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MAXIMAL COHEN–MACAULAY MODULES AND REPRESENTATION THEORY*

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0. Introduction

A very old and fundamental theorem concerning principal ideal domains states that a finitely generated module over such a ring must be a direct sum of cyclic modules. In the more restrictive setting of complete discrete valuation rings, one gets even better results, at least in the case of torsion free modules. In fact Kaplansky [8] shows that a countably generated torsion free module over a complete discrete valuation ring is necessarily isomorphic with a direct sum of a free module and a divisible module (that is, a vector space over the field of fractions). The important point here is that a countably generated, torsion free module over such a local ring is necessarily free as long as it is separated in its maximal ideal topology. It is this latter statement that is the source of our motivation in the beginning of Section 1. With a rather mild adjustment of hypothesis, we prove an analogous theorem for countably generated, torsion free modules over complete regular local rings. This result was overlooked in our first paper [3] on this subject. In addition we give an example as to why the strengthening of the topological condition is necessary. Other properties of maximal Cohen–Macaulay modules are also discussed. For example, if the local ring A is a module finite extension of the regular local ring R and if C is an A-module which is R-free, then every system of parameters of A is shown to be a regular sequence on C. These results as well as others that follow are an outgrowth of the author's paper [3] on representations of complete local rings that followed the landmark paper of M. Hochster [4], in which the existence of maximal Cohen–Macaulay modules was established for equicharacteristic local rings.

In Section 2, a module-theoretic technique is given for constructing representations of complete local rings (over complete regular local rings) assuming that one has representations modulo powers of a parameter.

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In Section 3, we give some applications of the existence of maximal Cohen–Macaulay modules that are free over a regular local base ring. For example, we give a short proof of H.-B. Foxby's result [2] that, if M is a module over an essentially equicharacteristic local ring $(A, \boldsymbol{m})$ and if $\boldsymbol{m}M \neq M$, then

$$\dim A \leq \dim M + \text{flat dim } M.$$

The crucial point here is that one can turn finite exact projective or flat A-complexes into exact projective or flat complexes over a regular local ring by tensoring with the aforementioned type of maximal Cohen–Macaulay A-module.

1. A generalization of a Theorem of Kaplansky

Let $(A, \boldsymbol{m})$ be a local ring and let $x_1, \ldots, x_d$ be a system of parameters for A. After Hochster [4], an A-module C is called a maximal Cohen–Macaulay module with respect to the system of parameters $x_1, \ldots, x_d$ provided $\boldsymbol{m}C \neq C$ and x_{i+1} is regular on $C/(x_1, \ldots, x_i)C$ for $i \geq 0$. In his remarkable paper [4], Hochster showed that every equicharacteristic local ring possessed such a countably generated module with respect to any prescribed system of parameters. In [3] we showed that, if A is complete and is a module finite extension of a regular local ring R, then one can construct from Hochster's module a new maximal Cohen–Macaulay A-module F which is free over R of countable rank. One is led to the question as to when countably generated, maximal Cohen–Macaulay modules over regular local rings are free. In case R is complete of dimension one (i.e. R is a complete discrete valuation ring), Kaplansky [8] gives a remarkably easy answer to this question. Namely, he shows that a countably generated, maximal Cohen–Macaulay module C is free provided it is separated in its maximal ideal topology. If the restrictions of completeness or countability are dropped, then there are all sorts of nonfree maximal Cohen–Macaulay modules. We shall show in Example 1.3 the condition that C be separated in its maximal ideal topology (i.e., the zero submodule of C is closed in the maximal ideal topology) is not sufficient to guarantee that C be free. However, the requirement that IC be closed in the maximal ideal topology on C for each ideal I turns out to be sufficient to guarantee that C be free (cf. Theorem 1.1). This result was essentially proven in our paper [3]. However, it was not precisely stated there, and so we shall do so here. All topological references will be with respect to the maximal ideal topology (i.e. "$\boldsymbol{m}$-adic" topology) unless otherwise stated.

Theorem 1.1. *Let $(R, \boldsymbol{m})$ be a complete regular local ring and suppose C is a countably generated R-module. Then the following are equivalent*:

(i) *The module C is a free R-module.*

(ii) *C is a maximal Cohen–Maculay R-module and IC is a closed submodule of C for each ideal I of R.*

(iii) *C is a flat R-module such that IC is a closed submodule for each ideal I of R.*

Proof. The implication (i) implies (ii) is obvious and (ii) implies (iii) is a result of Proposition 2.10 [3]. Finally, the implication (iii) implies (i) is simply Theorem 2.4 [3].

