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Homology representations arising from the half cube, II

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ABSTRACT

In a previous work, we defined a family of subcomplexes of the n-dimensional half cube by removing the interiors of all half cube shaped faces of dimension at least k, and we proved that the reduced homology of such a subcomplex is concentrated in degree $k-1$. This homology module supports a natural action of the Coxeter group $W(D_n)$ of type D. In this paper, we explicitly determine the characters (over $\mathbb{C}$) of these homology representations, which turn out to be multiplicity free. Regarded as representations of the symmetric group $\mathfrak{S}_n$ by restriction, the homology representations turn out to be direct sums of certain representations induced from parabolic subgroups. The latter representations of $\mathfrak{S}_n$ agree (over $\mathbb{C}$) with the representations of $\mathfrak{S}_n$ on the $(k-2)$-nd homology of the complement of the k-equal real hyperplane arrangement.

1. Introduction

As a graph, the n-dimensional hypercube is bipartite and connected. This induces a partition of its vertex set $V = V_n = \{\pm 1\}^n$ into two pieces, $V^e \cup V^o = V_n^e \cup V_n^o$, where V_n^e (respectively, V_n^o) consists of those vertices whose coordinates contain an even (respectively, odd) number of occurrences of -1. We define the *half cube*, Γ_n, to be the convex hull of the 2^{n-1} points in V_n^e. Using V_n^o in place of V_n^e in this construction gives rise to an isometric copy of the half cube.

The half cube is sometimes known as the *demihypercube*. It is described in Coxeter's classic text [7], where it is denoted by $\mathrm{h}\gamma_n$. In the case $n = 5$, it was described by Gosset in his seminal work on polytopes [9].

In a previous work [11], we showed that the k-faces of the half cube Γ_n are of two types: regular simplices and, for $k \geqslant 3$, isometric copies of half cubes of lower dimensions. These faces assemble naturally into a regular CW complex, C_n, which is homeomorphic to a ball. Furthermore, for each

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$3 \leqslant k \leqslant n$, there is an interesting subcomplex $C_{n,k}$ of C_n obtained by deleting the interiors of all the half cube shaped faces of dimensions $l \geqslant k$. We also showed in [11, Theorem 3.3.2] that the reduced homology of $C_{n,k}$ is free over $\mathbb{Z}$ and concentrated in degree $k-1$.

The nonzero Betti numbers $B(n,k)$ of $C_{n,k}$ can be characterized by simple recurrence relations: $B(n,0) = B(n,n) = 1$ and, for $0 < k < n$,

$$B(n,k) = 2B(n-1,k) + B(n-1,k-1).$$

There are also nonrecursive formulae for $B(n,k)$; for example, Björner–Welker [6, Theorem 1.1(c)] prove that

$$B(n,k) = \sum_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1},$$

where we interpret $\binom{-1}{-1}$ to mean 1. The numbers $B(n,k)$ are interesting because they occur in a diverse range of contexts, such as:

(i) in the problem of finding, given n real numbers, a lower bound for the complexity of determining whether some k of them are equal ([4,5], [6, §1]),
(ii) as the $(k-2)$-nd Betti numbers of the k-equal real hyperplane arrangement in $\mathbb{R}^n$ [6],
(iii) as the ranks of A-groups appearing in combinatorial homotopy theory [1,2],
(iv) as the number of nodes used by the Kronrod–Patterson–Smolyak cubature formula in numerical analysis [17, Table 3], and
(v) (when $k = 3$) in engineering, as the number of three-dimensional block structures associated to n joint systems in the construction of stable underground structures [14].

The connections between (i)–(iii) above are now well understood. Although the half cube polytope has no obvious direct relationship with any of the phenomena in (i)–(v), its associated homology modules share an intriguing feature in common with those appearing in (ii) and (iii): they all support natural actions of the symmetric group $\mathfrak{S}_n$. One possible way to forge a link between the half cube and the situations in (ii) or (iii) is to try to understand the various homology modules in terms of group representations.

The k-equal real hyperplane arrangement $V_{n,k}^{\mathbb{R}}$ is the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for some set of indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. The complement $\mathbb{R}^n - V_{n,k}^{\mathbb{R}}$, denoted by $M_{n,k}^{\mathbb{R}}$, is a manifold whose homology is concentrated in degrees $t(k-2)$, where $t \in \mathbb{Z}$ satisfies $0 \leqslant t \leqslant \lfloor \frac{n}{k} \rfloor$ (see [6, Theorem 1.1(b)]). It is clear from the definition that $M_{n,k}^{\mathbb{R}}$ supports an action of $\mathfrak{S}_n$ via permutation of coordinates, and this action endows the nonzero homology modules with the structure of $\mathfrak{S}_n$-modules. The characters of these modules (over $\mathbb{C}$) were computed explicitly by Peeva, Reiner and Welker in [16, Theorem 4.4].

The signed permutation group $G = \mathbb{Z}_2 \wr \mathfrak{S}_n$ acts naturally on the symbols $\{\pm 1, \pm 2, \ldots, \pm n\}$. The group G is sometimes called the octahedral group, because it is the automorphism group of the dual of the n-dimensional hypercube. It is also the automorphism group of the hypercube itself, where the signed objects may be identified with the possible coordinate entries of the vertices. The group G also acts as automorphisms of the hypercube regarded as a graph, and the subsets V_n^e and V_n^o are blocks of imprimitivity for this action. The subgroup H fixing V_n^e setwise has index 2 in G. It can be shown that the groups G and H coincide with the Coxeter groups $W(B_n)$ and $W(D_n)$ of types B_n and D_n, respectively.

