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Positive solutions to a system of adjointable operator equations over Hilbert C^*-modules ☆

Qing-Wen Wang [a,b,*], Chang-Zhou Dong [a]

[a] Department of Mathematics, Shanghai University, Shanghai 200444, PR China
[b] Division of Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore

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We present necessary and sufficient conditions for the existence of a positive solution to the system of adjointable operator equations $A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3$ over Hilbert C^*-modules. We also derive a representation for a general positive solution to this system when the solvability conditions are satisfied. The results of this paper extend some known results in the literature.

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* Corresponding author at: Department of Mathematics, Shanghai University, Shanghai 200444, PR China.
E-mail address: wqw858@yahoo.com.cn (Q.-W. Wang).

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1. Introduction

It is a very active research topic to investigate Hermitian positive semidefinite solutions to matrix equations or positive solutions to operator equations. For instance, Hermitian positive semidefinite solution to the matrix equation

$$AXB = C \tag{1.1}$$

were studied by Khatri and Mitra [9] in 1976 and Zhang [20] in 2004, respectively. In 2007, Cvetković-Ilić et al. [3] considered the positive solution to the special case of (1.1)

$$AXA^* = B \tag{1.2}$$

in C^*-algebras. In 2008, under the assumption that the underlying space is finite-dimensional or the range of B is contained in the range of A^*, Xu et al. [18] proposed a solvability condition for the operator equation (1.1) to have a positive solution, and gave an expression of the general positive solution to (1.1) in the general setting of Hilbert C^*-modules.

In 1976, Khatri [9] established necessary and sufficient conditions for the existence of Hermitian positive semidefinite solution to the system of matrix equations

$$\begin{cases} A_1X = C_1, \\ XB_2 = C_2 \end{cases} \tag{1.3}$$

and presented an expression for Hermitian positive semidefinite solution to this equation in terms of generalized inverse of some matrices when the solvability conditions are satisfied. In 2008, Dajić and Koliha [5] proposed necessary and sufficient conditions for the existence of a positive solution to (1.3) in rings and rings with an involution, and gave an expression of such a solution when the solvability conditions are met, which generalized the main results in [4] for Hilbert space operators. Moreover, Xu [19], Fang et al. [6] investigated system (1.3) of operator equations in the framework of Hilbert C^*-modules.

As is known to us that a Hilbert C^*-module is a natural generalization of a Hilbert space and a C^*-algebra. Therefore investigating operator equations over Hilbert C^*-modules is very meaningful. Note that (1.1)–(1.3) are some special cases of the following system of matrix equations or operator equations

$$\begin{cases} A_1X = C_1, \\ XB_2 = C_2, \\ A_3XB_3 = C_3, \end{cases} \tag{1.4}$$

whose general solution was first investigated over the complex number field by Bhimasankaram [2], and over the real quaternion field by Wang and Lin, etc. [11,13–16], respectively. So far, to our knowledge, there has been little information on either the positive semidefinite solution to system (1.4) of matrix equations over the complex field or the positive solution to system (1.4) of adjointable operator equations over Hilbert C^*-modules.

Motivated by the work mentioned above, we in this paper aim to give some necessary and sufficient conditions for the system of adjointable operator equations (1.4) to have a positive solution over the Hilbert C^*-modules, as well as present an expression for the general positive solution to this system when the solvability conditions are satisfied. Our approach is different from that in dealing with the complex matrix case.

The paper is organized as follows. In Section 2, we begin with some basic concepts and results about adjointable operators and generalized inverse of adjointable operators over Hilbert C^*-modules. In Section 3 we give some necessary and sufficient conditions for the existence of a positive solution to the system (1.4) of adjointable operator equations over Hilbert C^*-modules. When the solvability conditions are met, we also present an expression for the general positive solution to (1.4). In Section 4 we consider some special cases of the system (1.4) of adjointable operator equations over Hilbert C^*-modules.

2. Preliminaries

Hilbert C^*-modules arose as generalizations of the notion Hilbert space. The basic idea is to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the

C^*-algebra. The structure was first used by Kaplansky [8] in 1952. For more details of C^*-algebra and Hilbert C^*-modules, we refer the reader to [10,17].

