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A note on fractional derivatives and fractional powers of operators

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ABSTRACT

Definitions of fractional derivatives and fractional powers of positive operators are considered. The connection of fractional derivatives with fractional powers of positive operators is presented. The formula for fractional difference derivative is obtained.

1. Fractional derivatives and fractional powers of positive operators.

Let us give definitions of fractional derivatives (see [1–4]) and fractional powers of positive operators (see [5,6]) that will be needed below.

Definition 1. If $f(x) \in C([a,b])$ and $a < x < b$, then

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt,$$

where $\alpha \in (-\infty, \infty)$, is called the Riemann–Liouville fractional integral of order α. In the same fashion for $\alpha \in (0,1)$ we let

$$D_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt,$$

which is called the Riemann–Liouville fractional derivative of order α.

Note that if $f(a) = 0$, then we can write

$$D_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\, dt.$$

Here

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s}\, ds \quad (\alpha > 0). \tag{1.1}$$

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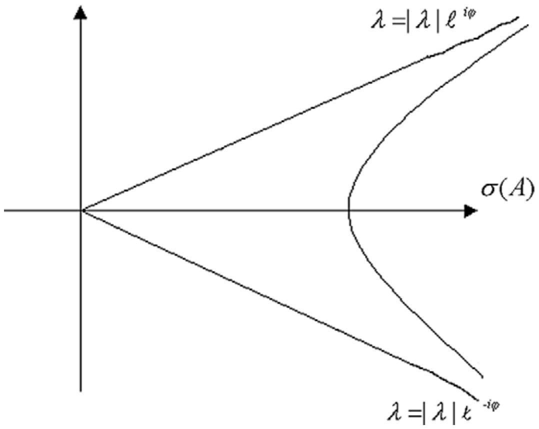

Fig. 1.

Definition 2. The operator A is said to be positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle φ, $0<2\varphi<2\pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1=[\rho\exp(i\varphi)\colon\ 0\leqslant\rho<\infty]$ and $S_2=[\rho\exp(-i\varphi)\colon\ 0\leqslant\rho<\infty]$, and outside it the resolvent $(\lambda I-A)^{-1}$ is subject to the bound

$$\left\|(\lambda I-A)^{-1}\right\|_{E\to E}\leqslant\frac{M(\varphi)}{1+|\lambda|}.$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A)=\varphi(A,E)$.

For positive operator A one can define negative fractional powers α by the formula

$$A^{-\alpha}=\frac{1}{2\pi i}\int_{\Gamma}\lambda^{-\alpha}R(\lambda)\,d\lambda\quad\big(0<\alpha<\infty,\ R(\lambda)=(A-\lambda I)^{-1},\ \Gamma=S_1\cup S_2\big).\tag{1.2}$$

The operators $A^{-\alpha}$ are bounded. By definition of $A^{-\alpha}$ the operators $A^{-\alpha}$ form a semigroup

$$A^{-(\alpha+\beta)}=A^{-\alpha}A^{-\beta}.$$

Using formula (1.2), we get

$$A^{-\alpha}=\frac{1}{2\pi i}\int_{-\infty}^{0}\lambda^{-\alpha}R(\lambda)\,d\lambda+\frac{1}{2\pi i}\int_{0}^{-\infty}\lambda^{-\alpha}R(\lambda)\,d\lambda,$$

where the integrals are taken along the lower and upper sides of the cut respectively: $\lambda=se^{-\pi i}$ and $\lambda=se^{\pi i}$. Hence

$$A^{-\alpha}=\frac{e^{\alpha\pi i}}{2\pi i}\int_{0}^{\infty}s^{-\alpha}R(-s)\,ds+\frac{e^{-\alpha\pi i}}{2\pi i}\int_{0}^{\infty}s^{-\alpha}R(-s)\,ds$$

or

$$A^{-\alpha}=\frac{\sin\alpha\pi}{\pi}\int_{0}^{\infty}s^{-\alpha}R(-s)\,ds=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{0}^{\infty}s^{-\alpha}R(-s)\,ds.\tag{1.3}$$