Before constructing our example that demonstrates that separation in the maximal ideal topology is not sufficient to guarantee freeness of a countably generated, maximal Cohen–Macaulay module over a complete regular local ring, we need a lemma on flatness. For a local ring R with a maximal ideal $\boldsymbol{m}$ and residue field k, the notation M^v denotes the Matlis dual of the R-module M, that is, $M^v = \operatorname{Hom}(M, E(k))$ where $E(k)$ denotes the injective envelope of the residue field of R. If R is complete; a standard duality formula (see Cartan and Eilenberg [1; Chapter VI]) gives that

$$\operatorname{Ext}^i_R(M, N) \simeq \operatorname{Tor}^R_i(M, N^v)^v,$$

for all $i > 0$ and finitely generated N.

Lemma 1.2. *Let $(A, \boldsymbol{m}, k)$ be a local domain and suppose that N is a torsion free A-module such that N_P is a flat A_P-module for each prime ideal P of height one in A. If $\operatorname{Ext}^i_A(N, k) \simeq \operatorname{Tor}^A_i(N, k)^v = 0$ for each $i > 0$, then N is a flat A-module.*

Proof. The proof follows that of Proposition 2.10 [3]. From our hypothesis, we get that $\operatorname{Tor}^A_i(N, L) = 0$ for all $i > 0$ and all modules L of finite length. We will show that $\operatorname{Tor}^A_i(N, M) = 0$ for all $i > 0$ and each finitely generated M by induction on the dimension of M. At this point we may suppose $\dim M > 0$ and that our claim holds for all finitely generated modules of lesser dimension. Since $\operatorname{Tor}^A_i(N, \;)$ vanishes on modules of finite length for $i > 0$, we may even suppose that depth $M > 0$. Let $a \in A$ be a regular element on M. From the exact sequence

$$0 \to M \xrightarrow{a} M \to M/aM \to 0$$

we have that

$$\operatorname{Tor}^A_i(N, M) \xrightarrow{a} \operatorname{Tor}^A_i(N, M)$$

is an isomorphism for $i > 0$, since $\operatorname{Tor}^A_i(N, M/aM) = 0$ by induction for $i > 0$. Consequently, for $i > 0$, $\operatorname{Tor}^A_i(N, M)$ is a-torsion free and a-divisible. On the other hand, for any prime ideal P of height one which contains a, we have that $\operatorname{Tor}^A_i(N, M)_P = 0$ if $i > 0$, since N_p is a flat A_P-module. It follows that

$$\operatorname{Tor}^A_i(N, M) = 0 \quad \text{for } i > 0,$$

and we conclude that N is a flat A-module.

Example 1.3. Let k be any countable field and let R be the localization of the full ring of polynomials $k[x_1, \ldots, x_n]$ at the maximal ideal $\boldsymbol{m} = (x_1, \ldots, x_n)$, where n is at least 2. Therefore, R is a regular local ring of dimension at least two. Now let π be a principal prime in R and let Q denote the field of fractions of the integral domain $R/\pi R$. We now observe that Q is necessarily a countably generated R-module and from [7] that Q has projective dimension 1 as an $R/\pi R$-module (denoted $\mathrm{pd}_{R/\pi R}(Q)=1$). We can now form an exact sequence

$$0 \to K \to F \to Q \to 0,$$

where F is a free R-module of countable rank and where the syzygy K is a countably generated R-module. since $\pi Q=0$, it follows that

$$\mathrm{Tor}_1^R(R/\pi R, Q) \simeq Q.$$

Consequently, after applying the functor $R/\pi R \otimes -$ to the above short exact sequence, we obtain the exact sequence

$$0 \to Q \to K/\pi K \to F/\pi F \to Q \to 0,$$

since $\mathrm{Tor}_1^R(R/\pi R, Q) \cong Q$. It follows that $K/\pi K \cong Q \oplus V$, where V is a nonzero free $R/\pi R$-module. This is a result of the fact that $\mathrm{pd}_{R/\pi R}(Q)=1$. Hence K cannot be a free R-module, since Q is not a free $R/\pi R$-module. However, K is separated in its m-adic topology and is free at prime ideals of height one, since K is isomorphic with a submodule of F. If $i>0$, then

$$\mathrm{Tor}_i^R(K, k) \cong \mathrm{Tor}_{i+1}^R(Q, k).$$

Moreover, if $a \in m$ and a is not divisible by π, then the map $Q \xrightarrow{a} Q$ is an isomorphism. Hence, so is the induced map

$$\mathrm{Tor}_{i+1}^R(Q, k) \xrightarrow{a} \mathrm{Tor}_{i+1}^R(Q, k).$$

However, this latter map is also zero since a annihilates k. Therefore, $\mathrm{Tor}_i^R(K, k) = 0$ for $i>0$, and thus by Lemma 1.2 K is a flat R-module. Of course K is also a maximal Cohen–Macaulay R-module which is not free.