The group $W(D_n)$ acts on the n-dimensional half cube by continuous transformations sending vertices to vertices and k-faces to k-faces. It follows that the group acts via cellular automorphisms on the CW complex associated to the half cube. In the special case $n = 4$, the half cube and the dual of the hypercube turn out to be isometric, so the 4-dimensional half cube has twice as many automorphisms as usual. It follows from the definitions that for all n, the group $W(D_n)$ contains a subgroup isomorphic to $\mathfrak{S}_n$.

Because the action of $W(D_n)$ on Γ_n is by cellular automorphisms, it follows that it induces, for each k, an action on the nonzero homology modules of $C_{n,k}$. In this paper, we will compute the

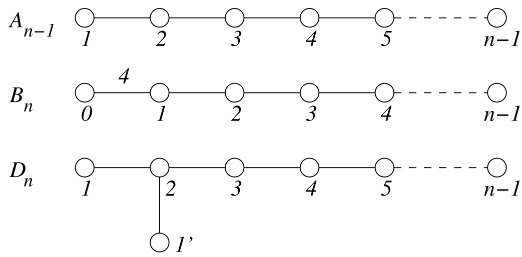

Fig. 1. Coxeter graphs of type A_{n-1}, B_n and D_n.

characters (over $\mathbb{C}$) of these homology representations. Regarded as modules for $W(D_n)$, the representations turn out to be multiplicity free (Theorem 4.4). These modules are not generally induced modules in any nontrivial sense. In contrast, if the homology representations are regarded as modules for the symmetric group $\mathfrak{S}_n$ by restriction, then they are no longer multiplicity free, but they do turn out to be isomorphic to direct sums of modules induced from maximal Young subgroups (Theorem 4.7). Furthermore, over the complex numbers, the action of $\mathfrak{S}_n$ on the $(k-1)$-st homology of $C_{n,k}$ agrees with the action of $\mathfrak{S}_n$ on the $(k-2)$-nd homology of $M^{\mathbb{R}}_{n,k}$. It would be interesting to have an explicit description of the corresponding integral homology representations in each case.

In particular, the characters of the homology representations of this paper, when considered as $\mathbb{C}\mathfrak{S}_n$-modules, agree precisely with those considered by Peeva, Reiner and Welker. (There are two obvious ways to regard $W(D_n)$-modules as modules for $\mathfrak{S}_n$, but one obtains isomorphic modules in each case.) It would be interesting to find a conceptual explanation for this, and for the fact that the degree in which these homology representations is shifted by 1. One consequence of Theorem 4.7 is that the $(k-2)$-nd homology of $M^{\mathbb{R}}_{n,k}$ admits an action of $W(D_n)$; it is not clear to me why this should be so.

Another consequence of our results (Corollary 4.6) is that if we restrict the representation of $W(D_n)$ on the $(k-1)$-st homology of $C_{n,k}$ to the subgroup $W(D_{n-1})$, then the corresponding branching rule categorifies the usual recurrence relation for the Betti numbers $B(n,k)$. Furthermore, if one computes the dimension of the homology representations from a knowledge of their characters, then one obtains a combinatorial proof of the Björner–Welker formula for $B(n,k)$ mentioned above.

2. Character theory of Coxeter groups of classical type

The main groups of interest in this paper are the finite Coxeter groups of classical types A_{n-1}, B_n and D_n. It will be convenient to number the vertices of the corresponding Coxeter graphs as shown in Fig. 1.

We now summarize some well-known properties of these groups. More details may be found in [13] or [3].

The *Coxeter group* $W = W(\Gamma)$ corresponding to a Coxeter graph Γ with vertices $S = S(\Gamma)$ is given by the presentation

$$\langle s_i\colon\ i \in S(\Gamma)\colon\ (s_i s_j)^{m_{ij}} = 1\rangle.$$

The numbers m_{ij} are defined to satisfy $m_{ii} = 1$ and $m_{ij} = m_{ji}$ for all $i, j \in S$. Furthermore, we have $m_{ij} = 2$ if i and j are not adjacent in the graph; $m_{ij} = 3$ if i and j are connected by an unlabeled edge; and $m_{ij} = k$ if i and j are connected by an edge labeled $k > 3$.

If $S' \subset S$, then we refer to the subgroup W' of W that is generated by S' as a *parabolic subgroup* of W. In this case, W' inherits the structure of a Coxeter group from W.

The Coxeter group $W(A_{n-1})$ is isomorphic (as an abstract group) to the symmetric group $\mathfrak{S}_n$, and the Coxeter generator s_i may be identified with the transposition $(i, i+1)$.

The Coxeter group $W(B_n)$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$ of order $2^n n!$. This may be regarded as a group of permutations of n signed objects, in which s_i acts by the transposition

$(i, i+1)$ for $1 \leqslant i < n$, and s_0 acts by changing the sign of the object numbered 1. As mentioned earlier, $W(B_n)$ is the group of automorphisms of the n-dimensional hypercube. The parabolic subgroup of $W(B_n)$ obtained by omitting the generator s_0 is canonically isomorphic (as a Coxeter group) to $W(A_{n-1})$.