Let $\mathfrak{A}$ be a C^*-algebra. An inner-product $\mathfrak{A}$-module is a linear space E which is a right $\mathfrak{A}$-module (with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in E, a \in \mathfrak{A}, \lambda \in \mathbb{C}$), together with a map $E \times E \to \mathfrak{A}$, $(x, y) \to \langle x, y\rangle$ such that

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(2) $\langle x, ya\rangle = \langle x, y\rangle a$;
(3) $\langle x, y\rangle = \langle y, x\rangle^*$;
(4) $\langle x, x\rangle \geqslant 0$, and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
for any $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$. An inner-product $\mathfrak{A}$-module E is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the induced norm $||x|| = \langle x, x\rangle^{1/2}$.

Throughout this paper H_1 and H_2 denote two Hilbert C^*-modules, and $\mathcal{B}(H_1, H_2)$ is the set of all maps $T : H_1 \to H_2$ for which there is a map $T^* : H_2 \to H_1$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for any $x \in H_1$ and $y \in H_2$. We know that any element T of $\mathcal{B}(H_1, H_2)$ is a bounded linear operator. We call $\mathcal{B}(H_1, H_2)$ the set of adjointable operators from H_1 into H_2. In case $H_1 = H_2, \mathcal{B}(H_1, H_1)$ which we abbreviate to $\mathcal{B}(H_1)$, is a C^*-algebra and we use the notation I to denote the identity operator. We write $R(A)$ and $N(A)$ for the range and null space of $A \in \mathcal{B}(H_1, H_2)$. An operator $A \in \mathcal{B}(H_1, H_2)$ is regular if there is an operator $A^- \in \mathcal{B}(H_2, H_1)$ such that $AA^-A = A, A^-$ is called an inner inverse of A. It is well known that A is regular if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, are closed and complemented subspaces of H_2 and H_1. For simplicity, we use $\mathcal{L}_A$ and $\mathcal{R}_A$ to stand for $I - A^-A$ and $I - AA^-$, respectively.

An operator $A \in \mathcal{B}(H)$ is called Hermitian (or self-adjoint) if $A^* = A$, and positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in H$, we write $A \geqslant 0$ if A is positive. The set $\mathcal{B}(H)^+$ of the positive operators is a subset of the Hermitian operators.

The Moore–Penrose inverse of $A \in \mathcal{B}(H_1, H_2)$ is defined as the operator $A^\dagger \in \mathcal{B}(H_2, H_1)$ satisfying the Penrose equations

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (A^\dagger A)^* = A^\dagger A, (AA^\dagger)^* = AA^\dagger.$$

An operator $A \in \mathcal{B}(H_1, H_2)$ has the (unique) Moore–Penrose inverse if and only if A has closed range, or equivalently if and only if it is regular. If a regular operator A is positive, then $A^\dagger \geqslant 0$ and $AA^\dagger = A^\dagger A$.

Lemma 2.1 (Lemma 2.1 in [4]). *Let $A \in \mathcal{B}(H_1, H_2)$ with closed range. Given a pair of topological complements M, N of $R(A), N(A)$ respectively, there exist a unique inner inverse $A^- \in \mathcal{B}(H_2, H_1)$ of A with $R(AA^-) = R(A), N(AA^-) = M, R(I - A^-A) = N(A^-A) = N(A)$ and $N(I - A^-A) = R(A^-A) = N$.*

Lemma 2.2 (Remark 1.2 in [19]). *The closeness of any one of the following sets implies the closeness of the remaining three sets $R(A), R(A^*), R(AA^*), R(A^*A)$.*

Lemma 2.3 (Lemma 3.6 in [18]). *Let H be a Hilbert C^*-module, A and B be two positive elements of $\mathcal{B}(H)$, and A, B and $A + B$ have closed ranges. Then*

(a) $R(A + B) = R(A) + R(B) = \{Ax + By | x, y \in H\}$,
(b) $(A + B)(A + B)^-A = A, (A + B)(A + B)^-B = B$,
(c) $A(A + B)^-B = A(A + B)^\dagger B$,
(d) $A(A + B)^-B = B(A + B)^-A$,
(e) $(A(A + B)^-B)^* = A(A + B)^-B$,
(f) $A(A + B)^-(A + B) = A, B(A + B)^-(A + B) = B$.

