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Vanishing of 3-loop Jacobi diagrams of odd degree

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Abstract

We prove the vanishing of the space of 3-loop Jacobi diagrams of odd degree. This implies that no 3-loop Vassiliev invariant can distinguish between a knot and its inverse.

Keywords: Vassiliev invariants; Jacobi diagram; Inverse knot

1. Introduction

A Jacobi diagram is a uni-trivalent graph with some extra structure. Such diagrams play a leading role in the theory of Vassiliev invariants and Kontsevich invariants of knots. Vassiliev invariants are defined by a filtration of the vector space spanned by knots, whose graded spaces are identified with vector spaces spanned by Jacobi diagrams subject to certain defining relations. The Kontsevich invariant of a knot is defined as an infinite linear sum of Jacobi diagrams. The physical background of these invariants is in the perturbative expansion of the Chern–Simons path integral, which is formulated in terms of uni-trivalent graphs; this is one explanation why Jacobi diagrams appear in this theory. The Kontsevich invariant is expected to classify knots, and from this point of view it is important to identify the vector space spanned by Jacobi diagrams subject to the defining relations.

It is conjectured that the space of Jacobi diagrams with an odd number of legs vanishes [1,9]. This would imply the claim that no Vassiliev invariant can distinguish a knot from its inverse, where the *inverse* of an oriented knot is the knot with the opposite orientation. In general, a knot

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and its inverse are not isotopic, the simplest counter-example being the knot 8_{17} with its two possible orientations. The consequences of the possibility that Vassiliev invariants cannot make this distinction are discussed in [5]. For the Lie algebra version of this claim, see Remark 3.3. Dasbach claimed to have proved the vanishing of n-loop Jacobi diagrams with an odd number of legs for $n \leqslant 6$, but his proof has a gap for $n \geqslant 3$; see Remark 3.2.

In the present paper, we prove the vanishing of 3-loop Jacobi diagrams with an odd number of legs (Theorem 3.1). In our proof, we consider the internal graph of a Jacobi diagram, which is the trivalent graph obtained from the Jacobi diagram by removing its legs, where a leg of a Jacobi diagram is an edge adjacent to a univalent vertex. Then, following Nakatsuru [7], we identify each Jacobi diagram with a polynomial whose variables correspond to the edges of the internal graph of the Jacobi diagram, and present the space of 3-loop Jacobi diagrams as a quotient space of a direct sum of polynomial algebras corresponding to 3-loop internal graphs. Here, the quotient is derived from the defining relations of Jacobi diagrams and from the symmetries of the internal graphs. Thus, the proof is reduced to calculating the image of the relations by the (skew) symmetrizer corresponding to the internal graph's symmetry. This approach provides in passing an alternative proof of [3, Theorem 7.4] in the 'even number of legs' case as well. The 4-loop, 5-loop, and 6-loop cases which Dasbach's result would have covered remain open. In these higher loop degrees, the techniques used here lead to more complicated calculations, which we have not been able to complete. New ideas seem necessary in order to make further progress.

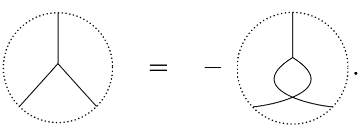
The paper is organized as follows. In Section 2, we review several definitions concerning Jacobi diagrams and related notions. In Section 3, we show how to identify the space of 3-loop Jacobi diagrams with a quotient space of a direct sum of polynomial algebras and prove the vanishing of 3-loop Jacobi diagrams with an odd number of legs, which is the main theorem of this paper. This proof requires the use of a certain lemma, which we prove in Section 4.

2. Jacobi diagrams

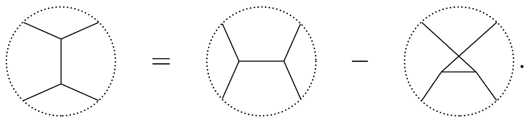
In this section we review definitions of Jacobi diagrams, the space of Jacobi diagrams, n-loop Jacobi diagrams, and define some notations. For general references on the theory of Jacobi diagrams see *e.g.* [1,8].

A *Jacobi diagram* is a graph whose vertices have valence 1 or 3 and whose trivalent vertices are oriented, *i.e.*, a cyclic order of 3 edges around each trivalent vertex is fixed. The *degree* of a Jacobi diagram is defined to be half the total number of vertices of the diagram. The *space of Jacobi diagrams* is the vector space over $\mathbb{Q}$ spanned by Jacobi diagrams subject to the AS (Anti-Symmetry) and IHX (written as "I" = "H" − "X") relations, which are local moves between Jacobi diagrams which differ inside a dotted circle as indicated below. The space of Jacobi diagrams is graded by degree. (A Jacobi diagram of the type we have just defined is sometimes called an *open Jacobi diagram*, and the space of these Jacobi diagrams is sometimes denoted $\mathcal{B}$ in the literature.)

