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Regularity criteria for almost every function in Sobolev spaces

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Abstract

In this paper we determine the multifractal nature of almost every function (in the prevalence setting) in a given Sobolev or Besov space according to different regularity exponents. These regularity criteria are based on local L^p regularity or on wavelet coefficients and give a precise information on pointwise behavior.

Keywords: Prevalence; Sobolev spaces; Besov spaces; Calderòn–Zygmund regularity criteria; Multifractal analysis; Wavelet bases

1. Introduction

The study of regularity, and more precisely of pointwise regularity of signals or functions raised a large amount of interest in scientific communities. This topic allows a better understanding of behavior of functions and it gives also a powerful classification tool in various domains. A recent theory, based on the study of pointwise smoothness is supplied by the multifractal analysis. The multifractal analysis was introduced in order to study the velocity of turbulent flows and was initially applied to understand the behavior of some invariant measures [14,32]. It was then used in several fields, such as signal or image processing [1,2]. But in each case, the criterium of

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regularity taken into account is the Hölder exponent, and this exponent is only well defined for locally bounded functions. It would be convenient to define new criteria on more general cases. For instance, the velocity of turbulent fluids is now known not to be bounded near vorticity filaments, see [3]. In the study of turbulent flows, in [27], Leray conjectured that self-similar weak solutions of Navier–Stokes equations with initial value in $L^2(\mathbb{R}^3)$ may develop singularities in a finite time. This problem was then widely studied [26] and different behaviors were produced following the initial value problem involved. Since [8] it is known that the set of singularities of these solutions is of vanishing Hausdorff dimension. In [9], an alternative definition of regularity was supplied which can give better results in elliptic PDEs and especially when viscous solutions occur. It would thus be natural to take this notion, which involves local L^p norms to study irregularities of Navier–Stokes solutions when initial data are supposed in L^p.

Furthermore, it would be convenient to establish regularity criteria in image processing, where those properties are widely used. A natural idea would be to determine properties of the characteristic function of sets. But Hölder regularity is not adapted for classification of natural images as it does not take into account the geometry of sets and takes only two values when it is applied to characteristic functions. Furthermore, most natural images, such as clouds images or medical images are discontinuous, see [3] and thus need to be studied in a more general framework.

Let us recall the principle of multifractal analysis. The natural notion of regularity used in the study of pointwise behavior is provided by the Hölder exponent, defined as follows.

Definition 1. Let $\alpha \geqslant 0$; a function $f : \mathbb{R}^d \to \mathbb{R}$ is $C^\alpha(x_0)$ if for each $x \in \mathbb{R}^d$ such that $|x - x_0| \leqslant 1$ there exists a polynomial P of degree less than $[\alpha]$ and a constant C such that,

$$\left|f(x) - P(x - x_0)\right| \leqslant C|x - x_0|^\alpha. \tag{1}$$

The *Hölder exponent* of f at x_0 is

$$h_f(x_0) = \sup\left\{\alpha\colon\ f \in C^\alpha(x_0)\right\}.$$

In some cases, functions may have an Hölder regularity which changes wildly from point to point. Rather than measure the exact value of the Hölder exponent, one studies the fractal dimension of sets where it takes a given value. The *spectrum of singularities*, also called multifractal spectrum and denoted $d(H)$, is the function which gives for each H the Hausdorff dimension of those sets. A function is then called *multifractal* if the support of its spectrum of singularities is an interval with no empty interior.

However, the Hölder exponent has some drawbacks that prevent from using it in any situation. First, it is only defined for locally bounded functions. If a function f belongs only to L^p_{loc} this exponent is no more defined. Furthermore, as pointed by Calderòn and Zygmund in [9], it is not preserved under pseudodifferential operator of order zero, and as stated in [30] cannot thus be characterized with conditions on wavelet coefficients.

Another drawback can be emphasized with the example of Raleigh–Taylor instability. This phenomenon occurs when two fluids which are not miscible are placed on top of each other. In this case, thin filaments appear giving to the interface between the two fluids a fractal structure, see [31] for a study. To study geometric properties of this interface, one would be interested on multifractal properties of its characteristic function. Nonetheless as such functions are not continuous and take only two values, their Hölder exponent is not define, and a multifractal approach cannot be carry out.

For all these reasons, it would be convenient to define a new kind of multifractal analysis constructed with more general exponents. Such construction is started in [23,24], where the authors proposed a multifractal formalism based on Calderòn–Zygmund exponents. These exponents were introduced in [9] as an extension of Hölder exponent to L^p_{loc} functions, invariant under pseudodifferential operator of order 0.

Definition 2. Let $p \in [1, \infty]$ and $u \geqslant -\frac{d}{p}$ be fixed. A function $f \in L^p_{loc}(\mathbb{R}^d)$ belongs to $T^p_u(x_0)$ if there exist a real $R > 0$ and a polynomial P, such that $\deg(P) < u + \frac{d}{p}$, and $c > 0$ such that:

$$\forall \rho \leqslant R\colon \quad \left(\frac{1}{\rho^d} \int\limits_{\|x-x_0\| \leqslant \rho} \left|f(x) - P(x)\right|^p dx\right)^{1/p} \leqslant c\rho^u. \tag{2}$$

The p-exponent of f at x_0 is $u^p_f(x_0) = \sup\{u\colon\ f \in T^p_u(x_0)\}$.