The Coxeter group $W(D_n)$ will be our main group of interest, because it is the group of automorphisms of the n-dimensional half cube (if $n > 4$). The group can be identified with the index 2 subgroup of $W(B_n)$ consisting of those elements effecting an even number of sign changes. As before, we may identify s_i with the permutation $(i, i+1)$ for $1 \leqslant i < n$. The other generator, $s_{1'}$, can be identified with the element $s_0 s_1 s_0$ of $W(B_n)$. It therefore acts by changing the sign of each of objects 1 and 2, followed by the transposition $(1, 2)$. The parabolic subgroup of $W(D_n)$ obtained by omitting the generator $s_{1'}$ is canonically isomorphic (as a Coxeter group) to $W(A_{n-1})$. We will often abuse notation slightly and refer to this subgroup as $\mathfrak{S}_n$.

For any Coxeter group (W, S), there is a unique homomorphism $\varepsilon : W \to \{\pm 1\}$ to the multiplicative group of 2 elements sending each element of S to -1. This homomorphism is known as the *sign representation*. We will write sgn_n (respectively, id_n) to denote the sign (respectively, trivial) representation of any of the Coxeter groups of types A_{n-1}, B_n or D_n.

The character theory of finite Coxeter groups of classical type is well understood. It is described in Geck and Pfeiffer's book [8, §5] and, more explicitly, in Stembridge's notes on the topic [19]. We now summarize some of the key properties of the theory for later use.

The irreducible representations of $W(A_{n-1})$ (over $\mathbb{C}$) are indexed by the partitions of n, or equivalently, the set of Young diagrams of size n. We will write the corresponding set of characters as

$$\{\chi^{\lambda} \colon |\lambda| = n\}.$$

The degree, $\chi^{\lambda}(1)$, of χ^{λ} is the number of standard Young tableaux of shape λ; that is, the number of ways of filling a Young diagram of shape λ with the numbers $1, 2, \ldots, n$ once each in such a way that the entries increase along rows and down columns. The identity character corresponds to the partition $\lambda = [n]$, whose Young diagram has one row, and the sign character corresponds to the partition $\lambda = [1^n]$, whose Young diagram has one column.

Another important character for our purposes corresponds to the partition

$$[n-1, 1],$$

which gives the character of the *reflection representation* associated to the Coxeter group of type A_{n-1}. This representation may also be constructed by first taking the n-dimensional representation of $\mathfrak{S}_n$ corresponding to the natural action of the group on n letters, and then quotienting by the 1-dimensional submodule spanned by the all-ones vector.

The irreducible characters of $W(B_n)$ are indexed by the set

$$\{\chi^{(\mu,\nu)} \colon |\mu| + |\nu| = n\}.$$

The dimensions of the irreducibles may be obtained from the corresponding dimensions in type A via the formula

$$\chi^{(\mu,\nu)}(1) = \binom{n}{|\mu|} \chi^{\mu}(1) \chi^{\nu}(1).$$

The identity character corresponds to the pair $([n], [0])$, and the sign character to the pair $([0], [1^n])$.

As described above, we may regard $W(D_n)$ as a subgroup of $W(B_n)$. Under this identification, the irreducible characters $\chi^{(\mu,\nu)}$ and $\chi^{(\nu,\mu)}$ (where $\mu \neq \nu$) both restrict to the same irreducible character of $W(D_n)$, which we denote by $\chi^{\{\mu,\nu\}}$. On the other hand, the irreducible character $\chi^{(\mu,\mu)}$ of $W(B_n)$ restricts to a sum of two nonisomorphic irreducible characters of $W(D_n)$ of the same degree; we denote the latter by $\chi_+^{\{\mu,\mu\}}$ and $\chi_-^{\{\mu,\mu\}}$. These exhaust all the irreducible characters of $W(D_n)$. In other words, the irreducible characters of $W(D_n)$ are indexed by the set

$$\{\chi^{\{\mu,\nu\}} \colon |\mu| + |\nu| = n\} \cup \{\chi_{\pm}^{\{\mu,\mu\}} \colon |\mu| = n/2\},$$

where the second subset is empty if n is odd. The identity character corresponds to the pair $\{[n],[0]\}$ and the sign character corresponds to the pair $\{[1^n],[0]\}$. It is immediate from the above remarks that the dimensions of the corresponding irreducibles are given by

$$\chi^{\{\mu,\nu\}}(1) = \binom{n}{|\mu|}\chi^{\mu}(1)\chi^{\nu}(1)$$

and

$$\chi_{\pm}^{\{\mu,\mu\}}(1) = \frac{1}{2}\binom{n}{|\mu|}\chi^{\mu}(1)^2.$$

The following two lemmas concerning characters of $W(D_n)$ will be important in the sequel. It will sometimes be convenient to write $\chi_{\varepsilon}^{\{\mu,\nu\}}$ to refer to the irreducible character $\chi^{\{\mu,\nu\}}$ if $\mu \neq \nu$, and to refer to either of the irreducible characters $\chi_{+}^{\{\mu,\nu\}}$ or $\chi_{-}^{\{\mu,\nu\}}$ if $\mu = \nu$.