The *AS relation*



The *IHX relation*



A Jacobi diagram is called *n-loop* if it is connected and its Euler number is equal to $1-n$; *i.e.*, its first Betti number is equal to n. (An n-loop Jacobi diagram is sometimes said to be of *loop degree* $n-1$ in the literature.) We denote by $\mathcal{A}_{n\text{-loop}}$ the space of n-loop Jacobi diagrams, *i.e.*, the vector space spanned by n-loop Jacobi diagrams subject to the AS and IHX relations. An edge adjacent to a univalent vertex is called a *leg*. We assume without loss of generality that a Jacobi diagram does not have a trivalent vertex which is adjacent to 2 legs, since a Jacobi diagram with such a trivalent vertex vanishes by the AS relation. The *internal graph* of a Jacobi diagram is the trivalent graph obtained from the Jacobi diagram by removing its legs. We denote by $\mathcal{A}(\Gamma)$ the space of Jacobi diagrams whose internal graph is Γ modulo the action of the symmetry of Γ.

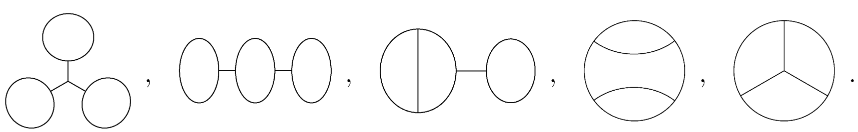
3. 3-loop Jacobi diagrams

In this section we identify the space of 3-loop Jacobi diagrams as a graded vector space. In Section 3.1 we present the space of 3-loop Jacobi diagrams in terms of spaces $\mathcal{A}(\Gamma)$ for 3-loop trivalent graphs Γ. In Section 3.2 we present the space of such diagrams using polynomial algebras. Using this presentation, we prove in Section 3.3 that the odd degree part of this space vanishes, which is the main theorem of this paper. In Section 3.4 we identify the even part of this space with some polynomial algebra (following [7]).

3.1. The space of 3*-loop Jacobi diagrams*

In this section, we present the space of 3-loop Jacobi diagrams in terms of spaces $\mathcal{A}(\Gamma)$ for 3-loop trivalent graphs Γ.

Ignoring orientations of internal vertices, the internal graph of a 3-loop Jacobi diagram may be one of the five graphs below,

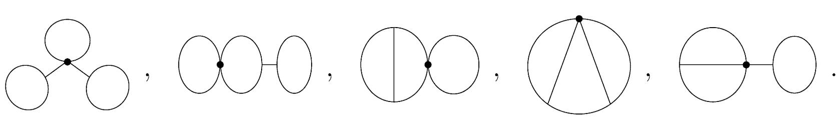


(3.1)

The space of 3-loop Jacobi diagrams is presented by

$$\mathcal{A}_{3\text{-loop}} \cong \Bigg(\bigoplus_{\Gamma \text{ in (3.1)}} \mathcal{A}(\Gamma) \Bigg) \Big/ \mathrm{IHX}, \tag{3.2}$$

where "IHX" implies the IHX relations among these Γ; all such relations are obtained by replacing a neighborhood of a 4-valent vertex of one of the following graphs with the defining graphs of the IHX relation,



(3.3)

We will see, in Sections 3.3 and 3.4 for the odd and even degree parts respectively, that (3.2) is isomorphic to

$$\mathcal{A}_{3\text{-loop}} \cong \big(\mathcal{A}(\;) \oplus \mathcal{A}(\;)\big)/\mathrm{IHX}, \tag{3.4}$$

where this "IHX" implies the IHX relation obtained from the fourth graph of (3.3). We describe $\mathcal{A}(\;)$ and $\mathcal{A}(\;)$ in terms of polynomial algebras in the next section.

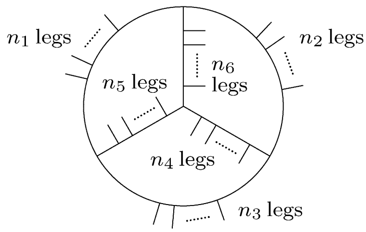
3.2. *Polynomial presentation of* 3*-loop Jacobi diagrams*

In this section we see that the space of 3-loop Jacobi diagrams is identified, as a graded vector space, with a quotient space of a direct sum of polynomial algebras.

We identify $\mathcal{A}(\;)$ with the polynomial algebra on six letters signifying legs on each of the arcs of the internal graphs, modulo the IHX relations on the legs, and modulo the action of $\mathfrak{S}_4$ the automorphism group of the tetrahedron. Thus:

$$\mathcal{A}(\;) \cong \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6]/(3.6), \mathfrak{S}_4,$$

where



is identified with $x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5} x_6^{n_6}$, (3.5)

and the following relations (as algebra relations) imply the IHX relations on the legs:

$$\begin{cases} x_1 - x_2 - x_6 = 0, \\ x_1 - x_3 + x_5 = 0, \\ x_4 + x_5 + x_6 = 0. \end{cases} \tag{3.6}$$