With this definition, the usual Hölder condition $f \in C^s(x_0)$ corresponds to $f \in T^p_u(x_0)$ where $p = \infty$. One can also check that the p-exponent is decreasing as a function of p. As it was done for the Hölder exponent one can define for each p the *p-spectrum of singularities* as the Hausdorff dimension of the set of points where the p-exponent takes a given value. In [23], the authors defined the weak accessibility exponent, given as a parameter of the geometry of the set. Specifically, this weak-scaling exponent deals with the local behavior of the boundary of a set. It is thus well adapted for fractal interfaces that might appear in experimental settings. They showed that this geometrical based exponent coincides with Calderòn–Zygmund exponents of the characteristic function of the boundary of the set.

Another regularity criterium, closely related to the previous ones is given by the following definition from [30]. With this exponent we can have a better understanding of the link between Calderòn–Zygmund exponents, Hölder exponent and the pointwise behavior of functions.

Definition 3. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function or a distribution and $x_0 \in \mathbb{R}^d$ be fixed. The *weak-scaling exponent* of f at x_0 is the smallest real number $\beta(f, x_0)$ satisfying:

1. $\beta(f, x_0) \geqslant u^p_f(x_0)\ \forall p \geqslant 1$.
2. $\beta(f, x_0) = s \Leftrightarrow \beta(\frac{\partial f}{\partial x_j}, x_0) = s - 1,\ j = 1, \ldots, d$.

Similarly, we define the *weak-scaling spectrum*, denoted by $d_{ws}(\beta)$ as the Hausdorff dimension of sets of points where $\beta(f, x)$ takes a given value β. As we will see later, the weak-scaling exponent can be fully characterized by conditions on wavelet coefficients.

In practical applications, the classical multifractal spectrum cannot be computed directly, as it takes into account intricate limits. Thus, some formulas, called *multifractal formalisms* were introduced in purpose to link the spectrum of singularities to some calculable quantities. There are indeed two formalisms based on conditions on wavelet coefficients. Historically the classical multifractal formalism stated in [13] was based directly on wavelet coefficients. Actually, this formula gave unexpected results and was shown to be false in several cases. It is nowadays known that it is the weak-scaling exponent which is involved in this formula. A second multifractal formalism, developed in [22] is based on "wavelet leaders", which can be seen as the theoretical counterpart of the "Wavelet Transform Modulus Maxima" used in [4]. This "wavelet leader" based formalism actually gives the spectrum of singularities in term of Hölder exponent. The weak-scaling exponent is thus more appropriated in order to understand the classical multifractal formalism. This exponent is also more stable under the action of differential operators. Furthermore it gives an additional information on the behavior of functions thanks to the following definition.

Definition 4. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}^d$. We say that x_0 is a *cusp* singularity for f if $\beta(f, x_0) = h_f(x_0)$. If $\beta(f, x_0) > h_f(x_0)$, x_0 is said to be an *oscillating singularity.*

An example of oscillating function at $x_0 = 0$ is given by $f(x) = |x|\sin(1/|x|)$. Here, $h_f(0) = 1$ while $\beta(f, 0) = +\infty$. And we have a cusp singularity when the behavior of the function at x_0 is like $|x|^\alpha$ but also like $|x|^\alpha + |x|\sin(1/|x|)$. Indeed we talk about a cusp singularity when the function does not have oscillations at a point, or if those oscillations are hidden by the Hölder behavior.

Many authors have studied generic values of the Hölder exponent in function spaces. In 1931 Banach [5], proved that the pointwise regularity of quasi-all, in a topological sense, continuous functions is zero. Here quasi-all means that this property is true in a countable intersection of dense open sets. Since then, Hunt in [15] showed that the same result is satisfied by measure theoretic almost every continuous functions. Recently, results such as those of [25] and [12] studied Hölder regularity of generic functions in Sobolev spaces in both senses. Whereas a large study of regularity properties for generic sets, there exists no result on genericity of Calderòn–Zygmund exponents or of weak-scaling exponent. Our purpose here is to provide a genericity result of those exponents in given Sobolev and Besov spaces, with the measure-theoretic notion of genericity supplied by prevalence.

Prevalence is a measure-theoretic notion of genericity on infinite dimensional spaces. In a finite dimensional space, the notion of genericity in a measure theoretic sense is supplied by the Lebesgue measure. The particular role played by this measure is justified by the fact that this is the only one which is σ-finite and invariant under translation. In a metric infinite dimensional space no measure enjoys these properties. The proposed alternative is to replace conditions on the measure by conditions on sets, see [6,10,17,16] and to take the following definition.

Definition 5. Let V be a complete metric vector space. A Borel set B in V is called Haar-null if there exists a probability measure μ with compact support such that

$$\mu(B + v) = 0 \quad \forall v \in V. \tag{3}$$

In this case the measure μ is said to be transverse to B.
A subset of V is called Haar-null if it is contained in a Borel Haar-null set.
The complement of a Haar-null set is called a prevalent set.

With a slight abuse of language we will say that a property is satisfied almost everywhere when it holds on a prevalent set.

Let us recall properties of Haar-null sets, see [10,17] and show how they generalize notion of Lebesgue measure zero sets.

Proposition 1.

1. *If S is Haar-null, then $\forall x \in V$, $x + S$ is Haar-null.*
2. *If* $\dim(V) < \infty$*, S is Haar-null if and only if* $\mathrm{meas}(S) = 0$ (*where* meas *denotes the Lebesgue measure*).
3. *Prevalent sets are dense.*
4. *If S is Haar-null and $S' \subset S$ then S' is Haar null.*
5. *The union of a countable collection of Haar-null sets is Haar null.*
6. *If* $\dim(V) = \infty$*, compact subsets of V are Haar-null.*

Remarks. Several kinds of measures can be used as transverse measures for a Borel set. Let us give two examples of transverse measure.

