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Hilbert's epsilon as an operator of indefinite committed choice

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Abstract

Hilbert and Bernays avoided overspecification of Hilbert's ε-operator. They axiomatized only what was relevant for their proof-theoretic investigations. Semantically, this left the ε-operator underspecified. After briefly reviewing the literature on semantics of Hilbert's epsilon operator, we propose a new semantics with the following features: We avoid overspecification (such as right-uniqueness), but admit indefinite choice, committed choice, and classical logics. Moreover, our semantics for the ε simplifies proof search and is natural in the sense that it mirrors some cases of referential interpretation of indefinite articles in natural language.

Keywords: Hilbert's epsilon operator; Logical foundations; Theories of truth and validity; Formal philosophy of language; Semantics of natural language; Formalized mathematics; Human-oriented interactive theorem proving; Automated theorem proving

1. Motivation, requirements specification, and overview

Hilbert's ε-symbol is a binder that forms terms; just like Peano's ι-symbol, which is sometimes attributed to Russell and written as $\bar{\iota}$ or as an inverted ι. Roughly speaking, the term $\varepsilon x.\,A$, formed from a variable x and a formula A, denotes *just some* object that is *chosen* such that —if possible— A (seen as a predicate on x) holds for it.

For Ackermann, Bernays, and Hilbert, the ε was an intermediate tool in proof theory, to be eliminated in the end. Instead of giving a model-theoretic semantics for the ε, they just specified those axioms which were essential in their proof transformations. These axioms did not provide a complete definition, but left the ε underspecified.

After reviewing the literature on extended semantics given to Hilbert's ε-operator in the 2nd half of the 20th century, we will propose a novel semantics for it.

Descriptive terms such as $\varepsilon x.\,A$ and $\iota x.\,A$ are of universal interest and applicability. We suppose that our novel treatment will turn out to be useful in many areas where logic is designed or applied as a tool for description and reasoning.

For the usefulness of such descriptive terms we consider the requirements listed below to be the most important ones. Our new indefinite ε-operator satisfies these requirements and—as it is defined by novel semantical techniques—may serve as the paradigm for the design of similar operators satisfying these requirements.

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Requirement I (Syntax): The syntax must clearly express where exactly a *commitment* to a choice of a special object is required, and where—to the contrary—different objects corresponding with the description may be chosen for different occurrences of the same descriptive term.

Requirement II (Reasoning): In a reductive proof step, it must be possible to replace a descriptive term with a term that corresponds with its description. The soundness of such a replacement must be expressible and should be verifiable in the original calculus.

Requirement III (Semantics): The semantics should be simple, straightforward, natural, formal, and model-based. Overspecification should be avoided carefully. Furthermore, the semantics should be modular and abstract in the sense that it adds the operator to a variety of logics, independently of the details of a concrete logic.

This paper is organized as follows: After a general introduction to the ε in § 2 and a review of the literature on the ε's semantics w.r.t. adequacy and Hilbert's intentions in § 3, we explain and formalize our novel approach to the ε's semantics: informally in § 4, more formally in § 5, summarized in § 6. We provide examples of possible application in the area of semantics of natural language in § 7, discuss some conceivable objections and variations in § 8, and conclude in § 9.

2. General introduction to Hilbert's ε

We introduce to the ι- and ε-operators (§ 2.1), to the ε's proof-theoretic origin (§ 2.2), and to our contrasting semantical objective (§ 2.3) with its emphasis on *indefinite* and *committed choice* (§ 2.4).

2.1. From the ι to the ε

The first occurrence of a descriptive ι-operator seems to be in Frege (1893/1903), Vol. I, where a boldface backslash is written instead of the ι. In Peano (1896f.), '$\bar{\iota}$' is written instead of 'ι'. In Peano (1899a), we find an alternative notation besides '$\bar{\iota}$', namely a ι-symbol upside-down. Both notations were to denote the inverse of Peano's ι-function, which constructs the singleton set of its argument. Today, we write '$\{y\}$' for Peano's 'ιy', the upside-down ι is not easily available in typesetting, and we write a simple non-inverted ι for the descriptive ι-operator.

All the slightly differing definitions of semantics for the ι-operator agree on the following: If there is a unique x such that the formula A holds (seen as a predicate on x), then the ι-term $\iota x.\,A$ denotes this unique object.

Example 1 *(ι-binder).* For an informal introduction to the ι-binder, consider Father to be a predicate for which $\mathsf{Father}(\mathsf{Heinrich\,III}, \mathsf{Heinrich\,IV})$ holds, i.e. "Heinrich III is father of Heinrich IV". Now, "*the* father of Heinrich IV" can be denoted by

$$\iota x.\,\mathsf{Father}(x, \mathsf{Heinrich\,IV}),$$

and because this is nobody but Heinrich III, i.e.

$$\mathsf{Heinrich\,III} = \iota x.\,\mathsf{Father}(x, \mathsf{Heinrich\,IV}),$$

we know that

$$\mathsf{Father}(\iota x.\,\mathsf{Father}(x, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV}).$$

Similarly,

$$\mathsf{Father}(\iota x.\,\mathsf{Father}(x, \mathsf{Adam}), \mathsf{Adam}), \tag{1.1}$$

and thus $\exists y.\,\mathsf{Father}(y, \mathsf{Adam})$, but, oops! Adam and Eve do not have any fathers. If you do not agree, you would probably appreciate the following problem that occurs when somebody has God as an additional father.

$$\mathsf{Father}(\mathsf{Holy\,Ghost}, \mathsf{Jesus}) \;\wedge\; \mathsf{Father}(\mathsf{Joseph}, \mathsf{Jesus}). \tag{1.2}$$

Then the Holy Ghost is *the* father of Jesus and Joseph is *the* father of Jesus:

$$\mathsf{Holy\,Ghost} = \iota x.\,\mathsf{Father}(x, \mathsf{Jesus}) \;\wedge\; \mathsf{Joseph} = \iota x.\,\mathsf{Father}(x, \mathsf{Jesus}) \tag{1.3}$$

This implies something *the* Pope may not accept, namely $\mathsf{Holy\,Ghost} = \mathsf{Joseph}$, and he anathematized Heinrich IV in the year 1076:

$$\mathsf{Anathematized}(\iota x.\,\mathsf{Pope}(x), \mathsf{Heinrich\,IV}, 1076). \tag{1.4}$$

□

From Frege (1893/1903) to Quine (1981), we find a multitude of ι-operators that are arbitrarily overspecified for the sake of completeness and syntactic eliminability. There are basically three ways of giving a semantics to the ι-terms without overspecification:

Russell's ι-operator: In Whitehead & Russell (1925ff.), the ι-terms do not refer to an object but make sense only in the context of a sentence. This was nicely described already in Russell (1905), without using any symbol for the ι, however.

Hilbert's ι-operator: To overcome the complex difficulties of that non-referential definition, in Hilbert & Bernays (1968/70), Vol. I, p. 392ff., a completed proof of $\exists! x.\, A$ was required to preceed each formation of the term $\iota x.\, A$, which otherwise could not be considered a well-formed term at all.

Peano's ι-operator: Since the inflexible treatment of Hilbert's ι-operator makes the ι quite impractical and the formal syntax of logic undecidable in general, in Vol. II of the same book, the ε, however, is already given a more flexible treatment. There, the simple idea is to leave the ε-terms uninterpreted, as will be described below. In this paper, we present this more flexible view also for the ι. Moreover, this view is already Peano's original one, cf. Peano (1896f.).

At least in non-modal classical logics, it is a well justified standard that *each term denotes*. More precisely—in each model or structure $\mathcal{S}$ under consideration—each occurrence of a proper term must denote an object in the universe of $\mathcal{S}$. Following that standard, to be able to write down $\iota x.\, A$ without further consideration, we have to treat $\iota x.\, A$ as an uninterpreted term about which we only know

$$\exists! x.\, A \;\Rightarrow\; A\{x \mapsto \iota x.\, A\} \qquad (\iota_0)$$

or in different notation

$$(\exists! x.\, (A(x))) \;\Rightarrow\; A(\iota x.\, (A(x))),$$

where, for some new y, we can define $\exists! x.\, A \;:=\; \exists y.\, \forall x.\, (x{=}y \;\Leftrightarrow\; A)$.

With (ι_0) as the only axiom for the ι, the term $\iota x.\, A$ has to satisfy A (seen as a predicate on x) only if there exists a unique object such that A holds for it. Moreover, the problems presented in Example 1 do not appear because (1.1) and (1.3) are not valid. Indeed, the description of (1.1) lacks existence and the descriptions of (1.3) and (1.4) lack uniqueness. The price we have to pay here is that—roughly speaking—the term $\iota x.\, A$ is of no use unless the unique existence $\exists! x.\, A$ can be derived.

2.2. *On the ε's proof-theoretic origin*

Compared to ι, the ε is more useful because—instead of (ι_0)—it comes with the stronger axiom

$$\exists x.\, A \;\Rightarrow\; A\{x \mapsto \varepsilon x.\, A\} \qquad (\varepsilon_0)$$

More precisely, as the formula $\exists x.\, A$ (which has to be true to guarantee a meaningful interpretation of the ε-term $\varepsilon x.\, A$) is weaker than the corresponding formula $\exists! x.\, A$ (for the resp. ι-term), the area of useful application is wider for the ε- than for the ι-operator. Moreover, in case of $\exists! x.\, A$, the ε-operator picks the same element as the ι-operator, i.e. $\exists! x.\, A \;\Rightarrow\; \big(\varepsilon x.\, A \,=\, \iota x.\, A\big)$.

As the basic methodology of David Hilbert's Programme is to treat all symbols as meaningless, he does not give us any semantics but only the axiom (ε_0). Although no meaning is required, it furthers the understanding. And therefore, in Hilbert & Bernays (1968/70), the fundamental work which summarizes the foundational contributions of David Hilbert and his group, Paul Bernays writes:

> $\varepsilon x.\, A$ ... „ist ein Ding des Individuënbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Uebersetzung der Formel (ε_0) ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, daß es überhaupt auf ein Ding des Individuënbereichs zutrifft."
>
> [Hilbert & Bernays (1968/70), Vol. II, p.12, modernized orthography]

> $\varepsilon x.\, A$ ... "is an object of the universe for which—according to the semantical translation of the formula (ε_0)—*the predicate A holds, provided that A holds for any object of the universe at all.*"

Example 2 *(ε instead of ι, part I).* *(continuing Example 1)*
Just as for the ι, for the ε we have $\mathsf{Heinrich\,III} \;=\; \varepsilon x.\,\mathsf{Father}(x, \mathsf{Heinrich\,IV})$ and
$$\mathsf{Father}(\varepsilon x.\,\mathsf{Father}(x, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV}).$$
But, from the contrapositive of (ε_0) and $\neg\mathsf{Father}(\varepsilon x.\,\mathsf{Father}(x, \mathsf{Adam}), \mathsf{Adam})$, we now conclude that
$$\neg\exists y.\,\mathsf{Father}(y, \mathsf{Adam}).$$ □

David Hilbert did not need any semantics or precise intention for the ε-symbol because it was introduced merely as a formal syntactical device to facilitate proof-theoretic investigations, motivated by the possibility to get rid of the existential and universal quantifiers via
$$\exists x.\,A \;\;\Leftrightarrow\;\; A\{x \mapsto \varepsilon x.\,A\} \tag{ε_1}$$
$$\forall x.\,A \;\;\Leftrightarrow\;\; A\{x \mapsto \varepsilon x.\,\neg A\} \tag{ε_2}$$
When we remove all quantifiers in a derivation of the (Hilbert-style) predicate calculus of Hilbert & Bernays (1968/70) along (ε_1) and (ε_2), the following transformations occur: Tautologies are turned into tautologies. The axioms
$$A\{x{\mapsto}t\} \;\;\Rightarrow\;\; \exists x.\,A \quad\text{and}\quad \forall x.\,A \;\;\Rightarrow\;\; A\{x{\mapsto}t\}$$
are turned into
$$A\{x{\mapsto}t\} \;\;\Rightarrow\;\; A\{x \mapsto \varepsilon x.\,A\} \tag{ε-formula}$$
and—roughly speaking w.r.t. two-valued logics—its contrapositive, respectively. The inference steps are turned into inference steps: modus ponens into modus ponens; instantiation of free variables as well as quantifier introduction into instantiation including ε-terms. Finally, the ε-formula is taken as a new axiom scheme instead of (ε_0) because it has the advantage of being free of quantifiers.

This argumentation is actually the start of the proof transformation which constructively proves the first of Bernays' two theorems on ε-elimination in first-order logic, the so-called *1st ε-theorem*. In its extended form, this theorem may be stated as follows:

Theorem 3 *(Extd. 1st ε-Theorem, Hilbert & Bernays (1968/70), Vol. II, p.79f.).*
From a derivation of $\exists x_1.\ldots\exists x_r.\,A$ (containing no bound variables besides the ones bound by the prefix $\exists x_1.\ldots\exists x_r.$) from the formulas $P_1,\ldots,P_k$ (containing no bound variables) in the predicate calculus (incl., as axiom schemes, ε-formula and (to specify equality) reflexivity and substitutability), we can construct a (finite) disjunction of the form $\bigvee_{i=0}^{s} A\{x_1,\ldots,x_r \mapsto t_{i,1},\ldots,t_{i,r}\}$ and a derivation of it, in which bound variables do not occur at all, from $P_1,\ldots,P_k$ in the elementary calculus (i.e. tautologies plus modus ponens and instantiation of free variables). □

Note that r, s range over natural numbers including 0, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound variables.

Moreover, the *2nd ε-Theorem* in Hilbert & Bernays (1968/70), Vol. II, states that the ε (just as the ι, cf. Hilbert & Bernays (1968/70), Vol. I) is a conservative extension of the predicate calculus in the sense that each formal proof of an ε-free formula can be transformed into a formal proof that does not use the ε at all. Generally, however, it is not a conservative extension when we add the ε either with (ε_0), with (ε_1), or with the ε-formula to other first-order logics—may they be weaker such as intuitionistic logic, or stronger such as set theories with axiom schemes over arbitrary terms including the ε, cf. § 3.1.3. Moreover, even in standard first-order logic there is no translation from the formulas containing the ε to formulas not containing it.

2.3. *Our objective*

While the historiographical and technical research on the ε-theorems is still going on and the methods of ε-elimination and ε-substitution did not die with Hilbert's Programme, this is not our subject here. We are less interested in Hilbert's formal programme and the consistency of mathematics than in the powerful use of logic in creative processes. And, instead of the tedious syntactical proof transformations, which easily lose their usefulness and elegance within their technical complexity and which—more importantly—can only refer to an already existing logic, we look for *semantical* means for finding new logics and new applications. And the question that still has to be answered in this field is: *What would be a proper semantics for Hilbert's ε?*

2.4. Indefinite and committed choice

Just as the ι-symbol is usually taken to be the referential interpretation of the *definite* articles in natural languages, it is our opinion that the ε-symbol should be that of the *indefinite* determiners (articles and pronouns) such as “a(n)” or “some”.

Example 4 *(ε instead of ι, part II).* *(continuing Example 1)*
It may well be the case that

$$\mathsf{Holy\,Ghost} \;=\; \varepsilon x.\,\mathsf{Father}(x,\mathsf{Jesus}) \;\;\wedge\;\; \mathsf{Joseph} \;=\; \varepsilon x.\,\mathsf{Father}(x,\mathsf{Jesus})$$

i.e. that “The Holy Ghost is a father of Jesus and Joseph is a father of Jesus.” But this does not bring us into trouble with the Pope because we do not know whether all fathers of Jesus are equal. This will become clearer when we reconsider this in Example 13. □

Closely connected to indefinite choice (also called “indeterminism” or “don’t care nondeterminism”) is the notion of *committed choice*. For example, when we have a new telephone, we typically *don’t care* which number we get, but once a number has been chosen for our telephone, we will insist on a *commitment to this choice*, so that our phone number is not changed between two incoming calls.

Example 5 *(Committed choice).*
Suppose we want to prove $\exists x.\,(x \neq x)$
According to (ε_1) from § 2.2 this reduces to $\varepsilon x.\,(x\neq x) \neq \varepsilon x.\,(x\neq x)$
Since there is no solution to $x\neq x$ we can replace
$\varepsilon x.\,(x\neq x)$ with anything. Thus, the above reduces to $\mathsf{0} \neq \varepsilon x.\,(x\neq x)$
and then, by exactly the same argumentation, to $\mathsf{0} \neq \mathsf{1}$
which is valid. Thus, we have proved our original formula $\exists x.\,(x \neq x)$, which, however, is invalid.
What went wrong? Of course, we have to commit to our choice for all occurrences of the ε-term introduced when eliminating the existential quantifier: If we choose $\mathsf{0}$ on the left-hand side, we have to commit to the choice of $\mathsf{0}$ on the right-hand side, too. □

3. Semantics for Hilbert’s ε in the literature

Let us briefly review the literature on the ε’s semantics with a an emphasis on practical adequacy and Hilbert’s intentions. To the best of our knowledge, we only omit Heusinger’s indexed ε-operator, which will be discussed in § 7.

3.1. Right-unique semantics

In contrast to the indefiniteness we suggested in § 2.4, in the literature nearly all semantics for Hilbert’s ε-operator are functional, i.e. [*right-*] *unique*; cf. Leisenring (1969) and the references there.