In order to better describe the action of $\mathfrak{S}_4$, following [7], we make the substitution

$$\begin{cases} y_1 = x_1 - x_5 + x_6, \\ y_2 = x_2 + x_4 - x_6, \\ y_3 = x_3 - x_4 + x_5, \\ y_4 = -x_1 - x_2 - x_3, \end{cases}$$

replacing variables corresponding with edges of the tetrahedron with variables corresponding with its faces. In these new variables,

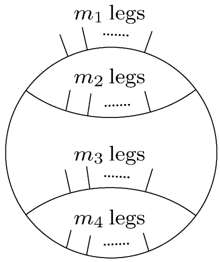
$$\begin{aligned} \mathcal{A}(\;) &\cong \mathbb{Q}[y_1, y_2, y_3, y_4]/(y_1 + y_2 + y_3 + y_4 = 0), \mathfrak{S}_4 \\ &\cong \mathbb{Q}[y_1, y_2, y_3, y_4]^{\mathfrak{S}_4}/(y_1 + y_2 + y_3 + y_4 = 0), \end{aligned}$$

where $\mathfrak{S}_4$ acts on $\mathbb{Q}[y_1, y_2, y_3, y_4]$ by permuting y_1, y_2, y_3, y_4 symmetrically in even degrees and skew-symmetrically in odd degrees.

We may identify $\mathcal{A}(\;)$ with the polynomial algebra on six letters modulo the IHX relations on the legs and modulo the action of the automorphism group of the -shape as above. Thus:

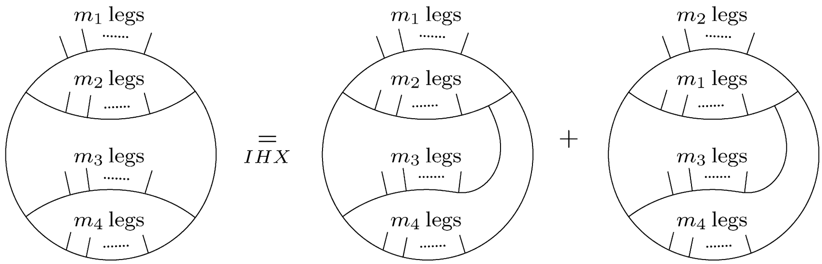
$$\mathcal{A}(\;) \cong \mathbb{Q}[z_1, z_2, z_3, z_4]/(z_1 + z_2 + z_3 + z_4 = 0), \mathrm{Aut}(\;),$$

where



is identified with $z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4}$. (3.7)

Jacobi diagrams whose internal graphs are and are related by the IHX relation which is obtained from the fourth graph of (3.3),



(3.8)

3.3. Odd degree part

The aim of this section is to prove the following theorem.

Theorem 3.1. *The space of* 3*-loop Jacobi diagrams of odd degree vanishes. That is,* $\mathcal{A}^{(\mathrm{odd})}_{3\text{-loop}} = 0$.

Proof. By (3.2),

$$\mathcal{A}^{(\mathrm{odd})}_{3\text{-loop}} \cong \Bigg(\bigoplus_{\Gamma \text{ in } (3.1)} \mathcal{A}(\Gamma)^{(\mathrm{odd})} \Bigg) \Big/ \mathrm{IHX}.$$

We show the vanishing of $\mathcal{A}(\Gamma)^{(\mathrm{odd})}$ for the first four graphs Γ in (3.1).

The vanishing of $\mathcal{A}(\;)^{(\mathrm{odd})}$ is shown as follows. It is shown by the IHX relation that this space is spanned by diagrams of the form (3.7). Such a diagram D is equal modulo the AS relation to $-D$ by reflection of the internal graph with respect to a vertical line, therefore $D = 0$. Hence, $\mathcal{A}(\;)^{(\mathrm{odd})} = 0$.

Similarly, reflection of the internal graph shows us that the spaces $\mathcal{A}(\;)^{(\mathrm{odd})}$ and $\mathcal{A}(\;)^{(\mathrm{odd})}$ also both vanish.

The vanishing of $\mathcal{A}(\;)^{(\mathrm{odd})}$ is shown as follows. Let D be a Jacobi diagram whose internal graph is . We can assume by the IHX relation that there are no legs adjacent to any separating arc. If there is a loop with an even number of legs, then the AS relation on the vertex connecting a separating arc with this loop gives $D = -D$ and therefore $D = 0$. Otherwise, by applying the IHX relation to a separating arc, D is equal to 2 times a Jacobi diagram in $\mathcal{A}(\;)^{(\mathrm{odd})} = 0$, and therefore $D = 0$. Hence, $\mathcal{A}(\;)^{(\mathrm{odd})} = 0$.