1. A finite dimensional space P is called a probe for a set $T \subset V$ if the Lebesgue measure on P is transverse to the complement of T. Those measures are not compactly supported probability measures. However one immediately checks that this notion can also be defined in the same way but stated with the Lebesgue measure defined on the unit ball of P. Note that in this case, the support of the measure is included in the unit ball of a finite dimensional subspace. The compactness assumption is therefore fulfilled.
2. If V is a function space, a probability measure on V can be defined by a random process X_t whose sample paths are almost surely in V. The condition $\mu(f + A) = 0$ means that the event $X_t - f \in A$ has probability zero. Therefore, a way to check that a property $\mathcal{P}$ holds only on a Haar-null set is to exhibit a random process X_t whose sample paths are in V and is such that

$$\forall f \in V, \text{ a.s.} \quad X_t + f \text{ does not satisfy } \mathcal{P}.$$

These properties, such as several examples of prevalent results can be found in the survey [16].

1.1. Statement of main results

The purpose of this paper is stated by the two following theorems which give the multifractal properties of almost every functions with regard to exponents defined in the previous section.

Theorem 1. *Let $s_0 \geqslant 0$ and $1 \leqslant p_0 < \infty$ be fixed.*

1. *For all $p \geqslant 1$ such that $s_0 - \frac{d}{p_0} > -\frac{d}{p}$ the p-spectrum of singularities of almost every function in $L^{s_0,p_0}(\mathbb{R}^d)$ is given by*

$$\forall u \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_p(u) = p_0(u - s_0) + d. \tag{4}$$

2. *For almost every function in $L^{s_0,q_0}(\mathbb{R}^d)$ the spectrum of singularities for the weak-scaling exponent is given by*

$$\forall \beta \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_{ws}(\beta) = p_0(\beta - s_0) + d. \tag{5}$$

This result in Sobolev spaces has an analogous in the Besov setting. Furthermore, Besov spaces are useful when wavelets are involved as it is the case here, those spaces having a simpler characterization.

Theorem 2. *Let $s_0 \geqslant 0$ and $0 < q, p_0 < \infty$ be fixed.*

1. *For all $p \geqslant 1$ such that $s_0 - \frac{d}{p_0} > -\frac{d}{p}$ the p-spectrum of singularities of almost every function in $B^{s_0,q}_{p_0}(\mathbb{R}^d)$ is given by*

$$\forall u \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_p(u) = p_0(u - s_0) + d. \tag{6}$$

2. *For almost every function in $B^{s_0,q}_{p_0}(\mathbb{R}^d)$ the spectrum of singularities for the weak-scaling exponent is given by*

$$\forall \beta \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_{ws}(\beta) = p_0(\beta - s_0) + d. \tag{7}$$

These theorems seem a bit surprising. Let us compare them with the following proposition from [12].

Proposition 2.

- *If $s - d/p \leqslant 0$, then almost every function in $L^{p,s}$ is nowhere locally bounded, and therefore its spectrum of singularities is not defined.*
- *If $s - d/p > 0$, then the Hölder exponent of almost every function f of $L^{p,s}$ takes values in $[s - d/p, s]$ and*

$$\forall H \in [s - d/p, s] \quad d_f(H) = Hp - sp + d. \tag{8}$$

Thus the main change from [12] is given by the fact that here β can take negative values. Indeed, our present theorems give a generic regularity in Sobolev or in Besov spaces that are not imbedded in global Hölder spaces. Even if in such spaces, the classical spectrum of singularities is not define for a prevalent set, we have an idea of the pointwise behavior of almost every distribution. In the other case, when $s_0 - \frac{d}{p_0} > 0$ and the spectrum of singularities exists, it coincides with the above spectra for almost every function in Besov spaces. Therefore, in the second case we generalize in this paper the result of [12] to more stable exponents.

In [28], it was also proved that in those spaces quasi-all functions, in the Baire's sense, have no oscillating singularities. Furthermore, presence of oscillating singularities is linked with the failure of the multifractal formalism in [33]. And in [11], it was already proven that almost every function in Besov spaces satisfies the multifractal formalism. The main result of this paper together with Definition 4 show that even if weak-scaling and Hölder exponents do not coincide they share the same spectrum. Thus, in the prevalence setting, oscillating singularities appear as an exceptional behavior in regular Sobolev or Besov spaces.

Another remark can be made thanks to the following proposition from [24] and from [34] that gives an upper bound for the p-spectrum.

Proposition 3. *Let $f \in B_{p_0}^{s_0,p_0}(\mathbb{R}^d)$, where $s_0 > 0$ and let $p \geqslant 1$ be such that $s_0 - \frac{d}{p_0} > -\frac{d}{p}$. Then*

$$\forall u \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_p(u) \leqslant p_0 u - s_0 p_0 + d. \tag{9}$$

This proposition together with Theorem 2 show that the generic regularity for p criteria is as bad as possible.

In Section 2 we will prove Theorems 1 and 2. For the sake of completeness, we first have to define our main tool which is given by wavelet expansions of functions. Wavelets are naturally present in multifractal analysis, see for instance [2]. Furthermore, in our case it allows a characterization of both functional spaces and pointwise regularities.

