alpha},d)$-trees which contain a specific d-vertex D_0, and for which D_0 has degree k is given by*

$$f(m)\binom{m-1}{m-k}(\Delta - 1)^{m-k} ,$$

where $f(m)$ is the number of $(\bar{\alpha},d)$-trees in which D_0 has maximal degree m.

Second Proof. We begin by expanding our remarks on the construction of trees, and making a few more definitions.

Suppose that an $(\bar{\alpha},d)$-tree K is constructed according to the rule described earlier by starting with a d-vertex D_1 (as a "root") and adding maximal simplices $A_1, ..., A_m$ in order. That is $A_1 \supseteq D_1$ and for $i > 1$,

$$A_i \cap \left(\bigcup_{j=1}^{i-1} A_j\right)$$

is a d-vertex D_i contained in some A_j with $j < i$. It can be shown that the sets $D_i \subset A_i$ which arise in this construction are independent of the ordering of the A_i's – any "admissible" ordering with root D_1 will associate D_i with A_i in every case. We can also define the sets $\tilde{A}_i = A_i - D_i$ without reference to a particular construction. Intuitively, D_i is the d-vertex "closest" to the root D_1; hence we refer to D_i as the *in-vertex* of A_i with respect to the root D_1. The other d-vertices of A_i will be called *out-vertices*. We say that A_i *is attached to* $\tilde{A}_j$ if $A_i \cap A_j$

is an out-vertex of A_j. Clearly, if $A_i \not\supseteq D_1$, there exists a unique $j < i$ such that A_i is attached to $\tilde{A}_j$. If $A_i \supseteq D_1$, we say that A_i *is attached to* D_1. For notational convenience, we also write $D_1 = \tilde{A}_0$. The sets $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_m$ form a partition of the vertices of K which we call the *partition of K with respect to* D_1 and denote by $\pi(K,D_1)$. (Actually, since the block $\tilde{A}_0$ plays a special role, we think of $\pi(K,D_1)$ as a *rooted partition*.) If π is any partition with root D, we say that K is *partitioned by* π if D is a d-vertex of K and $\pi = \pi(K,D)$.

The essential idea of the proof is to associate with K an ordinary tree whose vertices are in one-to-one correspondence with the sets $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_m$. We define the *skeleton of K with respect to* D_1 (denoted $\tau(K,D_1)$) to be the tree whose vertices are the integers $0, 1, \ldots, m$ and whose edges are the pairs $\{i,j\}$ such that A_i is attached to $\tilde{A}_j$. It is easy to verify that this indeed defines a tree by considering its construction in parallel with the construction of K.

Now let τ be a tree with vertex set $\{0, 1, \ldots, m\}$ and let $\pi = \{\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_m\}$ be a partition (of n labelled vertices) with $|\tilde{A}_0| = d$, $|\tilde{A}_i| = a_i - d$, $i = 1, \ldots, m$. We compute the number of $(\bar{\alpha},d)$-trees K such that $\pi(K,\tilde{A}_0) = \pi$ and $\tau(K,\tilde{A}_0) = \tau$.

Suppose that, in τ, the vertex i has degree e_i. If we regard τ as rooted at 0, then every vertex $j \neq 0$ has exactly $e_j - 1$ "outward" edges. If K is an $(\bar{\alpha},d)$-tree with partition π and skeleton τ, and $j \neq 0$, there must be exactly $e_j - 1$ indices i such that A_i is attached to $\tilde{A}_j$. There are exactly $\binom{a_j}{d} - 1$ ways to choose D_i so that A_i is attached to $\tilde{A}_j$. If $j = 0$, then there is only one possible choice – that is, if A_i is attached to $\tilde{A}_0$, then D_i must be $\tilde{A}_0$. The collection of all K's with $\pi(K,\tilde{A}_0) = \pi$ and $\tau(K,\tilde{A}_0) = \tau$ is obtained by choosing the D_i's in all admissible ways. Hence the total number of K's is

$$\xi_0^{e_0-1} \xi_1^{e_1-1} \ldots \xi_m^{e_m-1},$$

$$\text{where } \xi_0 = 1,\ \xi_1 = \binom{a_1}{d} - 1, \ldots, \xi_m = \binom{a_m}{d} - 1 .$$