Therefore, the space of 3-loop Jacobi diagrams of odd degree is presented by

$$\mathcal{A}^{(\mathrm{odd})}_{\text{3-loop}} \cong \mathcal{A}(\Theta)^{(\mathrm{odd})} \big/ \big(\big(\text{the right-hand side of (3.8)}\big) = 0\big).$$

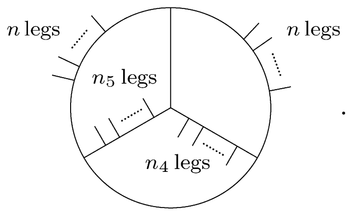
The vector space spanned by the right-hand side of (3.8) is spanned by

$$\big(x_1^{m_1} x_5^{m_2} + x_1^{m_2} x_5^{m_1}\big) x_4^{m_3} (-x_2)^{m_4}$$

in terms of polynomials under the identification (3.5). This space is spanned by

$$(x_1 + x_5)^m (x_1 x_5)^n x_4^{m_3} (-x_2)^{m_4}.$$

Noting that $x_1 + x_5 = x_3 = x_2 - x_4$, this space is further spanned by diagrams of the following form:



$$. \tag{3.9}$$

Hence,

$$\mathcal{A}^{(\mathrm{odd})}_{\text{3-loop}} \cong \mathcal{A}(\Theta)^{(\mathrm{odd})} \big/ \big((3.9) = 0\big).$$

In order to show that $\mathcal{A}^{(\mathrm{odd})}_{\text{3-loop}} = 0$, it is sufficient to show that $\mathcal{A}(\Theta)^{(\mathrm{odd})}$ is spanned by diagrams of the form (3.9). As mentioned in Section 3.2, the space of 3-loop Jacobi diagrams of odd degree is presented by

$$\mathcal{A}(\Theta)^{(\mathrm{odd})} \cong \big(\mathbb{Q}[y_1, y_2, y_3, y_4]^{(\mathrm{odd})}\big)^{\mathfrak{S}_4} / (y_1 + y_2 + y_3 + y_4 = 0),$$

where the action of $\mathfrak{S}_4$ on $\mathbb{Q}[y_1, y_2, y_3, y_4]^{(\mathrm{odd})}$ is skew symmetric. Since a skew symmetric polynomial is presented by the product of a symmetric polynomial and the discriminant $\Delta = \prod_{i<j}(y_i - y_j)$,

$$\mathcal{A}(\Theta)^{(\mathrm{odd})} \cong \Delta \cdot \mathbb{Q}[\sigma_2, \sigma_3, \sigma_4]^{(\mathrm{odd})} \cong \Delta\sigma_3 \cdot \mathbb{Q}\big[\sigma_2, \sigma_3^2, \sigma_4\big],$$

recalling that σ_i denotes the ith symmetric polynomial in y_1, y_2, y_3, y_4. Hence, the vector space spanned by the diagrams of the form (3.9) in $\mathcal{A}(\Theta)^{(\mathrm{odd})}$ is presented by the image of the following map:

$$\mathbb{Q}[x_1 x_2, x_4, x_5]^{(\mathrm{odd})} \to \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6]^{(\mathrm{odd})} / (3.6), \mathfrak{S}_4 \cong \Delta\sigma_3 \cdot \mathbb{Q}\big[\sigma_2, \sigma_3^2, \sigma_4\big].$$

By Lemma 4.1, this map is surjective, noting that

$$\begin{aligned}
x_1 &= (y_1 - y_4)/4, \\
x_2 &= (y_2 - y_4)/4, \\
x_4 &= (y_2 - y_3)/4, \\
x_5 &= (y_3 - y_1)/4.
\end{aligned}$$

Therefore, $\mathcal{A}(\Theta)^{(\mathrm{odd})}$ is spanned by the diagrams of the form (3.9), which implies the theorem. $\square$

Remark 3.2. Dasbach [2] claimed to have proved the vanishing of n-loop Jacobi diagrams with odd number of legs for $n \leqslant 6$ (cited in two of his subsequent papers—in [3] as Theorem 2.2 and half of Theorem 7.4, and in [4], although the focus of both papers is on the 'even number of legs' case). There is however a gap in the proof of his Theorem 5.4.3(iii) (the second equation on page 58 is wrong, since he is using 'modulo greater CW-vectors' to go one way but not the other).

Remark 3.3. It is known that no quantum invariant can distinguish a knot and its inverse. Hence, if there existed a counter-example to the conjecture that Jacobi diagrams with an odd number of legs vanish, such a Jacobi diagram would not be detectable by weight systems derived from Lie algebras. It is known [6,10] how to construct elements which cannot be detected by weight systems derived from Lie algebras, but the method employed in these papers would not give non-trivial diagrams with an odd number of legs, as it involves constructing non-trivial diagrams by multiplying particular elements of Vogel's algebra Λ, and the action of Λ does not change the number of legs.

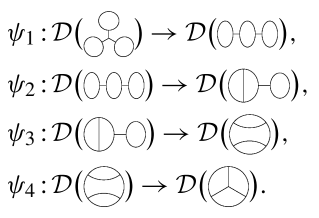
3.4. Even degree part

In this section, we review the identification of the space of 3-loop Jacobi diagrams of even degree with a polynomial algebra, following Nakatsuru [7]. This identification recovers [3, Theorem 7.4].