1.2. Wavelet expansions

There exist $2^d - 1$ oscillating functions $(\psi^{(i)})_{i\in\{1,\dots,2^d-1\}}$ in the Schwartz class such that the functions

$$2^{dj}\psi^{(i)}\left(2^j x - k\right), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$, see [29]. Wavelets are indexed by dyadic cubes $\lambda = [\frac{k}{2^j}; \frac{k+1}{2^j}[^d$. Thus, any function $f \in L^2(\mathbb{R}^d)$ can be written:

$$f(x) = \sum c_{j,k}^{(i)}\psi^{(i)}\left(2^j x - k\right)$$

where

$$c_{j,k}^{(i)} = 2^{dj}\int f(x)\psi^{(i)}\left(2^j x - k\right)dx.$$

(Note that we use an L^∞ normalization instead of an L^2 one, which simplifies the formulas.) If $p > 1$ and $s > 0$, Sobolev spaces have thus the following characterization, see [29]:

$$f \in L^{s,p}\left(\mathbb{R}^d\right) \iff \left(\sum_{\lambda\in\Lambda}|c_\lambda|^2\left(1 + 4^{js}\right)\chi_\lambda(x)\right)^{1/2} \in L^p\left(\mathbb{R}^d\right), \tag{10}$$

where $\chi_\lambda(x)$ denotes the characteristic function of the cube λ and Λ is the set of all dyadics cubes. Homogeneous Besov spaces, which will also be considered, are characterized (for $p, q > 0$ and $s \in \mathbb{R}$) by

$$f \in B_p^{s,q}\left(\mathbb{R}^d\right) \iff \sum_j\left(\sum_{\lambda\in\Lambda_j}|c_\lambda|^p 2^{(sp-d)j}\right)^{q/p} \leqslant C \tag{11}$$

where Λ_j denotes the set of dyadics cubes at scale j, see [29].

Hölder pointwise regularity can also be expressed in term of wavelet coefficients, see [18].

Proposition 4. *Let x be in $\mathbb{R}^d$. If f is in $C^\alpha(x)$ then there exists $c > 0$ such that for each λ:*

$$|c_\lambda| \leqslant c2^{-\alpha j}\big(1 + \big|2^j x - k\big|\big)^\alpha. \tag{12}$$

This proposition is not a characterization. If for any $\varepsilon > 0$, a function does not belong to $C^\varepsilon(\mathbb{R}^d)$ one cannot express its pointwise Hölder regularity in term of condition on wavelet coefficients. This is an advantage of Calderòn–Zygmund exponent since, as showed in [21], it can be linked to wavelet expansion without global regularity assumption.

Definition 6. Let x_0 be in $\mathbb{R}^d$ and $j \geqslant 0$. We denote by $\lambda_j(x_0)$ the unique dyadic cube of width 2^{-j} which contains x_0. And we denote

$$3\lambda_j(x_0) = \lambda_j(x_0) + \left[-\frac{1}{2^j}, \frac{1}{2^j}\right]^d.$$

Furthermore, we define the local square function by

$$S_f(j, x_0)(x) = \left(\sum_{\lambda \subset 3\lambda_j(x_0)} |c_\lambda|^2 \mathbf{1}_\lambda(x)\right)^{1/2}.$$

Proposition 5. *Let $p \geqslant 1$ and $s \geqslant 0$; if $f \in T^p_{s-\frac{d}{p}}(x_0)$, then $\exists C > 0$ such that wavelet coefficients of f satisfy for all $j \geqslant 0$*

$$\big\|S_f(j, x_0)\big\|_{L^p} \leqslant c2^{-j(u+d/p)}. \tag{13}$$

Conversely if (13) *holds and if $s - \frac{d}{p} \notin \mathbb{N}$ then $f \in T^p_{s-\frac{d}{p}}(x_0)$.*

As far as we are concerned, we do not need a characterization but a weaker condition which is given by the following proposition from [23].

Proposition 6. *Let $p \geqslant 1$ and $s \geqslant 0$; if $f \in T^p_{s-\frac{d}{p}}(x_0)$, then $\exists A, C > 0$ such that wavelet coefficients of f satisfy*

$$\exists C\ \forall j \quad 2^{j(sp-d)} \sum_{|k-2^j x_0| \leqslant A2^j} |c_{j,k}|^p \big(1 + \big|k - 2^j x_0\big|\big)^{-sp} \leqslant Cj. \tag{14}$$

Furthermore, it is also proved in [23] that the p-exponent can be derived from wavelet coefficients.

Proposition 7. *Let $p \geqslant 1$ and $f \in L^p_{loc}$. Define*

$$\varSigma^p_j(s, A) = 2^{j(sp-d)} \sum_{|k-2^j x_0| \leqslant A2^j} |c_{j,k}|^p \big(1 + \big|k - 2^j x_0\big|\big)^{-sp}, \tag{15}$$

for $A > 0$ small enough. And denote

$$i_p(x_0) = \sup\left\{s\colon\ \liminf \frac{\log(\Sigma_j^p(s, A)^{1/p})}{-j\log 2} \geqslant 0\right\}. \tag{16}$$

Then the following inequality always holds

$$u_f^p(x_0) \leqslant i_p(x_0) - \frac{d}{p}. \tag{17}$$

If furthermore there exists $\delta > 0$ such that $f \in B_p^{\delta,p}$ then the p-exponent of f satisfies

$$u_f^p(x_0) = i_p(x_0) - \frac{d}{p}. \tag{18}$$

As seen previously, the p-exponent is also related to the weak-scaling exponent. This one can also be expressed in term of wavelet coefficients, thanks to its relation with two-microlocal spaces, defined in [7].

Definition 7. Let s and s' be two real numbers. A distribution $f : \mathbb{R}^d \to \mathbb{R}$ belongs to the two-microlocal space $C^{s,s'}(x_0)$ if its wavelet coefficients satisfy that there exists $c > 0$ such that

$$\forall j, k \quad |c_{j,k}| \leqslant c2^{-sj}\left(1 + \left|2^j x_0 - k\right|\right)^{-s'}. \tag{19}$$

In [30] the following characterization of the weak-scaling exponent is given.