Lemma 2.1. *Let $\mathfrak{S}_n$ be the parabolic subgroup of $G = W(D_n)$ obtained by omitting the generator $s_{1'}$. Let $m = \lfloor \frac{n}{2} \rfloor$.*

(i) *If $\mu \neq \nu$, then we have*

$$\chi^{\{\mu,\nu\}}\downarrow^{G}_{\mathfrak{S}_n} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\chi^{\lambda},$$

where the $c^{\lambda}_{\mu\nu}$ are the Littlewood–Richardson coefficients.

(ii) *If n is odd, then we have*

$$\mathrm{id}_n\uparrow^{G}_{\mathfrak{S}_n} = \sum_{l \leqslant m} \chi^{\{[l],[n-l]\}}.$$

(iii) *If n is even, then we have*

$$\mathrm{id}_n\uparrow^{G}_{\mathfrak{S}_n} = \chi_{+}^{\{[m],[m]\}} + \sum_{l<m} \chi^{\{[l],[n-l]\}}.$$

Proof. Part (i) appears in [19, §3A]. Under the hypotheses of part (ii), we must have $\mu \neq \nu$ because n is odd. The conclusion of (ii) then follows from (i) by using Frobenius reciprocity and the Pieri rule. Part (iii) appears in [19, §3C]. □

Lemma 2.2. *Let $G = W(D_n)$, let $3 \leqslant k \leqslant n$, and let D_k (respectively, D_{n-k}) be the parabolic subgroup of G generated by the set*

$$\{s_{1'}\} \cup \{s_i\colon\ 1 \leqslant i < k\}$$

(respectively, $\{s_i\colon\ i > k\}$). Denote the usual inner product on characters by $\langle\ ,\ \rangle$. Suppose below that the ordered pairs (α,ψ) and (β,θ) are not equal.

(i) *If $\mu \neq \nu$ then we have*

$$\bigl\langle \chi^{\{\mu,\nu\}}\downarrow^{G}_{D_k\times D_{n-k}}, \chi_{\varepsilon}^{\{\alpha,\beta\}} \times \chi_{\varepsilon'}^{\{\psi,\theta\}} \bigr\rangle = c^{\mu}_{\alpha\psi}c^{\nu}_{\beta\theta} + c^{\mu}_{\alpha\theta}c^{\nu}_{\beta\psi} + c^{\mu}_{\beta\psi}c^{\nu}_{\alpha\theta} + c^{\mu}_{\beta\theta}c^{\nu}_{\alpha\psi}.$$

(ii) *We have*

$$\bigl\langle \chi_{\pm}^{\{\mu,\mu\}}\downarrow^{G}_{D_k\times D_{n-k}}, \chi_{\varepsilon}^{\{\alpha,\beta\}} \times \chi_{\varepsilon'}^{\{\psi,\theta\}} \bigr\rangle = c^{\mu}_{\alpha\psi}c^{\mu}_{\beta\theta} + c^{\mu}_{\alpha\theta}c^{\mu}_{\beta\psi}.$$

Proof. After applying Frobenius reciprocity, this becomes a restatement of results in [19, §3A, §3D]. □

The irreducible characters in type B_n have the following well-known branching rule, which will be useful in the sequel.

Lemma 2.3. *Let $\chi^{(\lambda,\mu)}$ be an irreducible character for $W(B_n)$. Then we have*

$$\chi^{(\lambda,\mu)}\downarrow^{W(B_n)}_{W(B_{n-1})} = \sum_{d\in I(\lambda)} \chi^{(\lambda^{(d)},\mu)} + \sum_{d\in I(\mu)} \chi^{(\lambda,\mu^{(d)})},$$

where $I(\lambda)$ is the set of removable boxes in the Young diagram corresponding to λ, and $\lambda^{(d)}$ is the result of removing box d from the Young diagram.

Proof. This appears in [8, §6.1.9]. □

Corollary 2.4. *Maintain the notation of Lemma* 2.3. *Suppose in addition that each character $\chi^{(\alpha,\beta)}$ appearing in Lemma* 2.3 *satisfies $\alpha \neq \beta$. Then we have*

$$\chi^{\{\lambda,\mu\}}\downarrow^{W(D_n)}_{W(D_{n-1})} = \sum_{d\in I(\lambda)} \chi^{\{\lambda^{(d)},\mu\}} + \sum_{d\in I(\mu)} \chi^{\{\lambda,\mu^{(d)}\}}.$$

Proof. Recall that each type B character $\chi^{(\alpha,\beta)}$ appearing in the statement restricts to the irreducible type D character $\chi^{\{\alpha,\beta\}}$. The result now follows from Lemma 2.3 and the fact that, under the usual identifications, we have $W(B_{n-1}) \cap W(D_n) = W(D_{n-1})$. □

3. The half cube

An n-dimensional (*Euclidean*) *polytope* Π_n is a closed, bounded, convex subset of $\mathbb{R}^n$ obtained by intersecting finitely many closed half-spaces associated to hyperplanes. The part of Π_n that lies in one of the hyperplanes is called a *facet*, and each facet is an $(n-1)$-dimensional polytope. A polytope is homeomorphic to an n-ball (which follows, for example, from [15, Lemma 1.1]), and the boundary of the polytope, which is equal to the union of its facets, is identified with the $(n-1)$-sphere by this homeomorphism.