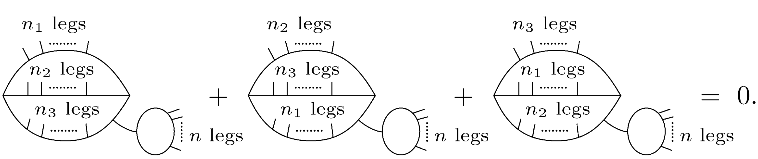
By (3.2),

$$\mathcal{A}^{(\text{even})}_{\text{3-loop}} \cong \Big(\bigoplus_{\Gamma \text{ in } (3.1)} \mathcal{A}(\Gamma)^{(\text{even})} \Big) \Big/ \text{IHX}.$$

Unlike the odd degree case, it is necessary to describe IHX relations among internal graphs Γ concretely, since $\mathcal{A}(\Gamma)^{(\text{even})}$ do not vanish for most Γ. Let $\mathcal{D}(\Gamma)$ denote the space of Jacobi diagrams whose internal graph is Γ, not divided by the action of the symmetry of Γ. Then, by definition, $\mathcal{A}(\Gamma) = \mathcal{D}(\Gamma)/\operatorname{Aut}(\Gamma)$. The IHX relations obtained from the first 4 graphs of (3.3) induce the maps



Here, for example, ψ_4 is the map taking the left-hand side of (3.8) to the right-hand side of (3.8). Further, the IHX relation obtained from the last graph of (3.3) is the relations,



(3.10)

By using these, the space of 3-loop Jacobi diagrams of even degree is presented by

$$\mathcal{A}^{(\text{even})}_{\text{3-loop}} \cong \Big(\bigoplus_{\Gamma \text{ in } (3.1)} \mathcal{D}(\Gamma)^{(\text{even})} \Big) \Big/ \big(\operatorname{Aut}(\Gamma) \text{ for } \Gamma \text{ in } (3.1),\ \psi_1, \psi_2, \psi_3, \psi_4,\ (3.10)\big).$$

Since $\mathcal{A}(\ )^{(\text{even})} = 0$ and ψ_1 induces the zero map $\mathcal{A}(\ )^{(\text{even})} \to \mathcal{A}(\ )^{(\text{even})}$, we can ignore the contribution from $\mathcal{A}(\ )^{(\text{even})}$. Further, since $\psi_3\psi_2$ descends to a map $\mathcal{A}(\ )^{(\text{even})} \to \mathcal{A}(\ )^{(\text{even})}$, we can ignore the contribution from $\mathcal{A}(\ )^{(\text{even})}$. Furthermore, since ψ_3 induces a map $\mathcal{A}(\ )^{(\text{even})} \to \mathcal{A}(\ )^{(\text{even})}$ and (3.10) vanishes in the image of $\psi_4\psi_3$, we can ignore the contribution from $\mathcal{A}(\ )^{(\text{even})}$. Hence,

$$\mathcal{A}^{(\text{even})}_{\text{3-loop}} \cong \big(\mathcal{D}(\ )^{(\text{even})} \oplus \mathcal{D}(\ )^{(\text{even})}\big)/\big(\mathrm{Aut}(\ ), \mathrm{Aut}(\ ), \psi_4\big).$$

It can be checked by concrete calculation that if Jacobi diagrams $D, D' \in \mathcal{D}(\ )^{(\text{even})}$ are related by $\mathrm{Aut}(\ )$, then $\psi_4(D)$ and $\psi_4(D')$ are related by $\mathrm{Aut}(\ )$. Hence, ψ_4 induces a map $\overline{\psi_4} : \mathcal{A}(\ )^{(\text{even})} \to \mathcal{A}(\ )^{(\text{even})}$. Therefore,

$$\mathcal{A}^{(\text{even})}_{\text{3-loop}} \cong \big(\mathcal{A}(\ )^{(\text{even})} \oplus \mathcal{A}(\ )^{(\text{even})}\big)/\overline{\psi}_4 \cong \mathcal{A}(\ )^{(\text{even})}.$$

Hence, by the identification of $\mathcal{A}(\ )$ with the polynomial algebra mentioned in Section 3.2,

$$\begin{aligned}\mathcal{A}^{(\text{even})}_{\text{3-loop}} &\cong \big(\mathbb{Q}[y_1, y_2, y_3, y_4]^{(\text{even})}\big)^{\mathfrak{S}_4}/(y_1 + y_2 + y_3 + y_4 = 0) \\ &\cong \mathbb{Q}[\sigma_2, \sigma_3, \sigma_4]^{(\text{even})} \cong \mathbb{Q}\big[\sigma_2, \sigma_3^2, \sigma_4\big],\end{aligned}$$

where σ_i denotes the ith symmetric polynomial in four variables y_1, y_2, y_3, y_4. It has as its generating function

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)} = \sum_{n \text{ even}} \left(\left\lfloor \frac{n^2+12n}{48} \right\rfloor + 1\right) x^n$$

recovering [3, Theorem 7.4] and agreeing with the results of [4].

