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Stability and bifurcation analysis on a discrete-time neural network☆

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Abstract

Using techniques developed by Kuznetsov to discrete-time systems, we study the stability of the equilibrium $(0, 0)$ and Neimark–Sacker bifurcation (also called Hopf bifurcation for map) of a discrete-time neural network system. The obtained results are less restrictive and improve upon the existing ones on Neimark–Sacker bifurcation of discrete-time neural network with special classes of transfer functions. The theoretical analyses are verified by numerical simulations. Our results have potential applications in neural networks.

MSC: 39A11; 92B20

Keywords: Neural network; Stability; Neimark–Sacker bifurcation

1. Introduction

The investigation of dynamic behavior for neural networks has been the subject of much recent activity since one of the models with electronic circuit implementation was proposed by Hopfield

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[7]. See, for example, [4,9,14]. Due to the networks of one or two neurons are prototypes to understand the dynamics of larger-scale networks, some progress has been made for such networks, for example [2,5,6,10,11,13,15–18] and the references therein.

In this paper, we consider the following discrete-time neural network model with self-connection in the form

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f_1(x_1(n)) + a_{12} f_2(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_1(x_1(n)) + a_{22} f_2(x_2(n)), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.1}$$

where x_i $(i = 1, 2)$ denotes the activity of the ith neuron, $\beta \in (0, 1)$ is internal decay of neurons. The constants a_{ij} $(i = 1, 2)$ denotes the connection weights. $f_i : \mathbb{R} \to \mathbb{R}$ is a continuous transfer function and $f_i(0) = 0$.

The discrete-time system (1.1) can be regarded as a discrete analogy of the differential system

$$\begin{aligned} \dot{x}_1(t) &= -\mu x_1(t) + w_{11} f_1(x_1(t)) + w_{12} f_2(x_2(t)), \\ \dot{x}_2(t) &= -\mu x_2(t) + w_{21} f_1(x_1(t)) + w_{22} f_2(x_2(t)) \end{aligned} \tag{1.2}$$

or the system with a piecewise constant arguments

$$\begin{aligned} \dot{x}_1(t) &= -\mu x_1(t) + w_{11} f_1(x_1([t])) + w_{12} f_2(x_2([t])), \\ \dot{x}_2(t) &= -\mu x_1(t) + w_{21} f_1(x_1([t])) + w_{22} f_2(x_2([t])), \end{aligned} \tag{1.3}$$

where $\mu > 0$ and $[\cdot]$ denotes the greatest integer function. One motivation of this research is system (1.1) includes the discrete version of systems (1.2) and (1.3). On the other hand, the wide application of differential equations with piecewise constant argument in certain biomedical models (see, example, [1]) and much progress has been made in the study of such as system (1.3) with the piecewise arguments since the pioneering work of Cooke and Wiener [3] and Shah and Wiener [12].

For the method of discrete analogy, we refer to [6,16,17].

For a special case of (1.1), with a transfer function $f_i(u) = \tanh(c_i u)$ and no self-connections ($a_{11} = a_{22} = 0$), Gopalsamy and Leung [6] gave some sufficient conditions to guarantee the stability of the equilibrium $(0, 0)$ and the existence of bifurcation. However, as Faria in [5], here we shall only assume $f_i(0) = 0$, $f_i \in C^1(\mathbb{R})$ for the stability analysis, and $f_i \in C^3(\mathbb{R})$, $f_i'(0) f_i'''(0) \neq 0$, $f_i(0) = f_i''(0) = 0$ for the bifurcation analysis. Also, we shall not assume any contains on the signs of the coefficients a_{ij} appearing in (1.1). In this paper, by the techniques developed by Kuznetsov [8], where using "project", the system into the critical eigenspace and its complement, we will study the stability of the, equilibrium $(0, 0)$ and Neimark–Sacker bifurcation (also called Hopf bifurcation for map). The conditions for asymptotical stability of the equilibrium $(0, 0)$ of (1.1) will be established. Moreover, when the bifurcation parameter exceeds a critical value, we find that the Neimark–Sacker bifurcation will occur and its direction and stability are determined completely by the sign of the value of $a(D^\star)$. The approach here is more general than the one considered in [6].