We now turn our attention to the following question. Let A be a complete local ring which is a module finite extension of the complete regular local ring R. Suppose that C is a maximal Cohen–Macaulay A-module which is free over R. What systems of parameters of A are regular sequences on C? Hochster [6] has shown how to modify his construction in order to obtain maximal Cohen–Macaulay modules so that every system of parameters of A is a regular sequence on C. Our next result establishes the fact that those maximal Cohen–Macaulay A-modules which are free over R already have this property.

Proposition 1.4. *Let A be a local ring which is a module finite extension of a Cohen–Macaulay local ring R. Suppose the A-module C is free as an R-module. Then every system of parameters for A is a regular sequence on C.*

Proof. It is clear that every system of parameters of R is a regular sequence on C. So we shall argue by way of induction on the number of parameters coming from R. Let $a_1, a_2, \ldots, a_d$ be the system of parameters under consideration. Since A is a module finite extension of R, we can find $b \in A$ such that $ba_1 = r_1 \in R$, and $(r_1, a_2, \ldots, a_d)$ is still a system of parameters for A. Since r_1 is necessarily regular on C, it follows that a_1 must be also. It remains to establish that, if $\sum_{i=1}^{t} a_i v_i = 0$, for $v_1, \ldots, v_t \in C$, then $v_t \in (a_1, \ldots, a_{t-1})C$. Multiplying the preceding equation through by b gives the equation

$$r_1 v_1 + \sum_{i=2}^{t} a_i(bv_i) = 0.$$

By induction, we have that $r_1, a_2, \ldots, a_d$ is a regular sequence on C. Hence $bv_t \in (r_1, a_2, \ldots, a_{t-1})C$, that is,

$$bv_t = r_1 w_1 + \sum_{i=2}^{t-1} a_i w_i.$$

Therefore, multiplying this equation by a_1, we obtain

$$r_1 v_t = r_1 a_1 w_1 + \sum_{i=2}^{t-1} a_i(a_1 w_i).$$

But then $v_t - a_1 w_1 \in (a_2, \ldots, a_{t-1})C$, since $r_1, a_2, \ldots, a_{t-1}$ is a regular sequence on C. Thus $v_t \in (a_1, \ldots, a_{t-1})C$, and the proof is complete.

Remark. (The notation is understood to be the same as the preceding notation.) All that one needs in order that the above argument holds is that every system of parameters for R is a regular sequence on C.

Our next result (Theorem 1.7) on maximal Cohen–Macaulay modules, which are free over a regular local base ring, was attributed to the author in Hochster's article [6]. Since no published proof has appeared, we shall present one here.

Lemma 1.5. *Let S and T be regular local rings and suppose further that S is a module finite extension of T. If K is an S-module that is free as a T-module, then K is necessarily free as an S-module.*

Proof. First of all, it easily follows that S is free as a T-module since a system of parameters of T will also be a system of parameters for S. Hence

$$\operatorname{depth}_S S = \operatorname{depth}_T S = \dim T$$

and consequently $\operatorname{pd}_T S = 0$. Since K is naturally an $S-T$ bimodule, we have the natural equivalence of functors on S-modules

$$\operatorname{Hom}_S(K \otimes_T S,\ ) \simeq \operatorname{Hom}_T(S, \operatorname{Hom}_S(K,\ )$$

via the Adjoint Isomorphism Theorem. If $\mathcal{S}$ represents a short exact sequence of S-modules, then the induced sequence $\mathrm{Hom}_S(K \otimes_T S, \mathcal{S})$ remains short exact, since both K and S are free over T. But then $\mathrm{Hom}_T(S, \mathrm{Hom}_S(K, \mathcal{S}))$ is also short exact by the above natural equivalence of functors. Since S is free as a T-module, we now have that the induced sequence $\mathrm{Hom}_S(K, \mathcal{S})$ is short exact for any short exact sequence $\mathcal{S}$ of S-modules. It follows that K is S-projective and, hence, S-free, since S is a local ring.