Lemma 2.4 (Lemma 3.7 in [18]). *Let H_1, H_2 be Hilbert C^*-modules, $A \in \mathcal{B}(H_1, H_2)$ with closed range. Then*

(a) $A^\dagger A$ and $AA^\dagger$ both are projections,
(b) $R(A^\dagger A) = R(A^\dagger) = R(A^*)$, so that $A^\dagger AA^* = A^*$,

(c) $R(AA^{\dagger}) = R(A)$, so that $A^*AA^{\dagger} = (AA^{\dagger}A)^* = A^*$,
(d) The restriction of $A^{\dagger}$ on $R(A)^{\perp}$ is identically zero.

For other important properties of operators and generalized inverses of operators see [7,12].

3. Positive solution to system (1.4) of adjointable operator equations

In this section, we present necessary and sufficient conditions for the existence of a positive solution, and give an expression for the general positive solution to system (1.4) of adjointable operator equations over Hilbert C^*-modules. We begin this section with the following lemmas, which can be deduced from [18,19].

Lemma 3.1 (Remark 3.6 in [18]). *Let H_1, H_2, H_3 be Hilbert C^*-modules, and $A \in \mathcal{B}(H_2, H_3), B \in \mathcal{B}(H_1, H_2)$ have closed ranges, and $C \in \mathcal{B}(H_1, H_3)$ such that $AA^{\dagger}CB^{\dagger}B = C$. Then for any $X \in \mathcal{B}(H_2)$, X is a solution to (1.1) if and only if $(A^{\dagger}A)X(BB^{\dagger}) = A^{\dagger}CB^{\dagger}$, where $A^{\dagger}A$ and $BB^{\dagger}$ are projections (hence positive elements) of $\mathcal{B}(H_2)$.*

Lemma 3.2 (Lemma 3.1 in [18]). *Let $A \in \mathcal{B}(H_1), B \in \mathcal{B}(H_2, H_1), D \in \mathcal{B}(H_2)$. Suppose that A has closed range, M is a Hermitian operator given by*

$$M = \begin{bmatrix} A & B \\ B* & D \end{bmatrix}.$$

Then $M \geqslant 0$ if and only if $A \geqslant 0, AA^{-}B = B, D - B^*A^{-}B \geqslant 0$.

Lemma 3.3 (Theorem 2.1 in [19]). *Let $A, C \in \mathcal{B}(H_1, H_2)$, and A and CA^* have closed ranges. Then the adjointable operator equation $AX = C$ has a positive solution $X \in \mathcal{B}(H_1)$ if and only if $CA^* \geqslant 0, R(C) \subseteq R(CA^*)$. In this case, the general positive solution is given by*

$$X = C^*(CA^*)^{-}C + \mathcal{L}_A S \mathcal{L}_A^*,$$

where $S \in \mathcal{B}(H_1)^+$ is arbitrary, $C^*(CA^*)^{-}C$ is a particular positive solution to $AX = C$, independent of the choice of the inner inverse $(CA^*)^{-}$.

Lemma 3.4 (Theorem 3.7 in [19]). *Let H, K, L be Hilbert C^*-modules, and $A_1, C_1 \in \mathcal{B}(H, K), B_2, C_2 \in \mathcal{B}(L, H)$,*

$$D = \begin{bmatrix} A_1 \\ B_2^* \end{bmatrix}, \quad E = \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix}, \quad F = \begin{bmatrix} C_1A_1^* & C_1B_2 \\ (A_1C_2)^* & C_2^*B_2 \end{bmatrix}$$

such that D, F be regular. Then (1.3) has a positive solution $X \in \mathcal{B}(H)$ if and only if F is positive and $R(E) \subseteq R(F)$. In this case, the general positive solution of (1.3) can be expressed as

$$X = E^*F^{-}E + \mathcal{L}_D T \mathcal{L}_D^*,$$

where $T \in \mathcal{B}(H)^+$ is arbitrary.

In 2007, Cvetković-Ilić et al. in [3] proposed a solvability condition for a positive solution to the operator equation (1.2), and derived an expression for the general positive solution to (1.2) in C^*-algebra, which can be generalized into Hilbert C^*-modules.