3.1.1. Ackermann’s (II,4) = Bourbaki’s (S7) = Leisenring’s (E2)

In Ackermann (1938) under the label (II,4), in Bourbaki (1954) under the label (S7) (where a τ is written for the ε, which must not be confused with Hilbert’s τ-operator, cf. Hilbert (1923)), and in Leisenring (1969) under the label (E2), we find the following axiom scheme:

$$\forall x.\,(A_0 \Leftrightarrow A_1) \;\;\Rightarrow\;\; \varepsilon x.\,A_0 = \varepsilon x.\,A_1 \tag{E2}$$

Contrary to our version (E2′) in Lemma 31 of § 5.6, in the standard framework the axiom (E2) imposes a right-unique behavior for the ε-operator, which is based on the extension of the predicate.

Axiom systems including (E2) are called *extensional* because—from a semantical point of view—the value of $\varepsilon x.\,A$ in each semantical structure $\mathcal{S}$ is functionally dependent on the extension of the formula A, i.e. on

$$\{\, o \mid \mathrm{eval}(\mathcal{S} \uplus \{x{\mapsto}o\})(A) = \mathsf{TRUE} \,\},$$

where ‘eval’ is the standard evaluation function that maps a structure (or algebra, interpretation) (including a valuation of the free variables) to a function mapping terms and formulas to values.

To get more freedom for the definition of a semantics of the ε, in Meyer-Viol (1995) and in Giese & Ahrendt (1999) the value of $\varepsilon x.\,A$ may additionally depend on the syntax besides the semantics. It is then given as a function depending on a semantical structure and on the syntactical details of the term $\varepsilon x.\,A$. In Giese & Ahrendt (1999), p.177, we read: "This definition contains no restriction whatsoever on the valuation of ε-terms." This is, however, not true because it imposes the restriction of a right-unique behavior, which denies the possibility of an indefinite behavior, as we will see below.

Note that (E2) has a disastrous effect in intuitionistic logic. This is already the case for its proper consequence $\forall x.\,A_0 \wedge \forall x.\,A_1 \;\Rightarrow\; \varepsilon x.\,A_0 = \varepsilon x.\,A_1$, which—together with (ε_0) and say "$0 \neq 1$"—turns each classical validity into an intuitionistic one; cf. Bell (1993a), § 3. For the strong consequences of the ε-formula in intuitionistic logic cf. Meyer-Viol (1995).

3.1.2. *Roots of the right-uniqueness requirement*

The omnipresence of the right-uniqueness requirement may have its historical justification in the fact that if we expand the dots "..." in the quotation preceding Example 2 in § 2.2, the full quotation reads:

„Das ε-Symbol bildet ſomit eine Art der Verallgemeinerung des μ-Symbols für einen beliebigen Individuënbereich. Der Form nach ſtellt es eine Funktion eines variablen Prädikates dar, welches außer demjenigen Argument, auf welches ſich die zu dem ε-Symbol gehörige gebundene Variable bezieht, noch freie Variable als Argumente („Parameter") enthalten kann. Der Wert dieſer Funktion für ein beſtimmtes Prädikat A (bei Feſtlegung der Parameter) iſt ein Ding des Individuënbereichs, und zwar iſt dieſes Ding gemäß der inhaltlichen Ueberſetzung der Formel (ε_0) *ein ſolches, auf das jenes Prädikat* A *zutrifft, vorausgeſetzt, daß es überhaupt auf ein Ding des Individuënbereichs zutrifft.*"
[Hilbert & Bernays (1968/70), Vol. II, p.12, modernized orthography]

"Thus, the ε-symbol forms a kind of generalization of the μ-symbol for arbitrary universes. Syntactically, it provides a function of a variable predicate, which—besides the argument to which the variable bound by the ε-symbol refers—may contain free variables as arguments ("parameters"). The value of this function for a given predicate A (for fixed values of the parameters) is an object of the universe for which—according to the semantical translation of the formula (ε_0)—*the predicate A holds, provided that A holds for any object of the universe at all.*"
("Syntactically" may be replaced with "Structurally")

Here the word "function" could be understood in its mathematical sense to denote a (right-) unique relation. And, what kind of function could it be but a choice function, choosing an element from the set of objects that satisfy A ? Accordingly, at a different place, we read:

„Darüber hinaus hat das ε die Rolle der Auswahlfunktion, d. h. im Falle, wo $A\,a$ auf mehrere Dinge zutreffen kann, iſt εA *irgendeines* von den Dingen a, auf welche $A\,a$ zutrifft." [Hilbert (1928), p. 68]

"Beyond that, the ε has the rôle of the choice function, i.e. in the case where $A(a)$ may hold for several objects, $\varepsilon x.\,(A(x))$ is *an arbitrary one* of the objects a for which $A(a)$ holds."

3.1.3. *Universal and generalized choice functions*

Since David Hilbert himself seems to have confused the consequences of the ε on the axiom of choice in the last but one paragraph of Hilbert (1923), we point out: Although the ε supplies us with a syntactical means for expressing a *universal choice function*, the axioms (E2), (ε_0), (ε_1), and (ε_2) do not imply the axiom of choice in set theories, unless the axiom schemes of replacement (collection) and comprehension (separation, subset) also range over expressions containing the ε; cf. Leisenring (1969), § IV 4.4.

Moreover, to be precise, the notion of a "choice function" must be generalized here because we need a *total* function on the power set of each (non-empty) universe. Thus, a value must be supplied even at the empty set: f is defined to be a *generalized choice function* if $f : \mathrm{dom}(f) \to \bigcup(\mathrm{dom}(f))$ and $\forall x \in \mathrm{dom}(f).\ (x = \emptyset \ \vee\ f(x) \in x)$.

3.1.4. Hans Hermes' (ε_5) and David DeVidi's (vext)

In Hermes (1965), p.18, the ε suffers from some overspecification in addition to (E2):

$$\varepsilon x.\,\mathsf{false} \;=\; \varepsilon x.\,\mathsf{true} \tag{ε_5}$$

This sets the value of the generalized choice function f at the empty set to the value of f at the whole universe. For classical logic, we can combine (E2) and (ε_5) into the following axiom of DeVidi (1995) for "very extensional" semantics:

$$\forall x.\left(\begin{array}{c}(\exists y.\,A_0\{x{\mapsto}y\} \;\Rightarrow\; A_0)\\ \Leftrightarrow\; (\exists y.\,A_1\{x{\mapsto}y\} \;\Rightarrow\; A_1)\end{array}\right) \;\Rightarrow\; \varepsilon x.\,A_0 = \varepsilon x.\,A_1 \tag{vext}$$

Indeed, (vext) implies (E2) and (ε_5). The other direction, however, does not hold for intuitionistic logic, where, roughly speaking, (vext) additionally implies that if the same elements make A_0 and A_1 as true as possible, then the ε-operator picks the same element for A_0 and A_1, even if the suprema $\exists y.\,A_0\{x{\mapsto}y\}$ and $\exists y.\,A_1\{x{\mapsto}y\}$ (in the complete Heyting algebra) are not equally true.

3.1.5. Completeness aspirations of Leisenring and Asser

Different possible choices for the value of the generalized choice function f at the empty set are discussed in Leisenring (1969), but as the consequences of any special choice are quite queer, the only solution that is found to be sufficiently adequate in Leisenring (1969) is to consider validity in *each* model given by *each* generalized choice function on the power set of the universe. Notice, however, that even in this case, in each single model, the value of $\varepsilon x.\,A$ is still *functionally* dependent on the extension of A. Roughly speaking, in Leisenring (1969) the axioms (ε_1), and (ε_2) from § 2.2 and (E2) from § 3.1.1 are shown to be complete w.r.t. this semantics of the ε in first-order logic.

This completeness makes it unlikely that this semantics exactly matches Hilbert's intentions: Indeed, if Hilbert's intended semantics for the ε could be completely captured by adding the single and straightforward axiom (E2), this axiom would not have been omitted in Hilbert & Bernays (1968/70). It is my opinion that the reason for this omission is that Hilbert's intentions for the ε were not right-unique but indefinite: If Hilbert had intended a right-unique behavior, it would not be impossible to derive (E2) from his axiomatization!

In Asser (1957) the objective is to find a semantics such that Hilbert's ε-calculus of Hilbert & Bernays (1968/70) is sound and *complete* for it. This semantics, however, has to depend on the details of the syntactical form of the ε-terms and, moreover, turns out to be necessarily so artificial that in Asser (1957) the author himself does not recommend it and admits not to believe that Hilbert could have intended it:

> „Allerdings ist dieser Begriff von Auswahlfunktion so kompliziert, daß sich seine Verwendung in der inhaltlichen Mathematik kaum empfiehlt."
> [Asser (1957), p. 59, modernized orthography]

"This notion of a choice function, however," (i.e. the type-3 choice function, providing a semantics for the ε-operator) "is so intricate that its application in contentual mathematics is hardly to be recommended."

> „Angesichts der Kompliziertheit des Begriffs der Auswahlfunktion dritter Art ergibt sich die Frage, ob bei Hilbert-Bernays (" . . . „) wirklich beabsichtigt war, diesen Begriff von Auswahlfunktion axiomatisch zu beschreiben. Aus der Darstellung bei Hilbert-Bernays glaube ich entnehmen zu können, daß das nicht der Fall ist,"
> [Asser (1957), p. 65, modernized orthography]

"The intricacy of the notion of the type-3 choice function puts up the question whether the intention in Hilbert & Bernays (1968/70) (" ... ") really was to describe this notion axiomatically. I believe I can draw from the presentation in Hilbert & Bernays (1968/70) that that is not the case,"

3.1.6. My assumption on Hilbert's intentions

The statements of Bernays and Hilbert in the original German language cited in § 3.1.2 are ambiguous with respect to the question of an intended (right-) unique behavior of the ε-operator. Hilbert probably wanted to have what we call today "committed choice", but simply used the word "function" for the following three reasons: Hilbert was not too much interested in semantics anyway. The technical term "committed choice" did not exist at Hilbert's time. Last but not least, right-uniqueness conveniently serves as a global commitment to any choice and thereby avoids the problem illustrated in Example 5 of § 2.4.

But the price for such an overspecification is high: Right-uniqueness restricts operationalization (cf. § 4.6) and applicability. For the price of right-uniqueness in capturing the semantics of sentences in natural language cf. § 7 and Geurts (2000).

And what we are going to show in this paper is that there is no reason to pay that price!

3.2. Indefinite semantics in the literature

The only occurrences of an indefinite semantics for Hilbert's ε in the literature seem to be Blass & Gurevich (2000) and the references there.

Consider the formula $\varepsilon x.\,(x = x) \;=\; \varepsilon x.\,(x = x)$ from Blass & Gurevich (2000) or the even simpler

$$\varepsilon x.\,\mathsf{true} \;=\; \varepsilon x.\,\mathsf{true} \qquad (\textsc{Reflex})$$

which may be valid or not, depending on the question whether the same object is taken on both sides of the equation or not. In natural language this is like "Something is equal to something.", whose truth is indefinite. If you do not think so, consider $\varepsilon x.\,\mathsf{true} \;\neq\; \varepsilon x.\,\mathsf{true}$ in addition, i.e. "Something is unequal to something.", and notice that the two sentences seem to be contradictory.

In Blass & Gurevich (2000), Kleene's strong three-valued logic is taken as a mathematically elegant means to solve the problems with indefiniteness. In spite of the theoretical and computational significance of this solution, however, from a practical point of view, Kleene's strong three-valued logic severely restricts its applicability. In applications, a logic is not an object of meta-logical investigation but a tool, and logical arguments are never made explicit because the presence of logic is either not realized at all or taken to be trivial, even by academics, unless they are formalists. Thus, regarding applications, we had better adhere to our common meta-logic, which in the western world is a subset of (modal) classical logic. A western court may accept that Lee Harvey Oswald killed John F. Kennedy as well as that he did not; but cannot accept a third possibility, a *tertium*, as required for Kleene's strong three-valued logic, and especially not the interpretation given in Blass & Gurevich (2000) that he *both* did and did not kill him, which directly contradicts any common sense.

4. Introduction to our novel indefinite free-variable semantics

4.1. Free γ- and free δ-variables

Before we can present our treatment of the ε, we have to provide some technical background. For a technically more detailed introduction cf. Wirth (2004).

We will now introduce free γ-, δ^--, and δ^+-variables. Free variables frequently occur in mathematical practice. Their logical function varies locally. It is typically determined implicitly by the context and the obviously intended semantics.

In this paper, however, we make this function explicit by using disjoint sets of variable-symbols for different functions. The classification of a free variable is indicated by adjoining the respective γ, δ^-, or δ^+ to the upper right of the symbol for the variable.

As already noted in Russell (1919), p.155, in mathematical practice, the free variables a^{free} and b^{free} in the (quasi-) formula

$$(a^{\mathrm{free}} + b^{\mathrm{free}})^2 \;=\; (a^{\mathrm{free}})^2 + 2\,a^{\mathrm{free}}\,b^{\mathrm{free}} + (b^{\mathrm{free}})^2$$

obviously have a universal intention and the quasi-formula itself is not meant to denote a propositional function but actually stands for the closed formula

$$\forall a, b.\,\big(\,(a+b)^2 \;=\; (a)^2 + 2\,a\,b + (b)^2\,\big)$$

In this paper, however, we indicate by

$$(a^{\delta^-} + b^{\delta^-})^2 \;=\; (a^{\delta^-})^2 + 2\,a^{\delta^-}\,b^{\delta^-} + (b^{\delta^-})^2$$

a proper formula with free δ^--variables, which—independently of its context—is logically equivalent to the universally quantified formula. Changing from universal to existential intention, it is somehow clear that the linear system

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathrm{free}} \\ y^{\mathrm{free}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

asks us to find solutions for x^{free} and y^{free}. We make this intention syntactically explicit by writing

$$\begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}\begin{pmatrix}x^\gamma\\y^\gamma\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}$$

instead. This formula with free γ-variables is not only logically equivalent to

$$\exists x, y.\ \left(\begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}\right)$$

but may additionally enable us to retrieve the solutions for x^γ and y^γ as the substitutions for x^γ and y^γ chosen in a formal proof.

Finally, the free δ^+-variables are to represent our ε-terms in the end. The names γ, δ^-, and δ^+ refer to the classification of reductive inference rules into α-, β-, γ-, and δ-rules of Smullyan (1968), as used in the following § 4.2.

4.2. γ- and δ-rules

Suppose we want to prove the existential property $\exists x.\, A$. The γ-rules of old-fashioned inference systems (such as Gentzen (1935) or Smullyan (1968), e.g.) require us to choose a *fixed* witnessing term t as a substitute for the bound variable *immediately* when eliminating the quantifier.

Let A be a formula. We do not permit binding of variables that already occur bound in a term or formula; that is: $\forall x.\ A$ is only a formula if no binder on x already occurs in A. The simple effect is that our formulas are easier to read and our γ- and δ-rules can replace *all* occurrences of x. Moreover, we assume that all binders have minimal scope, e.g. $\forall x, y.\ A \wedge B$ reads $(\forall x.\ \forall y.\ A) \wedge B$. Let Γ and Π be *sequents*, i.e. disjunctive lists of formulas.

γ**-rules:** Let t be any term: $\dfrac{\Gamma \quad \exists x.A \quad \Pi}{A\{x\mapsto t\} \quad \Gamma \quad \exists x.A \quad \Pi}$ $\qquad \dfrac{\Gamma \quad \neg\forall x.A \quad \Pi}{\overline{A\{x\mapsto t\}} \quad \Gamma \quad \neg\forall x.A \quad \Pi}$

Note that $\overline{A}$ is the *conjugate* of the formula A, i.e. B if A is of the form $\neg B$, and $\neg A$ otherwise. Moreover, in the good old days when trees grew upwards, Gerhard Gentzen (1909–1945) would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downwards.

More modern inference systems, however, (such as the ones in Fitting (1996)) enable us to delay the crucial choice of the term t until the state of the proof attempt may provide more information to make a successful decision. This delay is achieved by introducing a special kind of variable, called "dummy" in Prawitz (1960), "free" in Fitting (1996) and in Footnote 11 of Prawitz (1960), and "meta" in the field of planning and constraint solving. We call these variables *free γ-variables* and write them like x^γ. When these additional variables are available, we can reduce $\exists x.\, A$ first to $A\{x\mapsto x^\gamma\}$ and then sometime later in the proof we may globally substitute x^γ with an appropriate term.

The addition of the free γ-variables changes the notion of a term but not the γ-rules, whereas it becomes visible in the δ-rules. δ-rules introduce free δ-variables. The free δ-variables are also called "parameters" or "eigenvariables" and typically stand for arbitrary objects of which nothing is known. Now the occurrence of such a free δ-variable must be disallowed in the terms that may be substituted for those free γ-variables which had already been in use when an application of a δ-rule introduced this free δ-variable. The reason for this restriction on replacement of free γ-variables is that the dependence or scoping of the quantifiers must somehow be reflected in a dependence of the free variables. This dependence is to be captured in a binary relation on the free variables, called *variable-condition*. Indeed, it is sometimes unsound to instantiate a free γ-variable x^γ with a term containing a free δ-variable y^δ that was introduced later than x^γ:

Example 6. The formula $\qquad \exists x.\ \forall y.\ (x = y)$

is not generally valid. We can start a proof attempt as follows:

γ-step: $\qquad \forall y.\ (x^\gamma = y),\quad \exists x.\ \forall y.\ (x = y)$

δ-step: $\qquad (x^\gamma = y^\delta),\quad \exists x.\ \forall y.\ (x = y)$

Now, if the free γ-variable x^γ could be substituted by the free δ-variable y^δ, we would get the tautology $(y^\delta = y^\delta)$, i.e. we would have proved an invalid formula. To prevent this, the δ-step has to record (x^γ, y^δ) in a variable-condition, where (x^γ, y^δ) means that x^γ is somehow "necessarily older" than y^δ, so that we must not instantiate the free γ-variable x^γ with a term containing the free δ-variable y^δ. □

Starting with an empty variable-condition, we extend the variable-condition during a proof by δ-steps and by steps that globally instantiate γ- and δ^{+}-variables. This kind of instantiation of *rigid* variables is only sound if the resulting variable-condition is still acyclic after adding, for each free variable x^{free} instantiated with a term t and for each free variable z^{free} occurring in t, the pair $(z^{\mathrm{free}}, x^{\mathrm{free}})$ to the variable-condition. For the advantages of our special version of variable-conditions compared to previous approaches, such as Bibel (1987) or Kohlhase (1998), cf. §§ 2.1.5 and 2.2.1 of Wirth (2004).