2. Stability and existence of Neimark–Sacker bifurcation

In this section, we discuss the local stability of the equilibrium $(0, 0)$ of system (1.1). For most of the models in the literature, including the ones in [6,10,15], the transfer function f_i is $f_i(u) = \tanh(c_i u)$.

However, here we only need the following hypothesis:

(H_1) For $i = 1, 2,\ f_i \in C^1(\mathbb{R})$ and $f_i(0) = 0$.

We define the parameters

$$T = \tfrac{1}{2}(a_{11} f_1'(0) + a_{22} f_2'(0)), \quad D = (a_{11}a_{22} - a_{12}a_{21}) f_1'(0) f_2'(0).$$

For $T \in (-1-\beta, 1-\beta)$, we let

$$X_0 = \{(T, D) \in \mathbb{R}^2; L_1 < 0, L_2 < 0 \quad \text{and} \quad L_3 > 0\},$$

where

$$L_1 = 2(1-\beta)T - (1-\beta)^2 - D, \quad L_2 = -2(1+\beta)T - (1+\beta)^2 - D,$$
$$L_3 = -2\beta T + 1 - \beta^2 - D.$$

Theorem 1. *Suppose that hypothesis* (H_1) *is satisfied and* $(T, D) \in X_0$. *Then the zero solution of* (1.1) *is asymptotically stable.*

Proof. The characteristic equation for the linearization of (1.1) at (0, 0) is

$$\lambda^2 - 2(\beta + T)\lambda + \beta^2 + 2\beta T + D = 0. \tag{2.1}$$

Here, we have two cases.

Case 1: $T^2 \geqslant D$. In this case, the root of characteristic equation (2.1) is given by

$$\lambda_{1,2} = \beta + T \pm \sqrt{T^2 - D}. \tag{2.2}$$

Obviously, the eigenvalue $\lambda_{1,2}$ in (2.2) is inside the unit circle if and only if

$$(T, D) \in X_1 \cap X_2, \tag{2.3}$$

where

$$X_1 \overset{\text{def}}{=} \{(T, D) \in R^2; D > 2(1-\beta)T - (1-\beta)^2, T < 1-\beta, T^2 \geqslant D\},$$
$$X_2 \overset{\text{def}}{=} \{(T, D) \in R^2; D > -2(1+\beta)T - (1+\beta)^2, T > -1-\beta, T^2 \geqslant D\}.$$

Case 2: $T^2 < D$. In this case, the characteristic equation (2.1) has a pair of conjugate complex roots

$$\lambda_{1,2} = \beta + T \pm \sqrt{D - T^2}\,\mathrm{i}. \tag{2.4}$$

It is easy to verify that $|\lambda_{1,2}| < 1$ if and only if

$$(T, D) \in X_3 \stackrel{\text{def}}{=} \{(T, D) \in R^2; D < -2\beta T + 1 - \beta^2, T^2 < D\}. \tag{2.5}$$

Combining with Cases 1 and 2, we know that $X_0 = (X_1 \cap X_2) \cup X_3$. Thus, the eigenvalues $\lambda_{1,2}$ of the characteristic equation (2.1) inside the unit circle for $(T, D) \in X_0$. This implies that the zero solution of (1.1) is asymptotically stable. $\square$

Now, we choose D as the bifurcation parameter to study the Neimark–Sacker bifurcation of (0, 0). For $T^2 < D$, let

$$\lambda(D) = \beta + T + \sqrt{D - T^2}\mathrm{i}, \tag{2.6}$$

then, the eigenvalues in (2.1) are conjugate complex pair $\lambda(D)$ and $\overline{\lambda(D)}$. The modulus of the eigenvalue is

$$|\lambda| = \sqrt{\beta^2 + 2\beta T + D}. \tag{2.7}$$

Then, $|\lambda| = 1$ if and only if

$$D = D^\star = -2\beta T + 1 - \beta^2. \tag{2.8}$$

Obviously, we have

$$|\lambda| < 1 \quad \text{for } T^2 < D < D^\star.$$

Since the modulus of eigenvalue $|\lambda(D^\star)| = 1$, we know $D^\star$ is a critical value which destroy the stability of (0, 0). The following lemma is helpful to study bifurcation of (0, 0).

