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ORIGINAL ARTICLE

On some new sequence spaces of non-absolute type and matrix transformations

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Abstract In the present paper, we introduce the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$, which are BK-spaces of non-absolute type and prove that these spaces are linearly isomorphic to the spaces c_0 and c, respectively. We also compute their α-, β- and γ-duals and construct their basis. Finally, we characterize some matrix classes concerning with these spaces.

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1. Preliminaries, background and notation

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let ω denote the space of all sequences (real or complex); l_∞, c and c_0 denotes the space of all bounded, convergent and null sequences, respectively. Also, by bs, cs, l_1 and l_p we denote the space of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A- transformation from X into Y, if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$ (see, [1]). For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X:Y)$, we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l, which is called as the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}. \tag{1}$$

We shall denote the collection of all finite subsets of $\mathbb{N}$ by $\mathscr{F}$. Also, we shall write $e^{(k)}$ for the sequence whose only non-zero term is 1 at the kth place for each $k \in \mathbb{N}$. The approach of constructing a new sequence space by means of matrix domain of a particular limitation mehtod has been studied by several authors. They introduced the sequence spaces $(l_\infty)_{N_q}$ and c_{N_q} (see, [2]), $(l_p)_{C_1} = X_p$ and $(l_\infty)_{C_1} = X_\infty$ (see, [3]), $(l_\infty)_{R^t} = r^t_\infty$, $(c)_{R^t} = r^t_c$ and $(c_o)_{R^t} = r^t_0$ (see, [4]), $(l_p)_{R^t} = r^t_p$ (see, [5]), $(c_0)_{E^r} = e^r_0$ and $(c)_{E^r} = e^r_c$ (see, [6]), $(l_p)_{E^r} = e^r_p$ and

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$(l_\infty)_{E^r} = e^r_\infty$ (see, [7,8]), $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ (see, [9]), $[c_0(u,p)]_{A^r} = a^r_0(u,p)$ and $[c(u,p)]_{A^r} = a^r_c(u,p)$ (see, [10], $(l_p)_{A^r} = a^r_p$ and $(l_\infty)_{A^r} = a^r_\infty$ (see, [11], $(c_0)_{C_1} = \hat{c}_0$, $c_{C_1} = \hat{c}$ (see, [12], $c^\lambda_0(\Delta) = (c^\lambda_0)_\Delta$ and $c^\lambda(\Delta) = (c^\lambda)_\Delta$ (see, [13], $\mu_G = Z(u,v,\mu)$ (see, [14]), Neyaz and Hamid $r^q(u,p) = \{l(p)\}_{R^q}$ (see, [15]); where N_q, C_1, R^t and E^r denotes the Nörland, Cesäro, Riesz and Eular means, respectively, A^r and C are respectively defined in [14,16], $\mu = \{c_0,\ c,\ l_p\}$ and $1 \leqslant p < \infty$. Also, $c_0(u,\ p)$ and $c(u,\ p)$ denote the sequence spaces generated from the Maddox's spaces $c_0(p)$ and $c(p)$ by Basarir (see, [16]). In the present paper, following (see, [2–12,14,15]), we introduce the sequence spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ and derive some inclusion relations. Furthermore, we determine the α-, β- and γ-duals of these spaces. In the last section of the paper we characterize some matrix classes concerning these spaces.

2. The sequence spaces $c_0(^\lambda_u)$ and $c(^\lambda_u)$ of non-absolute type

In the present section we introduce the sequence spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ and show that these spaces are BK-spaces of non-absolute type which are linearly isomorphic to the spaces c_0 and c, respectively. A sequence space X with a linear topology is called a K-space if each map $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space X is called an FK-space provided X is complete linear metric space. An FK-space whose topology is normable is called a BK-space.

Let $\lambda = (\lambda_k)^\infty_{k=0}$ be a strictly increasing sequence of positive reals tending to infity, i.e.,

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and } \lambda_k \to \infty \quad \text{as } k \to \infty. \tag{2}$$

The sequence c^λ and c^λ_0 have been introduced by Mursaleen and Noman (see, [17]) as follows:

$$c^\lambda = \left\{x \in \omega : \lim_n \wedge_n(x) \text{ exists}\right\}$$

and

$$c^\lambda_0 = \left\{x \in \omega : \lim_n \wedge_n(x) = 0\right\},$$

where

$$\wedge_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k, \quad (k \in \mathbb{N}).$$

With the notation of (1) that $c^\lambda = (c)_\wedge$ and $c^\lambda_0 = (c_0)_\wedge$.