Lemma 3.5 (Corollary 2.3 in [3]). *Let H, K be Hilbert C^*-modules. Assume that $A \in \mathcal{B}(H, K), C \in \mathcal{B}(K)$ such that A has closed range, C is Hermitian and $R(C) \subset R(A)$. Then Eq. (1.2) has a positive solution $X \in \mathcal{B}(H)$ if and only if C is positive. If, in addition, C has closed range,then the general positive solution of (1.2) can be expressed as*

$$\begin{aligned} X = &X_0 + X_0X_0^{\dagger}E(I - A^{\dagger}A) + (I - A^{\dagger}A)E^*X_0X_0^{\dagger} \\ &+ (I - A^{\dagger}A)E^*X_0^{\dagger}E(I - A^{\dagger}A) + (I - A^{\dagger}A)F(I - A^{\dagger}A), \end{aligned}$$

where $X_0 = A^\dagger C(A^\dagger)^*$ *is a particular positive solution, E is arbitrary operator in* $\mathcal{B}(H)$*, and F is an arbitrary positive operator in* $\mathcal{B}(H)$.

Now, we turn our attention to consider the positive solution to system (1.4) of adjointable operator equations.

The solvability conditions and an expression for the general solution to the system (1.4) of matrix equations were once given over the complex number field by Bhimasankaram [2], as well as over the quaternion algebra by Wang [13], respectively. The results can be generalized into Hilbert C^*-modules. In the following theorem, we suppose the system of adjointable operator equations (1.4) is consistent. For simplicity, put

$$D = \begin{bmatrix} A_1 \\ B_2^* \end{bmatrix}, \quad E = \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix}, \quad F = \begin{bmatrix} C_1A_1^* & C_1B_2 \\ (A_1C_2)^* & C_2^*B_2 \end{bmatrix},$$

$$Y_0 = E^*F^-E, \quad M = A_3\mathcal{L}_D, \quad N = \mathcal{L}_D^*B_3, \quad L = M^\dagger(C_3 - A_3Y_0B_3)N^\dagger, \quad T = M^\dagger M + NN^\dagger,$$

$$P = T^-NN^\dagger, \quad Q = M^\dagger MT^-, \quad R = L + L^* + Y_1 + Y_2, \quad S = NN^\dagger T^-LT^-M^\dagger M.$$

We now give the main theorem of this paper as follows.

Theorem 3.6. *Let* H_i $(i = 1, 2, \ldots, 5)$ *be Hilbert* C^**-modules,* $A_1 \in \mathcal{B}(H_1, H_2), A_3 \in \mathcal{B}(H_1, H_4), B_2 \in \mathcal{B}(H_3, H_1), B_3 \in \mathcal{B}(H_5, H_1), C_1 \in \mathcal{B}(H_1, H_2), C_2 \in \mathcal{B}(H_3, H_1), C_3 \in \mathcal{B}(H_5, H_4)$. *Suppose that* $D, F, M, N, P, Q, R, S, T, NN^\dagger T^-L, M^\dagger MT^-L^*$ *have closed ranges. Then the consistent system of adjointable operator equations (1.4) has a positive solution in* $\mathcal{B}(H_1)$ *if and only if*

$$F \geqslant 0,\ S \geqslant 0,\ R(E) \subseteq R(F),\ R(NN^\dagger T^-L) \subseteq R(S),\ R(M^\dagger MT^-L^*) \subseteq R(S), \tag{3.1}$$

in which case an expression of the general positive solution of (1.4) can be expressed as

$$\begin{aligned} X = {} & Y_0 + \mathcal{L}_D(X_0 + X_0X_0^\dagger U(I - T^\dagger T) + (I - T^\dagger T)U^*X_0X_0^\dagger \\ & +(I - T^\dagger T)U^*X_0^\dagger U(I - T^\dagger T) + (I - T^\dagger T)V(I - T^\dagger T))\mathcal{L}_D^*, \end{aligned} \tag{3.2}$$

where Y_0 *is a particular positive solution of (1.3),* $X_0 = T^\dagger R(T^\dagger)^*$*,* Y_1 *and* Y_2 *are arbitrary positive solutions to*

$$Y_1P = LT^-M^\dagger M, \quad QY_2 = NN^\dagger T^-L \tag{3.3}$$

such that R is positive, U is arbitrary and V is an arbitrary positive operator in $\mathcal{B}(H_1)$.