It is also possible to define positive fractional powers A^{α} ($\alpha>0$) of A as the operators inverse to the negative powers. If $x\in D(A)$ we obtain a formula for the positive fractional powers ($0<\alpha<1$) of the operator A:

$$A^{\alpha}x=A^{\alpha-1}Ax=\frac{\sin\alpha\pi}{\pi}\int_{0}^{\infty}s^{\alpha-1}R(-s)Ax\,ds=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{0}^{\infty}s^{\alpha-1}R(-s)Ax\,ds.\tag{1.4}$$

In recent paper [7] the definitions of fractional derivatives as fractional powers of derivative operators $-i\frac{d}{dt}$ are suggested. The Taylor series and Fourier series are used to define fractional power of self-adjoint derivative operator. The Fourier integrals and Weyl quantization procedure are applied to derive the definition of fractional derivative operator.

In present paper the connection of fractional derivatives with fractional powers of positive operators of the first order is presented. The formula for fractional difference derivative is obtained. Well-posedness of fractional differential equations is obtained.

2. The connection of fractional derivatives with fractional powers of positive operators

Theorem 2.1. *Let A be the operator acting in $E = C[a,b]$ defined by the formula $Av(x) = v'(x)$, with the domain $D(A) = \{v(x)\colon\ v'(x) \in C[a,b],\ v(a) = 0\}$. Then A is a positive operator in the Banach space $E = C[a,b]$ and*

$$A^\alpha f(x) = D^\alpha_{a+} f(x)$$

for all $f(x) \in D(A)$.

Proof. Evidently, the operator $\lambda I + A$ has a bounded inverse for any $\lambda \geqslant 0$, and formula

$$\left[(\lambda I + A)^{-1} f\right](x) = \int_a^x e^{-\lambda(x-z)} f(z)\,dz \tag{2.1}$$

holds. From this formula it follows that A is a positive operator in the Banach space $E = C[a,b]$. Applying formulas (1.4) and (2.1), we get

$$\begin{aligned}
A^\alpha f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{\alpha-1}(sI + A)^{-1} f'(x)\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{\alpha-1} \int_a^x e^{-s(x-z)} f'(z)\,dz\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^x \left\{ \int_0^\infty s^{\alpha-1} e^{-s(x-z)}\,ds \right\} f'(z)\,dz.
\end{aligned}$$

Making the substitution $s(x-z) = p$ and using formula (1.1), we obtain

$$\int_0^\infty s^{\alpha-1} e^{-s(x-z)}\,ds = \frac{1}{(x-z)^\alpha} \int_0^\infty p^{\alpha-1} e^{-p}\,dp = \frac{\Gamma(\alpha)}{(x-z)^\alpha}.$$

Therefore

$$A^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{1}{(x-z)^\alpha} f'(z)\,dz = D^\alpha_{a+} f(x).$$

Theorem 2.1 is proved. □

Theorem 2.2. *Let A be the operator acting in $E = C[a,b]$ defined by the formula $Av(x) = v'(x)$, with the domain $D(A) = \{v(x)\colon\ v'(x) \in C[a,b],\ v(a) = 0\}$. Then*

$$A^{-\alpha} f(x) = I^\alpha_{a+} f(x)$$

for all $f(x) \in C[a,b]$.

Proof. Applying formulas (1.3) and (2.1), we get

$$\begin{aligned}
A^{-\alpha} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha}(sI + A)^{-1} f(x)\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \int_a^x e^{-s(x-z)} f(z)\,dz\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^x \left\{ \int_0^\infty s^{-\alpha} e^{-s(x-z)}\,ds \right\} f(z)\,dz.
\end{aligned}$$

Making the substitution $s(x-z) = p$ and using formula (1.1), we get

$$\int_0^\infty s^{-\alpha} e^{-s(x-z)}\,ds = \frac{1}{(x-z)^{1-\alpha}} \int_0^\infty p^{-\alpha} e^{-p}\,dp = \frac{\Gamma(1-\alpha)}{(x-z)^{1-\alpha}}.$$

Thus

$$A^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{(x-z)^{1-\alpha}} f(z)\,dz = I_{a+}^{\alpha} f(x).$$