Corollary 1.6. *Let S and T be regular local rings with S a module finite extension of T. Let M be any S-module. Then* $\mathrm{pd}_S M = \mathrm{pd}_T M$.

Proof. As was noted in the proof of Lemma 1.5, S is necessarily free as a T-module. So let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be an S-projective resolution of M and let $K_{i+1} = \mathrm{Ker}(P_i \to P_{i-1})$ be the $(i+1)$*st* syzygy in this resolution. Since S is free as a T-module, we have immediately that $\mathrm{pd}_T M \leq \mathrm{pd}_S M$. On the other hand, if some syzygy K_i is free as a T-module, then it is also free as an S-module by Lemma 1.5. Therefore, $\mathrm{pd}_S M \leq \mathrm{pd}_T M$ and thus $\mathrm{pd}_S M = \mathrm{pd}_T M$.

Theorem 1.7. *Let A be an equicharacteristic, complete local ring which is a module finite extension of the complete regular local ring R. Suppose that C is a maximal Cohen–Macaulay A-module which is free as an R-module. Let S be an equicharacteristic regular local ring such that $A = S/I$. Then* $\mathrm{pd}_S C = \dim S - \dim A$.

Proof. It is sufficient to establish the result when all complete rings in question have the same coefficient field. Let $R = k\,[[x_1, \ldots, x_d]]$. Then one can construct a ring homomorphism ϕ: $R \to S$ such that the triangle

$$\begin{array}{ccc} S & \longrightarrow & A \\ {\scriptstyle\phi}\nwarrow & & \nearrow \\ & R & \end{array}$$

commutes. Moreover, there are elements $y_1, \ldots, y_t$ in I such that $\phi(x_1), \ldots, \phi(x_d), y_1, \ldots, y_t$ is a system of parameters for S and S is a module finite extension of the regular local ring

$$T = k\,[[\phi(x_1), \ldots, \phi(x_d), y_1, \ldots, y_t]].$$

Thus, we obtain a commutative square of ring homomorphisms

$$\begin{array}{ccc} S & \longrightarrow & A \\ \uparrow & & \uparrow \\ T & \longrightarrow & R \end{array}$$

where the vertical maps are monic and the horizontal homomorphisms are surjections. Since the module C is R-free and since T and R are regular, standard "change of rings" results yield that

$$\mathrm{pd}_T C = \dim T - \dim R = \dim S - \dim A.$$

But by Corollary 1.6, we have that $\mathrm{pd}_T C = \mathrm{pd}_S C$ which completes our argument.

Corollary 1.8. *Let S be a complete regular, equicharacteristic local ring and let P be a prime ideal of S. Then there is a countably generated, maximal Cohen–Macaulay S/P-module C such that* $\mathrm{pd}_S C = \text{height } P$.

The proof of this result is the immediate consequence of Hochster's construction of maximal Cohen–Maculay modules [4], Theorem 3.1 [3] and Theorem 1.6.

2. Lifting of representations of local rings

As in the previous section, let A be a local ring which is a module finite extension of a regular local ring R. With regard to finding a finitely generated maximal Cohen–Macaulay A-module having rank n over R, it has been explained in [5] or [3] that this is equivalent to obtaining an R-algebra homomorphism $A \to M_n(R)$, where $M_n(R)$ denotes the ring of $n \times n$ matrices over R. This of course means that there is an action of A on a free R-module of rank n which extends the action of R. In [3] it was shown that every complete, equicharacteristic local ring A has a representation $A \to M_\infty(R)$, where $M_\infty(R)$ denotes the ring of countable, column finite matrices over R. In addition, Hochster [5] has established that, if $x \neq 0 \in R$ and if A/x^nA has a representation $A/x^nA \to M_t(R/x^nR)$ for each positive integer n, then A itself has a representation $A \to M_t(R)$. Hochster's method is to make use of his techniques for solving polynomial equations over commutative rings as set forth in [4] or [6]. Here we wish to demonstrate a module theoretic technique for the lifting of "noncoherent" representations of A/x^nA over R/x^nR (assuming such exist for infinitely many n) in order to obtain a representation of A over R. Suppose that V_n is the free R/x^nR-module for which A/x^nA has an action. By noncoherent, we simply mean that it need not be the case that V_{n+1}/x^nV_{n+1} is isomorphic with V_n. If this were true, then the construction could simply take the form of an inverse limit.