To make things more complicated, there are basically two different versions of the δ-rules: standard δ^{-}-rules (also simply called "δ-rules") and δ^{+}-rules (also called "*liberalized* δ-rules"). They differ in the kind of free δ-variable they introduce and—crucially—in the way they enlarge the variable-condition, depicted to the lower right of the bar:

δ^{-}**-rules:** Let x^{δ^-} be a new free δ^{-}-variable:

$$\frac{\Gamma \quad \forall x.A \quad \Pi}{A\{x\mapsto x^{\delta^-}\} \quad \Gamma \quad \Pi} \quad \mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi) \times \{x^{\delta^-}\} \qquad\qquad \frac{\Gamma \quad \neg\exists x.A \quad \Pi}{\overline{A\{x\mapsto x^{\delta^-}\}} \quad \Gamma \quad \Pi} \quad \mathcal{V}_{\gamma\delta^+}(\Gamma\ \neg\exists x.A\ \Pi) \times \{x^{\delta^-}\}$$

δ^{+}**-rules:** Let x^{δ^+} be a new free δ^{+}-variable:

$$\frac{\Gamma \quad \forall x.A \quad \Pi}{A\{x\mapsto x^{\delta^+}\} \quad \Gamma \quad \Pi} \quad \begin{array}{l}\{(x^{\delta^+}, \overline{A\{x\mapsto x^{\delta^+}\}})\} \\ \mathcal{V}_{\mathrm{free}}(\forall x.A) \times \{x^{\delta^+}\}\end{array} \qquad\qquad \frac{\Gamma \quad \neg\exists x.A \quad \Pi}{\overline{A\{x\mapsto x^{\delta^+}\}} \quad \Gamma \quad \Pi} \quad \begin{array}{l}\{(x^{\delta^+}, A\{x\mapsto x^{\delta^+}\})\} \\ \mathcal{V}_{\mathrm{free}}(\neg\exists x.A) \times \{x^{\delta^+}\}\end{array}$$

Notice that $\mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi)$ denotes the set of the free γ- and δ^{+}-variables occurring in the whole upper sequent, whereas $\mathcal{V}_{\mathrm{free}}(\forall x.A)$ denotes the set of all free (γ-, δ^{-}-, δ^{+}-) variables, but only the ones occurring in the *principal formula* $\forall x.A$. The smaller variable-conditions generated by the δ^{+}-rules mean more proofs. Indeed, the δ^{+}-rules enable additional proofs on the same level of γ*-multiplicity* (i.e. the number of repeated γ-steps applied to the identical principal formula); cf. e.g. Example 2.8, Wirth (2004), p. 21. For certain classes of theorems, some of these proofs are exponentially and even non-elementarily shorter than the shortest proofs which apply only δ^{-}-rules; for a survey cf. Wirth (2004), § 2.1.5. Moreover, additional proofs are possible with the δ^{+}-rules, which are not only shorter but also more natural and easier to find both by human beings and by automated systems; see the discussion on the design goals for inference systems in Wirth (2004), § 1.2.1, and the proof of the limit theorem for $+$ in Wirth (2006a). Summing it all up, the name "liberalized" for the δ^{+}-rules is indeed justified: They provide more freedom to the prover.[1]

Moreover, note that the singleton sets indicated to the upper right of the bar of the above δ^{+}-rules are to augment another global binary relation besides the variable-condition, namely a function called the *choice-condition*. This will be explained in § 4.5f.

There is a popular alternative to variable-conditions, namely Skolemization, where the free δ-variables become functions (i.e. their order is incremented) and the δ^{-}- and δ^{+}-rules give them the free γ-variables of $\mathcal{V}_{\gamma}(\Gamma\ \forall x.A\ \Pi)$ and $\mathcal{V}_{\gamma}(\forall x.A)$, resp., as initial arguments. Then, the occur-check of unification implements the restrictions on substitution of free γ-variables. In some inference systems, however, Skolemization is unsound (e.g. for higher-order systems such as the one in Kohlhase (1998) or the system in Wirth (2004) for *descente infinie*) or inappropriate (e.g. in the matrix systems of Wallen (1990)). We prefer inference systems with variable-conditions because they are simpler, more general, and not less efficient than Skolemizing inference systems. Notice that variable-conditions do not add unnecessary complexity: Firstly, if variable-conditions are superfluous we can work with an empty variable-condition as if there would be no variable-condition at all. Secondly, we will need the variable-conditions anyway for our choice-conditions, which again are needed to formalize our novel approach to Hilbert's ε-operator.

4.3. Quantifier elimination and subordinate ε-terms

The elimination of $\forall$- and $\exists$-quantifiers with the help of ε-terms (cf. § 2.2) may be more difficult than expected when some ε-terms become "subordinate" to others. In Hilbert & Bernays (1968/70), Vol. II, p. 24, these subordinate ε-terms, which are responsible for the difficulty to prove the ε-theorems constructively, are called "untergeordnete ε-Ausdrücke".

Definition 7 *(Subordinate).* An ε-term $\varepsilon v.\,B$ (or, more generally, a binder on v together with its scope B) is *superordinate* to an (occurrence of an) ε-term $\varepsilon x.\,A$ if

1. $\varepsilon x.\,A$ is a subterm of B and
2. an occurrence of the variable v in $\varepsilon x.\,A$ is free in B
   (i.e. the binder on v binds an occurrence of v in $\varepsilon x.\,A$).

An (occurrence of an) ε-term a is *subordinate* to an ε-term $\varepsilon v.\,B$ (or, more generally, a binder on v together with its scope B) if $\varepsilon v.\,B$ is superordinate to a. □

Example 8 *(Quantifier elimination and subordinate ε-terms).*
Consider the formula $\forall x.\,\exists y.\,\forall z.\,\mathsf{P}(x,y,z)$. Let us apply (ε_1) and (ε_2) from § 2.2 to remove the three quantifiers completely. We introduce the following abbreviations:

$$\begin{array}{ll|ll}
z_a(x)(y) & = \varepsilon z.\,\neg\mathsf{P}(x,y,z) & x_a & = \varepsilon x.\,\neg\mathsf{P}(x,y_a(x),z_a(x)(y_a(x))) \\
 & & x_b & = \varepsilon x.\,\neg\mathsf{P}(x,y_a(x),z_a(x)(y_b(x))) \\
y_a(x) & = \varepsilon y.\,\mathsf{P}(x,y,z_a(x)(y)) & x_c & = \varepsilon x.\,\neg\mathsf{P}(x,y_b(x),z_a(x)(y_b(x))) \\
y_b(x) & = \varepsilon y.\,\forall z.\,\mathsf{P}(x,y,z) & x_d & = \varepsilon x.\,\neg\forall z.\,\mathsf{P}(x,y_b(x),z) \\
 & & x_e & = \varepsilon x.\,\neg\exists y.\,\forall z.\,\mathsf{P}(x,y,z)
\end{array}$$

When we eliminate inside–out (i.e. start with the elimination of $\forall z.$) the transformation is
$\forall x.\,\exists y.\,\mathsf{P}(x,y,z_a(x)(y)),\qquad \forall x.\,\mathsf{P}(x,y_a(x),z_a(x)(y_a(x))),\qquad \mathsf{P}(x_a,y_a(x_a),z_a(x_a)(y_a(x_a))).$
When we eliminate outside–in (i.e. start with the elimination of $\forall x.$) the transformation is
$\exists y.\,\forall z.\,\mathsf{P}(x_e,y,z),\quad \forall z.\,\mathsf{P}(x_e,y_b(x_e),z),\quad \mathsf{P}(x_e,y_b(x_e),z_a(x_e)(y_b(x_e))),\quad \ldots,\quad \mathsf{P}(x_a,y_a(x_a),z_a(x_a)(y_a(x_a))),$
where the dots represent the rewritings of x_e over x_d, x_c, x_b to x_a (four times) and of y_b to y_a (twice in addition). Note that the resulting formula is the same in both cases. Indeed, it does not depend on the order in which we eliminate the quantifiers. Moreover, notice that this formula is quite deep. Indeed, in general n nested quantifiers result in an ε-nesting depth of 2^n-1 and huge ε-terms (such as x_a) occur up to n times with commitment to their choice.
If we write the resulting formula as

$$\mathsf{P}(x_a,y_c,z_d) \tag{8.1}$$

by setting $y_c = y_a(x_a)$, and $z_d = z_a(x_a)(y_a(x_a))$, then we have

$$\begin{array}{rll}
z_d & = \varepsilon z.\,\neg\mathsf{P}(x_a,y_c,z) & (8.2)\\
y_c & = \varepsilon y.\,\mathsf{P}(x_a,y,z_c(y)) & (8.3)\\
 & \quad\text{with}\quad z_c(y) = \varepsilon z.\,\neg\mathsf{P}(x_a,y,z) & (8.4)\\
x_a & = \varepsilon x.\,\neg\mathsf{P}(x,y_a(x),z_b(x)) & (8.5)\\
 & \quad\text{with}\quad z_b(x) = \varepsilon z.\,\neg\mathsf{P}(x,y_a(x),z) & (8.6)\\
 & \quad\text{and}\quad y_a(x) = \varepsilon y.\,\mathsf{P}(x,y,z_a(x)(y)) & (8.7)\\
 & \qquad\qquad\text{with}\quad z_a(x)(y) = \varepsilon z.\,\neg\mathsf{P}(x,y,z) & (8.8)
\end{array}$$

Firstly, note that the free variables x and y in the ε-terms $z_c(y)$, $z_b(x)$, $y_a(x)$, $z_a(x)(y)$ are actually bound by the next ε to the left, to which the respective ε-terms thus become subordinate. For example, the ε-term $z_c(y)$ is subordinate to the ε-term y_c. Secondly, the top ε-binders on the right-hand sides of the defining equations are exactly those that require a commitment to their choice. This means that each of z_a, z_b, z_c, and z_d may be chosen differently without affecting soundness of the equivalence transformation. The same holds for y_a and y_c. Note that the variables are strictly nested into each other. Thus we must choose in the order of z_a, y_a, z_b, x_a, z_c, y_c, z_d. Moreover, for z_c, z_b, y_a, z_a we actually have to choose functions instead of a simple objects.
In Hilbert's view, however, there are neither functions nor objects at all, but only terms, where x_a reads

$$\varepsilon x.\ \neg\mathsf{P}\left(\begin{array}{l} x,\\ \varepsilon y_\alpha.\,\mathsf{P}(x,y_\alpha,\varepsilon z_\alpha.\,\neg\mathsf{P}(x,y_\alpha,z_\alpha)),\\ \varepsilon z_\beta.\,\neg\mathsf{P}\Big(x,\ \ \varepsilon y_\alpha.\,\mathsf{P}(x,y_\alpha,\varepsilon z_\alpha.\,\neg\mathsf{P}(x,y_\alpha,z_\alpha)),\ \ z_\beta\Big), \end{array}\right)$$

and y_c and z_d take several lines more to write them down. □

The exponential explosion of term depth of Example 8 can be avoided by an outside–in removal of δ-quantifiers *without removing the quantifiers below ε-binders* and by a replacement of γ-quantified variables with free γ-variables. For the case of Example 8, this yields $\mathsf{P}(x_e,y^\gamma,z_e)$ with $z_e = \varepsilon z.\,\neg\mathsf{P}(x_e,y^\gamma,z)$ and $x_e = \varepsilon x.\,\neg\exists y.\,\forall z.\,\mathsf{P}(x,y,z)$.

Thus, in general, the nesting of binders for the complete elimination of a prenex of n quantifiers does not become deeper than $\frac{1}{4}(n+1)^2$. If we are only interested in reduction and not in equivalence transformation of a formula, we can abstract Skolem functions from the ε-terms and just reduce to the formula $\mathsf{P}(x^\delta, y^\gamma, z^\delta(y^\gamma))$. In a non-Skolemizing inference system with a variable-condition we get $\mathsf{P}(x^\delta, y^\gamma, z^\delta)$ instead, with $\{(y^\gamma, z^\delta)\}$ as an extension to the variable-condition. The ideas of Skolemization and of treating γ- and δ-quantifiers differently go back to the Peirce–Schröder tradition (cf. Anellis (2004)), Löwenheim (1915) (read charitably), Skolem (1928), and Herbrand (1930). Note that with Skolemization or variable-conditions we have no growth of nesting depth at all, and the same will be the case for our term-sharing approach to ε-terms.

4.4. *Do not be afraid of indefiniteness!*

From the discussion in §§ 2.4 and 3, one could get the impression that an indefinite logical treatment of the ε is not easy to find. Indeed, on first sight, there is the problem that some standard axiom schemes cannot be taken for granted, such as substitutability $s=t \;\;\Rightarrow\;\; f(s)=f(t)$ (note that this is similar to (E2) of § 3.1.1 when we take logical equivalence as equality!) and such as reflexivity $t=t$ (note that (REFLEX) of § 3.2 is an instance of this!). This means that it may not be sound to replace a subterm with an equal term and that even syntactically equal terms may not be definitely equal.

In computer programs, however, we are quite used to committed choice and to an indefinite behavior of choosing, and the violation of substitutability and even reflexivity is no problem:

Example 9 *(Violation of substitutability and reflexivity in programs).*

```
fun choose s = case s of Set (i :: _) => i | _ => raise Empty;
```

is the ML code of a function that chooses an element from a set implemented as a list. It simply returns the first element of the list. For another set that is equal—but where the list may have another order—the result may be different. Thus, the behavior of the function `choose` is indefinite for a given set, but each time it is called for an implemented set, it chooses a special element and *commits to this choice*, i.e. when called again, it returns the same value. In this case we have `choose s = choose s`, but `s = t` does not imply `choose s = choose t`. In an implementation where some parallel reordering of lists may take place, even `choose s = choose s` may be wrong. □

From this example we may learn that the question of `choose s = choose s` may be indefinite until the choice steps have actually been performed. *This is exactly how we will treat our* ε. The steps that are performed in logic are proof steps. Thus, on the one hand, when we want to prove

$$\varepsilon x.\,\mathsf{true} \;=\; \varepsilon x.\,\mathsf{true}$$

we can choose $\mathsf{0}$ for both occurrences of $\varepsilon x.\,\mathsf{true}$, get $\mathsf{0}=\mathsf{0}$, and the proof is successful. On the other hand, when we want to prove

$$\varepsilon x.\,\mathsf{true} \;\neq\; \varepsilon x.\,\mathsf{true}$$

we can choose $\mathsf{0}$ for one occurrence and $\mathsf{1}$ for the other, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again.
This procedure may seem wondrous, but is very similar to something quite common with free γ-variables, cf. § 4.1:
On the one hand, when we want to prove

$$x^\gamma = y^\gamma$$

we can choose $\mathsf{0}$ to substitute for both x^γ and y^γ, get $\mathsf{0}=\mathsf{0}$, and the proof is successful.
On the other hand, when we want to prove

$$x^\gamma \neq y^\gamma$$

we can choose $\mathsf{0}$ to substitute for x^γ and $\mathsf{1}$ to substitute for y^γ, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again.

4.5. *Replacing ε-terms with free δ^+-variables*

There is an important difference between the inequations $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ and $x^\gamma \neq y^\gamma$ at the end of the previous § 4.4: The latter does not violate the reflexivity axiom! And we are going to cure the violation of the former immediately with the help of a special kind of free variables, namely our *free δ^+-variables*, cf. § 4.1. Now, instead of $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ we write $x^{\delta^+} \neq y^{\delta^+}$ and remember what these free δ^+-variables stand for by storing this into a function C, called a *choice-condition*:

$$C(x^{\delta^+}) := \mathsf{true},$$
$$C(y^{\delta^+}) := \mathsf{true}.$$

For a first step, suppose that our ε-terms are not subordinate to any outside binder, cf. Definition 7. Then, we can replace an ε-term $\varepsilon z.\, A$ with a new free δ^+-variable z^{δ^+} and extend the partial function C by

$$C(z^{\delta^+}) := A\{z \mapsto z^{\delta^+}\}.$$

By this procedure we can eliminate all ε-terms without loosing any syntactical information.

As a first consequence of this elimination, the substitutability and reflexivity axioms are immediately regained, and the problems discussed in § 4.4 disappear.

A second reason for replacing the ε-terms with free δ^+-variables is that the latter can solve the question whether a committed choice is required: We can express—on the one hand—a committed choice by using a single free δ^+-variable and—on the other hand—a choice without commitment by using several variables with the same choice-condition.

Indeed, this also solves our problems with committed choice of Example 5 of § 2.4: Now, again using (ε_1), $\exists x.\,(x \neq x)$ reduces to $x^{\delta^+} \neq x^{\delta^+}$ with

$$C(x^{\delta^+}) := (x^{\delta^+} \neq x^{\delta^+})$$

and the proof attempt immediately fails due to the now regained reflexivity axiom.