Now, following Basar and Altay (see [18]), Ayden and Basar (see, [19]) and Mursaleen and Mohiuddine (see, [20–23]), we treat slightly more different than Kizmaz (see, [24]) and the other authors following him and employ the technique obtaining a new sequence space by means of the matrix domain of a triangle limitation method. Let $u = (u_k)$ be a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$. We thus introduce the sequence spaces $c(\Delta^\lambda_u)$ and $c_0(\Delta^\lambda_u)$ as follows:

$$c(\Delta^\lambda_u) = \left\{x \in \omega : \lim_n \hat{\wedge}_n(x) \text{ exists}\right\}$$

and

$$c_0(\Delta^\lambda_u) = \left\{x \in \omega : \lim_n \hat{\wedge}_n(x) = 0\right\},$$

where

$$\hat{\wedge}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) u_k (x_k - x_{k-1}), \ (k \in \mathbb{N}). \tag{3}$$

Here and in sequel, we shall use the convention that any term with a negative subscript is equal to naught.

With the notation of (1) that, $c(\Delta^\lambda_u) = (c)_{\hat{\wedge}}$ and $c_0(\Delta^\lambda_u) = (c_0)_{\hat{\wedge}}$.

If $u_k = (1,1,\ldots)$, these spaces reduces to $c(\Delta^\lambda)$ and $c_0(\Delta^\lambda)$ (see, [15]).

We define,

$$\hat{\lambda}_{nk} = \begin{cases} \frac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n} u_k, & \text{if } k < n, \\ \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} u_n, & \text{if } k = n, \\ 0, & \text{if } k > n. \end{cases} \tag{4}$$

It is clear that the matrix $\hat{\wedge} = \hat{\lambda}_{nk}$ is a triangle. We shall assume throughout the text that the sequences $x = (x_k)$ and $y = (y_k)$ are cocnnected by the relation, that is y is $\hat{\wedge}$-transform of x, where

$$y_k(\lambda) = \sum_{i=0}^{k-1} \frac{(\lambda_i - \lambda_{i-1}) - (\lambda_{i+1} - \lambda_i)}{\lambda_k} u_i + \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} u_k x_k \quad \text{for } k \in \mathbb{N}, \tag{5}$$

where here and in what follows, the summation running from 0 to k-1 is equal to zero when $k = 0$. It is clear from (3) that the relation (5) can be written as follows:

$$y_k(\lambda) = \sum_{i=0}^k \left(\frac{\lambda_i - \lambda_{i-1}}{\lambda_k}\right) u_i (x_i - x_{i-1}) \quad \text{for } k \in \mathbb{N}.$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ are BK-spaces with the norm*

$$\|x\|_{c_0(\Delta^\lambda_u)} = \|x\|_{c(\Delta^\lambda_u)} = \|\hat{\wedge}(x)\|_{l_\infty} = \sup_n \|\hat{\wedge}_n(x)\|.$$

Proof. The proof is a routine rerificaion, so is left as an easy exercise to the reader. □

Remark. One can easily check that the absolute property does not hold on the spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$, that is $\|x\|_{c_0(\Delta^\lambda_u)} \neq \||x|\|_{c_0(\Delta^\lambda_u)}$ and $\|x\|_{c(\Delta^\lambda_u)} \neq \||x|\|_{c(\Delta^\lambda_u)}$ for alteast one sequence in the spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ and this shows that $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ are sequence spaces of non-absolute type, where $|x| = (|x|)$.

Theorem 2.2. *The spaces $c_0(\Delta^\lambda_u)$ and $c(\Delta^\lambda_u)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c, respectively, that is $c_0(\Delta^\lambda_u) \cong c_0$ and $c(\Delta^\lambda_u) \cong c$.*

Proof. We only consider the case $c_0(\Delta^\lambda_u) \cong c_0$ and the case $c(\Delta^\lambda_u) \cong c$ will follow similarly. Thus, to prove the theorem, we must show the existence of linear bijection between $c_0(\Delta^\lambda_u)$ and c_0. For, consider the transformation T defined, with the notation (5), from $c_0(\Delta^\lambda_u)$ to c_0 by $x \to y(\lambda) = Tx$. Then $Tx = y(\lambda) = \hat{\wedge}(x) \in c_0$ for every $x \in c_0(\Delta^\lambda_u)$. Also, the linearty of T is obvious. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Furthermore, let $y = (y_k) \in c_0$ and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=0}^{k}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{u_j(\lambda_j-\lambda_{j-1})}y_i \quad \text{for } k\in\mathbb{N}. \tag{6}$$

Then, we obtain that

$$x_k(\lambda)-x_{k-1}(\lambda) = \sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{u_k(\lambda_k-\lambda_{k-1})}y_i.$$

Thus, for every $k\in\mathbb{N}$, we have by (3) that

$$\hat{\wedge}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}\sum_{i=k-1}^{k}(-1)^{k-i}\lambda_i y_i = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k y_k - \lambda_{k-1}y_{k-1}) = y_n.$$

This shows that $\hat{\wedge}(x)=y$ and since $y\in c_0$, we obtain that $\hat{\wedge}(x)\in c_0$. Thus, we deduce that $x\in c_0(\Delta_u^\lambda)$ and that $Tx = y$. Hence, T is surjective.