Let A be a complete local ring that is a module finite extension of a complete Gorenstein local ring R. Let x be a regular nonunit of R, and suppose for each integer $n > 0$ there is an A/x^nA-module V_n which is free as an R/x^nR-module. Put $V = \prod_n V_n$ and let

$$T = \{v \in V \mid x^l v = 0 \text{ for some } l > 0\}.$$

There is an exact sequence

$$0 \to T \to V \to C \to 0.$$

Since V is an A-module, it is easy to see that T is also an A-module from which it follows that C is an A-module. Since A is a module finite over R, it suffices to show that a system of parameters for R is a regular sequence on C, in order to establish that C is both a maximal Cohen–Macaulay A-module and maximal Cohen–Macaulay R-module. In order to facilitate the computations in our proofs to follow, we let F_n be a free R-module, for $n>0$, such that $F_n/x^nF_n \cong V_n$.

Lemma 2.1. (We use the notation above.) *An element $v=\langle v_n\rangle$ in V is in T if and only if there is a positive integer l and w_n in V_n such that $v_n = x^{n-l}w_n$ for $n \geq l$.*

Proof. The condition above is clearly sufficient. So suppose v in T. Then there is $l>0$ such that

$$x^lv = \langle x^lv_n\rangle = 0.$$

Let f_n in F_n be such that

$$v_n = f_n + x^nF_n.$$

Then $x^lv_n = 0$ implies that

$$x^lf_n = x^ng_n \quad \text{for } g_n \in F_n.$$

Since x is regular on R and since F_n is a free R-module, this means that

$$f_n = x^{n-l}g_n \quad \text{for } n \geq l.$$

Consequently, we may take

$$w_n = g_n + x^nF_n \quad \text{for } n \geq l.$$

The reverse implication is clear.

Lemma 2.2. *The module C (see definition above) is a maximal Cohen–Macaulay R-module.*

Proof. Let $x_1, x_2, \ldots, x_d$ be a system of parameters of R with $x = x_1$. Suppose that v in V has the property that it is an x^l-torsion element in V/x_2V, that is, suppose there is $w = \langle w_n\rangle$ in V with $x^lv = x_2w$. Since $x^lv_n = x_2w_n$ for each n, there are elements $f_n, g_n, h_n \in F_n$ with

$$v_n = f_n + x^nF_n, \qquad w_n = g_n + x^nF_n$$

such that

$$x^lf_n = x_2g_n + x^nh_n.$$

For $n \geq l$, one has that $g_n = x^l g_n^1$, since x_1, x_2 is a regular sequence on F_n. Therefore,

$$f_n = x_2 g_n^1 + x^{n-l} h_n \quad \text{for } n \geq l.$$

Let w_n^1 and z_n be elements of V_n such that

$$w_n^1 = g_n^1 + x^n F_n \quad \text{and} \quad z_n = h_n + x^n F_n \quad \text{for } n \geq l.$$

Also put

$$t_n = x^{n-l} z_n \quad \text{for } n \geq l.$$

For $n < l$ choose elements w_n^1 and t_n of V_n so that $v_n = x_2 w_n^1 + t_n$. Now let $w^1 = \langle w^1 \rangle$ and $t = \langle t_n \rangle$ in V. From the preceding equations we see that $v = x_2 w^1 + t$ and, by Lemma 2.1, that $t \in T$. Hence, if v in V represents an x^l-torsion element in $V/x_2 V$, then $v \equiv t$ (module $x_2 V$), where $t \in T$. This means that the image of the natural map $T/x_2 T$ in $V/x_2 V$ is the set of all elements in $V/x_2 V$ that are annihilated by some x^l, for $l > 0$. One can also easily check that x_2 is regular on both V and C, since (x, x_2) is a regular sequence on F_n for each n. Thus, it follows that the sequence

$$0 \to T/x_2 T \to V/x_2 V \to C/x_2 C \to 0$$

is exact and represents the same construction over $R/x_2 R$ as the original sequence did over R. Since x will always be regular on the right hand end of any such construction, we see that $x_2, \ldots, x_d, x$ is a regular sequence on C by induction.

It remains only to check that $\boldsymbol{m} C \neq C$ where $\boldsymbol{m}$ is the maximal ideal of R. For each $n > 0$ let v_n be a free generator of V_n and put $v = \langle v_n \rangle$. Suppose v represents an element of $\boldsymbol{m} C$. Then there are elements $a_1, \ldots, a_s$ in $\boldsymbol{m}$ and $w_1, \ldots, w_s$ in V such that

$$v = \left(\sum_{i=1}^{s} a_i w_i \right) + t,$$

where $t \in T$. But from the description of T in Lemma 2.1, we observe that v_n is necessarily in $\boldsymbol{m} V_n$ for $n > l$, where $x^l t = 0$. However, this is a contradiction to our choice of the v_n for $n > l$. Thus, $\boldsymbol{m} C \neq C$ and C is a maximal Cohen–Macaulay R-module.