Iterating this construction gives rise to a set of k-dimensional polytopes Π_k (called *k-faces*) for each $0 \leqslant k \leqslant n$. The elements of Π_0 are called *vertices* and the elements of Π_1 are called *edges*. It is not hard to show that a polytope is the convex hull of its set of vertices, and that the boundary of a polytope is precisely the union of its k-faces for $0 \leqslant k < n$. What is less obvious, but still true [20, Theorem 1.1], is that the convex hull of an arbitrary finite subset of $\mathbb{R}^n$ is a polytope in the above sense. It follows that a polytope is determined by its vertex set, and we write $\Pi(V)$ for the polytope whose vertex set is V. In particular, the n-dimensional hypercube is $\Pi(V_n)$, and the half cube Γ_n is by definition $\Pi(V_n^e)$.

The *dimension* of a face is the dimension of its affine hull. The *interior* of a face refers to its interior with respect to the induced topology on its affine hull.

Definition 3.1. Let $n \geqslant 4$ be an integer, and let $\mathbf{n} = \{1, 2, \ldots, n\}$.

Let $v' \in V_n^o$ and $S \subseteq \mathbf{n}$. We define the subset $K(v', S)$ of V_n^e by the condition that $v \in K(v', S)$ if and only if there exists $i \in S$ such that v and v' differ only in the i-th coordinate.

Let $v \in V_n^e$ and let $S \subseteq \mathbf{n}$. We define the subset $L(v, S)$ of V_n^e by the condition that $v' \in L(v, S)$ if and only if for all $i \notin S$, v and v' agree in the i-th coordinate. The set S is characterized as the set of coordinates at which not all points of $L(v, S)$ agree.

The k-faces of the half cube were classified in [11].

Theorem 3.2. *(See [11].) The k-faces of Γ_n for $k \leqslant n$ are as follows*:

(i) 2^{n-1} 0*-faces* (*vertices*) *given by the elements of* V_n^e;
(ii) $2^{n-2}\binom{n}{2}$ 1*-faces* $\Pi(K(v', S))$, *where* $v' \in V_n^o$ *and* $|S| = 2$;
(iii) $2^{n-1}\binom{n}{3}$ *simplex shaped* 2*-faces* $\Pi(K(v', S))$, *where* $v' \in V_n^o$ *and* $|S| = 3$;

(iv) $2^{n-1}\binom{n}{k+1}$ *simplex shaped k-faces* $\Pi(K(v', S))$, *where* $v' \in V_n^o$ *and* $|S| = k+1$ *for* $3 \leqslant k < n$;
(v) $2^{n-k}\binom{n}{k}$ *half cube shaped k-faces* $\Pi(L(v, S))$, *where* $v \in V_n^e$ *and* $|S| = k$ *for* $3 \leqslant k \leqslant n$.

Furthermore, two faces are conjugate under the action of $W(D_n)$ *if and only if they have the same dimension and the same shape.*

Proof. The classification of the k-faces is given in [11, Theorem 2.3.6], and the classification of the orbits under the action of $W(D_n)$ is given in [11, Theorem 4.2.3(ii)]. □

The unique n-face in (v) above corresponds to the interior of the polytope. Notice that a k-dimensional half cube has $2k$ facets, and a k-dimensional simplex has $k+1$ facets. The k-faces assemble naturally into a regular CW complex, C_n.

Definition 3.3. For each integer k with $3 \leqslant k \leqslant n$, let X_k (respectively, Y_k) be the set of simplex shaped (respectively, half cube shaped) faces of dimension k. We write $\mathbb{C}X_k$ (respectively, $\mathbb{C}Y_k$, $\mathbb{C}G$) for the $\mathbb{C}$-span of X_k (respectively, Y_k, G).

Let Z_k be the k-th chain module in the complex C_n with coefficients in $\mathbb{C}$. For $k \geqslant 3$, we have $Z_k = \mathbb{C}X_k \oplus \mathbb{C}Y_k$, with the conventions that if $-1 \leqslant k < 3$, we define $\mathbb{C}X_k = Z_k$ and $\mathbb{C}Y_k = 0$.

We now recall some of the key properties of this complex; the reader is referred to [11] for full details.

For any fixed k such that $3 \leqslant k \leqslant n$, one may form a CW subcomplex $C_{n,k}$ by removing the interiors of all the half cube shaped l-faces for $l \geqslant k$. In other words, the l-th chain module of $C_{n,k}$ is equal to Z_k if $l < k$, and to $\mathbb{C}X_k$ if $l \geqslant k$. The reduced (cellular) homology of $C_{n,k}$ is free over $\mathbb{Z}$ and concentrated in degree $k-1$ [11, Theorem 3.3.2].

The Coxeter group $W(D_n)$ acts naturally on V_n^e (and also on V_n^o) via signed permutations of the coordinates. This induces an action of $W(D_n)$ on the half cube Γ_n via cellular automorphisms. In particular, elements of $W(D_n)$ send k-faces of Γ_n to other k-faces of the same type (i.e., simplex shaped or half cube shaped), which means that the $\mathbb{C}$-modules $\mathbb{C}X_k$ and $\mathbb{C}Y_k$ of Definition 3.3 acquire the structure of $W(D_n)$-modules. In turn, there is an induced action of $W(D_n)$ on the subcomplex $C_{n,k}$ via cellular automorphisms, as well as on the homology modules of $C_{n,k}$ [11, Theorem 4.2.3].