As the second step, we still have to explain what to do with subordinate ε-terms. If the ε-term $\varepsilon z.\, A$ contains free occurrences of exactly the distinct variables $v_0, \ldots, v_{l-1}$, then we have to replace this ε-term with the application term $z^{\delta^+}(v_0)\cdots(v_{l-1})$ of the same type as z (for a new free δ^+-variable z^{δ^+}) and to extend the choice-condition C by

$$C(z^{\delta^+}) := \lambda v_0.\ \ldots \lambda v_{l-1}.\ A\{z \mapsto z^{\delta^+}(v_0)\cdots(v_{l-1})\}.$$

Example 10 *(Higher-order choice-condition).* *(continuing Example 8 of § 4.3)*
In our framework, the complete elimination of ε-terms in (8.1) of Example 8 results in

$$\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \qquad \text{(cf. (8.1)!)} \qquad (10.1)$$

with the following higher-order choice-condition:

$$C(z_d^{\delta^+}) := \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \qquad \text{(cf. (8.2)!)} \qquad (10.2)$$
$$C(y_c^{\delta^+}) := \mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_c^{\delta^+}(y_c^{\delta^+})) \qquad \text{(cf. (8.3)!)} \qquad (10.3)$$
$$C(z_c^{\delta^+}) := \lambda y.\ \neg\mathsf{P}(x_a^{\delta^+}, y, z_c^{\delta^+}(y)) \qquad \text{(cf. (8.4)!)} \qquad (10.4)$$
$$C(x_a^{\delta^+}) := \neg\mathsf{P}(x_a^{\delta^+}, y_a^{\delta^+}(x_a^{\delta^+}), z_b^{\delta^+}(x_a^{\delta^+})) \qquad \text{(cf. (8.5)!)} \qquad (10.5)$$
$$C(z_b^{\delta^+}) := \lambda x.\ \neg\mathsf{P}(x, y_a^{\delta^+}(x), z_b^{\delta^+}(x)) \qquad \text{(cf. (8.6)!)} \qquad (10.6)$$
$$C(y_a^{\delta^+}) := \lambda x.\ \mathsf{P}(x, y_a^{\delta^+}(x), z_a^{\delta^+}(x)(y_a^{\delta^+}(x))) \qquad \text{(cf. (8.7)!)} \qquad (10.7)$$
$$C(z_a^{\delta^+}) := \lambda x.\ \lambda y.\ \neg\mathsf{P}(x, y, z_a^{\delta^+}(x)(y)) \qquad \text{(cf. (8.8)!)} \qquad (10.8)$$

Notice that this representation of (8.1) is smaller and easier to understand than all previous ones. Indeed, by combination of λ-abstraction and term sharing via free δ^+-variables, in our framework the ε becomes practically feasible for the first time. □

4.6. Instantiating free δ^+-variables ("ε-substitution")

Having realized Requirement I (Syntax) of § 1 in the previous § 4.5, in this § 4.6 we are now going to explain how to satisfy Requirement II (Reasoning). To this end, we have to explain how to replace free δ^+-variables with terms that satisfy their choice-conditions.

The first thing to know about free δ^+-variables is: Just like the free γ-variables and contrary to free δ^--variables, the free δ^+-variables are *rigid* in the sense that the only way to replace a free δ^+-variable is to do it *globally*, i.e. in all formulas and all choice-conditions in an atomic transaction.

In *reductive* theorem proving, such as in sequent, tableau, or matrix calculi, we are in the following situation: While a free γ-variable x^γ can be replaced with nearly everything, the replacement of a free δ^+-variable y^{δ^+} requires some proof work, and a free δ^--variable z^{δ^-} cannot be instantiated at all.

Contrariwise, when formulas are used as tools instead of tasks, free δ^--variables can indeed be replaced—and this even locally (i.e. non-rigidly). This is the case not only for purely *generative* calculi, such as resolution and paramodulation calculi and Hilbert-style calculi such as the predicate calculus of Hilbert & Bernays (1968/70), but also for the lemma and induction hypothesis application in the otherwise reductive calculi of Wirth (2004), cf. Wirth (2004), § 2.5.2.

More precisely—again considering *reductive* theorem proving, where formulas are proof tasks—a free γ-variable x^{γ} may be instantiated with any term (of appropriate type) that does not violate the current variable-condition, cf. § 5.2 for details. The instantiation of a free δ^{+}-variable y^{δ^+} additionally requires some proof work depending on the current choice-condition C, which also puts some requirements on the variable-condition R and thus is formally called an R*-choice-condition*, cf. Definition 22 for the formal details. In general, if a substitution σ replaces—possibly among other free γ-variables and free δ^{+}-variables—the free δ^{+}-variable y^{δ^+} in the domain of the R-choice-condition C, then—to know that the global instantiation of the whole proof forest with σ preserves its soundness—we have to prove $(Q_C(y^{\delta^+}))\sigma$, where Q_C is given as follows:

Definition 11 *(Q_C).*
For an R-choice-condition C, we let Q_C be a total function from $\mathrm{dom}(C)$ into the set of single-formula sequents such that for each $y^{\delta^+} \in \mathrm{dom}(C)$ with

$$C(y^{\delta^+}) \quad = \quad \lambda v_0.\ \ldots \lambda v_{l-1}.\ B$$

for a formula B, we have

$$Q_C(y^{\delta^+}) \quad = \quad \forall v_0.\ \ldots \forall v_{l-1}.\ \big(\ \exists y.\ B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\} \ \ \Rightarrow\ \ B\ \big)$$

for an arbitrary fresh bound variable $y \in \mathrm{V}_{\mathrm{bound}} \setminus \mathcal{V}(C(y^{\delta^+}))$. □

Note that $Q_C(y^{\delta^+})$ is nothing but a formulation of axiom (ε_0) from § 2.1 in our framework, and Lemma 32 states its validity.

It is an essential property of our choice-conditions that all occurrences of y^{δ^+} in B necessarily are of the form $y^{\delta^+}(v_0)\cdots(v_{l-1})$, cf. Definition 22(2). [2] Therefore, the formula $Q_C(y^{\delta^+})$ is logically equivalent to the formula

$$\forall v_0.\ \ldots \forall v_{l-1}.\ \big(\ \exists z.\ B\{y^{\delta^+}\mapsto z\} \ \ \Rightarrow\ \ B\ \big)$$

for a new bound variable z of the same type as y^{δ^+}.

Example 12 *(Predecessor function).*
Suppose that our domain is natural numbers and that $y^{\delta^+}_{(\mathsf{p1})}$ has the choice-condition

$$C(y^{\delta^+}_{(\mathsf{p1})}) = \lambda v.\ (v = y^{\delta^+}_{(\mathsf{p1})}(v)+1).$$

Then, before we may instantiate $y^{\delta^+}_{(\mathsf{p1})}$ with the symbol p for the predecessor function specified by $\forall x.\ \big(\mathsf{p}(x+1) = x\big)$, we have to prove $(Q(y^{\delta^+}_{(\mathsf{p1})}))\{y^{\delta^+}_{(\mathsf{p1})} \mapsto \mathsf{p}\}$, which reads as

$$\forall v.\ \big(\ \exists y.\ (v = y+1) \Rightarrow (v = \mathsf{p}(v)+1)\ \big),$$

and is valid in arithmetic. □

Example 13 *(Canossa 1077).* *(continuing Example 4)*
The situation of Example 4 now reads

$$\mathsf{Holy\,Ghost} \ = \ z_0^{\delta^+} \quad \wedge \quad \mathsf{Joseph} \ = \ z_1^{\delta^+} \tag{13.1}$$

with $C(z_0^{\delta^+}) = \mathsf{Father}(z_0^{\delta^+}, \mathsf{Jesus})$,
and $C(z_1^{\delta^+}) = \mathsf{Father}(z_1^{\delta^+}, \mathsf{Jesus})$.
This avoids the previous trouble with the Pope because nobody knows whether $z_0^{\delta^+} = z_1^{\delta^+}$ holds. On the one hand, knowing (1.2) from Example 1 of § 2.1, we can prove (13.1) as follows: We first substitute $z_0^{\delta^+}$ with Holy Ghost because, for $\sigma_0 := \{z_0^{\delta^+} \mapsto \mathsf{Holy\,Ghost}\}$, we have $(C(z_0^{\delta^+}))\sigma_0$ and—a fortiori— $(Q_C(z_0^{\delta^+}))\sigma_0$, which reads

$$\exists z.\ \mathsf{Father}(z, \mathsf{Jesus}) \ \ \Rightarrow \ \ \mathsf{Father}(\mathsf{Holy\,Ghost}, \mathsf{Jesus});$$

and, analogously, substitute $z_1^{\delta^+}$ with Joseph because, for $\sigma_1 := \{z_1^{\delta^+} \mapsto \mathsf{Joseph}\}$, we have $(C(z_1^{\delta^+}))\sigma_1$ and—a fortiori— $(Q_C(z_1^{\delta^+}))\sigma_1$. After these substitutions, (13.1) becomes the tautology

$$\mathsf{Holy\,Ghost} \ = \ \mathsf{Holy\,Ghost} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}$$

On the other hand, if we want to have trouble, we can apply the substitution

$$\sigma' \ = \ \{z_0^{\delta^+} \mapsto \mathsf{Joseph},\ z_1^{\delta^+} \mapsto \mathsf{Joseph}\}$$

to (13.1) because of $(Q_C(z_0^{\delta^+}))\sigma' = (Q_C(z_1^{\delta^+}))\sigma_1 = (Q_C(z_1^{\delta^+}))\sigma'$. Then our task is to show

$$\mathsf{Holy\,Ghost} \ = \ \mathsf{Joseph} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}$$

To prove (13.1), this is a stupid trial, however. □

5. Formal presentation of our indefinite semantics

To satisfy Requirement III (Semantics) of § 1, we now present our novel semantics for the ε more formally, which is required for precision and consistency. As consistency of our new semantics is not trivial at all, technical rigor cannot be avoided. From § 4 the reader should have a good intuition of our intended representation and semantics of the ε, free δ^+-variables, and choice-conditions in our framework.

This § 5 is organized as follows: In §§ 5.2 and 5.4 we formalize variable-conditions and explain how to deal with free γ-variables syntactically and semantically. In § 5.3 we introduce a preliminary semantics that does not treat free δ^+-variables properly, and in § 5.6 we provide a proper semantics. We discuss choice-conditions in § 5.5. Our interest goes beyond soundness in that we want "*preservation of solutions*". By this we mean the following: All *closing substitutions* for the free γ-variables and free δ^+-variables—i.e. all solutions that transform a proof attempt (to which a proposition has been reduced) into a closed proof—are also solutions of the original proposition. This is similar to a proof in the programming language PROLOG (cf. Kowalski (1979)), computing answers to a query proposition that contains free γ-variables. Therefore, in § 5.7 we discuss this solution-preserving notion of *reduction*, especially under the aspect of global instantiation of free δ^+-variables. All in all, in this § 5, we extend and simplify the presentation of Wirth (2004), which contains also a comparative discussion, compatible extensions for *descente infinie*, and those proofs that are omitted here.

5.1. Basic notions and notation

'$\mathbf{N}$' denotes the set of natural numbers and '$\prec$' the ordering on $\mathbf{N}$. Let $\mathbf{N}_+ := \{\, n \in \mathbf{N} \mid 0 \neq n \,\}$. We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. For classes R, A, and B we define:

$\mathrm{dom}(R)$	$:= \{\, a \mid \exists b.\ (a,b) \in R \,\}$	*domain*
${}_A{\upharpoonleft} R$	$:= \{\, (a,b) \in R \mid a \in A \,\}$	*restriction to A*
$\langle A \rangle R$	$:= \{\, b \mid \exists a \in A.\ (a,b) \in R \,\}$	*image of A*, i.e. $\langle A \rangle R = \mathrm{ran}({}_A{\upharpoonleft} R)$

And the dual ones:

$\mathrm{ran}(R)$	$:= \{\, b \mid \exists a.\ (a,b) \in R \,\}$	*range*
$R{\upharpoonright}_B$	$:= \{\, (a,b) \in R \mid b \in B \,\}$	*range-restriction to B*
$R\langle B \rangle$	$:= \{\, a \mid \exists b \in B.\ (a,b) \in R \,\}$	*reverse-image of B*, i.e. $R\langle B \rangle = \mathrm{dom}(R{\upharpoonright}_B)$

Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function. Functions are (right-) unique relations and the meaning of '$f \circ g$' is extensionally given by $(f \circ g)(x) = g(f(x))$. The *class of total functions from A to B* is denoted as $A \rightarrow B$. The *class of (possibly) partial functions from A to B* is denoted as $A \rightsquigarrow B$. Both $\rightarrow$ and $\rightsquigarrow$ associate to the right, i.e. $A \rightsquigarrow B \rightarrow C$ reads $A \rightsquigarrow (B \rightarrow C)$.

Let R be a binary relation. R is a relation *on A* if $\mathrm{dom}(R) \cup \mathrm{ran}(R) \subseteq A$. R is *irreflexive* if $\mathrm{id} \cap R = \emptyset$. It is *A-reflexive* if ${}_A{\upharpoonleft}\mathrm{id} \subseteq R$. Speaking of a *reflexive* relation we refer to the largest A that is appropriate in the local context, and referring to this A we write R^0 to ambiguously denote ${}_A{\upharpoonleft}\mathrm{id}$. With $R^1 := R$, and $R^{n+1} := R^n \circ R$ for $n \in \mathbf{N}_+$, R^m denotes the m-step relation for R. The *transitive closure* of R is $R^+ := \bigcup_{n \in \mathbf{N}_+} R^n$. The *reflexive & transitive closure* of R is $R^* := \bigcup_{n \in \mathbf{N}} R^n$. A relation R (on A) is *well-founded* if each non-empty class B ($\subseteq A$) has an R-minimal element, i.e. $\exists a \in B.\ \neg\exists a' \in B.\ a' R\, a$.

5.2. Variables and R-substitutions

We assume the following four sets of symbols to be disjoint:

V_γ	*free γ-variables*, i.e. the free variables of Fitting (1996)
V_δ	*free δ-variables*, i.e. nullary parameters, instead of Skolem functions
$\mathrm{V}_{\mathrm{bound}}$	*bound variables*, i.e. variables to be bound, cf. below
Σ	*constants*, i.e. the function and predicate symbols from the signature

As explained in § 4.1, we partition the free δ-variables into *free δ^--variables* and *free δ^+-variables*: $\mathrm{V}_\delta = \mathrm{V}_{\delta^-} \uplus \mathrm{V}_{\delta^+}$. We define the *free variables* by $\mathrm{V}_{\mathrm{free}} := \mathrm{V}_\gamma \uplus \mathrm{V}_\delta$ and the *variables* by $\mathrm{V} := \mathrm{V}_{\mathrm{bound}} \uplus \mathrm{V}_{\mathrm{free}}$. Finally, the *rigid* variables by $\mathrm{V}_{\gamma\delta^+} := \mathrm{V}_\gamma \uplus \mathrm{V}_{\delta^+}$. We use '$\mathcal{V}_k(\Gamma)$' to denote the set of variables from V_k occurring in Γ.

Let σ be a substitution. σ is a *substitution on X* if $\mathrm{dom}(\sigma) \subseteq X$. We denote with '$\Gamma\sigma$' the result of replacing each occurrence of a variable $x \in \mathrm{dom}(\sigma)$ in Γ with $\sigma(x)$. (Actually, we may have to rename some of the bound variables in $\sigma(x)$ when we exclude the binding of a variable within the scope of a bound variable of the same name.) Unless otherwise stated, we tacitly assume that all occurrences of variables from $\mathrm{V}_{\mathrm{bound}}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a variable $x \in \mathrm{V}_{\mathrm{bound}}$ occurs only in the scope of a binder on x) and that each substitution σ satisfies $\mathrm{dom}(\sigma) \subseteq \mathrm{V}_{\mathrm{free}}$, so that no bound occurrences of variables can be replaced and no additional variable occurrences can become bound (i.e. captured) when applying σ.

Several binary relations on free variables will be introduced. The overall idea is that when (x, y) occurs in such a relation this means something like "x is necessarily older than y" or "the value of y depends on x or is described in terms of x".

Definition 14 *(Variable-condition).* A *variable-condition* is a subset of $\mathrm{V}_{\mathrm{free}} \times \mathrm{V}_{\mathrm{free}}$. □

Definition 15 *(σ-update).* Let R be a variable-condition and σ be a substitution.
The *σ-update of R* is $R \cup \{ (z^{\mathrm{free}}, x^{\mathrm{free}}) \mid x^{\mathrm{free}} \in \mathrm{dom}(\sigma) \wedge z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}})) \}$. □

Definition 16 *(R-substitution).* Let R be a variable-condition.
σ is an *R-substitution* if σ is a substitution and the σ-update of R is well-founded. □

Syntactically, $(x^{\mathrm{free}}, y^{\mathrm{free}}) \in R$ is to express that an R-substitution σ must not replace x^{free} with a term in which y^{free} could ever occur. This is guaranteed when the σ-updates R' of R are always required to be well-founded. For $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}}))$, we get $z^{\mathrm{free}}\ R'\ x^{\mathrm{free}}\ R'\ y^{\mathrm{free}}$, blocking z^{free} against terms containing y^{free}. Note that in practice a σ-update of R can always be chosen to be finite. In this case, it is well-founded iff it is acyclic.