Lemma 2.3. *The module* Hom(C, R) *is nonzero and is also a maximal Cohen–Macaulay R-module.*

Proof. From [3; Proposition 2.6(b)] one has that $\mathrm{Ext}_R^i(C, R) = 0$ for $i > 0$. Moreover, the above claim clearly holds when dim $R = 0$. So suppose y is a regular nonunit of R which is also regular on C. From the exact sequence

$$0 \to R \xrightarrow{y} R \to R/yR \to 0,$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}(C, R) \xrightarrow{y} \mathrm{Hom}(C, R) \to \mathrm{Hom}(C/yC, R/yR) \to 0,$$

since $\operatorname{Ext}^{\cdot}(C, R) = 0$. By induction on dim R we obtain the desired conclusions.

We now return to the exact sequence

$$0 \to T \to V \to C \to 0.$$

There is a flat resolution of $V = \prod V_n$,

$$0 \to \prod F_n \xrightarrow[\langle x^n \rangle]{} \prod F_n \to \prod V_n \to 0,$$

where $\langle x^n \rangle$ denotes the map: $\langle f_n \rangle \mapsto \langle x^n f_n \rangle$. Composing the homomorphisms $\prod F_n \to V$ and $V \to C$, one obtains an epimorphism $\prod F_n \to C$ with kernel K. Then

$$\prod x^n F_n \subseteq K \quad \text{and} \quad K/(\prod x^n F_n) \cong T.$$

If $v = \langle v_n \rangle \in T$, we have by Lemma 2.1 that

$$v_n = x^{n-l} w_n, \quad \text{for } w_n \in V_n \quad \text{and} \quad n \geq l,$$

where $x^l v = 0$. Consequently, $K = \bigcup_l K_l$ where

$$K_l = F_1 \oplus \cdots \oplus F_l \oplus x F_{l+1} \oplus x^2 F_{l+2} \oplus \cdots \text{ (direct product).}$$

Hence K is the ascending union of flat R-modules, and therefore K is a flat R-module. Consequently, the module C has flat dimension ≤ 1 over R. By a result of Raynaud and Gruson [9], it follows that C has finite projective dimension as well.

Lemma 2.4. *The module* $\operatorname{Hom}(C, R) = C^*$ *is naturally an* A*-module. As an* R*-module* C^* *is flat and can be embedded as a pure submodule in a direct product of copies of* R.

Proof. Since C is an A-module, one puts an A-module structure on C^* via $(af)(c) =: f(c)$. From above we have that C has a finite R-projective resolution

$$0 \to P_s \to \to P_{s-1} \to \cdots \to P_1 \to P_0 \to C \to 0.$$

Also as noted above, we have that

$$\operatorname{Ext}^i_R(C, R) = 0 \quad \text{for } i > 0.$$

Hence we obtain an exact sequence

$$0 \to C^* \to \operatorname{Hom}(P_0, R) \to \operatorname{Hom}(P_1, R) \to \cdots \to \operatorname{Hom}(P_s, R) \to 0.$$

Since each $\operatorname{Hom}(P_i, R)$ is a flat R-module, it follows that C^* is also flat. For the same reason, C^* is isomorphic with a pure submodule of $\operatorname{Hom}(P_0, R) \cong \prod R$, that is, the natural map

$$C^* \otimes_R M \to \operatorname{Hom}(P_0, R) \otimes_R M$$

is monic for each R-module M.

Lemma 2.5. *The A-module C^* is a direct limit of countably generated A-modules which are pure and free in C^* as R-modules.*

Proof. From the preceding discussion, we have that there is a pure exact sequence

$$0 \to C^* \to P_0^* \to P_1^* \to \cdots \to P_s^* \to 0,$$

where the P_i^* are each direct products of copies of R. From [3; Lemma 1.5] it follows that each P_i^* is a direct limit of its countably generated pure R-submodules.