Lemma 1. *Suppose that* (H_1) *is satisfied and* $-\beta < T < 1 - \beta$. *Then*

(i) $(\frac{\mathrm{d}}{\mathrm{d}D}|\lambda(D)|)_{D=D^\star} > 0$;
(ii) $\lambda^k(D^\star) \neq 1$ *for* $k = 1, 2, 3, 4$,
where $\lambda(D)$ *and* $D^\star$ *are given by* (2.6) *and* (2.8), *respectively*.

Proof. Obviously, we see $T^2 < D^\star$ from the assumption $T \in (-\beta, 1 - \beta)$. By direct calculation, we obtain from (2.7) and (2.8) that

$$\left(\frac{\mathrm{d}}{\mathrm{d}D}|\lambda(D)|\right)_{D=D^\star} = \frac{1}{2} > 0.$$

This means property (i) is true.

In what follows, we will deal with the property of (ii). Clearly, $\lambda^k(D^\star) = 1$ for some $k \in \{1, 2, 3, 4\}$ if and only if the argument $\arg\lambda(D^\star) \in \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. Since

$$|\lambda(D^\star)| = 1, \quad \operatorname{Re}\lambda(D^\star) > 0, \quad \operatorname{Im}\lambda(D^\star) > 0,$$

it follows that $\arg\lambda(D^\star) \notin \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. We complete the proof of (ii). $\square$

By Lemma 1 and the results in [13], we have the following

Theorem 2. *Suppose that* (H_1) *is satisfied and* $T \in (-\beta, 1-\beta)$. *Then we have*

(i) *if* $T^2 < D < D^{\star}$, *then the equilibrium* $(0, 0)$ *of* (1.1) *is asymptotically stable*;
(ii) *if* $D > D^{\star}$, *then the equilibrium* $(0, 0)$ *of* (1.1) *is unstable*;
(iii) *the Neimark–Sacker bifurcation occurs at* $D = D^{\star}$, *that is, system* (1.1) *has a unique closed invariant curve bifurcating from the equilibrium* $(0, 0)$ *near* $D = D^{\star}$,

where $D^{\star}$ *is given by* (2.8).

3. Direction and stability of the Neimark–Sacker bifurcation

In the above section, we have shown that Neimark–Sacker bifurcation occurs at some value $D = D^{\star}$ for system (1.1). In this section, by using the normal form method and the center manifold theory for discrete-time system developed by Kuznetsov [8], we will give an algorithm to study the direction, stability of the Neimark–Sacker bifurcation. For most of the models in the literature, for example [6,10,13,15], the transfer function f_i is $f_i(u) = \tanh(c_i u)$ for $i = 1, 2$. Thus, for $i = 1, 2$, we may assume that the transfer functions in (1.1) satisfy:

(H_2) $f_i \in C^{(3)}(R, R)$, $f_i(0) = f_i''(0) = 0$ and $f_i'(0) f_i'''(0) \neq 0$.

Now (1.1) can be rewritten as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \beta + a_{11} f_1'(0) & a_{12} f_2'(0) \\ a_{21} f_1'(0) & \beta + a_{22} f_2'(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} F_1(x, D) \\ F_2(x, D) \end{pmatrix}, \tag{3.1}$$

where $x = (x_1, x_2)^{\mathrm{T}} \in R^2$. We denote

$$A = A(D) = \begin{pmatrix} \beta + a_{11} f_1'(0) & a_{12} f_2'(0) \\ a_{21} f_1'(0) & \beta + a_{22} f_2'(0) \end{pmatrix} \tag{3.2}$$

and

$$r_j = T + \sqrt{D - T^2}\,\mathrm{i} - a_{jj} f_j'(0), \quad j = 1, 2, \tag{3.3}$$

then, from the definition of *T*, we can obtain

$$\bar{r}_1 = -r_2, \quad |r_j|^2 = -r_1 r_2 = -a_{12} a_{21} f_1'(0) f_2'(0), \quad j = 1, 2. \tag{3.4}$$

Claim. *For* $j = 1, 2$, *the inequations* $r_j \neq 0$ *and* $a_{12} a_{21} f_1'(0) f_2'(0) < 0$ *hold.*