Proof. Suppose that the consistent system of adjointable operator equations (1.4) has a positive solution X_0, then X_0 is a positive solution of the system of adjointable operator equations (1.3). It follows from Lemma 3.4 that F is positive and X_0 can be expressed as

$$X_0 = Y_0 + \mathcal{L}_DY\mathcal{L}_D^*, Y \in \mathcal{B}(H_1)^+ \tag{3.4}$$

where Y_0 is a particular positive solution of (1.3). Taking (3.4) into $A_3XB_3 = C_3$ yields that

$$MYN = C_3 - A_3Y_0B_3 \tag{3.5}$$

has a positive solution. By Lemma 3.1,

$$M^\dagger MYNN^\dagger = L \tag{3.6}$$

has a positive solution. It can be verified that

$$(NN^\dagger T^-M^\dagger M)^* = M^\dagger M(T^-)^*NN^\dagger = M^\dagger MT^-NN^\dagger = NN^\dagger T^-M^\dagger M.$$

Note that (3.6) is consistent, and

$$S = NN^\dagger T^-M^\dagger MYNN^\dagger T^-M^\dagger M = NN^\dagger T^-M^\dagger MY(NN^\dagger T^-M^\dagger M)^*.$$

Hence, if Y is positive, so is S. Now, we show that $N(S^*) \subseteq N((NN^\dagger T^-L)^*)$. Let $S^*x = 0$, by Reid's inequality for the positive operator $NN^\dagger YNN^\dagger$,

$$\begin{aligned}\|(NN^{\dagger}T^{-}L)^{*}x\|^{2} &= \|(NN^{\dagger}T^{-}M^{\dagger}MYNN^{\dagger})^{*}x\|^{2}\\ &= \|NN^{\dagger}YNN^{\dagger}T^{-}M^{\dagger}Mx\|^{2}\\ &= \|NN^{\dagger}Y(NN^{\dagger})^{*}T^{-}M^{\dagger}Mx\|^{2}\\ &\leqslant \|NN^{\dagger}Y(NN^{\dagger})^{*}\|\langle NN^{\dagger}Y(NN^{\dagger})^{*}T^{-}M^{\dagger}Mx, T^{-}M^{\dagger}Mx\rangle\\ &= \|NN^{\dagger}Y(NN^{\dagger})^{*}\|\langle M^{\dagger}MT^{-}NN^{\dagger}Y(NN^{\dagger})^{*}T^{-}M^{\dagger}Mx, x\rangle\\ &= \|NN^{\dagger}Y(NN^{\dagger})^{*}\|\langle NN^{\dagger}T^{-}M^{\dagger}MYNN^{\dagger}T^{-}M^{\dagger}Mx, x\rangle\\ &= \|NN^{\dagger}Y(NN^{\dagger})^{*}\|\langle Sx, x\rangle\\ &= 0,\end{aligned}$$

which means $(NN^{\dagger}T^{-}L)^{*}x = 0$, implying $N(S^{*}) \subseteq N((NN^{\dagger}T^{-}L)^{*})$. Hence $R(NN^{\dagger}T^{-}L) \subseteq R(S)$. Similarly, we can get $R(M^{\dagger}MT^{-}L^{*}) \subseteq R(S)$.

Suppose (3.1) is satisfied. By F is positive, the system of adjointable operator equations (1.3) has a positive solution and this positive solution can be expressed as

$$X = Y_0 + \mathcal{L}_D Y \mathcal{L}_D^{*}, \quad Y \in \mathcal{B}(H_1)^{+} \tag{3.7}$$

where Y_0 is a particular positive solution of (1.3). Taking (3.7) into $A_3XB_3 = C_3$, we can get (3.5). Now, we want to show that (3.5) has a positive solution. By Lemma 3.1, we get that (3.6) has a positive solution. We first show that when S is positive and $R(NN^{\dagger}T^{-}L) \subseteq R(S)$, $R(M^{\dagger}MT^{-}L^{*}) \subseteq R(S)$, positive solutions Y_1, Y_2 can be so determined that R is positive. We rewrite (3.3) as $Y_1P = L_1$, $QY_2 = L_2$. It is easy to verify that $L_2Q^{*} = S = P^{*}L_1$. By (3.1), $R(S) = R(L_1^{*}) = R(L_2)$. It follows from Lemma 3.3 that the general positive solutions to every equation of (3.3) can be written respectively as

$$Y_1 = L_1S^{-}L_1^{*} + \mathcal{R}_P^{*}V\mathcal{R}_P, \quad Y_2 = L_2^{*}S^{-}L_2 + \mathcal{L}_Q W \mathcal{L}_Q^{*}.$$