5.3. *R-validity*

Instead of defining validity from scratch, we require some abstract properties typically holding in two-valued semantics. Validity is given relative to some Σ-structure $\mathcal{S}$, assigning a non-empty universe (or "carrier") to each type. For $\mathrm{X} \subseteq \mathrm{V}$ we denote the set of total $\mathcal{S}$-valuations of X (i.e. functions mapping variables to objects of the universe of $\mathcal{S}$ (respecting types)) with

$$\mathrm{X} \to \mathcal{S}$$

and the set of (possibly) partial $\mathcal{S}$-valuations of X with

$$\mathrm{X} \rightsquigarrow \mathcal{S}$$

For $\delta : \mathrm{X} \to \mathcal{S}$ we denote with '$\mathcal{S} \uplus \delta$' the extension of $\mathcal{S}$ to the variables of X. More precisely, we assume some evaluation function 'eval' such that $\mathrm{eval}(\mathcal{S} \uplus \delta)$ maps each term whose constants and free occurring variables are from $\Sigma \uplus \mathrm{X}$ into the universe of $\mathcal{S}$ (respecting types) such that for all $x \in \mathrm{X}$: $\mathrm{eval}(\mathcal{S} \uplus \delta)(x) = \delta(x)$. Moreover, $\mathrm{eval}(\mathcal{S} \uplus \delta)$ maps each formula B whose constants and free occurring variables are from $\Sigma \uplus \mathrm{X}$ to TRUE or FALSE, such that B is valid in $\mathcal{S} \uplus \delta$ iff $\mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}$.

Notice that we leave open what our formulas and what our Σ-structures exactly are. The latter can range from a first-order Σ-structure to a higher-order modal Σ-model, provided that the following two standard textbook lemmas hold for a term or formula B (possibly with some *unbound* occurrences of variables from $\mathrm{V}_{\mathrm{bound}}$) and a Σ-structure $\mathcal{S}$ with valuation $\delta : \mathrm{V} \rightsquigarrow \mathcal{S}$.

EXPLICITNESS LEMMA
The value of the evaluation function on B depends only on the valuation of those variables that actually occur free in B; formally: For X being the set of variables that occur free in B, if $\mathrm{X} \subseteq \mathrm{dom}(\delta)$:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B) \;=\; \mathrm{eval}(\mathcal{S} \uplus {}_{\mathrm{X}}\!\upharpoonright\!\delta)(B). \qquad \square$$

SUBSTITUTION [VALUE] LEMMA
Let σ be a substitution. If the variables that occur free in $B\sigma$ belong to $\mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) \;=\; \mathrm{eval}\Big(\mathcal{S} \uplus \big((\sigma \uplus {}_{\mathrm{V} \setminus \mathrm{dom}(\sigma)}\!\upharpoonright\!\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \delta) \big)\Big)\big(B \big). \qquad \square$$

We are now going to define a new notion of validity of sets of sequents, i.e. sets of lists of formulas.

As this new kind of validity depends on a variable-condition R, it is called "R-validity". It provides the free γ-variables with an existential semantics given by their valuation $\epsilon(e)(\delta) : \mathrm{V}_\gamma \to \mathcal{S}$, and the free δ-variables with a universal semantics by $\delta : \mathrm{V}_\delta \to \mathcal{S}$.

The definition is top-down and the function ϵ (having nothing to do with Hilbert's ε) and the notion of an $(\mathcal{S}, R)$-valuation are to be explained in § 5.4, which also contains examples illustrating R-Validity.

Definition 17 *(R-validity, K).*
Let R be a variable-condition. Let $\mathcal{S}$ be a Σ-structure with valuation $\delta : \mathrm{V} \rightsquigarrow \mathcal{S}$. Let G be a set of sequents.
G is *R-valid in $\mathcal{S}$* if there is an $(\mathcal{S}, R)$-valuation e such that G is $(e, \mathcal{S})$-valid.
G is *$(e, \mathcal{S})$-valid* if G is $(\delta', e, \mathcal{S})$-valid for all $\delta' : \mathrm{V}_\delta \to \mathcal{S}$.
G is *$(\delta, e, \mathcal{S})$-valid* if G is valid in $\mathcal{S} \uplus \epsilon(e)(\delta) \uplus \delta$.
G is *valid in $\mathcal{S}\uplus\delta$* if Γ is valid in $\mathcal{S}\uplus\delta$ for all $\Gamma \in G$.
A sequent Γ is *valid in $\mathcal{S}\uplus\delta$* if there is some formula listed in Γ that is valid in $\mathcal{S}\uplus\delta$.
Validity in a class of Σ-structures is understood as validity in each of the Σ-structures of that class. If we omit the reference to a special Σ-structure we mean validity in some fixed class K of Σ-structures, such as the class of all Σ-structures or the class of Herbrand Σ-structures. □

5.4. $(\mathcal{S}, R)$-valuations

Let $\mathcal{S}$ be some Σ-structure. We now define semantical counterparts of our R-substitutions on V_γ, which we will call "$(\mathcal{S}, R)$-valuations".

As an $(\mathcal{S}, R)$-valuation plays the rôle of a *raising function* (a dual of a Skolem function as defined in Miller (1992)), it does not simply map each free γ-variable directly to an object of $\mathcal{S}$ (of the same type), but may additionally read the values of some free δ-variables under an $\mathcal{S}$-valuation $\delta : \mathrm{V}_\delta \to \mathcal{S}$. More precisely, an $(\mathcal{S}, R)$-valuation e takes some restriction of δ as a second argument, say $\delta' : \mathrm{V}_\delta \rightsquigarrow \mathcal{S}$ with $\delta' \subseteq \delta$. In short:

$$e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}.$$

Moreover, for each free γ-variable x^γ, we require that the set $\mathrm{dom}(\delta')$ of free δ-variables read by $e(x^\gamma)$ is identical for all δ. This identical set will be denoted with $S_e\langle\!\{x^\gamma\}\!\rangle$ below. Technically, we require that there is some "semantical relation" $S_e \subseteq \mathrm{V}_\delta \times \mathrm{V}_\gamma$ such that for all $x^\gamma \in \mathrm{V}_\gamma$:

$$e(x^\gamma) \ : \ (S_e\langle\!\{x^\gamma\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$

This means that $e(x^\gamma)$ can read the value of y^δ if and only if $(y^\delta, x^\gamma) \in S_e$.
Note that, for each $e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, at most one semantical relation exists, namely

$$S_e \ := \ \{\ (y^\delta, x^\gamma) \mid x^\gamma \in \mathrm{V}_\gamma \wedge y^\delta \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(e(x^\gamma))))\ \}.$$

In some of the following definitions we are slightly more general because we want to apply the terminology not only to free γ-variables but also to free δ^+-variables.

Definition 18 *(Semantical relation (S_e)).*
The *semantical relation for e* is

$$S_e \ := \ \{\ (y, x) \mid x \in \mathrm{dom}(e) \wedge y \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(e(x))))\ \}.$$

e is *semantical* if e is a partial function on V such that for all $x \in \mathrm{dom}(e)$:

$$e(x) : (S_e\langle\!\{x\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$ □

Definition 19 *($(\mathcal{S}, R)$-valuation).*
Let R be a variable-condition and let $\mathcal{S}$ be a Σ-structure.
e is an *$(\mathcal{S}, R)$-valuation* if $e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, e is semantical, and $R \cup S_e$ is well-founded. □

Finally, we need the technical means to turn an $(\mathcal{S}, R)$-valuation e together with a valuation δ of the free δ-variables into a valuation $\epsilon(e)(\delta)$ of the free γ-variables:

Definition 20 *(ϵ).*
We define the function $\quad \epsilon \quad : \quad (\mathrm{V} \rightsquigarrow (\mathrm{V} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \quad \to \quad (\mathrm{V} \rightsquigarrow \mathcal{S}) \quad \to \quad \mathrm{V} \quad \rightsquigarrow \quad \mathcal{S}$

for $\qquad e : \mathrm{V} \rightsquigarrow (\mathrm{V} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}, \qquad \delta : \mathrm{V} \rightsquigarrow \mathcal{S}, \qquad x \in \mathrm{V}$

by $\qquad \epsilon(e)(\delta)(x) := e(x)({}_{S_e\langle\{x\}\rangle}{\upharpoonright}\delta).$ □

Example 21 *(R-validity).*
For $x^\gamma \in \mathrm{V}_\gamma$, $y^\delta \in \mathrm{V}_\delta$, the sequent $x^\gamma{=}y^\delta$ is $\emptyset$-valid in each $\mathcal{S}$ because we can choose $S_e := \mathrm{V}_\delta \times \mathrm{V}_\gamma$ and $e(x^\gamma)(\delta) := \delta(y^\delta)$ for $\delta : \mathrm{V}_\delta \to \mathcal{S}$, resulting in $\epsilon(e)(\delta)(x^\gamma) = e(x^\gamma)({}_{S_e\langle\{x^\gamma\}\rangle}{\upharpoonright}\delta) = e(x^\gamma)({}_{\mathrm{V}_\delta}{\upharpoonright}\delta) = \delta(y^\delta)$. This means that $\emptyset$-validity of $x^\gamma{=}y^\delta$ is the same as validity of $\forall y.\ \exists x.\ x{=}y$. Moreover, note that $\epsilon(e)(\delta)$ has access to the δ-value of y^δ just as a raising function f for x in the raised (i.e. dually Skolemized) version $f(y^\delta){=}y^\delta$ of $\forall y.\ \exists x.\ x{=}y$.

Contrary to this, for $R := \mathrm{V}_\gamma \times \mathrm{V}_\delta$, the same formula $x^\gamma{=}y^\delta$ is not R-valid in general because then the required well-foundedness of $R \cup S_e$ (cf. Definition 19) implies $S_e = \emptyset$, and the value of x^γ cannot depend on $\delta(y^\delta)$ anymore, due to $e(x^\gamma)({}_{S_e\langle\{x^\gamma\}\rangle}{\upharpoonright}\delta) = e(x^\gamma)({}_{\emptyset}{\upharpoonright}\delta) = e(x^\gamma)(\emptyset)$. This means that $(\mathrm{V}_\gamma \times \mathrm{V}_\delta)$-validity of $x^\gamma{=}y^\delta$ is the same as validity of $\exists x.\ \forall y.\ x{=}y$. Moreover, note that $\epsilon(e)(\delta)$ has no access to the δ-value of y^δ just as a raising function c for x in the raised version $c{=}y^\delta$ of $\exists x.\ \forall y.\ x{=}y$.

For a more general example let $G = \{\ A_{i,0} \ldots A_{i,n_i-1} \mid i \in I\ \}$, where, for $i \in I$ and $j \prec n_i$, the $A_{i,j}$ are formulas with free γ-variables from $\boldsymbol{e}$ and free δ-variables from $\boldsymbol{u}$. Then $(\mathrm{V}_\gamma \times \mathrm{V}_\delta)$-validity of G means

$$\exists \boldsymbol{e}.\ \forall \boldsymbol{u}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$$

whereas $\emptyset$-validity of G means

$$\forall \boldsymbol{u}.\ \exists \boldsymbol{e}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$$

Also each other sequence of universal and existential quantifiers can be represented by a variable-condition R, starting from the empty set and applying the δ-rules from § 4.2. A translation of a variable-condition R into a sequence of quantifiers may, however, require a strengthening of dependences, in the sense that a backwards translation would result in a variable-condition R' with $R \subsetneq R'$. This means that our variable-conditions can express logical dependences more fine-grained than standard quantifiers. □

5.5. Choice-conditions

Definition 22 *(Choice-condition).*
C is an *R-choice-condition* if R is a well-founded variable-condition and C is a partial function from V_{δ^+} into the set of formula-valued λ-terms, such that for all $y^{\delta^+} \in \mathrm{dom}(C)$:

(1) $z^{\mathrm{free}}\ R^*\ y^{\delta^+}$ for all $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(C(y^{\delta^+}))$, and

(2) $C(y^{\delta^+})$ is of the form $\lambda v_0.\ \ldots \lambda v_{l-1}.\ B$, where
B is a formula whose free occurring variables from $\mathrm{V}_{\mathrm{bound}}$ are among $\{v_0, \ldots, v_{l-1}\} \subseteq \mathrm{V}_{\mathrm{bound}}$
and where, for $v_0 : \alpha_0, \ \ldots,\ v_{l-1} : \alpha_{l-1}$, we have
$y^{\delta^+} : \alpha_0 \to \cdots \to \alpha_{l-1} \to \alpha_l$ for some type α_l,
and each occurrence of y^{δ^+} in B is of the form $y^{\delta^+}(v_0) \cdots (v_{l-1})$. □

Example 23 *(Choice-condition).* *(continuing Example 10)*

(a) If R is a well-founded variable-condition that satisfies

$$z_a^{\delta^+}\ R\ y_a^{\delta^+}\ R\ z_b^{\delta^+}\ R\ x_a^{\delta^+}\ R\ z_c^{\delta^+}\ R\ y_c^{\delta^+}\ R\ z_d^{\delta^+},$$

then the C of Example 10 is an R-choice-condition, indeed.

(b) If some clever person would like to do the complete quantifier elimination of Example 10 by

$$\begin{array}{rcl} C'(z_d^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \\ C'(y_c^{\delta^+}) & := & \mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \\ C'(x_a^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \end{array}$$

then he would—among other things—need $z_d^{\delta^+}\ R^+\ y_c^{\delta^+}\ R^+\ z_d^{\delta^+}$, by Definition 22(1) due to the values of C' at $y_c^{\delta^+}$ and $z_d^{\delta^+}$. This renders R non-well-founded. Thus, this C' cannot be an R-choice-condition for any R. Note that the choices required by C' for $y_c^{\delta^+}$ and $z_d^{\delta^+}$ are in an unsolvable conflict, indeed.

(c) For a more elementary example, take

$$\begin{array}{rcl} C''(x^{\delta^+}) & := & (x^{\delta^+} = y^{\delta^+}) \\ C''(y^{\delta^+}) & := & (x^{\delta^+} \neq y^{\delta^+}) \end{array}$$

Then x^{δ^+} and y^{δ^+} form a vicious circle of conflicting choices for which no valuation can be found that is compatible with C'', cf. Definition 24 and Lemma 25 below. But C'' is no choice-condition at all because there is no well-founded variable-condition R that could turn it into an R-choice-condition. □

We now split our valuation $\delta : \mathrm{V}_\delta \to \mathcal{S}$; while $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$ valuates the free δ^--variables, π valuates the remaining free δ^+-variables. As the choices of π may depend on τ, the technical realization is similar to that of the dependence of the $(\mathcal{S}, R)$-valuations on the free δ-variables, as described in § 5.4.

Definition 24 *(Compatibility).*
Let C be an R-choice-condition, $\mathcal{S}$ a Σ-structure, and e an $(\mathcal{S}, R)$-valuation.
π is $(e, \mathcal{S})$-*compatible with* (C, R) if

(1) $\pi : \mathrm{V}_{\delta^+} \to (\mathrm{V}_{\delta^-} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ is semantical (cf. Definition 18) and $R \cup S_e \cup S_\pi$ is well-founded.

(2) For all $y^{\delta^+} \in \mathrm{dom}(C)$ with $C(y^{\delta^+}) = \lambda v_0.\ \ldots\lambda v_{l-1}.\ B$ for a formula B, for all $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$, for all $\eta : \{y^{\delta^+}\} \to \mathcal{S}$, and for all $\chi : \{v_0, \ldots, v_{l-1}\} \to \mathcal{S}$, setting $\delta := \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ and $\delta' := \eta \uplus {}_{\mathrm{V}\setminus\{y^{\delta^+}\}}\!\upharpoonright\!\delta$ (i.e. δ' is the η-variant of δ):
   If B is $(\delta', e, \mathcal{S})$-valid, then B is also $(\delta, e, \mathcal{S})$-valid. □

Roughly speaking, Item (1) of this definition requires—for similar reasons as before—that the flow of information between variables expressed in R, e, and π is acyclic.
To understand Item (2), consider an R-choice-condition $C := \{(y^{\delta^+},\ \lambda v_0.\ \ldots\lambda v_{l-1}.\ B)\}$, which restricts the value of y^{δ^+} with the formula-valued λ-term $\lambda v_0.\ \ldots\lambda v_{l-1}.\ B$. Then C simply requires that a different choice for the $\epsilon(\pi)(\tau)$-value of y^{δ^+} cannot give rise to the validity of the formula B in $\mathcal{S} \uplus \epsilon(e)(\delta) \uplus \delta$. Or—in other words—that $\epsilon(\pi)(\tau)(y^{\delta^+})$ is chosen such that B becomes valid, whenever such a choice is possible. This is closely related to Hilbert's ε-operator in the sense that y^{δ^+} is given the value of

$$\lambda v_0.\ \ldots\lambda v_{l-1}.\ \varepsilon y.\ B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\}$$

for a fresh bound variable y.

As the choice for y^{δ^+} depends on the other free variables of $\lambda v_0.\ldots\lambda v_{l-1}.B$ (i.e. the free variables of $\lambda v_0.\ \ldots\lambda v_{l-1}.\ \varepsilon y.\ B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\}$), we included this dependence into the transitive closure of the variable-condition R in Definition 22(1). Therefore, the well-foundedness of R avoids the conflict of Example 23(c).

Note that the empty function $\emptyset$ is an R-choice-condition for each well-founded variable-condition R. Furthermore, each π with $\pi : \mathrm{V}_{\delta^+} \to \{\emptyset\} \to \mathcal{S}$ is $(e, \mathcal{S})$-compatible with $(\emptyset, R)$ due to $S_\pi = \emptyset$. Indeed, as stated in the following lemma, a compatible π always exists. This is due to Definition 22(1) and the well-foundedness of $R \cup S_e$ (according to Definition 19) and due to the restriction on the occurrence of y^{δ^+} in B in Definition 22(2).