Proposition 8. *A tempered distribution f belongs to $\Gamma^s(x_0)$ if and only if there exists $s' < 0$ such that f belongs to $C^{s,s'}(x_0)$.*

The weak-scaling exponent of f is

$$\beta(f, x_0) = \sup\left\{s\colon\ f \in \Gamma^s(x_0)\right\}. \tag{20}$$

But we will rather take the following alternative characterization from [23] that gives a simpler condition in term of wavelet coefficients.

Proposition 9. *Let f be a tempered distribution. The weak-scaling exponent of f at x_0 is the supremum of $s > 0$ such that*:

$$\forall \varepsilon > 0\ \exists c > 0\ \forall (j, k) \text{ such that } \quad \left|2^j x_0 - k\right| < 2^{\varepsilon j}, \qquad |c_{j,k}| \leqslant c2^{-(s-\varepsilon)j}. \tag{21}$$

2. Proofs of Theorems 1 and 2

2.1. The p-spectrum

In this section, we only prove the first point of Theorem 2. We will see how this proof can be adapted to Theorem 1 in a second time.

In a first part, the result that we prove is more precise than the one stated. Indeed, we prove that for each $\alpha \in (1,\infty)$ and for each $p \geqslant 1$, the p-exponent of almost every function of $B^{s_0,p_0}_{p_0}(\mathbb{R}^d)$ is smaller than

$$s - \frac{d}{p} + \frac{d}{\alpha p} \tag{22}$$

on a set of Hausdorff dimension greater than $\frac{d}{\alpha}$.

These fractal sets are closely related to the dyadic approximation of points.

Definition 8. Let $\alpha \in (1,\infty)$ be fixed. We denote

$$F_\alpha = \left\{x\colon\ \exists \text{ a sequence } \big((k_n, j_n)\big)_{n\in\mathbb{N}} \left|x - \frac{k_n}{2^{j_n}}\right| \leqslant \frac{1}{2^{\alpha j_n}}\right\}. \tag{23}$$

This set F_α can also be defined as

$$\limsup_{i\to\infty} \bigcup_{l\in\mathbb{N}^d} F_\alpha^{i,l}$$

where $F_\alpha^{i,l}$ denotes the cube $\frac{l}{2^i} + [-\frac{1}{2^{\alpha i}}; \frac{1}{2^{\alpha i}}]^d$.

If $x \in F_\alpha$ it is said α-approximable by dyadics. The dyadic exponent of x is defined by $\alpha(x_0) = \sup\{\alpha\colon\ x_0 \text{ is } \alpha\text{-approximable by dyadics}\}$.

As stated in [20], the Hausdorff dimension of F_α is at least $\frac{d}{\alpha}$.

In order to prove our result we show that the set of functions where for α and $p \geqslant 1$ given, the p-exponent is larger than (22) at a point of F_α is included in a countable union of Haar-null Borel sets.

Let $p \geqslant 1$ be given such that $s_0 - \frac{d}{p_0} > -\frac{d}{p}$. For $\alpha \geqslant 1$ fixed we denote $s(\alpha) = s_0 - \frac{d}{p_0} + \frac{d}{\alpha p_0} + \frac{d}{p}$. For $\varepsilon > 0$ fixed, let $\beta = s(\alpha) + \varepsilon$. We first check that the set of functions in $B^{s_0,p_0}_{p_0}$ satisfying (14) with exponent β at a point in F_α is a Haar-null Borel set. This set can be included in a countable union over $A > 0$ and $c > 0$ of sets $M(A,c)$ which are sets of functions in $B^{s_0,p_0}_{p_0}(\mathbb{R}^d)$ satisfying

$$\exists x \in F_\alpha\ \forall j \quad 2^{j(\beta p - d)} \sum_{|k-2^j x| \leqslant A2^j} |c_\lambda|^p \big(1 + \big|k - 2^j x\big|\big)^{-\beta p} \leqslant c.$$

And for each $i \in \mathbb{N}$ these sets can be included in the countable union over $l \in \{0,\dots,2^i - 1\}^d$ of $M_{i,l}(A,c)$, defined by the set of f such that

$$\exists x \in F_\alpha^{i,l}\ \forall j \quad 2^{j(\beta p - d)} \sum_{|k-2^j x| \leqslant A2^j} |c_\lambda|^p \big(1 + \big|k - 2^j x\big|\big)^{-\beta p} \leqslant c.$$

Each $M_{i,l}(A,c)$ is a closed set. Indeed, suppose that a sequence (f_n) of elements of $M_{i,l}(A,c)$ converges to f in $B^{s_0,p_0}_{p_0}(\mathbb{R}^d)$. Denote $c^n_{j,k}$ the wavelet coefficients of f_n, for each $n \in \mathbb{N}$, and $c_{j,k}$ those of f. The mapping giving the wavelet coefficients of a function f in a Besov space is continuous, thus for each j,k $c^n_{j,k}$ converge to $c_{j,k}$. Furthermore for each n there exists $x_n \in F^{i,l}_\alpha$ such that f_n satisfies (14) at x_n. Thus

$$\forall j \quad 2^{j(\beta p-d)} \sum_{|k_n-2^j x_n|\leqslant A2^j} \left|c^n_\lambda\right|^p \left(1+\left|k_n-2^j x_n\right|\right)^{-\beta p} \leqslant c. \tag{24}$$

As $F^{i,l}_\alpha$ is a compact set, there exists an accumulation point $x \in F^{i,l}_\alpha$ of x_n. Furthermore, if k_n is such that $|k_n - 2^j x_n| \leqslant A2^j$ for a subsequence $x_{\phi(n)}$ such that $\lim x_{\phi(n)} = x$, the corresponding $k_{\phi(n)}$ converges to k with $|k-2^j x| \leqslant A2^j$. Thus up to a subsequence, when n tends to infinity, (24) becomes

$$\forall j \quad 2^{j(\beta p-d)} \sum_{|k-2^j x|\leqslant A2^j} |c_\lambda|^p \left(1+\left|k-2^j x\right|\right)^{-\beta p} \leqslant c.$$

Consequently f belongs to $M_{i,l}(A,c)$ and $M(A,c)$ is a Borel set.