Further, we have for every $x\in c_0(\Delta_u^\lambda)$ that

$$\|Tx\|_{c_0} = \|Tx\|_{l_\infty} = \|\hat{\wedge}(x)\|_{l_\infty} = \|x\|_{c_0(\Delta_u^\lambda)},$$

which means that $c_0(\Delta_u^\lambda)$ and c_0 are linearly isomorphic.

It can similarly shown that if the spaces $c_0(\Delta_u^\lambda)$ and c_0 are respectively replaced by the spaces $c(\Delta_u^\lambda)$ and c, then we obtain the fact that $c(\Delta_u^\lambda)\cong c$ and this concludes the proof. □

3. The inclusion relations

In the present section, we establish some inclusion relations concerning with the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$.

Theorem 3.1. *The inclusion $c_0(\Delta_u^\lambda)\subset c(\Delta_u^\lambda)$ strictly holds.*

Proof. It is obvious that $c_0(\Delta_u^\lambda)\subset c(\Delta_u^\lambda)$ holds defined by $x_k=\frac{k+1}{u_k}$ for all $k\in\mathbb{N}$. Then, we have by (3) that

$$\hat{\wedge}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1}) = 1$$

for all $k\in\mathbb{N}$, which shows that $\hat{\wedge}(x)=e$ and hence $\hat{\wedge}(x)=c-c_0$ where $e=(1,1,1,\ldots)$. Thus, the sequence x is in $c(\Delta_u^\lambda)$ but not in $c_0(\Delta_u^\lambda)$. Hence, the inclusion $c_0(\Delta_u^\lambda)\subset c(\Delta_u^\lambda)$ is strict and the proof is complete. □

4. The bases for the spaces $c_0\binom{\lambda}{u}$ and $c\binom{\lambda}{u}$

In the present section, we give two sequences of the points of the spaces $c_0(\Delta_u^\lambda)$ $c(\Delta_u^\lambda)$ which form the bases for these spaces.

If the normed space X contains a sequence (b_n) with the property that for every $x\in X$, there is a unique sequence of scalars (α_n) such that

$$\lim_n\|x-\sum_{k=0}^{n}\alpha_k b_k\| = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum_k\alpha_k b_k$ which has the sum x is then called the expansion of x with respect to b_n and is written as $x=\sum_k\alpha_k b_k$. Now, because of the isomorphism T defined from $c_0(\Delta_u^\lambda)$to c_0, in the proof of Theorem 2.2, is onto, the inverse image of the bases $e^{(k)}$ of the space c_0 is the bases for the new space $c_0(\Delta_u^\lambda)$. Therefore, we have the following result:

Theorem 4.1. *Define the sequence $b^k(\lambda)=\left\{b_n^{(k)}(\lambda)\right\}_{n\in\mathbb{N}}$ of the elements of the space $c_0(\Delta_u^\lambda)$ for every fixed $k\in\mathbb{N}$ by*

$$b_n^k(\lambda) = \begin{cases} \left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}-\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}\right)u_k, & \text{if } k<n,\\ \left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}\right)u_k, & \text{if } k=n,\\ 0, & \text{if } k>n. \end{cases} \tag{7}$$

Then, the sequence $\{b_n^k(\lambda)\}$ is a bases for the space $c_0(\Delta_u^\lambda)$ for any $x\in c_0(\Delta_u^\lambda)$ has a unique representation of the form

$$x=\sum_k\alpha_k(\lambda)b^k(\lambda) \tag{8}$$

where $\alpha_k(\lambda)=\hat{\wedge}_k(x)$ for all $k\in\mathbb{N}$.

Theorem 4.2. *The set $\{b,\ b_k(\lambda)\}$ is a bases for the space $c(\Delta_u^\lambda)$ for any $x\in c(\Delta_u^\lambda)$ has a unique representation of the form*

$$x = lb+\sum_k[\alpha_k(\lambda)-l]b^k(\lambda) \tag{9}$$

where $l=\lim_n\hat{\wedge}_k(x)$ and $\alpha_k(\lambda)=\hat{\wedge}(x)$ for all $k\in\mathbb{N}$.