It now suffices to show that, if one has a pure exact sequence $0 \to K \to M \xrightarrow{\phi} N \to 0$ of flat R-modules, where both M and N are direct limits of their countably generated pure (necessarily flat) R-submodules, then K is also such a direct limit. Then by induction it will follow that C^* is a direct limit of its countably generated pure R-submodules. Let K_0 be any countably generated R-submodule of K. Then one may choose countably generated pure R-submodules $\{M_i\}_{i=0}^{\infty}$ and $\{N_i\}_{i=0}^{\infty}$ of M and N, respectively, such that

$$K_0 \subseteq M_i \subseteq M_{i+1}, \quad \text{for each } i \geq 0,$$

and such that

$$N_i \subseteq \phi(M_i) \subseteq N_{i+1}, \quad \text{for } i \geq 0.$$

Put $M_\infty = \bigcup_i M_i$ and $N_\infty = \bigcup_i N_i$. Then M_∞ and N_∞ are countably generated pure R-submodules of M and N, respectively, and moreover we have that $\phi(M_\infty) = N_\infty$ with $K_0 \subseteq M_\infty$. Let $K_\infty = K \cap M_\infty$. It easily follows that K_∞ is a countably generated, pure flat R-submodule of K containing K_0. By using the "interlacing" technique above and the fact that A is a module finite extension of R, it follows that a cofinal subset of the countably generated pure R-submodules in the direct limit of C^* are in fact A-modules as well. The remaining fact that these countably generated pure R-submodules are free as R-modules follows from the fact that C^* is pure in $P_0^* \cong \prod R$ and a result of Raynaud and Gruson [9] (see also [3; Corollary 1.6]).

We have now established the main theorem of this section.

Theorem 2.6. *Let the complete local ring A be a module finite extension of the complete local Gorenstein ring R. Suppose that x is a regular nonunit of R and that some nonzero A/x^nA-module is free as an R/x^nR-module for each positive integer n. Then some countably generated nonzero A-module is free as an R-module.*

Perhaps the crucial point in the proof of the above theorem is the fact that, in order to establish that C is a Cohen–Macaulay module (Lemma 2.2), it was necessary to be working with free R/x^nR-modules so that one could lift them back to R when doing computations involving R-sequences. To replace the free R/x^nR-modules by Cohen–Macaulay R-modules does not seem to work. Moreover, even with Theorem 2.6 in hand, it appears to be nontrivial to reduce the existence (in all characteristics) of maximal Cohen–Macaulay modules down to dimension zero. As Hochster points out in [5], there are module finite extensions of zero dimensional Gorenstein rings without representations over the Gorenstein ring.

3. Applications

In this section we shall discuss some applications of maximal Cohen–Macaulay modules which are free modules over a base regular local ring. The advantage of these modules is that they turn projective complexes into projective complexes over the base regular ring. It is often the case that the conclusions desired can be easily obtained over the regular ring and then translated back to the original ring. The basis for these results are Lemma 3.1 (due to H.-B. Foxby [2]) and Theorem 3.2. Although our statement of the next lemma differs slightly from Foxby's [2; Proposition 6.3], the proof is exactly the same.

Lemma 3.1 (Foxby). *Let A be a local ring having an ideal I with* $\dim A = \dim A/I$ *such that A/I is a module finite extension of a regular local ring R. Moreover, assume that some A/I-module C is free over R. If M is an A-module of finite flat dimension over A, then* $\mathrm{Tor}_i^A(C, M) = 0$ *for all $i > 0$.*

Theorem 3.2. *Let the local ring A be a module finite extension of the regular local ring R and suppose that C is a maximal Cohen–Macaulay A-module which is free over R. Suppose that M is an A-module and that the complex*

$$\mathbb{F}: \cdots \to F_2 \to F_1 \to F_0$$

represents an A-free resolution of M.

(a) *The complex $C \otimes_A \mathbb{F}$ is then an R-free complex with homology* $\mathrm{Tor}_i^A(C, M)$ *for $i \geq 0$. Moreover, $H_0(C \otimes \mathbb{F}) = C \otimes_A M \neq 0$.*

(b) *If M has finite flat dimension over A, then $C \otimes_A \mathbb{F}$ is an acyclic free complex over R.*

(c) *If K is an A-module, then $C \otimes_A K$ is R-free if and only if K is A-free. Consequently, if* $\mathrm{pd}_A M < \infty$, *then* $\mathrm{pd}_A M = \mathrm{pd}_R(C \otimes M)$.

Proof. Part (a) is clear except perhaps for the fact that $C \otimes_A M \neq 0$. However, this is a consequence of the isomorphism

$$A \otimes_R (C \otimes_A M) \cong (A \otimes_R C) \otimes_A M$$

and the fact that $A \otimes_R C$ is isomorphic to a free A-module, since C is R-free. Part (b) follows easily from Lemma 3.1.