In fact, if $r_1=0$ or $r_2=0$, From (3.4), we have $r_1r_2=a_{12}a_{21}f_1'(0)f_2'(0)=0$. Thus, by the expression $D=(a_{11}a_{22}-a_{12}a_{21})f_1'(0)f_2'(0)$, we see $D=a_{11}a_{22}f_1'(0)f_2'(0)$ and

$$T^2=\left[\frac{a_{11}f_1'(0)+a_{22}f_2'(0)}{2}\right]^2 \geqslant a_{11}a_{22}f_1'(0)f_2'(0)=D,$$

this is a contradiction to $T^2<D$. Hence $r_j\neq 0$ $(j=1,2)$ and it follows from (3.4) that $a_{12}a_{21}f_1'(0)$ $f_2'(0)<0$.

Let $q(D)\in\mathbb{C}^2$ be an eigenvector of $A(D)$ corresponding to eigenvalue $\lambda(D)$ given by (2.6). Then

$$A(D)q(D)=\lambda(D)q(D).$$

Again let $p(D)\in C^2$ be an eigenvector of the transposed matrix $A^{\mathrm{T}}(D)$ corresponding to its eigenvalue, that is $\overline{\lambda(D)}$

$$A^{\mathrm{T}}(D)p(D)=\overline{\lambda(D)}p(D).$$

By direct calculation we obtain

$$q\sim\left(1,\frac{a_{21}f_1'(0)}{r_2}\right)^{\mathrm{T}},\quad p\sim\left(1,\frac{a_{12}f_2'(0)}{\overline{r}_2}\right)^{\mathrm{T}},$$

where r_j $(j=1,2)$ is given by (3.3). For the eigenvector $q=(1,a_{21}f_1'(0)/r_2)^{\mathrm{T}}$, to normalize p, let

$$p=\frac{\overline{r}_2}{\overline{r}_2-r_2}\left(1,\frac{a_{12}f_2'(0)}{\overline{r}_2}\right)^{\mathrm{T}},$$

we have $\langle p,q\rangle=1$, where $\langle\cdot,\cdot\rangle$ means the standard scalar product in $\mathbb{C}^2:\langle p,q\rangle=\overline{p}_1q_1+\overline{p}_2q_2$. Any vector $x\in\mathbb{R}^2$ can be represented for D near $D^\star$ as

$$x=zq(D)+\overline{zq(D)}$$

for some complex z. Obviously

$$z=\langle p(D),x\rangle.$$

Thus, system (3.1) can be transformed for D near $D^\star$ into the following form:

$$z\longmapsto\lambda(D)z+g(z,\overline{z},D), \tag{3.5}$$

where $\lambda(D)$ can be written as $\lambda(D)=(1+\varphi(D))\mathrm{e}^{\mathrm{i}\theta(D)}$ ($\varphi(D)$ is a smooth function with $\varphi(D^\star)=0$) and

$$g(z,\overline{z},D)=\sum_{k+l\geqslant 2}\frac{1}{k!l!}g_{kl}(D)z^k\overline{z}^l. \tag{3.6}$$

From assumption (H_2), we know that F_i $(i = 1, 2)$ in (3.1) can be expanded as

$$F_1(\xi, D) = \frac{a_{11}}{6} f_1^{'''}(0)\xi_1^3 + \frac{a_{21}}{6} f_2^{'''}(0)\xi_2^3 + \mathrm{O}(\|\xi\|^4),$$

$$F_2(\xi, D) = -\frac{a_{21}}{6} f_1^{'''}(0)\xi_1^3 + \frac{a_{22}}{6} f_2^{'''}(0)\xi_2^3 + \mathrm{O}(\|\xi\|^4).$$

It follows that:

$$B_i(x, y) := \sum_{j,k}^{2} \left.\frac{\partial^2 F_i(\xi, D^\star)}{\partial\xi_j \partial\xi_k}\right|_{\xi=0} x_j y_k = 0, \quad i = 1, 2 \tag{3.7}$$

and

$$\begin{aligned} C_i(x, y, u) &:= \sum_{j,k,l}^{2} \left.\frac{\partial^3 F_i(\xi, D^\star)}{\partial\xi_j \partial\xi_k \partial\xi_l}\right|_{\xi=0} x_j y_k u_l \\ &= a_{i1} f_1^{'''}(0) x_1 y_1 u_1 + a_{i2} f_2^{'''}(0) x_2 y_2 u_2, \quad i = 1, 2. \end{aligned} \tag{3.8}$$