Lemma 25. *If C is an R-choice-condition, $\mathcal{S}$ a Σ-structure, and e an $(\mathcal{S}, R)$-valuation, then there is some π that is $(e, \mathcal{S})$-compatible with (C, R).* □

Just as the variable-condition R, the R-choice-condition C may grow during proofs. This kind of extension together with a simple soundness condition plays an important rôle in inference:

Definition 26 *(Extension).* (C', R') is an *extension of* (C, R) if C is an R-choice-condition, C' is an R'-choice-condition, $C \subseteq C'$, and $R \subseteq R'$. □

Lemma 27 *(Extension). Let (C', R') be an extension of (C, R).*
If e is an $(\mathcal{S}, R')$-valuation and π is $(e, \mathcal{S})$-compatible with (C', R'),
then e is also an $(\mathcal{S}, R)$-valuation and π is also $(e, \mathcal{S})$-compatible with (C, R). □

After global application of an R-substitution σ we now have to update both R and C:

Definition 28 *(Extended σ-update).* Let C be an R-choice-condition and let σ be a substitution.
The *extended σ-update* (C', R') *of* (C, R) is given by:

$C' := \{ (x, B\sigma) \mid (x, B) \in C \wedge x \notin \mathrm{dom}(\sigma) \}$,
R' is the σ-update of R, cf. Definition 15. □

Lemma 29 *(Extended σ-update). If C is an R-choice-condition, σ an R-substitution, and if (C', R') is the extended σ-update of (C, R), then C' is an R'-choice-condition.* □

5.6. (C, R)-validity

While the notion of R-validity (cf. Definition 17) already provides the free γ-variables with an existential semantics, it fails to give the free δ^+-variables the proper semantics according to an R-choice-condition C. This deficiency is overcome in the following notion of "(C, R)-validity", which—roughly speaking—requires the following: For arbitrary values of the free δ^--variables, we must be able to choose values for the free δ^+-variables satisfying C, and then we must be able to choose values for the free γ-variables, such that the sequents become valid. Note that the dependences of these choices are restricted by R. In a formal top down representation, this reads:

Definition 30 *((C, R)-validity).*
Let C be an R-choice-condition, let $\mathcal{S}$ be a Σ-structure, and let G be a set of sequents.
G is *(C, R)-valid in $\mathcal{S}$* if
G is $(\pi, e, \mathcal{S})$-valid for some $(\mathcal{S}, R)$-valuation e and some π that is $(e, \mathcal{S})$-compatible with (C, R).
G is *$(\pi, e, \mathcal{S})$-valid* if G is $\big(\epsilon(\pi)(\tau) \uplus \tau,\ e,\ \mathcal{S}\big)$-valid for each $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$. □

Notice that the notion of $(\pi, e, \mathcal{S})$-validity with $\pi : \mathrm{V}_{\delta^+} \to (\mathrm{V}_{\delta^-} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ differs from $(\delta, e, \mathcal{S})$-validity with $\delta : \mathrm{V} \rightsquigarrow \mathcal{S}$ as given in Definition 17, so that the last line of Definition 30 is indeed an explicit definition. Moreover, notice that (C, R)-validity treats the free δ^+-variables properly, whereas R-validity of Definition 17 does not.

In our framework the formula (E2) of § 3.1.1 looks like (E2′) in the following lemma.

Lemma 31 *((C, R)-validity of (E2′)). Let C be an R-choice-condition.*
For $i \in \{0, 1\}$, let A_i be a formula in which $x \in \mathrm{V}_{\mathrm{bound}}$ may occur free, and assume $x_i^{\delta+} \in \mathrm{V}_{\delta^+}$ with $C(x_i^{\delta+}) = A_i\{x \mapsto x_i^{\delta+}\}$, $x_i^{\delta+} \notin \mathcal{V}(A_0, A_1)$, $x_i^{\delta+} \notin \mathrm{dom}(R)$, and $x_i^{\delta+} \notin \mathcal{V}(\langle \mathrm{V} \backslash \{x_i^{\delta+}\} \rangle C)$.
The formula

$$\forall x.\ (A_0 \Leftrightarrow A_1) \ \Rightarrow\ x_0^{\delta+} = x_1^{\delta+} \tag{E2′}$$

is (C, R)-valid. □

Note that the pre-conditions of Lemma 31 may simply be achieved by taking *fresh* free δ^+-variables $x_0^{\delta+}$ and $x_1^{\delta+}$ and adding $(\mathcal{V}_{\mathrm{free}}(A_i\{x \mapsto x_i^{\delta+}\}) \backslash \{x_i^{\delta+}\}) \times \{x_i^{\delta+}\}$ to the current variable-condition. Roughly speaking, Lemma 31 holds because after choosing a value for $x_0^{\delta+}$ we can take the same value for $x_1^{\delta+}$, simply because $x_1^{\delta+}$ is new and can read all free δ^--variables, and especially those that $x_0^{\delta^+}$ reads. The reader may try to find a semantical proof of Lemma 31 or look it up in Wirth (2006b). Moreover, in Example 36 we will do a formal, nice, and short proof of Lemma 31 in our calculus.

As already noted in § 4.6, the single-formula sequent $Q_C(y^{\delta^+})$ of Definition 11 is a formulation of axiom (ε_0) of § 2.1 in our framework.

Lemma 32 *((C, R)-validity of $Q_C(y^{\delta^+})$).*
Let C be an R-choice-condition. Let $y^{\delta^+} \in \mathrm{dom}(C)$.
The single-formula sequent $Q_C(y^{\delta^+})$ is (C, R)-valid.
Moreover, $Q_C(y^{\delta^+})$ is $(\pi, e, \mathcal{S})$-valid for each Σ-structure $\mathcal{S}$, each $(\mathcal{S}, R)$-valuation e, and each π that is $(e, \mathcal{S})$-compatible with (C, R). □

Proof. Let $C(y^{\delta^+}) = \lambda v_0. \ \ldots \lambda v_{l-1}. \ B$ for a formula B. Then we have
$$Q_C(y^{\delta^+}) \;=\; \forall v_0. \ \ldots \forall v_{l-1}. \ \big(\ \exists y. \ B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\} \ \Rightarrow \ B \ \big)$$
for some $y \in \mathrm{V}_{\mathrm{bound}} \backslash \mathcal{V}(C(y^{\delta^+}))$. For π being $(e, \mathcal{S})$-compatible with (C, R), the $(\pi, e, \mathcal{S})$-validity follows now directly from Definition 24(2), according to the short discussion following Definition 11. □

5.7. Reduction

Reduction is the reverse of consequence. It is the backbone of logical reasoning, especially of abduction and goal-directed deduction. Our version of reduction does not only reduce a set of problems to another set of problems but also guarantees that the solutions of the latter also solve the former; where "solutions" means the valuations for the rigid variables, i.e. for the free γ-variables and the free δ^+-variables.

Definition 33 *(Reduction).*
Let C be an R-choice-condition.
Let $\mathcal{S}$ be a Σ-structure, and let G_0 and G_1 be sets of sequents.
G_0 *(C, R)-reduces to* G_1 *in* $\mathcal{S}$ if
for each $(\mathcal{S}, R)$-valuation e and each π that is $(e, \mathcal{S})$-compatible with (C, R):
if G_1 is $(\pi, e, \mathcal{S})$-valid, then G_0 is $(\pi, e, \mathcal{S})$-valid. □

Theorem 34 *(Reduction).*
Let C be an R-choice-condition; $\mathcal{S}$ a Σ-structure; G_0, G_1, G_2, and G_3 sets of sequents.

1. (Validity) *If G_0 (C, R)-reduces to G_1 in $\mathcal{S}$ and G_1 is (C, R)-valid in $\mathcal{S}$, then G_0 is (C, R)-valid in $\mathcal{S}$, too.*

2. (Reflexivity) *In case of $G_0 \subseteq G_1$: G_0 (C, R)-reduces to G_1 in $\mathcal{S}$.*

3. (Transitivity) *If G_0 (C, R)-reduces to G_1 in $\mathcal{S}$ and G_1 (C, R)-reduces to G_2 in $\mathcal{S}$, then G_0 (C, R)-reduces to G_2 in $\mathcal{S}$.*

4. (Additivity) *If G_0 (C, R)-reduces to G_2 in $\mathcal{S}$ and G_1 (C, R)-reduces to G_3 in $\mathcal{S}$, then $G_0 \cup G_1$ (C, R)-reduces to $G_2 \cup G_3$ in $\mathcal{S}$.*

5. (Monotonicity) *For (C', R') being an extension of (C, R):*
(a) If G_0 is (C', R')-valid in $\mathcal{S}$, then G_0 is (C, R)-valid in $\mathcal{S}$.
(b) If G_0 (C, R)-reduces to G_1 in $\mathcal{S}$, then G_0 (C', R')-reduces to G_1 in $\mathcal{S}$.

6. (Instantiation) *For an R-substitution σ on $\mathrm{V}_{\gamma\delta^+}$, for the extended σ-update (C', R') of (C, R), and for $O := \mathrm{dom}(C) \cap \mathrm{dom}(\sigma) \cap R^*\langle \mathcal{V}_{\delta^+}(G_0, G_1)\rangle$:*
(a) If $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is (C', R')-valid in $\mathcal{S}$, then G_0 is (C, R)-valid in $\mathcal{S}$.
(b) If G_0 (C, R)-reduces to G_1 in $\mathcal{S}$, then $G_0\sigma$ (C', R')-reduces to $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$ in $\mathcal{S}$. □

Proof. Items 1 to 5 are the Items 1 to 5 of Lemma 2.31 of Wirth (2004).
Item 6 follows from Lemma B.6 of Wirth (2004) when we set the meta variable N of Lemma B.6 to $\mathrm{dom}(C) \cap \langle(\mathrm{dom}(C) \cap \mathrm{dom}(\sigma)) \setminus O\rangle R^*$. □

Items 1 to 5 of Theorem 34 are straightforward. Item 6 is only technically complicated. Roughly speaking, the idea behind Item 6 is that reduction stays invariant under global application of the substitution σ on rigid variables, provided that we change from (C, R) to its extended σ-update (C', R') and that, in case that σ replaces some free δ^+-variable $y^{\delta+}$ constrained by the choice-condition C, we can establish that this is a proper choice by showing $(Q_C(y^{\delta+}))\sigma$, cf. Definition 11. The rest of this § 5.7 will give further explanation for the application of Theorem 34 and especially for Item 6.

Example 35 *(Instantiation with* $\mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \emptyset$*).*
For a simple application of Theorem 34(6b), where no free δ^+-variables occur and only a free γ-variable is instantiated, let us have a glimpse at the example proof of Wirth (2004), § 3.3. Let G_0 be the proposition we want to prove, namely $\{\, z_0^{\gamma}(x_0^{\delta-})(y_0^{\delta-}) \prec \mathsf{ack}(x_0^{\delta-}, y_0^{\delta-}) \,\}$, which says that Ackermann's function has a lower bound that is to be determined during the proof. Moreover, let G_1 —together with variable-condition R and R-choice-condition $\emptyset$— represent the current state of the proof. Then G_0 $(\emptyset, R)$-reduces to G_1. Moreover, in the example, G_1 reduces to a known lemma when we apply the substitution $\sigma := \{z_0^{\gamma} \mapsto \lambda x.\, \lambda y.\, y\}$. Now, Theorem 34(6b) says that the instantiated (and $\lambda\beta$-reduced) theorem $\{\, y_0^{\delta-} \prec \mathsf{ack}(x_0^{\delta-}, y_0^{\delta-}) \,\}$ $(\emptyset, R)$-reduces to the instantiated proof state $G_1\sigma$ and thus is $(\emptyset, R)$-valid by Theorem 34(3,1). Note that in this case the extended σ-update of $(\emptyset, R)$ is $(\emptyset, R)$ itself, and we have $O = \emptyset$ due to $\mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \emptyset$. Moreover, by Theorem 34(6a), also the original $\{\, z_0^{\gamma}(x_0^{\delta-})(y_0^{\delta-}) \prec \mathsf{ack}(x_0^{\delta-}, y_0^{\delta-}) \,\}$ is known to be $(\emptyset, R)$-valid, but who would be interested in this weaker result now? □

Example 36 *((C, R)-validity of (E2′)).* *(continuing Lemma 31)*
Let us give a formal proof of (E2′) in our framework on a very abstract level by applying Theorem 34. We will reduce the set containing the single-formula sequent of the formula (E2′) to a valid set. This will complete our proof by Theorem 34(1). In the following, be aware of the requirements on occurrence of the variables as described in Lemma 31. We extend (C, R) with a fresh variable $y^{\delta+}$ with

$$C'(y^{\delta+}) \;:=\; \left(\begin{array}{c} (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y^{\delta+} = x_0^{\delta+}) \\ \wedge\ (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y^{\delta+}\}) \end{array}\right).$$

Of course, to satisfy Definition 22(1), the current variable-condition R must be extended to

$$R' \;:=\; R \cup (\mathcal{V}_{\mathrm{free}}(A_0, A_1) \cup \{x_0^{\delta+}\}) \times \{y^{\delta+}\}.$$

Note that, if we had done this extension during the proof, we would have needed Item 5b of Theorem 34 to keep reduction invariant, but as there is no reduction sequence given yet, it suffices to use Item 5a instead. Similarly, instead of Item 6b, we apply Item 6a, with $\sigma := \{x_1^{\delta+} \mapsto y^{\delta+}\}$. Then we have $O \subseteq \mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \{x_1^{\delta+}\}$.
For (C'', R'') being the extended σ-update of (C', R'), Item 6a says that it suffices to show (C'', R'')-validity of the set with the two single-formula sequents $\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow x_0^{\delta+} = y^{\delta+}$ and $(Q_{C'}(x_1^{\delta+}))\sigma$. The latter sequent reads $(\exists x.\ A_1\{x \mapsto x_1^{\delta+}\}\{x_1^{\delta+} \mapsto x\} \Rightarrow A_1\{x \mapsto x_1^{\delta+}\})\sigma$, i.e. $\exists x.\ A_1 \Rightarrow A_1\{x \mapsto y^{\delta+}\}$. But a simple case analysis on $\forall x.\ (A_0 \Leftrightarrow A_1)$ shows that the whole set (C'', R'')-reduces to

$$\left\{\begin{array}{cc} \exists x.\ A_0 \;\Rightarrow\; A_0\{x \mapsto x_0^{\delta+}\}; & \left(\begin{array}{l} \exists y.\ \left(\begin{array}{c} (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y = x_0^{\delta+}) \\ \wedge\ (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y\}) \end{array}\right) \\ \Rightarrow \left(\begin{array}{c} (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y^{\delta+} = x_0^{\delta+}) \\ \wedge\ (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y^{\delta+}\}) \end{array}\right) \end{array}\right) \end{array}\right\},$$

i.e. to $\{Q_{C''}(x_0^{\delta+});\ \ Q_{C''}(y^{\delta+})\}$, which is (C'', R'')-valid by Lemma 32. (Note that by Item 4 of Theorem 34 it would have been sufficient to show that each of the formulas of the set (C'', R'')-reduces to some (C'', R'')-valid set.) Thus, $(\mathrm{E2}')\sigma$ is (C'', R'')-valid. By Item 6a this means that (E2′) is (C', R')-valid, and by Item 5a this means that (E2′) is (C, R)-valid, as was to be shown. Finally, note that we have $R'' = R' \cup \{(y^{\delta+}, x_1^{\delta+})\}$, so that σ is an R'-substitution. Indeed, the graph of R'' is acyclic:

$$\begin{tikzcd}
x_0^{\delta+} \arrow[r, "R'"'] & y^{\delta+} \arrow[r, "R''"'] & x_1^{\delta+} \\
\mathcal{V}_{\mathrm{free}}(A_0) \arrow[u, "R^+"'] \arrow[ur, "R'"'] & & \mathcal{V}_{\mathrm{free}}(A_1) \arrow[u, "R^+"'] \arrow[ul, "R'"]
\end{tikzcd}$$

□

Example 37 *(Instantiation with higher-order choice-condition).* *(continuing Example 12)*
Suppose that

(p1) $\forall v.\ \big(\,\exists y.\ (v = y{+}1)\ \Rightarrow\ (v = \mathsf{p}(v){+}1)\,\big)$

is one of our lemmas for the predecessor function p in the arithmetic of natural numbers, and that we want to use this lemma as justification for replacing $y^{\delta^+}_{(\mathsf{p1})}$ under R-choice-condition $C(y^{\delta^+}_{(\mathsf{p1})}) = \lambda v.\ (v = y^{\delta^+}_{(\mathsf{p1})}(v){+}1)$ globally with p. Note that this was required in Example 12.

By Theorem 34(6), for $\sigma := \{y^{\delta^+}_{(\mathsf{p1})} \mapsto \mathsf{p}\}$, we have to show $(Q_C(y^{\delta^+}_{(\mathsf{p1})}))\sigma$, which is just the above lemma (p1). □

6. Summary of our free-variable framework for Hilbert's ε

According to Smullyan's classification of (problem-) reduction rules into α, β, γ, and δ, we call the quantifiers eliminated and the variables introduced by γ- and δ-rules, γ- and δ-*quantifiers* and *free* γ- and *free* δ-*variables*, respectively. Splitting the free δ-variables into simple (δ^-) and liberalized ones (δ^+), we get three kinds of *free variables* (written x^{free}) in our *free-variable framework*, namely δ^--, δ^+-, and γ-variables. Regarding validity, free δ^--variables (written x^{δ^-}) are implicitly universally quantified, whereas free δ^+-variables (x^{δ^+}) and free γ-variables (x^{γ}) are implicitly existentially quantified. The structure of quantification is represented globally in a *variable-condition* R, which is a directed acyclic graph on free variables, roughly speaking in such a way that a free δ-variable y^{δ} is put into the scope of another free variable x^{free} by an edge $(x^{\mathrm{free}}, y^{\delta})$. The value of a free δ^+-variable y^{δ^+} given by π may depend on the value of free δ^--variables x^{δ^-}. In that case, the *semantical relation* S_π, contains an edge $(x^{\delta^-}, y^{\delta^+})$. Similarly, the value of a free γ-variable y^{γ} given by e may depend on the value of a free δ-variable x^{δ} if $(x^{\delta}, y^{\gamma}) \in S_e$. The quantificational structure is enforced by the requirement that $R \cup S_e \cup S_\pi$ must be acyclic, cf. Definitions 18 and 24.