5. The α-, β- and γ-duals of the spaces $c_0\binom{\lambda}{u}$ and $c\binom{\lambda}{u}$

In the present section, we state and prove the theorems determining α-, β- and γ- duals of the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$of non-absolute type.

For the sequence space X and Y, define the set

$$S(X:Y)=\{z=(z_k): xz=(x_kz_k)\in Y\}. \tag{10}$$

With the notation of (10), the α-, β- and γ- duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ and are defined by

$$X^\alpha = S(X:l_1),\quad X^\beta=S(X:cs)\quad\text{and } X^\gamma=S(X:bs).$$

We now state some lemmas which are used in proving the theorems.

Lemma 5.1 (*see*, 15). *$A\in(c_0:l_1)=(c:l_1)$ if and only if*

$$\sup_{K\in\mathbb{F}}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty.$$

Lemma 5.2 (*see*, 15). *$A\in(c_0:c)$ if and only if*

$$\lim_n a_{nk} \text{ exists } \textit{for all } k\in\mathbb{N}, \tag{11}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty. \tag{12}$$

Lemma 5.3 (*see*, 15). *$A\in(c:c)$ if and only if (11) and (12) hold and*

$$\lim_n\sum_k a_{nk} \text{ exists.} \tag{13}$$

Lemma 5.4 (*see*, 15). *$A\in(c_0:l_\infty)=(c:l_\infty)$ if and only if (12) holds. We now prove the following:*

Theorem 5.5. *The α-dual of the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ is the set*

$$a_1^\lambda = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathbb{F}} \sum_n \left| \sum_{k \in K} b_{nk}^\lambda \right| < \infty \right\},$$

where the matrix $B^\lambda = (b_{nk}^\lambda)$ is defined via the sequence $a = (a_n)$ by

$$b_n^k(\lambda) = \begin{cases} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k}\right)\frac{a_n}{u_k}, & \text{if } k < n, \\ \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)\frac{a_n}{u_n}, & \text{if } k = n; \quad (n, k \in \mathbb{N}), \\ 0, & \text{if } k > n. \end{cases}$$

Proof. Let $a = (a_n) \in \omega$. Then, by bearing in mind the relations (5) and (6), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{u_k(\lambda_k - \lambda_{k-1})}, \quad (n \in \mathbb{N}). \tag{14}$$

Thus, we observe by (14) that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in c_0(\Delta_u^\lambda)$ or $c(\Delta_u^\lambda)$ if and only if $B^\lambda y \in l_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_n)$ is in the α-dual of the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ if and only if $B^\lambda \in (c_0 : l_1) = (c : l_1)$. We, therefore, obtain by Lemma 5.1 with B^λ instead of A that $a \in \{c_0(\Delta_u^\lambda)\}^\alpha = \{c(\Delta_u^\lambda)\}^\alpha$ if and only if

$$\sup_{K \in \mathbb{F}} \sum_n \left| \sum_{k \in K} b_{nk}^\lambda \right| < \infty,$$

which leads us to the consequence that $\{c_0(\Delta_u^\lambda)\}^\alpha = \{c(\Delta_u^\lambda)\}^\alpha = a_1^\lambda$. This completes the proof of the theorem. □

Theorem 5.6. *Define the sets a_2^λ, a_3^λ, a_4^λ and a_5^λ as follows:*

$$a_2^\lambda = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$a_3^\lambda = \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n-1} |\hat{a}_k(n)| < \infty \right\},$$

$$a_4^\lambda = \left\{ a = (a_k) \in \omega : \sup_k \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} u_k^{-1} a_k \right| < \infty \right\}$$

and

$$a_5^\lambda = \left\{ a = (a_k) \in \omega : \lim_k (k+1) a_k \text{ converges} \right\},$$

where

$$\hat{a}_k(n) = \lambda_k u_k^{-1} \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{n} a_j \right];$$

$(k < n)$.

Then $\{c_0(\Delta_u^\lambda)\}^\beta = a_2^\lambda \cap a_3^\lambda \cap a_4^\lambda$ and $\{c(\Delta_u^\lambda)\}^\beta = a_3^\lambda \cap a_4^\lambda \cap a_5^\lambda$.