By (3.6)–(3.8) and the formulas

$$g_{20}(D^\star) = \langle p, B(q, q)\rangle, \quad g_{11}(D^\star) = \langle p, B(q, \overline{q})\rangle, \quad g_{02}(D^\star) = \langle p, B(\overline{q}, \overline{q})\rangle$$

and

$$g_{21}(D^\star) = \langle p, C(q, q, \overline{q})\rangle,$$

we obtain

$$g_{20}(D^\star) = g_{11}(D^\star) = g_{02}(D^\star) = 0$$

and

$$\begin{aligned} g_{21}(D^\star) &= \overline{p}_1 C_1(q, q, \overline{q}) + \overline{p}_2 C_2(q, q, \overline{q}) \\ &= \frac{r_2}{r_2 - \overline{r}_2} \left\{ a_{11} f_1^{'''}(0) - \frac{a_{21}^2 f_1'(0)^2 f_2^{'''}(0)}{r_2 f_2'(0)} + \frac{a_{12}a_{21} f_2'(0) f_1^{'''}(0)}{r_2} - \frac{a_{21}^2 a_{22} f_1'(0)^2 f_2^{'''}(0)}{r_2^2} \right\} \\ &= \frac{1}{a_{12}(r_1 + r_2) f_2'(0)} \{ a_{11}a_{12} r_2 f_2'(0) f_1^{'''}(0) + a_{21}a_{22}\overline{r}_2 f_1'(0) f_2^{'''}(0) \\ &\quad + a_{12}^2 a_{21} f_2'(0)^2 f_1^{'''}(0) - a_{12}a_{21}^2 f_1'(0)^2 f_2^{'''}(0) \} \\ &= \frac{1}{2a_{12}\sqrt{D^\star - T^2} f_2'(0) i} \{ a_{11}a_{12} f_2'(0) f_1^{'''}(0)(T + \sqrt{D^\star - T^2} i - a_{22} f_2'(0)) \\ &\quad + a_{21}a_{22} f_1'(0) f_2^{'''}(0)(T - \sqrt{D^\star - T^2} i - a_{22} f_2'(0)) \\ &\quad + a_{12}^2 a_{21} f_2'(0)^2 f_1^{'''}(0) - a_{12}a_{21}^2 f_1'(0)^2 f_2^{'''}(0) \}, \end{aligned}$$

which, together with $e^{-i\theta(D^\star)} = \overline{\lambda(D^\star)}$ and the expression $D = (a_{11}a_{22} - a_{12}a_{21}) f_1'(0) f_2'(0)$, implies that