To prove that it is also a Haar-null set, we construct a probe as transverse measure, in this way the compactness assumption is clearly satisfied. This probe is based on a slight modification of the "saturating function" introduced in [20].

Let $i \in \mathbb{N}$ and $l \in \{0,\dots,2^i-1\}^d$ be fixed. Let $n \in \mathbb{N}$ be fixed large enough such that $N = 2^{dn} > \frac{d}{p\alpha\varepsilon} + 1$. Each dyadic cube λ is split into M subcubes of size $2^{-d(j+n)}$. For each index $m \in \{1,\dots,N\}$, we choose a subcube $i(\lambda)$ and the wavelet coefficient of g_i is given by:

$$d^m_\lambda = \begin{cases} \frac{1}{j^a} 2^{(\frac{d}{p_0}-s_0)j} 2^{-\frac{d}{p_0}J} & \text{if } m = i(\lambda), \\ 0 & \text{else,} \end{cases} \tag{25}$$

where $a = \frac{2}{p_0}$ and $J \leqslant j$ and $K \in \{0,\dots 2^J-1\}^d$ are such that

$$\frac{k}{2^j} = \frac{K}{2^J}$$

is an irreducible form. It is proven in [12] that these functions belong to $B^{s_0,p_0}_{p_0}$.

Furthermore, if a point $x \in (0,1)^d$ is α-approximable by dyadics, there exists a subsequence (j_n,k_n) where $j_n = [J_n\alpha]$, J_n and K_n being defined in (23) and k_n is such that $\frac{k_n}{2^{j_n}} = \frac{K_n}{2^{J_n}}$. The corresponding wavelet coefficients of all functions g_m satisfy that there exists a constant $c>0$ such that if (j,k) satisfy $|x_0 - \frac{k}{2^j}| < A$:

$$d^m_{j,k} > c(A) \frac{2^{(\frac{d}{p_0}-s_0)j} 2^{-\frac{d}{\alpha p_0}j}}{j^a}. \tag{26}$$

Let $f = \sum c_{j,k}\psi_{j,k}$ be an arbitrary function in $B^{s_0,p_0}_{p_0}(\mathbb{R}^d)$. Suppose that there exist two points $\gamma_1 \in \mathbb{R}^N$ and $\gamma_2 \in \mathbb{R}^N$ such that for $a = 1,2$, $f + \sum_m \gamma^m_a g^m$ belong to $M_{i,l}(A,c)$. By definition there also exist two points x_1 and x_2 in $F^{i,l}_\alpha$ such that, for $a=1,2$,

$$\forall j \quad 2^{j(\beta p-d)} \sum_{|k-2^j x_a|\leqslant A2^j} \left|c_\lambda + \sum_{m=1}^N \gamma^m_a d^m_\lambda\right|^p \left(1+\left|k-2^j x\right|\right)^{-\beta p} \leqslant c.$$

As $\beta > 0$, this condition implies:

$$\forall j \quad 2^{j(\beta p-d)} \sum_{|k-2^j x_a| \leqslant A2^j} \left| c_\lambda + \sum_{m=1}^{N} \gamma_a^m d_\lambda^m \right|^p \left(1 + A2^j\right)^{-\beta p} \leqslant c.$$

But x_1 and x_2 belong to same dyadic cubes of size $j > i$. Thus the same k satisfies $|k - 2^j x_a| \leqslant A2^j$ for $a = 1, 2$ and wavelet coefficients of $f_1 - f_2$ are such that for all $j > \alpha i$

$$2^{j(\beta p-d)} \sum_{|k-2^j x_a| \leqslant A2^j} \left| \sum_{m=1}^{N} (\gamma_1^m - \gamma_2^m) d_\lambda^m \right|^p \left(1 + A2^j\right)^{-\beta p} \leqslant 2c.$$

It is obvious that

$$\begin{aligned} &2^{j(\beta p-d)} \sum_{|\frac{k}{2^j} - x_1| \leqslant A} \left| \sum_m (\gamma_1^m - \gamma_2^m) d_\lambda^m \right|^p 2^{-\beta p j} \left(2^{-j} + \left|\frac{k}{2^j} - x_1\right|\right)^{-\beta p} \\ &\quad \geqslant 2^{j(\tilde{s} p-d)} \sup_{|\frac{k}{2^j} - x_1| \leqslant A} \left| \sum_m (\gamma_1^m - \gamma_2^m) d_\lambda^m \right|^p 2^{-\tilde{s} p j} \left(2^{-j} + \left|\frac{k}{2^j} - x_1\right|\right)^{-\tilde{s} p}. \end{aligned}$$

Using definition of function g_m, if for each j we define $j' = j + n$, at scale j' there is only one function g_m with nonzero coefficient. And with (26) one finally obtains that there exists a subsequence j such that

$$\begin{aligned} &2^{n(\beta p-d)} 2^{j(\beta p-d)} \sup_{|\frac{k}{2^j} - x_1| \leqslant A} \left| \sum_m (\gamma_1^m - \gamma_2^m) d_\lambda^m \right|^p 2^{-\beta p j} \left(2^{-j} + \left|\frac{k}{2^j} - x_1\right|\right)^{-\tilde{s} p} \\ &\quad \geqslant \left|\gamma_1^i - \gamma_2^i\right|^p \tilde{c}^p \frac{1}{j^{pa}} 2^{p\varepsilon j}, \end{aligned}$$

where $\tilde{c}$ depends only on n and A.