Next, we compute the total number of $(\bar{\alpha},d)$-trees with partition π. By the above argument, this is equal to

$$\sum_{\tau} \xi_0^{e_0(\tau)-1} \xi_1^{e_1(\tau)-1} \ldots \xi_m^{e_m(\tau)-1} ,$$

where the sum is taken over all trees τ on $m + 1$ labelled vertices. We now apply Cayley's formula in its "generating function" form (see

Moon [7]). This is a refinement of Cayley's result which asserts that

$$\sum_{\tau} x_0^{e_0(\tau)-1} x_1^{e_1(\tau)-1} \dots x_m^{e_m(\tau)-1} = (x_0+x_1+\dots+x_m)^{n-1} .$$

Under the substitution $x_i \leftrightarrow \xi_i$, the sum $(x_0 + \dots + x_m)$ becomes $1 + \sum_{i=1}^{m}(\binom{e_i}{d} - 1) = \Delta$. Hence the total number of (α, d)-trees with partition π is Δ^{m-1}.

The number of rooted partitions $\pi = \{\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_m\}$ with $|\tilde{A}_0| = d$ and $|A_i| = a_i - d$, $i = 1, \dots, m$, is easily seen to be $T_0(\bar{\alpha}, d)$. On the other hand, every $T_0(\bar{\alpha}, d)$-tree can be partitioned in exactly Δ ways (one for each choice of the root $\tilde{A}_0$). Hence the total number of labelled $(\bar{\alpha}, d)$-trees $T(\bar{\alpha}, d) = T_0(\bar{\alpha}, d)\Delta^{m-2}$.

Because our main intermediate result has independent interest, we state it as a separate theorem.

Theorem 2.4. *Let $\pi = \{A_0, A_1, \dots, A_m\}$ be a rooted partition of n labelled vertices such that $|\tilde{A}_0| = d$ and $|\tilde{A}_i| = a_i - d$, $i = 1, \dots, m$. Then the number of $(\bar{\alpha}, d)$-trees partitioned by π is Δ^{m-1}.*

In the case of ordinary trees this reduces to the usual formula n^{n-2}. The partition π must be the trivial partition into singletons, and *every* tree is partitioned by π. Hence, in this case the expression Δ^{m-1} enumerates all labelled trees. In the case of k-trees a partition π consists of a d-vertex $\tilde{A}_0$ chosen as root and singletons for the remaining points. Hence $\binom{n}{k}\Delta^{m-1}$ enumerates rooted k-trees, and dividing by Δ yields $T(\bar{\alpha}, d) = \binom{n}{k}\Delta^{m-2}$.

3. Coding $(\bar{\alpha}, d)$-trees

For ordinary trees, a purely "combinatorial" proof of Cayley's formula can be given by associating trees with $(n-2)$-tuples of integers from the set $\{1, \dots, n\}$. The resulting correspondence is known as the Prüfer code for trees [7, 8]. It provides a compact way of representing trees, and also contains explicitly information as to the degree of each vertex. If T is a tree whose vertices are labelled $1, \dots, n$, then the Prüfer code for T can be described as follows:

Make a list $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_{n-1}, b_{n-1}\}$ of the edges of T with the property that a_i is the smallest endpoint (in the case of a rooted tree the root is never considered an endpoint) of the tree obtained from

T by removing $a_1, \ldots, a_{i-1}$ and all incident edges. This can be done by finding a_1, removing it, then finding a_2, and so forth. The *Prüfer code* of T is the sequence $(b_1, \ldots, b_{n-2})$.

The Rényis [9] extended the idea of the Prüfer correspondence to k-trees, and their methods extend further to $(\bar{\alpha},d)$-trees with only minor modifications. However, the situation is more complicated, and the results are only partially satisfactory.

Ideally, a generalized Prüfer code would consist of a rule for associating $(\bar{\alpha},d)$-trees having a fixed partition π with $(m-1)$-tuples of integers $1, 2, \ldots, \Delta$. However, the principal difficulty is this: while the ordinary code is made up of actual vertices, it is difficult to predict which set of Δ d-subsets will form the d-vertices of an $(\bar{\alpha},d)$-tree. There are always Δ of them, but they are different for each tree. The Renyis' procedure leads to an $(m-1)$-tuple of d-vertices from which the tree can be reconstructed. However, since the d-sets are not chosen from a fixed domain, it cannot be immediately deduced that the total number of $(\bar{\alpha},d)$-trees with partition π is Δ^{m-1}. One of the main results of the Rényis' paper is to derive this by careful analysis of which $(m-1)$-tuples of d-sets are "admissible" – that is, which ones arise from a tree. Another problem is that since the condition for admissibility is rather complicated, it is hard to generate trees from scratch – for example, to list all trees with certain degree sequences.