$$\begin{aligned}
a(D^\star) &= \mathrm{Re}\left(\frac{e^{-i\theta(D^\star)} g_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1-2e^{i\theta(D^\star)})e^{-2i\theta(D^\star)}}{2(1-e^{i\theta(D^\star)})} g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \\
&= \mathrm{Re}\left(\frac{e^{-i\theta(D^\star)}}{2} g_{21}\right) \\
&= \frac{1}{4a_{12} f_2'(0)} \{(\beta + T)[a_{11}a_{12} f_2'(0) f_1'''(0) - a_{21}a_{22} f_1'(0) f_2'''(0)] \\
&\quad - [a_{11}a_{12} f_2'(0) f_1'''(0) + a_{21}a_{22} f_1'(0) f_2'''(0)](T - a_{22} f_2'(0)) \\
&\quad - a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) + a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\} \\
&= \frac{1}{4a_{12} f_2'(0)} \{a_{11}a_{12} f_2'(0) f_1'''(0)[\beta + a_{22} f_2'(0)] - a_{21}a_{22} f_1'(0) f_2'''(0)[\beta + a_{11} f_1'(0)] \\
&\quad - a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) + a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\} \\
&= \frac{1}{4a_{12} f_2'(0)} \{\beta[a_{11}a_{12} f_2'(0) f_1'''(0) - a_{21}a_{22} f_1'(0) f_2'''(0)] \\
&\quad + [a_{11}a_{22} - a_{12}a_{21}][a_{12} f_2'(0)^2 f_1'''(0) - a_{21} f_1'(0)^2 f_2'''(0)]\} \\
&= \frac{1}{4a_{12} f_2'(0)} \Big\{\beta[a_{11}a_{12} f_2'(0) f_1'''(0) - a_{21}a_{22} f_1'(0) f_2'''(0)] \\
&\quad + \frac{-2\beta T + 1 - \beta^2}{f_1'(0) f_2'(0)} [a_{12} f_2'(0)^2 f_1'''(0) - a_{21} f_1'(0)^2 f_2'''(0)]\Big\} \\
&= \frac{1}{4a_{12} f_1'(0) f_2'(0)^2} \{\beta f_1'(0) f_2'(0)[a_{11}a_{12} f_2'(0) f_1'''(0) - a_{21}a_{22} f_1'(0) f_2'''(0)] \\
&\quad + [-\beta(a_{11} f_1'(0) + a_{22} f_2'(0)) + 1 - \beta^2][a_{12} f_2'(0)^2 f_1'''(0) - a_{21} f_1'(0)^2 f_2'''(0)]\} \\
&= \frac{1}{4a_{12} f_1'(0) f_2'(0)^2} \{\beta[a_{11}a_{21} f_1'(0)^3 f_2'''(0) - a_{12}a_{22} f_2'(0)^3 f_1'''(0)] \\
&\quad + (1 - \beta^2)[a_{12} f_2'(0)^2 f_1'''(0) - a_{21} f_1'(0)^2 f_2'''(0)]\} \\
&= \frac{a_{21} f_1'(0)}{4a_{12} f_2'(0)} \cdot \frac{f_2'''(0)}{f_2'(0)} [\beta a_{11} f_1'(0) - 1 + \beta^2] + \frac{f_1'''(0)}{4 f_1'(0)} [1 - \beta^2 - \beta a_{22} f_2'(0)].
\end{aligned} \tag{3.9}$$

From the above argument, we have the following result.

Theorem 3. *Suppose that* (H_2) *is satisfied and* $T \in (-\beta, 1-\beta)$. *Then the direction and stability of Neimark–Sacker bifurcation of* (1.1) *can be determined by the sign of* $a(D^\star)$. *Indeed, if* $a(D^\star) < 0 (> 0)$, *then the Neimark–Sacker bifurcation of* (1.1) *at* $D = D^\star$ *is supercritical* (*subcritical*) *and unique closed invariant curve bifurcating from* $(0,0)$ *is asymptotically stable* (*unstable*), *where* $D^\star$ *is given by* (2.8).

Remark 1. If the two neuron network (1.1) without self-connections modelled by a discrete-time system of the from (1.1) with $a_{11} = a_{22} = 0$:

$$x_1(n+1) = \beta x_1(n) + a_{12} f_2(x_2(n)),$$

$$x_2(n+1) = \beta x_2(n) + a_{21} f_1(x_1(n)), \quad n = 0, 1, 2, \ldots . \tag{3.10}$$

From (3.9), we can obtain

$$a(D^\star) = \frac{1-\beta^2}{4}\left(\frac{f_1'''(0)}{f_1'(0)} - \frac{a_{21} f_1'(0)}{a_{12} f_2'(0)} \cdot \frac{f_2'''(0)}{f_2'(0)}\right). \tag{3.11}$$

For $\mathrm{sgn}(f_1'(0) f_1'''(0)) = \mathrm{sgn}(f_2'(0) f_2'''(0))$, recalling for $a_{12}a_{21} f_1'(0) f_2'(0) < 0$ from the analysis of previous, we have $\mathrm{sgn}(a(D^\star)) = \mathrm{sgn}(f_1'(0) f_1'''(0)) = \mathrm{sgn}(f_2'(0) f_2'''(0))$ from (3.11). Thus we can obtain the following result.