Those two inequalities imply that

$$\|\gamma_1 - \gamma_2\|_\infty^p \leqslant 2cc(N) i^{1/p_0} 2^{-\varepsilon \alpha p i}. \tag{27}$$

Therefore the set of γ such that $f + \sum_i \gamma^m g^m$ belongs to $M_{i,l}(A, c)$ is included in a ball of radius less than $(2cc(N))^N i^{N/p_0} 2^{-\varepsilon \alpha p N i}$. Taking the countable union over l, we obtain that for each i_0 fixed, the set of γ satisfying

$$\exists x \in F_\alpha^{i_0} \text{ such that } \quad f + \sum_m \gamma^m g^m \text{ satisfies (24) at } x$$

is of Lebesgue measure bounded by

$$\sum_{i=i_0}^{\infty}\bigl(2cc(N)\bigr)^N i^{N/p_0}2^{di-\varepsilon\alpha pNi}.$$

As N is large enough, this measure tends to zero when i_0 tends to infinity. And $M(A,c)$ is then a Haar-null set.

As this result does not depend on c or on A, we can take the union over countable $c_n > 0$ and $A_n > 0$. Then the set of functions in $B_{p_0}^{s_0,p_0}(\mathbb{R}^d)$ belonging to $T_\beta^p(x)$ at a point $x \in F_\alpha$ is a Haar-null set.

Thus,

$$\forall p \geqslant 1,\ \forall\alpha \geqslant 1\ \forall\beta > s(\alpha) \text{ a.s. in } B_{p_0}^{s_0,p_0}\ \forall x \in F_\alpha \quad u_f^p(x) \leqslant \beta.$$

Taking $\varepsilon \to 0$ it follows by countable intersection that

$$\forall p \geqslant 1,\ \forall\alpha \geqslant 1 \text{ a.s. in } B_{p_0}^{s_0,p_0}\ \forall x \in F_\alpha \quad u_f^p(x) \leqslant s(\alpha).$$

Therefore, if α_n is a dense sequence in $(1,\infty)$, using the same argument, one obtains that

$$\forall p \geqslant 1,\ \text{a.s. in } B_{p_0}^{s_0,p_0}\ \forall n \in \mathbb{N}\ \forall x \in F_{\alpha_n} \quad u_f^p(x) \leqslant s(\alpha_n). \tag{28}$$

Let f be a function satisfying (28) and $\alpha \geqslant 1$ be fixed. Let $\alpha_{\phi(n)}$ be a nondecreasing subsequence of α_n converging to α. Then the intersection E_α of F_{α_n} contains F_α and for all $x \in E_\alpha$, and thus for all $x \in F_\alpha$, $u_f^p(x) \leqslant s(\alpha)$. Furthermore, see [19], there exists a measure m_α positive on F_α but such that every set of dimension less than $\frac{d}{\alpha}$ is of measure zero. Let us denote by G_H the set of points where $u_p(x) < H$. According to Proposition 3, this set can be written as a countable union of sets of m_α measure zero. Thus, we obtain

$$m_\alpha\bigl(\bigl\{x\colon u_p(x) = H\bigr\}\bigr) = m_\alpha(F_\alpha \backslash G_H) > 0.$$

Which gives us the p spectrum of singularities

$$\forall u \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_p(u) = p_0 u + d - s_0 p_0.$$

This proof does not depend on the choice of q. It can then be extended in the same way for any Besov space $B_{p_0}^{s_0,q}$ for $0 \leqslant q < \infty$.

The proof for the Sobolev case is similar. The functions g_m defined in (25) also belong to $B_{p_0}^{s_0,1}$. Since $B_{p_0}^{s_0,1} \hookrightarrow L^{s_0,p_0}$, the g_m belong to L^{s_0,p_0} and the remaining of the proof is unchanged.

2.2. Generic values of the weak-scaling spectrum

We now prove of the second point of Theorems 1 and 2. As in the previous case, we prove Theorem 2 using the same argument as in the previous part giving the Sobolev case.

Proposition 10. *Let $s_0 > 0$ and $0 \leqslant p_0, q < \infty$ be fixed. For almost every function in $B^{s_0,q}_{p_0}$ the spectrum of singularities for the weak-scaling exponent is given by*

$$\forall \beta \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_{ws}(\beta) = p_0(\beta - s_0) + d. \tag{29}$$

Proof. Let $\alpha \geqslant 1$ be fixed and denote by F_α the set of Definition 8. Let $\varepsilon > 0$ be fixed and define $\beta = s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha} + \varepsilon$.

According to Proposition 9, we first have to show that for a given $c > 0$ the set:

$$M_{\alpha,c} = \left\{ f = \sum c_\lambda \psi_\lambda \in B^{s_0,q}_{p_0}\colon\ \exists x \in F_\alpha\ \forall \varepsilon' > 0\ \forall (j,k)\ \left|2^j x - k\right| \leqslant 2^{\varepsilon' j}\ |c_\lambda| \leqslant c 2^{-(\beta-\varepsilon')j} \right\} \tag{30}$$

is a Borel Haar-null set.