The following basic result will be of key importance in the sequel.

Lemma 3.4. *Let* $n \geqslant 3$ *and let s be a Coxeter generator of the group* $G = W(D_n)$. *The element s acts on the half cube* Γ_n *by a reflection in a hyperplane through the origin. The induced action of s on* $H_{n-1}(C_{n,n})$ *and on the n-th chain module of* C_n *is negation.*

Proof. The first assertion follows from [10, Proposition 3.6, Lemma 5.3].

The CW space corresponding to the subcomplex $C_{n,n}$ is obtained from C_n by deleting the (interior of the) unique n-cell. It follows that this space is homeomorphic to S^{n-1}. A well-known result [12, 2.2(e)] then shows that s acts on $H_{n-1}(S^{n-1}; \mathbb{Z})$ by negation, proving the first part of the second assertion. For the final assertion, we use the fact that s acts continuously on Γ_n, which means that it acts on the chain complex of C_n by a chain map. Since the n-th chain module of C_n has rank 1, the fact that s acts by negation on $H_{n-1}(S^{n-1}; \mathbb{Z})$ forces it to act by negation on C_n, which completes the proof. □

Lemma 3.5. *Let* $n \geqslant 4$, $G = W(D_n)$ *and let k satisfy* $3 \leqslant k \leqslant n$. *Let* D_k *denote the parabolic subgroup of* $W(D_n)$ *generated by the set*

$$\{s_{1'}\} \cup \{s_i\colon\ 1 \leqslant i < k\},$$

and let $\mathfrak{S}_{n-k}$ *denote the parabolic subgroup of* $W(D_n)$ *generated by the set*

$$\{s_i\colon\ i > k\}.$$

Regarding $\mathbb{C}Y_k$ as the natural $\mathbb{C}G$ permutation module, we have

$$\mathbb{C}Y_k \cong_G (\mathrm{sgn}_k \otimes \mathrm{id}_{n-k})\uparrow^G_{D_k \times \mathfrak{S}_{n-k}}.$$

Proof. By Theorem 3.2, there is one orbit of half cube shaped faces for each $3 \leqslant k \leqslant n$. One of these has vertex set $L(v, S)$, where

$$v = (1, 1, \dots, 1)$$

and

$$S = \{1, 2, \dots, k\}.$$

Let e be the k-cell of the CW complex C_n corresponding to $L(v, S)$. It is clear from the definitions of the action of G as signed permutations that the set $L(v, S)$ is fixed setwise by all the s_i other than s_k. The group generated by this subset of the generators is $G_k := D_k \times \mathfrak{S}_{n-k}$, which has order $2^{k-1}k!(n-k)!$ and index

$$2^{n-k}\binom{n}{k}$$

in G. It now follows from Theorem 3.2(v) that G_k is the full set stabilizer of $L(v, S)$.

The Coxeter generators $s_{1'}, s_1, \dots, s_{k-1}$ of D_k act as reflections in hyperplanes through the origin. Lemma 3.4 then shows that each of these generators sends e to $-e$. In contrast, the Coxeter generators $s_{k+1}, s_{k+2}, \dots, s_n$ fix $L(v, S)$ (and its convex hull) pointwise. These generators fix e.

The assertion follows from the above observations. $\square$

Lemma 3.6. *Maintain the notation of Lemma* 3.5*, and let $\eta(k, e)$ denote the partition $[e+1, 1, \dots, 1]$ of $k+e$. The character of the module $\mathbb{C}Y_k$ is given by*

$$\sum_{e \leqslant n-k} \chi^{\{\eta(k,e),[n-k-e]\}} + \sum_{e' \leqslant n-(k+1)} \chi^{\{\eta(k+1,e'),[n-(k+1)-e']\}}.$$

Proof. By transitivity of induction and Lemma 3.5, we have

$$\mathbb{C}Y_k \cong_G \left((\mathrm{sgn}_k \otimes \mathrm{id}_{n-k})\uparrow^{D_k \times D_{n-k}}_{D_k \times \mathfrak{S}_{n-k}}\right)\uparrow^G_{D_k \times D_{n-k}}.$$

Let $m = \lfloor \frac{n-k}{2} \rfloor$. By Lemma 2.1, we have

$$(\mathrm{sgn}_k \otimes \mathrm{id}_{n-k})\uparrow^{D_k \times D_{n-k}}_{D_k \times \mathfrak{S}_{n-k}} = \sum_{l \leqslant m} \chi^{\{[1^k],[0]\}} \times \chi^{\{[l],[n-k-l]\}}$$

if $n-k$ is odd, and

$$(\mathrm{sgn}_k \otimes \mathrm{id}_{n-k})\uparrow^{D_k \times D_{n-k}}_{D_k \times \mathfrak{S}_{n-k}} = \chi^{\{[1^k],[0]\}} \times \chi_+^{\{[m],[m]\}} + \sum_{l < m} \chi^{\{[1^k],[0]\}} \times \chi^{\{[l],[n-k-l]\}}$$

if $n-k$ is even. Lemma 2.2 is applicable in this situation, because neither of the partitions $[1^k]$ or $[0]$ has one row. The assertion now follows from Lemma 2.2 and the Pieri rule. (Observe that the numbers appearing in Lemma 2.2(ii) are always zero in this case.) $\square$

Remark 3.7. The methods used in Lemma 3.6 to determine the characters of the modules $\mathbb{C}Y_k$ can be extended to compute the characters of the modules $\mathbb{C}X_k$, as well as the characters of all the representations corresponding to cycles and to boundaries in the subcomplexes $C_{n,k}$.