Note that

$$P^{*}(L - L_1S^{-}L_2) = L_2 - SS^{-}L_2 = 0; \quad (L - L_1S^{-}L_2)Q^{*} = L_1 - L_1S^{-}S = 0.$$

Hence

$$\begin{aligned}R &= (L_1 + L_2^{*})S^{-}(L_1^{*} + L_2) + [\mathcal{R}_P^{*}V\mathcal{R}_P + \mathcal{L}_Q W\mathcal{L}_Q^{*} + L - L_1S^{-}L_2 + L^{*} - L_2^{*}S^{-}L_1^{*}]\\ &= (L_1 + L_2^{*})S^{-}(L_1^{*} + L_2) + \begin{bmatrix}\mathcal{R}_P^{*} & \mathcal{L}_Q\end{bmatrix}\begin{bmatrix} V & L - L_1S^{-}L_2\\ L^{*} - L_2^{*}S^{-}L_1^{*} & W\end{bmatrix}\begin{bmatrix}\mathcal{R}_P\\ \mathcal{L}_Q^{*}\end{bmatrix}.\end{aligned}$$

When we choice S^{-} be $S^{\dagger}$, then $(L_1 + L_2^{*})S^{\dagger}(L_1^{*} + L_2)$ is positive, any other inner inverse of S is of the form $S^{-} = S^{\dagger} + Y - S^{\dagger}SYSS^{\dagger}$ for some Y. Then

$$(L_1 + L_2S^{*})S^{-}(L_1^{*} + L_2) = (L_1 + L_2^{*})(S^{\dagger} + Y - S^{\dagger}SYSS^{\dagger})(L_1^{*} + L_2) = (L_1 + L_2^{*})S^{\dagger}(L_1^{*} + L_2),$$

which show that $(L_1 + L_2S^{*})S^{-}(L_1^{*} + L_2)$ is positive for any inner inverse of S. If we choose

$$V = I, \quad W = (L^{*} - L_2^{*}S^{-}L_1^{*})(L - L_1S^{-}L_2),$$

by Lemma 3.2, R is positive, and we can get

$$Y_1 = L_1S^{-}L_1^{*} + \mathcal{R}_P^{*}\mathcal{R}_P, \quad Y_2 = L_2^{*}S^{-}L_2 + (L^{*} - L_2^{*}S^{-}L_1^{*})(L - L_1S^{-}L_2).$$

It is easy to verify that (3.2) is a positive solution to (1.4).

Suppose that $\overline{X}$ is a positive solution of (1.4). It follows from Lemma 3.4 that $\overline{X}$ can be express as

$$\overline{X} = Y_0 + \mathcal{L}_D Y \mathcal{L}_D^{*}, \quad Y \in \mathcal{B}(H_1)^{+}$$

where Y_0 is a particular positive solution of (1.3). Then

$$\overline{X} - Y_0 = \mathcal{L}_D W \mathcal{L}_D^{*} \tag{3.8}$$

for a positive operator W. Putting

$$Y_1 = M^\dagger MWM^\dagger M, Y_2 = NN^\dagger WNN^\dagger,$$

it follows from

$$\begin{aligned} M^\dagger MWM^\dagger MT^- NN^\dagger &= M^\dagger MWNN^\dagger T^- M^\dagger M \\ &= M^\dagger (A_3\mathcal{L}_D W\mathcal{L}_D^* B_3)N^\dagger T^- M^\dagger M \\ &= LT^- M^\dagger M, \\ M^\dagger MT^- NN^\dagger WNN^\dagger &= NN^\dagger T^- M^\dagger MWNN^\dagger \\ &= NN^\dagger T^- M^\dagger (A_3\mathcal{L}_D W\mathcal{L}_D^* B_3)N^\dagger \\ &= NN^\dagger T^- L, \end{aligned}$$

that Y_1 and Y_2 are positive solutions to the adjointable operator equations in (3.3), respectively. Note that

$$\begin{aligned} R &= L + Y_1 + Y_2 + L^* \\ &= M^\dagger (C_3 - A_3Y_0B_3)N^\dagger + M^\dagger MWM^\dagger M + NN^\dagger WNN^\dagger + (M^\dagger (C_3 - A_3Y_0B_3)N^\dagger)^* \\ &= M^\dagger MWNN^\dagger + M^\dagger MWM^\dagger M + NN^\dagger WNN^\dagger + NN^\dagger WMM^\dagger \\ &= (M^\dagger M + NN^\dagger)W(M^\dagger M + NN^\dagger) \\ &= TWT \\ &= TWT^*. \end{aligned}$$