Proof. Consider the equation,

$$\begin{aligned} \sum_{k=0}^{n} a_k x_n &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \left[\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{u_j(\lambda_j - \lambda_{j-1})} y_i \right] \right\} a_k \\ &= \sum_{k=0}^{n-1} \frac{\lambda_k}{u_k} \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{n} a_j \right] y_k \\ &\quad + \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n y_n \\ &= \sum_{k=0}^{n-1} \hat{a}_k(n) y_k + \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n y_n = D_n^\lambda(y), \end{aligned} \tag{15}$$

where $D^\lambda = (d_{nk}^\lambda)$ is defined by

$$d_{nk}^\lambda = \begin{cases} \hat{a}_k(n) & \text{if } k < n, \\ \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})}) a_n, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Then, we deduce by (15) $ax = (a_k x_k) \in cs$ whenever $ax = (a_k x_k) \in c_0(\Delta_u^\lambda)$ if and only if $D^\lambda y \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \{c_0(\Delta_u^\lambda)\}^\beta$ if and only if $D^\lambda \in (c_0 : c)$. Therefore, by using Lemma 5.2, we derive from (11) and (12) that

$$\sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N}, \tag{16}$$

$$\sup_n \sum_{k=0}^{n-1} |\hat{a}_k(n)| < \infty, \tag{17}$$

$$\sup_k \left| \frac{\lambda_k}{u_k(\lambda_k - \lambda_{k-1})} a_k \right| < \infty. \tag{18}$$

Hence, we conclude that $\{c_0(\Delta_u^\lambda)\}^\beta = a_2^\lambda \cap a_3^\lambda \cap a_4^\lambda$.

Similarly, we deduce from Lemma 5.3 with (15) that $a = (a_k) \in \{c(\Delta_u^\lambda)\}^\beta$ if and only if $D^\lambda \in (c : c)$. Therefore, we derive from (11) and (12) that (16)–(18) hold. Further, it can easily be seen that the equality,

$$\sum_{k=0}^{n} (k+1) a_k = \sum_{k=0}^{n-1} \hat{a}_k(n) + \frac{\lambda_n}{u_n(\lambda_n - \lambda_{n-1})} a_n; \quad (n \in \mathbb{N}), \tag{19}$$

holds which can be written as follows:

$$\sum_{k=0}^{n} (k+1) a_k = \sum_k d_{nk}^\lambda, \quad (n \in \mathbb{N}).$$

Consequently, we obtain from (13) that

$\{(k+1) a_k\} \in cs$.

Hence, we deduce that, $\{c(\Delta_u^\lambda)\}^\beta = a_3^\lambda \cap a_4^\lambda \cap a_5^\lambda$. □

Theorem 5.7. *The γ dual of the spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ is the set $a_3^\lambda \cap a_4^\lambda$.*

Proof. This result can be proved similarly as the proof of the Theorem 5.6 with Lemma 5.4 instead of Lemma 5.2. □

6. Certain matrix mappings on the spaces $c_0\binom{\lambda}{u}$ and $c\binom{\lambda}{u}$

For brevity in notation, we shall write

$$\hat{a}_{nk}(m) = \frac{\lambda_k}{u_k}\left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right)\sum_{j=k+1}^{m} a_{nj}\right], \quad (k < m)$$

and

$$\hat{a}_{nk} = \frac{\lambda_k}{u_k}\left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k}\right)\sum_{j=k+1}^{\infty} a_{nj}\right],$$

for all $n, k, m \in \mathbb{N}$ provided the convergence of the series.

We first state some lemmas which are used in proving the Theorems 6.5, 6.6, 6.7.

Lemma 6.1 (*see,* 25). *The matrix mappings between the BK-spaces are continuous.*

Lemma 6.2 (*see,* 26). *$A \in (c{:}l_p)$ if and only if*

$$\sup_{F\in\mathbb{F}}\sum_n\left|\sum_{k\in F}a_{nk}\right|^p < \infty; \quad (1 \leqslant p < \infty).$$

Lemma 6.3 (*see,* 26). *$A \in (c{:}c_0)$ if and only if*

$$\sup_n\sum_k a_{nk} < \infty, \tag{20}$$

$$\lim_n a_{nk} = 0, \quad \textit{for all } k \in \mathbb{N}, \tag{21}$$

$$\lim_n\sum_k a_{nk} = 0. \tag{22}$$

Lemma 6.4 (*see,* 26). *$A \in (c_0{:}c_0)$ if and only if* (20) and (21) *holds.*

We now prove the following results on matrix tranformations:

Theorem 6.5.