Suppose that an ε-term $\varepsilon z.\,A$ has free occurrences of exactly the variables $v_0, \ldots, v_{l-1}$ which are not free variables of our framework, but are bound in the context of this ε-term. Then we replace it in its context with the application term $z^{\delta^+}(v_0)\cdots(v_{l-1})$ for a fresh free δ^+-variable z^{δ^+} and set the value of a global function C, called the *choice-condition*, at z^{δ^+} according to

$$C(z^{\delta^+}) \ := \ \lambda v_0.\ \ldots \lambda v_{l-1}.\ A\{z \mapsto z^{\delta^+}(v_0)\cdots(v_{l-1})\},$$

and extend R with an edge $(y^{\mathrm{free}}, z^{\delta^+})$ for each free variable y^{free} occurring in A. A free δ^+-variable must take a value that makes its choice-condition $C(z^{\delta^+})$ true—if such a choice is possible: Let 'eval' be the standard evaluation function. Let $\mathcal{S}$ be any of the semantical structures under consideration. Let χ be an arbitrary $\mathcal{S}$-valuation of the variables $v_0, \ldots, v_{l-1}$. Then z^{δ^+} has to denote a function f such that, whenever there is some $\mathcal{S}$-valuation δ' of z with $\mathrm{eval}(\mathcal{S} \uplus \chi \uplus \delta')(A) = \mathsf{TRUE}$, we also have $\mathrm{eval}(\mathcal{S} \uplus \chi \uplus \delta)(A) = \mathsf{TRUE}$ for $\delta := \{z \mapsto f(\chi(v_0), \ldots, \chi(v_{l-1}))\}$.

We take a *sequent* to be a list of formulas which denotes the disjunction of these formulas. The only logical inference we need is (problem-) *reduction*, the backbone of abduction and goal-directed deduction, cf. § 5.7. In reduction, free δ^--variables behave as constant parameters, free γ-variables may be globally instantiated with each term, and the instantiation of free δ^+-variables must additionally satisfy the current choice-condition. In addition, the applied substitution σ must be an R-*substitution*, which means that the current variable-condition R remains acyclic when we extend R to its so-called σ-*update*, which additionally consists of the edges from the free variables in $\sigma(z^{\gamma\delta^+})$ to $z^{\gamma\delta^+}$, for each free variable $z^{\gamma\delta^+}$ in $\mathrm{dom}(\sigma)$. In case of a free δ^+-variable $z^{\delta^+} \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$, σ satisfies the current choice-condition C if $(Q_C(z^{\delta^+}))\sigma$ can be shown in the context of the updated variable-condition and choice-condition. Here, for a choice-condition $C(z^{\delta^+})$ given as above, $Q_C(z^{\delta^+})$ denotes the formula

$$\forall v_0.\ \ldots \forall v_{l-1}.\ \big(\,\exists z.\,A \Rightarrow A\{z \mapsto z^{\delta^+}(v_0)\cdots(v_{l-1})\}\,\big)$$

which is nothing but our version of Hilbert's axiom (ε_0), cf. Definition 11. Moreover, the global choice-condition C must be updated by removing z^{δ^+} from its domain $\mathrm{dom}(C)$ and by applying σ to the C-values of the free δ^+-variables remaining in $\mathrm{dom}(C)$. Under these conditions, the invariance of reduction under substitution is stated in Theorem 34(6b). Finally, note that $Q_C(z^{\delta^+})$ itself is always valid in our framework, cf. Lemma 32.

The major difference to Hilbert's original underspecified ε-operator is that we do not have the requirement of globally committed choice for any ε-term: Different free δ^+-variables with the same choice-condition may take different values. A minor difference is that the term-sharing of ε-terms with the help of free δ^+-variables improves the readability of our formulas considerably. Finally, opposed to the semantics of Leisenring (1969) and all other classical semantics for the ε, the implicit quantification of our free δ^+-variables is existential instead of universal, which simplifies theorem proving because we only have to find an arbitrary solution instead of checking all of them.

7. Linguistic examples

Due to the close relation of descriptive terms to semantics of determiners of articles and anaphoric pronouns in natural language, we exemplify our novel version of Hilbert's ε with several linguistic standard examples.

The general task in our examples will be to find a logical representation for a problematic sentence in natural language, and possibly to refine this representation in a reductive proof attempt with instantiation of free variables.

Indeed, our free-variable framework with its reduction and instantiation can be extended to a framework of *weighted abduction* as found in Hobbs (2003ff.), Chapter 3. The "typical elements" of Hobbs (1996) should then be modeled by a new form of free δ^+-variables obtained by changing "some π" in Definition 30 into "each π" as in Wirth (1998), Definition 5.7 (Definition 4.4 in short version); cf. also our § 8.3. Two different "typical elements" of the same set (cf. Hobbs (1996), p. 6 of WWW version) can then be modeled as two variables with the same choice-condition.

I lack the expertise to claim that the following examples have any relevance to the area of computer linguistics, but the following aspects of our framework may be interesting to philosophers of language and to computer linguists:

1. It combines the two major requirements for the usefulness of operators for descriptive terms in the description of the semantics of determiners of natural language, namely *non-right-uniqueness* (cf. § 7.1) and *committed choice* (cf. § 7.2).

2. It comes with a free-variable framework, which may help to overcome problems with quantifiers and their scope limits, cf. § 7.2.

3. Our novel and disentangled semantical framework for Hilbert's ε-operator may serve as a platform for the design of similar operators for descriptive terms tailored to the special requirements resulting from the description and computation of the semantics of natural language, cf. § 8.3.

7.1. Problems with right-uniqueness and extensionality

Right-uniqueness of descriptive terms means that different occurrences of syntactically equal terms must denote identical objects, typically given by a (right-unique) choice function applied to the extension of the description, cf. § 3.1. In such an extensional semantics, even syntactically different terms must denote the same object if the extensions of their descriptions are equal.

The problematic aspect of the following example (U1) is that the two occurrences of the same phrase "a bishop" probably denote two different bishops.

A bishop met *a bishop*. (U1)

Thus, to be appropriate for the description of indefinite determiners in natural language, right-uniqueness of descriptive terms must be avoided and a non-right-unique representation must be chosen.

We model (U1) as $\mathsf{Met}(b_1^{\delta+}, b_2^{\delta+})$ with choice-condition $C(b_1^{\delta+}) := \mathsf{Bishop}(b_1^{\delta+})$ and $C(b_2^{\delta+}) := \mathsf{Bishop}(b_2^{\delta+})$. Even if we found out that Met may be reflexive, we still could apply the substitution $\sigma := \{b_2^{\delta+} \mapsto b_1^{\delta+}\}$, because $(Q_C(b_2^{\delta+}))\sigma$ is equal to $Q_C(b_1^{\delta+})$, i.e. the substitution σ satisfies the choice-condition C according to Theorem 34(6) due to Lemma 32.

In Geurts (2000), the use of Hilbert's ε in form of choice functions for the semantics of indefinites is attacked in several ways and, roughly speaking, the following thesis is put up:

> There is no way to interpret indefinites *in situ*, but some form of "movement" is necessary, which may be interpreted as changing scopes of quantifiers.

Although the examples given in Geurts (2000) are perfectly convincing in the given setting, we point out that these problems with the ε disappear when one uses a non-right-unique version such as ours. The following three example sentences and their labels are the ones of Geurts (2000).

All bicycles were stolen by *a German*. (1a)

The problematic aspect here is the following: (1a) has the weak and strong readings of

$$\forall x.\ \big(\mathsf{Bicycle}(x)\ \Rightarrow\ \exists y.\ (\mathsf{German}(y)\ \wedge\ \mathsf{StolenBy}(x,y))\big) \tag{1a-weak}$$

and

$$\exists y.\ \big(\mathsf{German}(y)\ \wedge\ \forall x.\ (\mathsf{Bicycle}(x)\ \Rightarrow\ \mathsf{StolenBy}(x,y))\big) \tag{1a-strong}$$

and switching between the two requires some "movement".

We model this as $\mathsf{Bicycle}(x^{\delta^-})\ \Rightarrow\ \mathsf{StolenBy}(x^{\delta^-}, y^{\delta^+})$ with choice-condition $C(y^{\delta^+}) := \mathsf{German}(y^{\delta^+})$. If—in a first step—we find a model for this sentence with an empty variable-condition, then—in a second step—we can check whether it also satisfies a variable-condition that contains $(y^{\delta^+}, x^{\delta^-})$ in addition. A success of the first step provides us with a model for the weaker reading; a success of the second step with one for the stronger reading, too; i.e. that all bicycles were stolen by the same German. And this without "moving" any quantifiers or the like.

For a more interesting problem with right-unique ε, let us consider:

Every girl gave *a flower* to *a boy she fancied*. (5)

As a (generalized) choice function must pick the identical element from an identical extension, in Geurts (2000) there is a problem with two girls who love all boys, but give their flowers to two different ones. Ignoring past tense, we model this as $\mathsf{Girl}(x^{\delta^-})\ \Rightarrow\ \mathsf{Give}(x^{\delta^-}, z^{\delta^+}, y^{\delta^+})$ with choice-condition

$$C(y^{\delta^+})\quad :=\quad \mathsf{Girl}(x^{\delta^-})\ \Rightarrow\ \mathsf{Boy}(y^{\delta^+}) \wedge \mathsf{Loves}(x^{\delta^-}, y^{\delta^+})$$
$$C(z^{\delta^+})\quad :=\quad \mathsf{Flower}(z^{\delta^+})$$

Geurts' problem does not appear in our modeling simply because our semantical relation (cf. Definition 18) does not depend on the common extension of their love, but—to admit the dependences of y^{δ^+} and z^{δ^+} on x^{δ^-}—only has to contain $(x^{\delta^-}, y^{\delta^+})$ and $(x^{\delta^-}, z^{\delta^+})$, which is in accordance with our variable-condition (which only has to contain $(x^{\delta^-}, y^{\delta^+})$ due to our above choice-condition for y^{δ^+}) in the sense that their union is acyclic, cf. Definition 24(1).

The same problem of a common extension but a different choice object—but now in all possible worlds and intensions—of the following example is again no problem for us.

Every odd number is followed by *an even number that is not equal to it*. (7)

We model this as $\quad \mathsf{Odd}(x^{\delta^-})\ \Rightarrow\ x^{\delta^-}{+}1 = y^{\delta^+}$

with choice-condition $\quad C(y^{\delta^+})\quad :=\quad \mathsf{Odd}(x^{\delta^-})\ \Rightarrow\ \mathsf{Even}(y^{\delta^+}) \wedge y^{\delta^+}{\neq}x^{\delta^-}$

7.2. *Donkey sentences and Heusinger's indexed ε-operator*

If *a man* has *a donkey*, *he* beats *it*. (D)

So-called *donkey sentences* demonstrate difficulties of interaction of indefinite noun phrases in a conditional ("a man", "a donkey" in (D)) and anaphoric pronouns referring to them in the conclusion ("he", "it"). For references on donkey sentences, cf. e.g. Heusinger (1997), § 7. If semantics is represented with the help of quantification, donkey sentences reveal difficulties resulting from quantifiers and their scopes.

If *a man* loves *a woman*, *she* loves *him*, too. (L0)

If we start by modeling (L0) tentatively as

$$\begin{array}{c}\exists y_0.\ \big(\mathsf{Male}(y_0) \wedge \exists x_0.\ \big(\mathsf{Female}(x_0) \wedge \mathsf{Loves}(y_0, x_0)\big)\big)\\ \Rightarrow\ \mathsf{Female}(x_1^{\gamma}) \wedge \mathsf{Loves}(x_1^{\gamma}, y_1^{\gamma}) \wedge \mathsf{Male}(y_1^{\gamma})\end{array} \tag{L1}$$

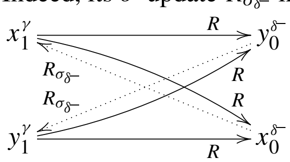
we have no chance to resolve the reference of the pronouns "she" and "him" (x_1^{γ} and y_1^{γ}) before we get rid of the quantifiers. If we apply δ^--rules (cf. § 4.2) (besides α- and β-rules), we end up with the three sequents

$$\begin{array}{l}\neg\mathsf{Male}(y_0^{\delta^-}),\ \neg\mathsf{Female}(x_0^{\delta^-}),\ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \mathsf{Female}(x_1^{\gamma})\\ \neg\mathsf{Male}(y_0^{\delta^-}),\ \neg\mathsf{Female}(x_0^{\delta^-}),\ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \mathsf{Loves}(x_1^{\gamma}, y_1^{\gamma})\\ \neg\mathsf{Male}(y_0^{\delta^-}),\ \neg\mathsf{Female}(x_0^{\delta^-}),\ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \mathsf{Male}(y_1^{\gamma})\end{array} \tag{L2}$$

and a variable-condition R including $\{x_1^{\gamma}, y_1^{\gamma}\} \times \{x_0^{\delta^-}, y_0^{\delta^-}\}$, which says that the substitution

$$\sigma_{\delta^-}\ :=\ \{x_1^{\gamma}{\mapsto}x_0^{\delta^-},\ \ y_1^{\gamma}{\mapsto}y_0^{\delta^-}\}$$

which turns the first and last sequents into tautologies and the middle one (L2) into the intended reading of (L0), must not be applied as it is not an R-substitution. Indeed, its σ-update $R_{\sigma_{\delta^-}}$ introduces cycles to R, cf. Definition 16:



Using δ^+-rules instead of the δ^--rules we get

$$\begin{array}{l}\neg\mathsf{Male}(y_0^{\delta+}),\ \neg\mathsf{Female}(x_0^{\delta+}),\ \neg\mathsf{Loves}(y_0^{\delta+}, x_0^{\delta+}),\ \mathsf{Female}(x_1^{\gamma})\\ \neg\mathsf{Male}(y_0^{\delta+}),\ \neg\mathsf{Female}(x_0^{\delta+}),\ \neg\mathsf{Loves}(y_0^{\delta+}, x_0^{\delta+}),\ \mathsf{Loves}(x_1^{\gamma}, y_1^{\gamma})\\ \neg\mathsf{Male}(y_0^{\delta+}),\ \neg\mathsf{Female}(x_0^{\delta+}),\ \neg\mathsf{Loves}(y_0^{\delta+}, x_0^{\delta+}),\ \mathsf{Male}(y_1^{\gamma})\end{array}$$

and a variable-condition R including $\{(y_0^{\delta+}, x_0^{\delta+})\}$ instead. After application of the R-substitution $\sigma_{\delta^+} := \{x_1^{\gamma}\mapsto x_0^{\delta+},\ y_1^{\gamma}\mapsto y_0^{\delta+}\}$, the instance of (L1) reduces to

$$\neg\mathsf{Male}(y_0^{\delta+}),\ \neg\mathsf{Female}(x_0^{\delta+}),\ \neg\mathsf{Loves}(y_0^{\delta+}, x_0^{\delta+}),\ \mathsf{Loves}(x_0^{\delta+}, y_0^{\delta+}) \tag{L3}$$

A closer look reveals that our σ_{δ^+}-updated variable-condition R now looks like

$$\begin{tikzcd} x_1^{\gamma} & & y_0^{\delta+} \arrow[d, "R"] \arrow[lld, dotted, "R_{\sigma_{\delta^+}}"'] \\ y_1^{\gamma} & & x_0^{\delta+} \arrow[llu, dotted, "R_{\sigma_{\delta^+}}"] \end{tikzcd}$$

while our (σ_{δ^+}-updated) R-choice-condition C generated by the δ^+-rules is

$$\begin{array}{l}C(y_0^{\delta+}) := \mathsf{Male}(y_0^{\delta+})\ \wedge\ \exists x_0.\ \big(\mathsf{Female}(x_0) \wedge \mathsf{Loves}(y_0^{\delta+}, x_0)\big)\\ C(x_0^{\delta+}) := \mathsf{Female}(x_0^{\delta+}) \wedge \mathsf{Loves}(y_0^{\delta+}, x_0^{\delta+})\end{array}$$

But even if (L3) may be true, this is not what we wanted to say in (L0), where "she" and "he" are obviously meant to be universal (strong, δ^-), whereas our free δ^+-variables here are implicitly *existentially* quantified, cf. Definition 30, contrary to the ones in Wirth (1998), Definition 5.7 (Definition 4.4 in short version).