By Lemma 3.5, W can be expressed as

$$\begin{aligned} W = & X_0 + X_0X_0^\dagger U(I - T^\dagger T) + (I - T^\dagger T)U^*X_0X_0^\dagger \\ & + (I - T^\dagger T)U^*X_0^\dagger U(I - T^\dagger T) + (I - T^\dagger T)V(I - T^\dagger T), \end{aligned} \tag{3.9}$$

where $X_0 = T^\dagger R(T^\dagger)^*$, Y_1 and Y_2 are arbitrary positive solution to

$$Y_1P = LT^- M^\dagger M, \quad QY_2 = NN^\dagger T^- L$$

such that R is positive, U is arbitrary and V is an arbitrary positive operator in $\mathcal{B}(H_1)$. Taking (3.9) into (3.8), we know $\overline{X}$ can be expressed as (3.2). □

4. Some special cases of the system of adjointable operator equations (1.4)

In this section, we consider some special cases of the system of adjointable operator equations (1.4).

The following corollary consider the adjointable operator equation (1.1) over the Hilbert C^*-modules. Without loss of generality, by Lemma 3.1, we assume that the coefficient operators A and B are both positive.

Corollary 4.1. *Let $A, B \in \mathcal{B}(H)^+, C \in \mathcal{B}(H)$. Suppose that $A + B$, $E = (A + B)^\dagger B, K = A(A + B)^\dagger, C + C^* + Y + Z, T = B(A + B)^\dagger C(A + B)^\dagger A$ have closed ranges. Then the adjointable operator equation (1.1) has a positive solution in $\mathcal{B}(H)$ if and only if*

$$AA^\dagger CB^\dagger B = C, \quad T \geqslant 0, \quad R[A(A + B)^\dagger C^*] \subset R(T), \quad R[B(A + B)^\dagger C] \subset R(T),$$

in which case the general positive solution of (1.1) is given by

$$X = X^* + X^*(X^*)^\dagger US + SU^*X^*(X^*)^\dagger + SU^*(X^*)^\dagger US + SVS,$$

where $X^* = (A + B)^\dagger (C + C^* + Y + Z)[(A + B)^\dagger]^*$ is a particular positive solution of (1.1), Y and Z are arbitrary positive solution of

$$Y(A+B)^{\dagger}B = C(A+B)^{\dagger}A, A(A+B)^{\dagger}Z = B(A+B)^{\dagger}C$$

such that $C + C^* + Y + Z$ is positive, $S = I - (A+B)^{\dagger}(A+B)$, U is arbitrary and V is an arbitrary positive operator in $\mathcal{B}(H)$.

Remark 4.1. In 2008, Xu et al. [18] proposed a solvability condition for the existence of a positive solution to the adjointable operator equation (1.1), and derived an expression for the general positive solution to (1.1) over Hilbert C^*-modules. However, their results are restricted by the assumption that the underlying space is finite-dimensional or $R(B)$ is contained in $R(A^*)$. Our result in Corollary 4.1 has no the constraint mentioned above.

We denote the complex number field by $\mathbb{C}$, the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathbb{C}^{m\times n}$. Next corollary considers positive semidefinite solutions to the matrix equation (1.1) over the complex number field. Without loss of generality, by [9], we assume that the coefficient matrices A and B are both positive semidefinite.

Corollary 4.2. *Let $A, B \in \mathbb{C}^{m\times m}$ be positive semidefinite matrices and $C \in \mathbb{C}^{m\times m}$. Then the matrix equation (1.1) has a positive semidefinite solution in $\mathbb{C}^{m\times m}$ if and only if*

$$AA^{\dagger}CB^{\dagger}B = C, \quad T \geqslant 0; \quad \operatorname{rank} T = \operatorname{rank}\{A(A+B)^{\dagger}C^*\} = \operatorname{rank}\{B(A+B)^{\dagger}C\},$$