4. A lemma on polynomial algebras

The aim of this section is to prove Lemma 4.1, which was used in the proof of the main theorem in the previous section.

The *skew symmetrizer*

$$\mathbb{Q}[y_1, y_2, y_3, y_4] \to \Delta \cdot \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$$

is the linear map sending

$$f(y_1, y_2, y_3, y_4) \text{ to } \frac{1}{4!} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau) f(y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}),$$

where σ_i is the ith symmetric polynomial in y_1, y_2, y_3, y_4 and $\Delta = \prod_{i<j}(y_i - y_j)$ as before. We consider the composition

$$\begin{aligned}&\mathbb{Q}\big[y_1 - y_3, y_2 - y_3, (y_1 - y_4)(y_2 - y_4)\big] \\ &\quad \to \mathbb{Q}[y_1, y_2, y_3, y_4]/(y_1 + y_2 + y_3 + y_4) \to \Delta \cdot \mathbb{Q}[\sigma_2, \sigma_3, \sigma_4]\end{aligned}$$

where the first map is the projection of the inclusion, and the second map is a quotient of the skew symmetrizer.

Lemma 4.1. *The odd degree part of the above map,*

$$\mathbb{Q}\big[y_1 - y_3, y_2 - y_3, (y_1 - y_4)(y_2 - y_4)\big]^{(\mathrm{odd})} \to \Delta\sigma_3 \cdot \mathbb{Q}\big[\sigma_2, \sigma_3^2, \sigma_4\big],$$

is surjective, where $\mathbb{Q}[\cdots]^{(\mathrm{odd})}$ *denotes the vector subspace of* $\mathbb{Q}[\cdots]$ *spanned by polynomials of odd degrees.*

Proof. We put

$$\begin{aligned}
&P_2(y_1, y_2, y_3) = (y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2,\\
&P_3(y_1, y_2, y_3) = (y_1 - y_2)(y_2 - y_3)(y_3 - y_1),\\
&P_4(y_1, y_2, y_3, y_4) = (y_1 - y_3)(y_2 - y_3)(y_1 - y_4)(y_2 - y_4).
\end{aligned}$$

By definition,

$$12P_2(y_1, y_2, y_3)^n P_3(y_1, y_2, y_3)^{2m+3} P_4(y_1, y_2, y_3, y_4)^k$$

belongs to $\mathbb{Q}[y_1 - y_3, y_2 - y_3, (y_1 - y_4)(y_2 - y_4)]^{(\mathrm{odd})}$ for any non-negative integers n, m, k. Since $P_2(y_1, y_2, y_3)$ and $P_3(y_1, y_2, y_3)$ are invariant under cyclic permutations of y_1, y_2, y_3, the above polynomial and

$$\begin{aligned}
&4P_2(y_1, y_2, y_3)^n P_3(y_1, y_2, y_3)^{2m+3}\\
&\quad \times \big(P_4(y_1, y_2, y_3, y_4)^k + P_4(y_1, y_3, y_2, y_4)^k + P_4(y_1, y_4, y_2, y_3)^k\big)
\end{aligned}$$

are taken to the same image by the skew symmetrizer. Further, since the last factor of the above formula is a symmetric polynomial, the skew symmetrizer takes the above formula to

$$\begin{aligned}
Q^{n,m,k} = \big(&P_2(y_1, y_2, y_3)^n P_3(y_1, y_2, y_3)^{2m+3} + P_2(y_4, y_3, y_2)^n P_3(y_4, y_3, y_2)^{2m+3}\\
&+ P_2(y_3, y_4, y_1)^n P_3(y_3, y_4, y_1)^{2m+3} + P_2(y_2, y_1, y_4)^n P_3(y_2, y_1, y_4)^{2m+3}\big)\\
&\times \big(P_4(y_1, y_2, y_3, y_4)^k + P_4(y_1, y_3, y_2, y_4)^k + P_4(y_1, y_4, y_2, y_3)^k\big).
\end{aligned}$$

Hence, it is sufficient to show that $\Delta\sigma_3 \cdot \mathbb{Q}[\sigma_2, \sigma_3^2, \sigma_4]$ is spanned by $Q^{n,m,k}$.

For a fixed non-negative integer d, we consider the vector subspace of $\Delta\sigma_3 \cdot \mathbb{Q}[\sigma_2, \sigma_3^2, \sigma_4]$ spanned by polynomials of degree $2d + 9$. Since it is spanned by $\Delta\sigma_3 \cdot \sigma_2^n \sigma_3^{2m} \sigma_4^k$ for non-negative integers n, m, k satisfying $n + 2k + 3m = d$, its dimension is equal to the number of such (n, m, k). Since the $Q^{n,m,k}$'s are such polynomials of this number, it is sufficient to show the linear independence of $Q^{n,m,k}$ for non-negative integers n, m, k satisfying that $n + 2k + 3m = d$.