Corollary 1. *Suppose that* (H_2) *is satisfied and* $\mathrm{sgn}\{f_1'(0) f_1'''(0)\} = \mathrm{sgn}\{f_2'(0) f_2'''(0)\}$. *Then the direction and stability of Neimark–Sacker bifurcation of* (3.10) *can be determined by the sign of* $f_k'(0) f_k'''(0)$. *Indeed, if* $f_k'(0) f_k'''(0) < 0 (> 0)$, *then the Neimark–Sacker bifurcation of* (3.10) *at* $D = D^\star$ *is supercritical* (*subcritical*) *and unique closed invariant curve bifurcating from* $(0,0)$ *is asymptotically stable* (*unstable*).

Remark 2. For the case the decay ratio β and connection weights a_{ij} in (1.1) are the functions $\beta(\tau)$, $a_{ij}(\tau)$ such that $\beta \in C(\mathbb{R}^+, (0,1))$, $a_{ij} \in C(\mathbb{R}^+, \mathbb{R})$. We can choose τ as the bifurcation parameter. From (2.8), we can obtain the critical value $\tau^\star$ of τ such that the modulus of eigenvalue $|\lambda(\tau^\star)| = 1$. If the derivative $(\frac{\mathrm{d}D(\tau)}{\mathrm{d}\tau}|)_{\tau=\tau^\star} \neq 0$, it follows that $(\frac{\mathrm{d}}{\mathrm{d}\tau}|\lambda(\tau)|)_{\tau=\tau^\star} \neq 0$. Thus, we conclude that Theorem 3 and Corollary 1 are available if $D^\star$ is replaced by $\tau^\star$. In [6], the authors consider the following system

$$x_1(n+1) = \mathrm{e}^{-\tau} x_1(n) + \alpha(1-\mathrm{e}^{-\tau}) \tanh[c_1 x_2(n)],$$

$$x_2(n+1) = \mathrm{e}^{-\tau} x_2(n) - \alpha(1-\mathrm{e}^{-\tau}) \tanh[c_2 x_1(n)], \quad n = 0, 1, 2, \cdots, \tag{3.12}$$

where $\tau, \alpha > 0$ and $c_k > 0$ for $k = 1, 2$. Obviously $T = 0$ and from (2.8), we can calculate $\tau^\star = \ln((\alpha^2 c_1 c_2 + 1)/(\alpha^2 c_1 c_2 - 1))$ and $(\frac{\mathrm{d}}{\mathrm{d}\tau}|\lambda(\tau)|)_{\tau=\tau^\star} > 0$. Thus Neimark–Sacker bifurcation occurs when $\tau = \tau^\star$ for the system (3.12). On the other hand, since $f_1(u) = \tanh(c_2 u)$ and $f_2(u) = \tanh(c_1 u)$, it follows that $f_k'(0) f_k'''(0) < 0$ for $k = 1, 2$. By corollary 1, we know that the Neimark–Sacker bifurcation of (3.12)

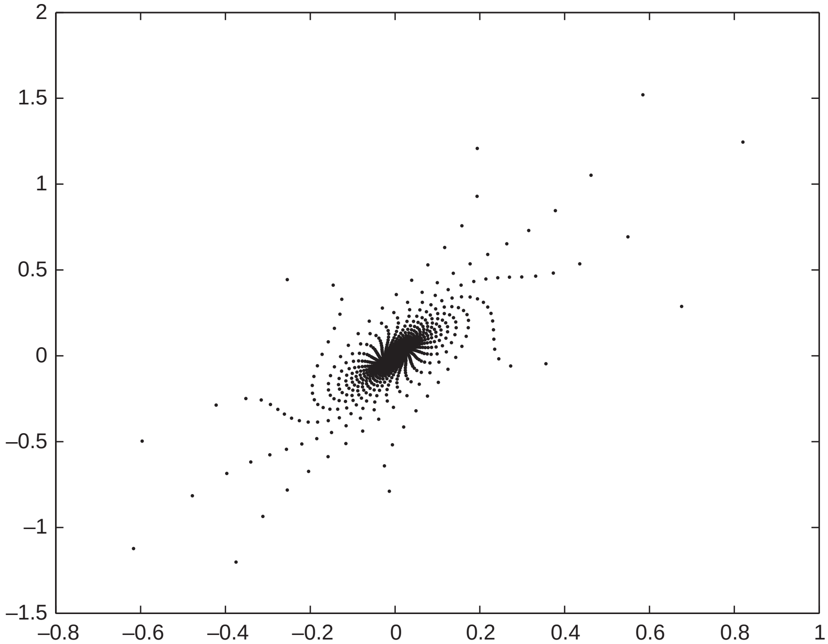

Fig. 1. The equilibrium (0, 0) is asymptotically stable.

at $\tau = \tau^{\star}$ is supercritical and the unique closed invariant curve bifurcating from (0, 0) is asymptotically stable.