in which case the general positive semidefinite solution of (1.1) is given by

$$X = X^* + X^*(X^*)^{\dagger}US + SU^*X^*(X^*)^{\dagger} + SU^*(X^*)^{\dagger}US + SVS,$$

where $X^* = (A+B)^{\dagger}(C + C^* + Y + Z)[(A+B)^{\dagger}]^*$ is a particular positive semidefinite of (1.1), Y and Z are arbitrary positive semidefinite solution of

$$Y(A+B)^{\dagger}B = C(A+B)^{\dagger}A, A(A+B)^{\dagger}Z = B(A+B)^{\dagger}C$$

such that $C + C^* + Y + Z$ is positive semidefinite, $S = I - (A+B)^{\dagger}(A+B)$, U is arbitrary and V is an arbitrary positive semidefinite matrix in $\mathbb{C}^{m\times m}$.

Remark 4.2. Khatri and Mitra [9] once presented the necessary and sufficient conditions for the existence of positive semidefinite solution to matrix equation (1.1) and established the expression of the general positive semidefinite solution in terms of generalized inverse of some matrices when the solvability conditions are satisfied. However, in 1984, Baksalary [1] pointed out that the general expression of positive semidefinite solution to matrix equation (1.1) in [9] did not involve all of the positive semidefinite solution. In 2008, Xu etc. gave a correct expression of the general positive semidefinite solution to matrix equation (1.1) in terms of generalized inverses of some matrices [18, Theorem 5.5]. In Corollary 4.2, we also give a new expression of this general positive semidefinite solution which is different from one in [18].

We now investigate the positive semidefinite solution to the system of matrix equations (1.4) in the following corollary. For simplicity, we put

$$D = \begin{bmatrix} A_1 \\ B_2^* \end{bmatrix}, \quad E = \begin{bmatrix} C_1 \\ C_2^* \end{bmatrix}, \quad F = \begin{bmatrix} C_1A_1^* & C_1B_2 \\ (A_1C_2)^* & C_2^*B_2 \end{bmatrix},$$

$$Y_0 = E^*F^-E, \quad M = A_3\mathcal{L}_D, \quad N = \mathcal{L}_D^*B_3, \quad L = M^{\dagger}(C_3 - A_3Y_0B_3)N^{\dagger}, \quad T = M^{\dagger}M + NN^{\dagger},$$

$$P = T^-NN^{\dagger}, \quad Q = M^{\dagger}MT^-, \quad R = L + L^* + Y_1 + Y_2, \quad S = NN^{\dagger}T^-LT^-M^{\dagger}M.$$

Corollary 4.3. *Suppose that $A_1 \in \mathbb{C}^{m\times n}, A_3 \in \mathbb{C}^{p\times n}, B_2 \in \mathbb{C}^{n\times l}, B_3 \in \mathbb{C}^{n\times q}, C_1 \in \mathbb{C}^{m\times n}, C_2 \in \mathbb{C}^{n\times l}$, $C_3 \in \mathbb{C}^{p\times q}$, then the consistent system of matrix equations (1.4) has an positive semidefinite solution in $\mathbb{C}^{m\times m}$ if and only if*

$$F \geqslant 0, \quad S \geqslant 0, \quad \operatorname{rank}(NN^{\dagger}T^-L) = \operatorname{rank}(S), \quad \operatorname{rank}(M^{\dagger}MT^-L^*) = \operatorname{rank}(S),$$

in which case the general positive semidefinite solution of (1.4) *is given by*

$$X = Y_0 + \mathcal{L}_D(X_0 + X_0X_0^{\dagger}U(I - T^{\dagger}T) + (I - T^{\dagger}T)U^*X_0X_0^{\dagger} + (I - T^{\dagger}T)U^*X_0^{\dagger}U(I - T^{\dagger}T) + (I - T^{\dagger}T)V(I - T^{\dagger}T))\mathcal{L}_D^*,$$

where Y_0 *is a particular positive semidefinite solution of* (1.3), $X_0 = T^{\dagger}R(T^{\dagger})^*$, Y_1 *and* Y_2 *are arbitrary positive semidefinite solution to*

$$Y_1P = LT^{-}M^{\dagger}M, \quad QY_2 = NN^{\dagger}T^{-}L$$

such that R *is positive semidefinite,* U *is arbitrary and* V *is an arbitrary positive semidefinite matrix in* $\mathbb{C}^{m\times m}$.

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