Now suppose that K is an A-module such that $C \otimes_A K$ is R-free. Then necessarily $A \otimes_R (C \otimes_A K)$ is A-free. But also

$$A \otimes_R (C \otimes_A K) \cong (A \otimes_R C) \otimes_A K$$

is isomorphic with a direct sum of copies of K, since $A \otimes_R C$ is necessarily A-free. Hence, K is isomorphic to an A-direct summand of a free A-module and thus is itself free.

The above theorem gives an easy proof of the result of Raynaud and Gruson [9] in a special case of complete local rings.

Corollary 3.3 (Raynaud–Gruson). *Suppose that A is an equicharacteristic complete local ring which is a module finite extension of the complete regular local ring R. If M is a flat A-module, then M has A-projective dimension $\leq \dim A$.*

Proof. From [3] we have that there is an A-module C which is free as an R-module. Let $\mathbb{F} \to M$ be an A-free resolution of M. Then $C \otimes_R \mathbb{F}$ is an R-free resolution of $C \otimes_R M$. By Theorem 3.2(c), we have that

$$\operatorname{pd}_A M = \operatorname{pd}_R C \otimes_R M \leq \dim R = \dim A.$$

We now apply Theorem 3.2 to give a rather short proof of Foxby's result [2; Theorem 6.2]. Let A be a local ring and let M be a (not necessarily finitely generated A-module). For the purposes of our next theorem, we define the dimension of M to be $\dim(A/\operatorname{ann}(M))$. If A is a module finite extension of the local ring R, we observe that $\dim_A M = \dim_R M$. The local ring A is called essentially equicharacteristic provided there is an ideal I in A with $\dim A = \dim(A/I)$ and such that A/I contains a field as a subring.

Theorem 3.4 (Foxby). *Let $(A, \mathbf{m})$ be an essentially equicharacteristic local ring and let M be an A-module such that $\mathbf{m}M \neq M$. Then* $\dim A \leq \dim M + \text{flat}\dim M$.

Proof. Of course we may assume that $\text{flat}\dim M < \infty$, that is, that M has a finite flat resolution $\mathbb{F} \to M$. As was observed by Foxby [2], we may assume that A is complete. We can now find a minimal prime P in A so that $\dim A = \dim A/P$ and such that A/P is a module finite extension of an equicharacteristic complete, regular local ring R. From [3], there is an A/P-module C such that C is free as an R-module. From Lemma 3.1, we have that

$$\operatorname{Tor}_i^A(C, M) = 0, \quad \text{for } i > 0,$$

and consequently that $C \otimes_A \mathbb{F}$ is a finite flat resolution of the R-module $C \otimes_A M$. We now have the inequalities

$$\dim R = \dim A, \qquad \dim_A M \leq \dim_R C \otimes_R M$$

and

$$\text{flat}\dim_R(C \otimes_A M) \leq \text{flat}\dim_A M.$$

Thus, it suffices to establish the result for A a complete regular local ring. If the maximal ideal $\boldsymbol{m}$ is in Ass M, then

$$\mathrm{Ext}_A^d(M, A) \cong \mathrm{Tor}_d^A(M, E)^v \neq 0,$$

where $d = \dim A$ and $E = A^v$ is the injective envelope of the residue field of A. Hence the conclusion holds in this case. Moreover, using localization, one can use that

$$\mathrm{Tor}_h^A(M, E(A/Q)) \neq 0, \quad \text{for } h = htQ,$$

where $E(A/Q)$ is the injective envelope of A/Q, for $Q \in \mathrm{Ass}\, M$. Thus,

$$\dim A = h + \dim A/Q \leq h + \dim M \leq \text{flat dim } M + \dim M,$$

which completes our proof.

Note added in proof

The hypothesis of Lemma 1.2 is too weak as stated. The hypothesis should read that "N_P is a flat A_P-module for each *nonmaximal* prime ideal P in A." With this strengthening of the hypothesis, Lemma 1.2 becomes correct with the proof given. Further, the application of Lemma 1.2 in Example 1.3 remains correct with this change. The same error is also committed in Proposition 1.10 [3]. Here the hypothesis should state that M is a maximal Cohen–Macaulay R-module for every system of parameters of R. The conclusion that M is a flat module over the regular local ring then follows. The author would like to thank H.-B. Foxby for pointing out these deficiencies in the aforementioned statements.

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