4. Main results

In order to prove our main results, we require a version of the Hopf trace formula that applies in contexts more general than simplicial complexes.

Theorem 4.1 *(Hopf trace formula). (See [15, Theorem 22.1].) Let K be a finite complex with chain modules $C_p(K)$ (over $\mathbb{C}$) and homology modules $H_p(K)$. Let $\phi : C_p(K) \to C_p(K)$ be a chain map, and let ϕ_* be the induced map on homology. Then we have*

$$\sum_p (-1)^p \operatorname{tr}\bigl(\phi, C_p(K)\bigr) = \sum_p (-1)^p \operatorname{tr}\bigl(\phi_*, H_p(K)\bigr). \qquad \square$$

Lemma 4.2. *Consider the CW complex C_n of Definition* 3.3; *its chain modules are the Z_l for $-1 \leqslant l \leqslant n$. Let ϕ be a chain map of this chain complex. Then we have*

$$\sum_p (-1)^p \operatorname{tr}(\phi, Z_p) = 0.$$

Proof. The chain complex C_n is a CW decomposition of the half cube, which is a contractible space and has trivial reduced homology. Theorem 4.1 applies to the complex C_n, and the previous observation shows that the right-hand side of the Hopf trace formula is zero, completing the proof. $\square$

Lemma 4.3. *Consider the CW subcomplex $C_{n,k}$ of C_n; its chain modules are Z_l for $-1 \leqslant l < k$ and $\mathbb{C}X_l$ for $k \leqslant l \leqslant n$, where $Z_l = \mathbb{C}X_l \oplus \mathbb{C}Y_l$ for $l \geqslant 3$. Let ϕ be a chain map of this chain complex. Then we have*

$$\operatorname{tr}\bigl(\phi_*, H_{k-1}(C_{n,k})\bigr) = \sum_{l \geqslant k} (-1)^{l-k} \operatorname{tr}(\phi, \mathbb{C}Y_l).$$

Proof. We first apply the Hopf trace formula to $C_{n,k}$ to obtain

$$\sum_p (-1)^p \operatorname{tr}\bigl(\phi, C_p(C_{n,k})\bigr) = \sum_p (-1)^p \operatorname{tr}\bigl(\phi_*, H_p(C_{n,k})\bigr).$$

Since, by [11, Theorem 3.3.2], the reduced homology of $C_{n,k}$ is concentrated in degree $k-1$, this simplifies to

$$(-1)^{k-1} \operatorname{tr}\bigl(\phi_*, H_{k-1}(C_{n,k})\bigr) = \sum_p (-1)^p \operatorname{tr}\bigl(\phi, C_p(C_{n,k})\bigr).$$

By Lemma 4.2, we have

$$\sum_p (-1)^p \operatorname{tr}\bigl(\phi, C_p(C_{n,k})\bigr) + \sum_{p \geqslant k} (-1)^p \operatorname{tr}(\phi, \mathbb{C}Y_k) = 0,$$

which, combined with the preceding equation, gives

$$(-1)^{k-1} \operatorname{tr}\bigl(\phi_*, H_{k-1}(C_{n,k})\bigr) = \sum_{p \geqslant k} (-1)^{p+1} \operatorname{tr}(\phi, \mathbb{C}Y_k) = 0.$$

The assertion now follows by multiplying both sides by $(-1)^{k-1}$. $\square$

Theorem 4.4. *Let $n \geqslant 4$, $G = W(D_n)$ and let k satisfy $3 \leqslant k \leqslant n$. Let $\eta(k,e)$ denote the partition $[e+1, 1, \ldots, 1]$ of $k+e$. The character of the representation of G on the $(k-1)$-st homology of the complex $C_{n,k}$ is given by*

$$\chi_D(n,k) = \sum_{e \leqslant n-k} \chi^{\{\eta(k,e),[n-k-e]\}}.$$

Proof. Let χ_k denote the character of the G-module $\mathbb{C}Y_k$. By Lemma 4.3, the character of the homology representation is given by the alternating sum

$$\chi_k - \chi_{k+1} + \chi_{k+2} - \chi_{k+3} \cdots.$$

The result now follows from Lemma 3.6: all of the terms appearing in the statement of that result cancel, except those involving a partition of the form $\eta(l, e)$ for $l = k$. □

Remark 4.5. It is known [18, §4] that the k-th exterior power of the (n-dimensional) reflection representation of $W(D_n)$ is irreducible and corresponds to the pair of partitions

$$\{[1^k], [n-k]\} = \{\eta(k, 0), [n-k-0]\}.$$

Theorem 4.4 shows that this is one of the constituents of the representation of $W(D_n)$ on the $(k-1)$-st homology of $C_{n,k}$.