In order to prove the linear independence of the $Q^{n,m,k}$'s we first make the substitution

$$\begin{aligned}
&y_1 = \big(3t^a - t^b - t^c\big)/4,\\
&y_2 = \big(-t^a + 3t^b - t^c\big)/4,\\
&y_3 = \big(-t^a - t^b + 3t^c\big)/4,\\
&y_4 = -\big(t^a + t^b + t^c\big)/4,
\end{aligned}$$

where t is a variable tending to ∞, and a, b, c are real numbers satisfying that $a > b > c > 0$ and $a - b < b - c < 2(a - b)$. Since

$$\begin{aligned}
&y_1 - y_4 = t^a,\\
&y_2 - y_4 = t^b,\\
&y_3 - y_4 = t^c,
\end{aligned}$$

we have that

$$P_2(y_1,y_2,y_3)=\left(t^a-t^b\right)^2+\left(t^b-t^c\right)^2+\left(t^c-t^a\right)^2=2t^{2a}\left(1-t^{-(a-b)}+o\left(t^{-(b-c)}\right)\right),$$

where $f(t)=g(t)+o(t^{\varepsilon})$ means that $(f(t)-g(t))/t^{\varepsilon}\to 0$ as $t\to\infty$. Hence,

$$P_2(y_1,y_2,y_3)^n=2^nt^{2an}\left(1-nt^{-(a-b)}+o\left(t^{-(b-c)}\right)\right).$$

Similarly,

$$\begin{aligned}
&P_2(y_4,y_3,y_2)^n=2^nt^{2bn}\left(1+o\left(t^0\right)\right),\\
&P_2(y_3,y_4,y_1)^n=2^nt^{2an}\left(1+o\left(t^0\right)\right),\\
&P_2(y_2,y_1,y_4)^n=2^nt^{2an}\left(1-nt^{-(a-b)}+o\left(t^{-(b-c)}\right)\right),\\
&P_3(y_1,y_2,y_3)^{2m+3}=-t^{(2a+b)(2m+3)}\left(1-(2m+3)\left(t^{-(a-b)}+t^{-(b-c)}\right)+o\left(t^{-(b-c)}\right)\right),\\
&P_3(y_4,y_3,y_2)^{2m+3}=t^{(2b+c)(2m+3)}\left(1+o\left(t^0\right)\right),\\
&P_3(y_3,y_4,y_1)^{2m+3}=-t^{(2a+c)(2m+3)}\left(1+o\left(t^0\right)\right),\\
&P_3(y_2,y_1,y_4)^{2m+3}=t^{(2a+b)(2m+3)}\left(1-(2m+3)t^{-(a-b)}+o\left(t^{-(b-c)}\right)\right),\\
&P_4(y_1,y_2,y_3,y_4)^k=t^{(2a+2b)k}\left(1+o\left(t^0\right)\right),\\
&P_4(y_1,y_3,y_2,y_4)^k=(-1)^kt^{(2a+b+c)k}\left(1+o\left(t^0\right)\right),\\
&P_4(y_1,y_4,y_2,y_3)^k=t^{(2a+b+c)k}\left(1+o\left(t^0\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&P_2(y_1,y_2,y_3)^nP_3(y_1,y_2,y_3)^{2m+3}+P_2(y_4,y_3,y_2)^nP_3(y_4,y_3,y_2)^{2m+3}\\
&\qquad+P_2(y_3,y_4,y_1)^nP_3(y_3,y_4,y_1)^{2m+3}+P_2(y_2,y_1,y_4)^nP_3(y_2,y_1,y_4)^{2m+3}\\
&\quad=-2^nt^{2an+(2a+b)(2m+3)}\left(1-(n+2m+3)t^{-(a-b)}-(2m+3)t^{-(b-c)}+o\left(t^{-(b-c)}\right)\right)\\
&\qquad+2^nt^{2bn+(2b+c)(2m+3)}\left(1+o\left(t^0\right)\right)\\
&\qquad-2^nt^{2an+(2a+c)(2m+3)}\left(1+o\left(t^0\right)\right)\\
&\qquad+2^nt^{2an+(2a+b)(2m+3)}\left(1-(n+2m+3)t^{-(a-b)}+o\left(t^{-(b-c)}\right)\right)\\
&\quad=2^n(2m+3)t^{2an+(2a+b)(2m+3)-(b-c)}\left(1+o\left(t^0\right)\right),
\end{aligned}$$

noting that we need $2m+3>1$ when we verify that

$$2an+(2a+b)(2m+3)-(b-c)>2an+(2a+c)(2m+3).$$

Further,

$$P_4(y_1,y_2,y_3,y_4)^k+P_4(y_1,y_3,y_2,y_4)^k+P_4(y_1,y_4,y_2,y_3)^k=\varepsilon t^{(2a+2b)k}\left(1+o\left(t^0\right)\right),$$

where $\varepsilon=1$ if $k>0$, and $\varepsilon=3$ if $k=0$. Therefore,

$$Q^{n,m,k}=\varepsilon 2^n(2m+3)t^{2(n+2m+k+3)a+2(m+k+1)b+c}\left(1+o\left(t^0\right)\right).$$