Our results are very convenient to determine the direction and stability of Neimark–Sacker bifurcation of (1.1) even if $a_{kk} \neq 0$ for $k = 1, 2$.

Example. Choose $\beta = \frac{1}{2}$, $a_{11} = 1$, $a_{12} = -1$, $a_{22} = -1$ and $f_1(u) = \sin(u)$ $f_2(u) = \arctan(u/2)$ in the system (1.1). Then $f_1'(0) = 1$, $f_2'(0) = \frac{1}{2}$, $f_1''(0) = f_2''(0) = 0$, $f_1'''(0) = -1 < 0$, $f_2'''(0) = -\frac{1}{4}$. By the simple calculation, we know

$$T = \frac{a_{11} f_1'(0) + a_{22} f_2'(0)}{2} = \frac{1}{4}, \quad L_1 + D = 2(1-\beta)T - (1-\beta)^2 = 0$$

$$L_2 + D = -2(1+\beta)T - (1+\beta)^2 = -3, \quad L_3 + D = -2\beta T + 1 - \beta^2 = \tfrac{1}{2}.$$

It follows from (2.8) that $D^{\star} = \frac{1}{2}$ (the corresponding value $a_{21} = 2$), that is the Neimark–Sacker bifurcation occurs when $D = \frac{1}{2}$. If we let $a_{21} = 1.99$, it is easy to obtain $D = \frac{99}{200}$. Obviously, $(T, D) \in X_0$, this implies (0, 0) is asymptotically stable. If $a_{21} = 2.01$, then $D = \frac{101}{200} > D^{\star}$ and we have $a(D^{\star}) = -\frac{3}{8} < 0$ from (3.9). Hence, using Theorem 3, we know that there exists an asymptotically stable invariant cycle bifurcating from (0, 0). This fact is verified by the numerical simulation in Figs. 1 and 2.

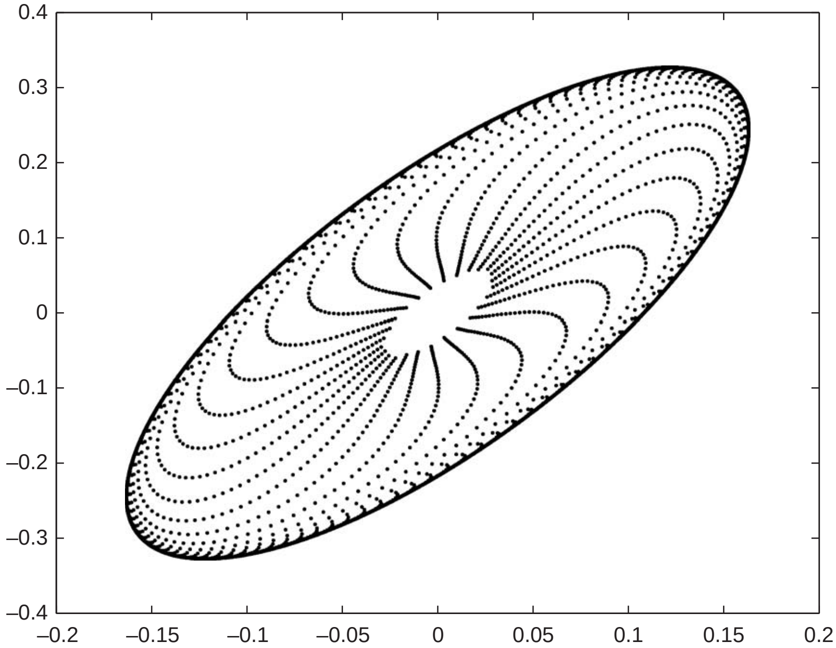

Fig. 2. An invariant closed circle bifurcates from equilibrium $(0, 0)$, where $(x_1(0), x_2(0)) = (0.01, 0.05)$.

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