Corollary 4.6. *Maintain the notation of Theorem* 4.4*, and assume that $k < n$. We have*

$$\chi_D(n, k)\downarrow^{W(D_n)}_{W(D_{n-1})} = 2\chi_D(n-1, k) + \chi_D(n-1, k-1).$$

Proof. Since $k \geqslant 3$, Corollary 2.4 shows that

$$\begin{aligned}\chi^{\{\eta(k,e),[n-k-e]\}}\downarrow^{W(D_n)}_{W(D_{n-1})} &= \chi^{\{\eta(k,e-1),[n-k-e]\}} + \chi^{\{\eta(k,e),[n-k-e-1]\}} + \chi^{\{\eta(k-1,e),[n-k-e]\}} \\ &= \chi^{\{\eta(k,e-1),[(n-1)-k-(e-1)]\}} + \chi^{\{\eta(k,e),[(n-1)-k-e]\}} \\ &\quad + \chi^{\{\eta(k-1,e),[(n-1)-(k-1)-e]\}},\end{aligned}$$

where we ignore any terms involving partitions with negative parts. The result now follows by summing over e, as in Theorem 4.4. □

Theorem 4.7. *Let $n \geqslant 4$ and let k satisfy $3 \leqslant k \leqslant n$. Let $\mathfrak{S}_n$ denote the parabolic subgroup of $W(D_n)$ corresponding to the omission of the generator $s_{1'}$. Let E_l be the $(l-1)$-dimensional reflection representation for $\mathfrak{S}_l$ described in Section* 2*.*

(i) *Regarded as a $\mathbb{C}\mathfrak{S}_n$-module by restriction, the $(k-1)$-st homology of the complex $C_{n,k}$ is isomorphic to*

$$\bigoplus_{e \leqslant n-k} \left(\mathrm{id}_{n-k-e} \otimes \bigwedge^{k-1} E_{k+e}\right)\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k-e} \times \mathfrak{S}_{k+e}}.$$

(ii) *As $\mathbb{C}\mathfrak{S}_n$-modules, the $(k-1)$-st homology of $C_{n,k}$ is isomorphic to the $(k-2)$-nd (co)homology of the complement, $M^{\mathbb{R}}_{n,k}$, of the k-equal real hyperplane arrangement.*

Proof. We first prove (i). By Theorem 4.4, it is enough to show that, for $e \leqslant n-k$, the restriction of the character $\chi^{\{\eta(k,e),[n-k-e]\}}$ of $W(D_n)$ to $\mathfrak{S}_n$ corresponds to the representation

$$\left(\mathrm{id}_{n-k-e} \otimes \bigwedge^{k-1} E_{k+e}\right)\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k-e} \times \mathfrak{S}_{k+e}}$$

of $\mathfrak{S}_n$.

By [8, Proposition 5.4.12], the character of the $\mathfrak{S}_{k+e}$-module $\bigwedge^{k-1} E_{k+e}$ is given by the partition $\mu = [e+1, 1^{k-1}]$. The character of the $\mathfrak{S}_{n-k-e}$-module id_{n-k-e} is given by the one-row partition $\nu = [n-k-e]$. Using standard results [8, Definition 6.1.1], the character of the induction product of these two characters to $\mathfrak{S}_n$ is

$$\sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

The proof of (i) is completed by Lemma 2.1(i), which shows that we also have

$$\chi^{\{\eta(k,e),[n-k-e]\}}\downarrow^{W(D_n)}_{\mathfrak{S}_n} = \chi^{\{\mu,\nu\}}\downarrow^{W(D_n)}_{\mathfrak{S}_n} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\chi^{\lambda}.$$

In [16, Theorem 4.4] Peeva, Reiner and Welker proved that

$$H^{s(k-2)}(M_{n,k};\mathbb{C}) \cong \bigoplus_{\substack{(i_0,i_1,\ldots,i_s)\\ sk+\sum_j i_j=n}} \mathcal{S}_{(i_0)*(i_1+1,1^{k-1})*\cdots*(i_s+1,1^{k-1})}.$$

The notation $\mathcal{S}_\lambda$ refers to the representation of $\mathfrak{S}_{|\lambda|}$ corresponding to the Schur function s_λ. The symbol $*$ can be identified with product (in the case of Schur functions) or induction product (in terms of representations). The special case $s=1$ gives

$$H^{k-2}(M_{n,k};\mathbb{C}) \cong \bigoplus_{\substack{(i_0,i_1)\\ k+i_0+i_1=n}} \mathcal{S}_{(i_0)*(i_1+1,1^{k-1})}.$$

The argument given in the second paragraph of this proof now shows that the complex character of the $(k-2)$-nd cohomology of $M^{\mathbb{R}}_{n,k}$, regarded as a $\mathbb{C}\mathfrak{S}_n$-module, agrees with the character of the representation described in part (i). This completes the proof of (ii). $\square$

Remark 4.8. It is natural to ask whether there is an analogue of Theorem 4.7 if one omits the generator s_1. Note that in Theorem 4.4, none of the irreducible representations of $W(D_n)$ appearing are of the form $\chi_+^{\{\mu,\mu\}}$ or $\chi_-^{\{\mu,\mu\}}$. Because of this, one obtains the same representations even if the generator s_1 is omitted. (Compare with Lemma 2.1(i).)

Remark 4.9. Note that, under the usual identifications, we have

$$W(D_{n-1}) \cap W(A_n) = W(A_{n-1}).$$

It follows that the type A homology representations described in Theorem 4.7 have a branching rule analogous to the type D branching rule of Corollary 4.6. However, this would not be such an obvious result in the absence of the wider context of the type D representations.

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