Instead of introducing a universally quantified kind of free δ^+-variables in addition to our most useful existentially quantified ones, we had better *start without quantifiers from the very beginning*, namely with

$$\begin{array}{l}\mathsf{Male}(y_0^{\delta-}) \wedge \mathsf{Loves}(y_0^{\delta-}, x_0^{\delta-}) \wedge \mathsf{Female}(x_0^{\delta-})\\ \Rightarrow\ \mathsf{Female}(x_1^{\gamma}) \wedge \mathsf{Loves}(x_1^{\gamma}, y_1^{\gamma}) \wedge \mathsf{Male}(y_1^{\gamma})\end{array} \tag{L4}$$

and empty variable-condition R', and then apply the R'-substitution σ_{δ^-} from above to reduce its instance to

$$\neg\mathsf{Male}(y_0^{\delta-}),\ \neg\mathsf{Loves}(y_0^{\delta-}, x_0^{\delta-}),\ \neg\mathsf{Female}(x_0^{\delta-}),\ \mathsf{Loves}(x_0^{\delta-}, y_0^{\delta-}) \tag{L5}$$

which captures the universal meaning of (L0) properly.

Instead of a donkey sentence such as (L0) that prefers a genuinely universal reading as in (L5), the following donkey sentence prefers a partial switch to an existential reading:

If *a bachelor* loves *a woman*, *he* marries *her*. (M0)

If a Catholic bachelor loves three women, he is loved by each of them, but may marry at most one. Thus

$$\begin{array}{l}\mathsf{Male}(y_0^{\delta-}) \wedge \mathsf{Loves}(y_0^{\delta-}, x_0^{\delta-}) \wedge \mathsf{Female}(x_0^{\delta-})\\ \Rightarrow\ \mathsf{Male}(y_1^{\gamma}) \wedge \mathsf{Marries}(y_1^{\gamma}, x_1^{\gamma}) \wedge \mathsf{Female}(x_1^{\gamma})\end{array} \tag{M1}$$

should be refined by application of $\{x_1^{\gamma} \mapsto x_1^{\delta+},\ y_1^{\gamma} \mapsto y_0^{\delta-}\}$ and simplification to

$$\neg\mathsf{Male}(y_0^{\delta-}),\ \neg\mathsf{Loves}(y_0^{\delta-}, x_0^{\delta-}),\ \neg\mathsf{Female}(x_0^{\delta-}),\ \mathsf{Female}(x_1^{\delta+}) \tag{M2a}$$

$$\neg\mathsf{Male}(y_0^{\delta-}),\ \neg\mathsf{Loves}(y_0^{\delta-}, x_0^{\delta-}),\ \neg\mathsf{Female}(x_0^{\delta-}),\ \mathsf{Marries}(y_0^{\delta-}, x_1^{\delta+}) \tag{M2b}$$

with choice-condition

$$C(x_1^{\delta+}) \quad := \quad \mathsf{Male}(y_0^{\delta-})\ \Rightarrow\ \mathsf{Loves}(y_0^{\delta-}, x_1^{\delta+}) \wedge \mathsf{Female}(x_1^{\delta+}) \tag{C2}$$

On the one hand, if there is no women loved by the bachelor $y_0^{\delta-}$, both (M2a) and (M2b) are valid. On the other hand, if there is at least one woman he loves, (M2a) is valid due to its last formula and (C2), and (M2b)+(C2) expresses the intended reading of (M0).

Notice that we indeed have the possibility to let "woman" be universal (strong, δ^-) and "her" existential (weak, δ^+), picking one of the women loved by the bachelor—if there are any. Our treatment is more flexible than a similar one of (D) along supposition theory in Parsons (1994). Moreover, both treatments are more lucid than that of a sentence analogous to (M0) in Heusinger (1997) which we discuss in the following:

As we have just seen, a major advantage of reference in natural language is the possibility to refer to an object a second time. Thus, the ε can hardly be of any use in semantics of natural language without the possibility to express committed choice, cf. § 2.4. Unless we introduce special concepts (such as our free δ^+-variables for ε-terms) we need right-uniqueness to express committed choice, cf. §§ 2.4 and 3.1.6. As we have seen in § 7.1, right-uniqueness of representation, however, must be abandoned in our linguistic context.

In Heusinger (1997), the right-uniqueness and the extensionality of the ε are kept, but the usefulness for describing the semantics of natural language is improved by adding a situational index to the ε-symbol which makes it possible to denote different choice functions explicitly. Right-uniqueness can then be avoided by representing different occurrences of syntactically equal descriptive terms with different situational indexes. We refer to this indexed ε as *Heusinger's indexed ε-operator*. It already occurs in the English draft paper Heusinger (1996). Heusinger (1997), however, is a German monograph on applying Hilbert's epsilon to the semantics of noun phrases and pronouns in natural language, with a focus on salience. Heusinger's indexed ε-operator is used to describe the definite as well as the indefinite article in specific as well as non-specific contexts, resulting in four different representations. Cf. Heusinger (1997) also for further reference on the ε in the semantics of natural language.

Mutandis mutatis and the readability improved, the modeling of (M0) according to (19a) on p.185 of Heusinger (1997) would be

$$\exists f.\ \forall i.\ \left(\begin{array}{l} \quad\ \mathsf{Loves}\big(\ \varepsilon_i\, y.\,\mathsf{Male}(y),\ \ \varepsilon_{f(i)} x.\,\mathsf{Female}(x)\ \big) \\ \Rightarrow \mathsf{Marries}\big(\ \varepsilon_{a^*} y.\,\mathsf{Male}(y),\ \ \varepsilon_{a^*} x.\,\mathsf{Female}(x)\ \big) \end{array}\right) \tag{19a$'$}$$

where the index a^* of Heusinger's indexed ε-operator seems to denote a choice function that chooses men as i does and women as $f(i)$ does. How a^* is to be formalized remains unclear in Heusinger (1997). The real problem, however, is that (19a$'$) does not represent the intended meaning of (M0): To wit, take an f such that $f(i)$ always chooses a woman not loved by the man chosen by i; then—ex falso quodlibet—our bachelors in love may remain unmarried, contradicting (M0). A comparison on precision and simplicity of (19a$'$) with our above (M2b)+(C2) should speak for itself.

For more linguistic example applications cf. Wirth (2006b).

8. Discussion

8.1. Where have all the ε-terms gone?

The ε-symbol does not occur anymore in our terms, and our formulas are much more readable than in the standard approach of in-line presentation of ε-terms, which was always just a theoretical presentation because in practical proofs the ε-terms would have grown so large that the mere size made them inaccessible to human inspection. To see this, compare our presentation in Example 10 to the one in Example 8, which is still hard to read although we have invested some efforts in a readable form of presentation. From a mathematical point of view, however, the original ε-terms are still present in our approach, up to isomorphism and with the exception of some irrelevant term sharing. To make these ε-terms explicit, we just have to apply the following small rewrite system: Let C be an R-choice-condition. For any $u^{\delta^+} \in \mathrm{dom}(C)$, say $C(u^{\delta^+}) = \lambda v_0.\ \ldots \lambda v_{l-1}.\ B$ for a formula B, consider the rewrite rule

$$u^{\delta^+} = \lambda v_0.\ \ldots \lambda v_{l-1}.\ \varepsilon u.\ B\{u^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto u\}$$

for a fresh bound variable u (of proper type). Note that u^{δ^+} does not occur in the right-hand side due to the restriction on the occurrence of u^{δ^+} in B to the form $u^{\delta^+}(v_0)\cdots(v_{l-1})$, according to Definition 22(2). Therefore, by the well-foundedness of our variable-condition R and Definition 22(1), the left-hand side is bigger than the right-hand side in the well-founded multiset extension of R. Thus, normalization of any formula A with these rewrite rules terminates with a formula B. As typed $\lambda\alpha\beta$-reduction is also terminating, we can apply it to remove the λ-terms introduced by the rewriting.

Example 38. *(continuing Example 10)*

By the procedure described above, the R-choice-condition C of Example 10 in § 4.5 becomes the following rewrite system, which—let it be noted in passing—is most convenient for computing Ackermann's degree and rank:

		degree	rank
$z_d^{\delta^+} = \varepsilon z_d.\ \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d)$	(cf. (10.2)!)	3	1
$y_c^{\delta^+} = \varepsilon y_c.\ \mathsf{P}(x_a^{\delta^+}, y_c, z_c^{\delta^+}(y_c))$	(cf. (10.3)!)	2	2
$z_c^{\delta^+} = \lambda y.\ \varepsilon z_c.\ \neg\mathsf{P}(x_a^{\delta^+}, y, z_c)$	(cf. (10.4)!)	—	1
$x_a^{\delta^+} = \varepsilon x_a.\ \neg\mathsf{P}(x_a, y_a^{\delta^+}(x_a), z_b^{\delta^+}(x_a))$	(cf. (10.5)!)	1	3
$z_b^{\delta^+} = \lambda x.\ \varepsilon z_b.\ \neg\mathsf{P}(x, y_a^{\delta^+}(x), z_b)$	(cf. (10.6)!)	—	1
$y_a^{\delta^+} = \lambda x.\ \varepsilon y_a.\ \mathsf{P}(x, y_a, z_a^{\delta^+}(x)(y_a))$	(cf. (10.7)!)	—	2
$z_a^{\delta^+} = \lambda x.\ \lambda y.\ \varepsilon z_a.\ \neg\mathsf{P}(x, y, z_a)$	(cf. (10.8)!)	—	1

And if we rewrite $\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+})$ (cf. (10.1)!) with this rewrite system plus $\lambda\alpha\beta$-reduction, we end up in a normal form which is the first-order ε-formula (8.1) of Example 8, with the exception of some renaming of ε-bound variables. Note that the normal form even preserves our information on committed choice when we consider any ε-term binding an occurrence of a variable of the same name to be committed to the same choice. In this sense, the representation of the normal form is isomorphic to our original one via (10.1), R, and C. □

8.2. *Are we breaking with the traditional treatment of Hilbert's ε?*

Our new semantical free-variable framework was actually developed to meet the requirements analysis for the combination of mathematical induction in the liberal style of Fermat's *descente infinie* with state-of-the-art logical deduction. The framework provides a formal system in which a working mathematician can straightforwardly develop his proofs supported by powerful automation, cf. Wirth (2004).

If traditionality meant restriction to expressional means of the early 20th century—with its foundational crisis and special emphasis on constructivism, intuitionism, and proof transformation—then our approach would not classify as traditional. But with its equivalents for the traditional ε-terms (cf. § 8.1) and for the ε-substitution methods (cf. §§ 4.6 and 5.7), our framework is deeply rooted in this tradition. In the meanwhile the fear of inconsistency should have been cured by Wittgenstein (1939). The main disadvantage of a constructive syntactical framework for the ε as compared to a semantical one is the following: Constructive proofs of practically relevant theorems become too huge and too tedious, whereas semantical proofs are of a better manageable size. More important is the possibility to invent *new and more suitable logics for new applications* with semantical means, whereas proof transformations can only refer to already existing logics, cf. § 2.3. It is our intention with this paper to pass the heritage of Hilbert's ε on to new generations interested in computational linguistics, automated theorem proving, and mathematics assistance systems; fields in which—with very few exceptions—the overall common opinion still is that the ε hardly can be of any practical benefit.

8.3. *On the design of similar operators*

In § 1 we already mentioned that the semantical free-variable framework for our ε may serve as the paradigm for the design of other operators similar to our version of the ε. In this § 8.3, we give some general hints on the two most obvious options for adjustment of the presented definitions to achieve the intended properties of such new operators.

One option for adjustment is the definition of (C, R)-validity. For instance, the "some π" in Definition 30 is something we can play around with. Indeed, in Wirth (1998), Definition 5.7 (Definition 4.4 in short version), we can read "any π" instead (i.e. "each π"), which is just the opposite extreme, for which (E2′) of Lemma 31 is valid iff $\exists!x.\,A_0 \vee \exists x.\forall y.(x{=}y)$. In between of both extremes, we could design operators tailored for generalized quantifiers (e.g. with cardinality specifications) or for the special needs of specification and computation of semantics of natural language. Note that the changes of our general framework for these operators would be quite moderate: In any case, it is "each π" what we read in the important Lemma 32 and the crucial Definition 33. Roughly speaking, only Theorem 34(6a) for the case of $O \neq \emptyset$ as well as Theorem 34(5a) would become false for a different choice on the quantification of π in Definition 30. The reason why we prefer "some π" to "each π" here and in Wirth (2004) is that "some π" results in additional valid formulas (e.g. (E2′)) and simplifies theorem proving. Contrary to "each π" and to all classical semantics in the literature, "some π" frees us from considering all possible choices: We just have to pick a single arbitrary one and fix it in a proof step. Moreover, "some π" is very close to Hilbert's intentions on ε-substitution as described best in Hilbert & Bernays (1968/70), Vol. II, § 2.4.

Another option for adjustment is the definition of compatibility. For instance, by modifying Item (2) of Definition 24 we can strengthen the notion of compatibility in such a way that either $\delta(y^{\delta^+})$ has to pick the *smallest* value such that B is $(\delta, e, \mathcal{S})$-valid or else, slightly weaker, has to pick a value v such that B is $(\delta, e, \mathcal{S})$-valid and v is not the successor of a natural number for which B becomes $(\delta, e, \mathcal{S})$-valid. The latter modification of compatibility, for instance, could be useful in an analysis of the failed trials of Hilbert's group to show termination of ε-substitution in arithmetic, as described in Hilbert & Bernays (1968/70), Vol. II, § 2.4.

All in all, in our conceptually disentangled framework for the ε, there are at least these two well-defined and conceptually simple options for a convenient adjustment to achieve similar operators for different purposes.

9. Conclusion

Our novel indefinite semantics for Hilbert's ε presented in this paper was developed to solve the difficult soundness problems that came up in the combination of mathematical induction (in the liberal style of Fermat's *descente infinie*) with state-of-the-art deduction. Thereby, it had passed an evaluation of its usefulness even before it was recognized as a candidate for the semantics that David Hilbert probably had in mind for his ε. While the speculation on this question will go on, the semantical framework for Hilbert's ε proposed in this paper definitely has the following advantages:

Syntax: As right-uniqueness is not required, different objects may be chosen for different free δ^+-variables with the same choice-condition. The requirement of a commitment to a choice, however, is expressed syntactically and most clearly by the sharing of a free δ^+-variable, cf. § 4.5.

Semantics: Our novel semantics for the ε is simple and straightforward in the sense that the ε-operator becomes similar to the referential use of the indefinite article in natural language, cf. § 7. Our semantics for the ε is based on an abstract formal approach that extends a semantics for closed formulas (satisfying only very weak requirements, cf. § 5.3) to a semantics with several kinds of free variables: existential (γ), universal (δ^-), and ε-constrained (δ^+).

Reasoning: In a reductive proof step, our representation of an ε-term $\varepsilon x.\,A$ can be replaced with *each* term t that satisfies the formula $\exists x.\,A \;\Rightarrow\; A\{x\mapsto t\}$, cf. § 4.6. Thus, the soundness of such a replacement is likely to be expressible and verifiable in the original calculus. Our free-variable framework for the ε is especially convenient for developing proofs in the style of a working mathematician, cf. Wirth (2004), Wirth (2006a). Indeed, our approach simplifies theorem proving because we do not have to consider all proper choices t for x (as in Leisenring (1969) and all other classical semantical approaches) but only a single arbitrary one, which is fixed in a proof step, just as choices are settled in program steps, cf. § 4.4.

Finally, we hope that our new semantical framework will help to solve further practical and theoretical problems and improve the applicability of descriptive terms as a logical tool for specification and reasoning. Although we have only touched the surface of the subject in § 8.3, a tailoring of operators similar to ours which meet the special demands of specification and computation in other areas (such as philosophy of language and computer linguistics, cf. § 7) seems to be especially promising.

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Notes

Note 1 One could object that $\mathcal{V}_{\mathrm{free}}(A)$ does not have to be a subset of $\mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi)$ in general. But the additional free δ^--variables blocked by the δ^+-rules (as compared to the δ^--rules) do not block proofs in practice. This has the following reason: With a reasonably minimal variable-condition R, the only additional cycles that could occur are of the form $y^\gamma\ R\ z^{\delta^-}\ R\ x^{\delta^+}\ R\ u^{\mathrm{free}}\ R^*\ y^\gamma$ with $y^\gamma, z^{\delta^-} \in \mathcal{V}(\Gamma\ \forall x.A\ \Pi)$; and in this case the corresponding δ^--rule would result in the cycle $y^\gamma\ R\ x^{\delta^-}\ R\ u^{\mathrm{free}}\ R^*\ y^\gamma$ anyway. Moreover, δ^--rules and free δ^--variables do not occur in inference systems with δ^+-rules before Wirth (2004), so that in the earlier systems $\mathcal{V}_{\mathrm{free}}(A)$ is indeed a subset of $\mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi)$.

Note 2 If the occurrences of y^{δ^+} in $C(y^{\delta^+})$ could differ in their arguments, there could be irresolvable conflicts on special arguments. And, in these conflicts, the choice of a *function as a whole* would essentially violate Hilbert's axiomatizations: As only object variables and no functions are considered for ε-binding in Hilbert & Bernays (1968/70), the axiom schemes (ε_0) and (ε-*formula*) (cf. § 2.2) seem to require us to choose the values of this function individually. For example, in case of $C(y^{\delta^+}) \;=\; \lambda b.\,\Big(\,y^{\delta^+}(b) \wedge \neg(y^{\delta^+}(\mathsf{true}) \wedge y^{\delta^+}(\mathsf{false}))\,\Big)$, for choosing y^{δ^+}, we are in conflict between

- $\lambda b'.\,(b' = \mathsf{false})$ (i.e. $\lambda b'.\,\neg b'$, for $C(y^{\delta^+})(\mathsf{false})$ to be true) and
- $\lambda b'.\,(b' = \mathsf{true})$ (i.e. $\lambda b'.\,b'$, for $C(y^{\delta^+})(\mathsf{true})$ to be true).

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