Theorem 2.2 is proved. □

Note that

$$I_{a+}^{\alpha} f(x) = D_{a+}^{-\alpha} f(x).$$

3. Fractional difference derivatives

Denote that

$$[a,b]_h = \{x_k = a + kh,\ 0 \leqslant k \leqslant N,\ Nh = b - a\}.$$

Theorem 3.1. *Let A_h be the operator acting in $E_h = C[a,b]_h$ defined by the formula $A_h v^h(x) = \{\frac{v_k - v_{k-1}}{h}\}_1^N$, with $v_0 = 0$. Then A_h is a positive operator in the Banach space $E_h = C[a,b]_h$ and*

$$A_h^{\alpha} f^h(x) = \left\{ \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} \frac{\Gamma(k-m-\alpha+1)}{(k-m)!} \frac{f_m - f_{m-1}}{h^{\alpha}} \right\}_1^N.$$

Proof. Evidently, the operator $\lambda I + A_h$ has a bounded inverse for any $\lambda \geqslant 0$, and formula

$$(\lambda I + A_h)^{-1} f^h(x) = \left\{ \sum_{m=1}^{k} R^{k-m+1} f_m h \right\}_{k=1}^N \tag{3.1}$$

holds. Here $R = (1 + h\lambda)^{-1}$. From this formula it follows that A_h is a positive operator in the Banach space $E_h = C[a,b]_h$. Applying formulas (1.4) and (3.1), we get

$$\begin{aligned} A_h^{\alpha} f^h(x) &= \left\{ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{\alpha-1} (sI + A_h)^{-1} \frac{f_k - f_{k-1}}{h}\,ds \right\}_1^N \\ &= \left\{ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{\alpha-1} \sum_{m=1}^{k} \frac{1}{(1+hs)^{k-m+1}} \frac{f_m - f_{m-1}}{h} h\,ds \right\}_1^N \\ &= \left\{ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{m=1}^{k} \left[\int_0^\infty s^{\alpha-1} \frac{1}{(1+hs)^{k-m+1}}\,ds \right] \frac{f_m - f_{m-1}}{h} h \right\}_1^N. \end{aligned}$$

Since

$$\frac{1}{(1+hs)^{k-m+1}} = \frac{1}{(k-m)!} \int_0^\infty t^{k-m} e^{-t(1+sh)}\,dt,$$

we have that

$$\int_0^\infty s^{\alpha-1} \frac{1}{(1+hs)^{k-m+1}} ds = \int_0^\infty s^{\alpha-1} \frac{1}{(k-m)!} \int_0^\infty t^{k-m} e^{-t(1+sh)}\,dt\,ds = \frac{1}{(k-m)!} \int_0^\infty t^{k-m} e^{-t} \int_0^\infty s^{\alpha-1} e^{-tsh}\,ds\,dt.$$

Making the substitution $tsh = p$ and using formula (1.1), we obtain

$$\int_0^\infty s^{\alpha-1}\frac{1}{(1+hs)^{k-m+1}}\,ds=\frac{1}{(k-m)!}\int_0^\infty t^{k-m}e^{-t}\frac{1}{(th)^\alpha}\int_0^\infty p^{\alpha-1}e^{-p}\,dp\,dt,$$

$$\int_0^\infty s^{\alpha-1}\frac{1}{(1+hs)^{k-m+1}}\,ds=\frac{1}{(k-m)!}\int_0^\infty t^{k-m-\alpha}e^{-t}\,dt\,\frac{1}{h^\alpha}\Gamma(\alpha).$$

Therefore

$$A_h^\alpha f^h(x)=\left\{\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\frac{1}{(k-m)!}\int_0^\infty t^{k-m-\alpha}e^{-t}\,dt\,\frac{1}{h^\alpha}\frac{f_m-f_{m-1}}{h}h\right\}_1^N.$$

Theorem 3.1 is proved. □

So it will be natural to note that

$$D_h^\alpha f^h(x):=\left\{\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\frac{\Gamma(k-m-\alpha+1)}{(k-m)!}\frac{f_m-f_{m-1}}{h^\alpha}\right\}_1^N.$$

It is called the Riemann–Liouville fractional difference derivative of order α.

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