Let us remark that for all $i \in \mathbb{N}$, this set is included in the countable union of:

$$M_{\alpha,c}(i,l) = \left\{ f \in B^{s_0,q}_{p_0}\colon\ \exists x \in F^{i,l}_\alpha\ \forall \varepsilon' > 0\ \forall (j,k)\ \left|2^j x - k\right| \leqslant 2^{\varepsilon' j}\ |c_\lambda| \leqslant c 2^{-(\beta-\varepsilon')j} \right\}. \tag{31}$$

One easily checks that $M_{\alpha,c}(i,l)$ is closed and therefore that $M_{\alpha,c}$ is a Borel set.

To prove that $M_{\alpha,c}$ is also Haar-null, we use a different transverse measure than in the previous section, by taking the measure induced by a stochastic process. As $M_{\alpha,c}$ depends only on the dyadic properties of points, we can also restrict the proof to $[0,1]^d$. Consider the following stochastic process on $[0,1]^d$:

$$X_x = \sum_{j=0}^{\infty} \sum_{\lambda \in [0,1]^d} \varepsilon_{j,k} \frac{2^{-(s_0 - \frac{d}{p_0})j} 2^{-\frac{d}{p_0}J}}{j^a} \psi\left(2^j x - k\right) \tag{32}$$

where J and a are defined as in (25) and $\{\varepsilon_{j,k}\}_{j,k}$ is a Rademacher sequence. That is the $\varepsilon_{j,k}$ are i.i.d. random variables such that

$$\mathbb{P}(\varepsilon_{j,k} = 1) = \mathbb{P}(\varepsilon_{j,k} = -1) = \frac{1}{2}.$$

This process belongs to $B^{s_0,q}_{p_0}$. Furthermore, the measure defined by this stochastic process is supported by the continuous image of a compact set. Thus, $(X_x)_{x \in [0,1]^d}$ defines a compactly supported probability measure on $B^{s_0,q}_{p_0}$.

Let f be an arbitrary function in $B^{s_0,q}_{p_0}(\mathbb{R}^d)$. Thanks to Fubini's theorem, it is sufficient to prove that for all $x \in F_\alpha$, almost surely, condition (21) is not satisfied by $f + X$.

Let $x_0 \in F_\alpha$ be fixed and suppose that $f + X$ satisfies condition (21) at x_0. Then for all $\varepsilon' > 0$ and for all (j,k) such that $|k - 2^j x_0| \leqslant 2^{\varepsilon' j}$,

$$\left| c_{j,k} + \varepsilon_{j,k} \frac{2^{-(s_0 - \frac{d}{p_0})j} 2^{-\frac{d}{p_0}J}}{j^a} \right| \leqslant c 2^{-(\beta-\varepsilon')j}.$$

Taking (J_n, K_n) the sequence of Definition 8, $j = [\alpha J_n]$ and $k = \frac{K_n 2^j}{2^{J_n}}$ one obtains that there exists a sequence (j,k) such that $|2^j x_0 - k| \leqslant 1$ and the following property holds:

$$\varepsilon_{j,k} = c_{j,k} j^a 2^{(s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha})j} + o\big(2^{-(\varepsilon-\varepsilon')j}\big).$$

Taking $\varepsilon' = \frac{\varepsilon}{2}$, one obtains that $\varepsilon_{j,k} \sim c_{j,k} j^a 2^{(s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha})j_n}$ when $j_n \to \infty$. Since the $c_{j,k}$ are deterministic, this result implies that there exists an infinite sequence of independent stochastic variables which are deterministic. This event is of probability zero and $M_{\alpha,c}$ is a Haar-null set.

Therefore, taking countable unions over $c > 0$ and $\varepsilon \to 0$, it follows that for all $\alpha \geqslant 1$, the set of functions in $B^{s_0,q}_{p_0}$ with a weak-scaling exponent greater than $s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha}$ at some point of F^α is a Haar-null set.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a dense sequence in $(1,\infty)$ and take a countable union over α_n. We finally obtain

$$\text{a.s. in } B^{s_0,q}_{p_0}\big(\mathbb{R}^d\big)\ \forall n \in \mathbb{N}\ \forall x \in F_{\alpha_n} \quad \beta(f,x) \leqslant s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha_n}.$$

With a similar argument as in Section 2.1, one can prove that:

$$\text{a.s. in } B^{s_0,q}_{p_0}\big(\mathbb{R}^d\big)\ \forall \alpha \geqslant 1\ \forall x \in F_{\alpha} \quad \beta(f,x) \leqslant s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha}. \tag{33}$$

Furthermore, we saw in Section 2.1 that there exists a measure m_α which is positive on F_α and such that

$$m_\alpha\left(\left\{x;\ u_p(x) = s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha}\right\}\right) > 0.$$

And by definition, $\forall p \geqslant 1$, $\beta(f,x) \geqslant u_p(x)$, thus

$$m_\alpha\left(\left\{x;\ \beta(f,x) = s_0 - \frac{d}{p_0} + \frac{d}{p_0\alpha}\right\}\right) > 0.$$

Which states that the spectrum of singularities for the weak-scaling exponent of almost every function in $B^{s_0,q}_{p_0}(\mathbb{R}^d)$ is given by

$$\forall \beta \in \left[s_0 - \frac{d}{p_0}, s_0\right] \quad d_{ws}(\beta) = p_0\beta + d - s_0 p_0. \qquad \square$$

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