We present now a slightly different coding scheme for $(\bar{\alpha},d)$-trees which circumvents most of these difficulties. In what follows, let $\pi = \{\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_m\}$ be a partition rooted at $\widetilde{A}_0$ with $|\widetilde{A}_0| = d$, $|\widetilde{A}_i| = a_i - d$, $i = 1, \ldots, m$. We ultimately associate $(\bar{\alpha},d)$-trees partitioned by π with $(m-1)$-tuples of symbols taken from a set $X = \{x_q^p\}$, $p = 0, \ldots, m$, $q = 1, \ldots, \xi_p$, where

$$\xi_p = \begin{cases} 1 & \text{if } p = 0, \\ \binom{a_p}{d} - 1 & \text{if } p > 0. \end{cases}$$

Clearly the set X has cardinality Δ as desired.

We suppose that the collection of all $\binom{n}{d}$ d-subsets of vertices is given some linear order – say lexicographic. We also assume that the blocks $\widetilde{A}_1, \ldots, \widetilde{A}_m$ are ordered so that $\widetilde{A}_i < \widetilde{A}_j$ if and only if $i < j$.

Now let K be an $(\bar{\alpha},d)$-tree which is partitioned by π. A block $\widetilde{A}_j$ of K will be called an *endpoint* if no out-vertex of A_j is the in-vertex of another simplex (equivalently, if $\widetilde{A}_j \cap A_i$ is empty for i different from j).

We construct a permutation σ of the indices 1, ..., m as follows: let $\widetilde{A}_{\sigma(1)}$ be the smallest endpoint of K. Remove $\widetilde{A}_{\sigma(1)}$, and let $\widetilde{A}_{\sigma(2)}$ be the smallest endpoint in the remaining tree. Continue in this fashion until each of the blocks $\widetilde{A}_1, \ldots, \widetilde{A}_m$ has been removed. If $D_{\sigma(i)}$ is the in-vertex of $A_{\sigma(i)}$, we call the sequence $(D_{\sigma(1)}, D_{\sigma(2)}, \ldots, D_{\sigma(m-1)})$ the *Rényi code* for K. (Note that while $\widetilde{A}_{\sigma(1)}, \ldots, \widetilde{A}_{\sigma(m)}$ is a permutation of the $\widetilde{A}_i$'s, the $D_{\sigma(i)}$'s need not be distinct.) This sequence is the result of extending the Rényi's coding scheme to $(\bar{\alpha},d)$-trees. Let us define the *degree* of a d-vertex to be the number of maximal simplices which contain it. Then the degree of a given d-vertex D in K is one greater than the number of occurrences of D in the Rényi code for K. However, as mentioned, not all sequences arise as codes for trees.

We proceed further, and associate with each $D_{\sigma(i)}$ a symbol x_q^p as follows. Suppose that for each $i = 1, \ldots, m$, $A_{\sigma(i)}$ is attached to $\widetilde{A}_{p(i)}$ — that is, $D_{\sigma(i)}$ is an out-vertex of $A_{p(i)}$ if $p(i) > 0$, and $D_{\sigma(i)} = \widetilde{A}_0$ if $p(i) = 0$. Moreover suppose that $D_{\sigma(i)}$ is the $q(i)$th out-vertex of $A_{p(i)}$ in the linear ordering of d-subsets. If $p(i) = 0$, we assume that $q(i) = 1$. The *Prüfer code for K* is defined to be the sequence $(x_{q(1)}^{p(1)}, x_{q(2)}^{p(2)}, \ldots, x_{q(m-1)}^{p(m-1)})$.

At this point we make two assertions. The first is that different $(\bar{\alpha},d)$-trees give rise to different Prüfer codes — that is, K can be reconstructed from its code. The second is that every possible $(m-1)$-tuple of x_q^p's occurs as the Prüfer code of a tree. By Theorem 2.4, these statements are equivalent. However, since proving them separately provides a new proof of Theorem 2.4, we will do so. The key to both assertions is the following lemma.