(i): *Let $1 \leqslant p < \infty$. Then $A \in (c(\Delta_u^\lambda) : l_p)$ if and only if*

$$\sup_{F\in\mathbb{F}}\sum_n\left|\sum_{k\in F}\hat{a}_{nk}\right|^p < \infty; \tag{23}$$

$$\sup_m\sum_{k=0}^{m-1}|\hat{a}_{nk}(m)| < \infty; \quad (n \in \mathbb{N}), \tag{24}$$

$$\{(k+1)a_{nk}\}_{k=0}^{\infty} \in cs; \quad (n \in \mathbb{N}), \tag{25}$$

$$\lim_n \frac{\lambda_k}{u_k(\lambda_k - \lambda_{k-1})}a_{nk} = a_n; \quad (n \in \mathbb{N}) \tag{26}$$

and

$$(a_n) \in l_p. \tag{27}$$

(ii) *$A \in (c(\Delta_u^\lambda) : l_\infty)$ if and only if* (25) and (26) *holds and*

$$\sup_n\sum_n|\hat{a}_{nk}| < \infty \tag{28}$$

and $(a_n) \in l_\infty$.

Proof. Suppose the conditions (23)–(27) holds and $x \in c(\Delta_u^\lambda)$. Then, we have by Theorem 5.6 that $\{a_{nk}\}_{n\in\mathbb{N}} \in [c(\Delta_u^\lambda)]^\beta$ for all $n \in \mathbb{N}$ and this implies the existence of the A-transform of x, i.e., Ax exists. Also, it is clear that the associated sequence $y = (y_k)$ is in the space c and hence $y_k \to l$ as $k \to \infty$ for some suitable l. Further, it follows by combining Lemma 6.2 with (23) that the matrix $\widehat{A} = (\hat{a}_{nk})$ is in the class $(c{:}l_p)$, where $1 \leqslant p < \infty$.

Let us now consider the following eqquality derived by using the relation (5) from the m^{th} partial sum of the series $\sum_k a_{nk}x_k$;

$$\sum_{k=0}^{m}a_{nk}x_k = \sum_{k=0}^{m-1}\hat{a}_{nk}(m)y_k + \frac{\lambda_m}{u_m(\lambda_m - \lambda_{m-1})}a_{nm}y_m; \quad (n, k \in \mathbb{N}). \tag{29}$$

Then, since $y \in c$ and $\widehat{A} \in (c : l_p)$, $\widehat{A}y$ exists and so the series $\sum_k \hat{a}_{nk}y_k$ converges for every $n \in \mathbb{N}$. Furthermore, it follows by (25) that the series $\sum_{j=k}^{\infty}a_{nj}$ converges for all $n, k \in \mathbb{N}$ and hence $\hat{a}_{nk}(m) \to \hat{a}_{nk}$ as $m \to \infty$. Therefore, if we pass to the limit in (29) as $m \to \infty$, then we obtain by (26) that

$$\sum_k a_{nk}x_k = \sum_k \hat{a}_{nk}y_k + la_n; \quad (n \in \mathbb{N}), \tag{30}$$

which can be written as follows:

$$A_n(x) = \widehat{A}_n(y) + la_n; \quad (n \in \mathbb{N}). \tag{31}$$

This yields by taking the l_p-norm that

$$\|Ax\|_{l_p} \leqslant \|\widehat{A}y\|_{l_p} + |l|\|a_n\|_{l_p} < \infty,$$

which leads us to the consequence that $Ax \in l_p$ and hence $A \in (c(\Delta_u^\lambda) : l_p)$.

Conversely, suppose that $A \in (c(\Delta_u^\lambda) : l_p)$. Then $\{a_{nk}\}_{n\in\mathbb{N}} \in [c(\Delta_u^\lambda)]^\beta$ for all $n \in \mathbb{N}$ and this with Theorem 5.6 implies the necessity of the conditions (24) and (25). On the hand, since $c(\Delta_u^\lambda)$ and l_p are BK-spaces; we have by Lemma 6.1 that there is a constant $M > 0$ such that

$$\|Ax\|_{l_p} \leqslant M\|x\|_{c(\Delta_u^\lambda)}, \tag{32}$$

holds for $x \in c(\Delta_u^\lambda)$. Now, let $F \in \mathbb{F}$. Then, the sequence $z = \sum_{k\in F}b^{(k)}(\lambda)$ is in $c(\Delta_u^\lambda)$, where, the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}}$ is defined by (7) for every fixed $k \in \mathbb{N}$. Further, we by (9) that

$$\|z\|_{l_\infty} = \|\wedge(z)\|_{l_\infty} = \|\sum_{k\in F}\wedge(b^{(k)}(\lambda))\|_{l_\infty} = \|\sum_{k\in F}e^k\|_{l_\infty}.$$

Furthermore, for ever $n \in \mathbb{N}$, we obtain by (7) that

$$A_n(z) = \sum_{k\in F}A_n(b^{(k)}(\lambda)) = \sum_{k\in F}\sum_j a_{nj}(b_j^{(k)}(\lambda)) = \sum_{k\in F}\hat{a}_{nk}.$$

Hence, since the inequality (32) is satisfied for the sequence $z \in c(\Delta_u^\lambda)$, we have for any $f \in \mathbb{F}$ that

$$\left[\sum_n\left|\sum_{k\in F}\hat{a}_{nk}\right|^p\right]^{\frac{1}{p}} \leqslant M,$$

which shows the necessity of (23). Thus, it follows by Lemma 6.2 that $\widehat{A} = (\hat{a}_{nk}) \in (c : l_p)$.

Now, let $y = (y_k) \in c/c_0$ and consider the sequence $x = (x_k)$ defined by (6) for every $k \in \mathbb{N}$. Then, $x \in c(\Delta_u^\lambda)$ and $y = \wedge(x)$, that is the sequences x and y are connected by the relation (5). Therefore, the transforms Ax and $\widehat{A}y$ exists. This leads us to the convergence of the series $\sum a_{nk}x_k$ and $\sum \hat{a}_{nk}y_k$ for every $n \in \mathbb{N}$. Thus, we deduce that

$$\lim_m \sum_{k=0}^{m-1} \hat{a}_{nk}(m)y_k = \sum_k \hat{a}_{nk}(m)y_k; \quad (n,k \in \mathbb{N}).$$

Consequently, we obtain from (29) as $m \to \infty$ that

$$\lim_m \frac{\lambda_m}{u_m(\lambda_m - \lambda_{m-1})} a_{nm} y_m; \quad (n \in \mathbb{N}),$$

exists and since $y \in c/c_0$, we conclude that $\lim_m \frac{\lambda_m}{u_m(\lambda_m - \lambda_{m-1})} a_{nm}$ exists, which shows the necessity of (26) and so the relation (31) holds, where $l = \lim_k y_k$.

Finally, since $Ax \in l_p$ and $\widehat{A}x \in {}_p$, the necessity of (27) is immediate by (31) and the proof of part (i) of the theorem is complete.

Since the part(ii) can be proved by using the similar way of that used in the proof of the part (i) with Lemma 5.4 instead of Lemma 6.2. □

Remark. It is obvious by (28) that the limit

$$\lim_m \sum_{k=0}^{m-1} |\hat{a}_{nk}(m)| = \sum_k |\hat{a}_{nk}(m)|;$$

exists for each $n \in \mathbb{N}$. This just says us that the condition (28) implies the condition (24).

Now, we may note that $(c_0:l_p) = (c:l_p)$ for $1 \leqslant p \leqslant \infty$ (see, [21]). Thus, by means of the Theorem 5.6 and Lemma 5.4, we immmediately conclude the following theorem:

Theorem 6.6.

(i) *Let* $1 \leqslant p < \infty$. *Then* $A \in (c_0(\Delta_u^\lambda) : l_p)$ *if and only if* (23) *and* (24) *holds and*

$$\sum_{j=k}^{\infty} a_{nj} \text{ exists}, \quad (n,k,m \in \mathbb{N}); \tag{33}$$

$$\left\{ \frac{\lambda_k}{u_k(\lambda_k - \lambda_{k-1})} a_{nk} \right\} \in l_\infty; \quad (n \in \mathbb{N}). \tag{34}$$

(ii) $A \in (c_0(\Delta_u^\lambda) : l_\infty)$ *if and only if* (25)–(27) *hold.*

Proof. It is natural thing that the present theorem can be proved by the same technique used in the proof of Theorem 6.5, above and so we omit the proof. □

Theorem 6.7.

(i) *Let* $1 \leqslant p < \infty$. *Then* $A \in (c(\Delta_u^\lambda) : c)$ *if and only if* (25),(26) *and* (28) *hold and*

$$\lim_n a_n = 0, \tag{35}$$

$$\lim_n \hat{a}_{nk} = \alpha_k; \quad (k \in \mathbb{N}), \tag{36}$$

and

$$\lim_n \sum_k \hat{a}_{nk} = \alpha. \tag{37}$$

Proof. Suppose A satisfies the conditions (25), (26), (28), (35), (36) and (37) and take any $x \in c(\Delta_u^\lambda)$. Then, since (28) implies (24), we have by Theorem 5.6 that $\{a_{nk}\}_{n\in\mathbb{N}} \in [c(\Delta_u^\lambda)]^\beta$ for all $n \in \mathbb{N}$ and this implies the existence of the A-transform of x, i.e., Ax exists. We also osberve from (28) and (36) that

$$\sum_{j=0}^{k} |\alpha_j| \leqslant \sup_n \sum_j |\hat{a}_{nj}| < \infty$$

holds for every $n \in \mathbb{N}$. This implies that $(\alpha_k) \in l_1$ and hence the series $\sum_k (y_k - l)$ converges, where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ by the relation (5)such that $y_k \to l$ as $k \to \infty$. Further, it is obvious by combining Lemma 5.3 with the conditions (28), (36) and (37) that the matrix $\widehat{A} = \hat{a}_{nk}$ is in the class $(c{:}c)$.