Lemma 3.1. *If* $(x_{q(1)}^{p(1)}, x_{q(2)}^{p(2)}, \ldots, x_{q(m-1)}^{p(m-1)})$ *is the generalized Prüfer code of an* $(\bar{\alpha},d)$*-tree* K *partitioned by* π, *then* $(p(1), p(2), \ldots, p(m-1))$ *is the ordinary Prüfer code of its skeleton* $\tau(K,\widetilde{A}_0)$ *rooted at* 0 *(thus* 0 *is never removed as an endpoint in constructing the code).*

The lemma follows easily from the definition of $\tau(K,\widetilde{A}_0)$. Clearly the edges of $\tau(K,\widetilde{A}_0)$ must be the pairs $\{\sigma(1), p(1)\}, \{\sigma(2), p(2)\}, \ldots, \{\sigma(m-1), p(m-1)\}, \{\sigma(m), 0\}$. We observe that $\sigma(i)$ is an endpoint of the remaining skeleton if and only if $\widetilde{A}_{\sigma(i)}$ is an endpoint of the remaining $(\bar{\alpha},d)$-tree, hence the *smallest* endpoints also correspond. Thus $\sigma(i)$ for $i = 1, \ldots, m-1$ gives the order of encoding edges in constructing the ordinary Prüfer code for $\tau(K,\widetilde{A}_0)$ rooted at 0, as required.

In order to prove that the correspondence is one-to-one, we produce a *decoding scheme*. Based on the lemma, we describe it as follows: use the sequence $(p(1), \ldots, p(m-1))$ to reconstruct $\tau(K, \tilde{A}_0)$. Then use the numbers $q(i)$ to construct K from its skeleton.

More specifically we use $(p(1), \ldots, p(m-1))$ to reconstruct the permutation $\sigma(1), \ldots, \sigma(m)$. Then define $D_{\sigma(m)} = \tilde{A}_0$, and for $j = m-1, \ldots, 1$, $D_{\sigma(j)}$ = the $q(j)$th out-vertex of $\tilde{A}_{p(j)} \cup D_{p(j)}$. Since $p(j) = \sigma(j')$ for some $j' > j$, the set $D_{p(j)}$ has already been determined, so the definition of $D_{\sigma(j)}$ makes sense. We attach each $\tilde{A}_{\sigma(j)}$ to $D_{\sigma(j)}$. If the Prüfer code comes from an $(\bar{\alpha}, d)$-tree K, this process clearly reconstructs it. Hence the correspondence is one-to-one.

To show that every $(m-1)$-tuple arises, we first observe that the above decoding procedure always produces $(\bar{\alpha},d)$-trees. If $(x^{p(1)}_{q(1)}, \ldots, x^{p(m-1)}_{q(m-1)})$ is an arbitrary sequence, then $(p(1), \ldots, p(m-1))$ decodes into an ordinary tree τ, and the rest of the decoding procedure gives a specific way of constructing an $(\bar{\alpha}, d)$-tree K with skeleton τ. It remains to show that the generalized Prüfer code of this K is the same as the sequence with which we started. Since K has skeleton τ, and the ordinary Prüfer code for τ (rooted at 0) is $(p(1), \ldots, p(m-1))$ the $p(j)$'s must be the same. Once these are determined it is clear that the $q(j)$'s must agree. This proves that each of the Δ^{m-1} possible $(m-1)$-tuples appears as the code for some $(\bar{\alpha},d)$-tree.

We can condense the above results into a single statement about the generating function for $(\bar{\alpha},d)$-trees with a fixed partition.

Theorem 3.2. *Let* $\bar{\alpha} = \{a_1, \ldots, a_m\}$, $d < \min \bar{\alpha}$, $n = d + \sum_{i=1}^{m}(a_i - d)$, $\xi_0 = 1$, $\xi_i = \binom{a_i}{d} - 1$ *for* $i = 1, \ldots, m$. *Let* $\{x^p_q\}$, $p = 0, \ldots, m$, $q = 1, \ldots, \xi_p$, *be a set of distinct indeterminates. Then*

$$\left(\sum_{p=0}^{m} \sum_{q=1}^{\xi_p} x^p_q \right)^{m-1} = \sum_K \left\{ \prod_{i=1}^{m-1} x^{p(i)}_{q(i)} \right\},$$

where the sum on the right is taken over all $(\bar{\alpha},d)$*-trees* K *with a fixed partition, and* $x^{p(i)}_{q(i)}$ *is the ith symbol appearing in the generalized Prüfer code for* K.

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