Now, by the similar proof used in the proof of the Theorem 6.5, we obtain that the relation (30) holds which can be written as follows:

$$\sum_k a_{nk}x_k = \sum_k \hat{a}_{nk}(y_k - l) + l\sum_k \hat{a}_{nk} + la_n; \ (n,k \in \mathbb{N}). \tag{38}$$

In this situation, we see by passing to the limit in (38) as $n \to \infty$ that the first term on the right tends to $\sum_k \alpha_k(y_k - l)$ by (28) and (36), the second term tends to $l\alpha$ by (37) and the last term tends to la by (35). Consequently, we have that

$$A_n(x) \to \sum_k \alpha_k(y_k - l) + l(\alpha - \alpha) \quad \text{as } n \to \infty,$$

which shows that $Ax \in c$ and hence $A \in (c(\Delta_u^\lambda) : c)$.

Conversely, suppose that $A \in (c(\Delta_u^\lambda) : c)$. Then, since the inclusion $c \subset l_\infty$ holds, we have that $A \in (c(\Delta_u^\lambda) : l_\infty)$. This leads us with Theorem 6.5 to the necessity of the conditions (25), (26) and (28). Furthermore, let $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}} \in c(\Delta_u^\lambda)$ and hence by (7) for every $k \in \mathbb{N}$. Then, it can be easily seen that $Ab^k(\lambda) = \{\hat{a}_{nk}\}_{n\in\mathbb{N}}$ and hence $\{\hat{a}_{nk}\}_{n\in\mathbb{N}} \in c$ for every $k \in \mathbb{N}$, which shows the necessity of (36). Next, let $z = \sum_k b^{(k)}(\lambda)$. Then, since the linear transformation $T : c(\Delta_u^\lambda) \to c$ defined by analogy as in the proof of the Theorem 22 is continuous, we obtain by (9) that

$$\wedge(z) = \sum_k \wedge(b^k(\lambda)) = \sum_k \delta_{nk} = 1; \quad (k \in \mathbb{N}),$$

which shows that $\wedge(z) = e \in c$ and hence $z \in c(\Delta_u^\lambda)$. On the other hand, sicne $c(\Delta_u^\lambda)$ and c are BK spaces, Lemma 6.1, implies the continuity of the matrix mapping $A : c(\Delta_u^\lambda) \to c$. Thus, we have for every $n \in \mathbb{N}$ that

$$A_n(z) = \sum_k A_n(b^k(\lambda)) = \sum_k \hat{a}_{nk},$$

which shows the necessity of (37).

Now, it follows by (28),(36) and (37) with Lemma 5.3 that $\widehat{A} = \hat{a}_{nk} \in (c : c)$. This leads us with (25) and (26) to the consequence that the relation (31) holds for all sequences $x \in c(\Delta_u^\lambda)$ and $y \in c$ which are connected by the relation (5) such that $y_k \to l$ as $k \to \infty$.

Further, since $Ax \in c$ and $\widehat{A}y \in c$, the necessity of (35) is immediate by (31) and this completes the proof. □

Corollary 6.8. *Let $1 \leqslant p < \infty$. Then $A \in (c(\Delta_u^\lambda) : c_0)$ if and only if (25),(26) and (28) hold and*

$$\lim_n a_n = a,$$

$$\lim_n \hat{a}_{nk} = 0; \quad (k \in \mathbb{N}), \tag{39}$$

and,

$$\lim_n \sum_k \hat{a}_{nk} = 0.$$

Proof. The proof can be obtained on similar lines as in Theorem 6.7 with Lemma 6.3 instead of Lemma 5.3. □

Corollary 6.9. *$A \in (c_0(\Delta_u^\lambda) : c)$ if and only if (28), (32), (33) and (36) hold*

Proof. This result can be proved similarly by using Lemma 5.2 and Theorems 5.6 and 6.6. □

Corollary 6.10. *$A \in (c_0(\Delta_u^\lambda) : c_0)$ if and only if (28), (33), (34) and (39) hold*

Proof. This result can be proved similarly by using Lemma 6.4 and Theorem 5.6 and Corollary 6.9. □

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