This implies that the only possible linear relations between the $Q^{n,m,k}$'s are between those having the same value of $(n+2m+k,m+k)$, and in particular the same value of $m+k$. In other words, the vector space which we are considering is presented by the direct sum:

$$\operatorname{span}\left\{Q^{n,m,k}\mid n+2k+3m=d\right\}=\bigoplus_{\ell}\operatorname{span}\left\{Q^{n,m,k}\mid n+2k+3m=d,\ m+k=\ell\right\}.$$

Next, in order to complete the proof of the linear independence of the $Q^{n,m,k}$'s, we make another substitution

$$\begin{aligned}
y_1 &= \left(2t^a - t^c\right)/4 + t^b/2,\\
y_2 &= \left(2t^a - t^c\right)/4 - t^b/2,\\
y_3 &= \left(-2t^a + 3t^c\right)/4,\\
y_4 &= -\left(2t^a + t^c\right)/4,
\end{aligned}$$

where t is as above, and a, b, c are real numbers satisfying that $a > b > c > 0$ and $b - c < a - b < 2(b - c)$. Since

$$\begin{aligned}
y_1 - y_4 &= t^a + t^b/2,\\
y_2 - y_4 &= t^a - t^b/2,\\
y_3 - y_4 &= t^c,\\
y_1 - y_2 &= t^b,
\end{aligned}$$

we have that

$$\begin{aligned}
P_2(y_1, y_2, y_3)^n &= 2^n t^{2an}\left(1 - 2nt^{-(a-c)} + o\left(t^{-(a-c)}\right)\right),\\
P_2(y_4, y_3, y_2)^n &= 2^n t^{2an}\left(1 - nt^{-(a-b)} - nt^{-(a-c)} + o\left(t^{-(a-c)}\right)\right),\\
P_2(y_3, y_4, y_1)^n &= 2^n t^{2an}\left(1 + nt^{-(a-b)} - nt^{-(a-c)} + o\left(t^{-(a-c)}\right)\right),\\
P_2(y_2, y_1, y_4)^n &= 2^n t^{2an}\left(1 + o\left(t^{-(a-c)}\right)\right),\\
P_3(y_1, y_2, y_3)^{2m+3} &= -t^{(2a+b)(2m+3)}\left(1 - 2(2m+3)t^{-(a-c)} + o\left(t^{-(a-c)}\right)\right),\\
P_3(y_4, y_3, y_2)^{2m+3} &= t^{(2a+c)(2m+3)}\left(1 - (2m+3)\left(t^{-(a-b)} + t^{-(a-c)}\right) + o\left(t^{-(a-c)}\right)\right),\\
P_3(y_3, y_4, y_1)^{2m+3} &= -t^{(2a+c)(2m+3)}\left(1 + (2m+3)\left(t^{-(a-b)} - t^{-(a-c)}\right) + o\left(t^{-(a-c)}\right)\right),\\
P_3(y_2, y_1, y_4)^{2m+3} &= t^{(2a+b)(2m+3)}\left(1 + o\left(t^{-(a-c)}\right)\right),\\
P_4(y_1, y_2, y_3, y_4)^k &= t^{4ak}\left(1 + o\left(t^0\right)\right),\\
P_4(y_1, y_3, y_2, y_4)^k &= (-1)^k t^{(2a+b+c)k}\left(1 + o\left(t^0\right)\right),\\
P_4(y_1, y_4, y_2, y_3)^k &= t^{(2a+b+c)k}\left(1 + o\left(t^0\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&P_2(y_1, y_2, y_3)^n P_3(y_1, y_2, y_3)^{2m+3} + P_2(y_4, y_3, y_2)^n P_3(y_4, y_3, y_2)^{2m+3}\\
&\qquad + P_2(y_3, y_4, y_1)^n P_3(y_3, y_4, y_1)^{2m+3} + P_2(y_2, y_1, y_4)^n P_3(y_2, y_1, y_4)^{2m+3}\\
&\quad = -2^n t^{2an+(2a+b)(2m+3)}\left(1 - 2(n+2m+3)t^{-(a-c)} + o\left(t^{-(a-c)}\right)\right)\\
&\qquad + 2^n t^{2an+(2a+c)(2m+3)}\left(1 - (n+2m+3)\left(t^{-(a-b)} + t^{-(a-c)}\right) + o\left(t^{-(a-c)}\right)\right)\\
&\qquad - 2^n t^{2an+(2a+c)(2m+3)}\left(1 + (n+2m+3)\left(t^{-(a-b)} - t^{-(a-c)}\right) + o\left(t^{-(a-c)}\right)\right)\\
&\qquad + 2^n t^{2an+(2a+b)(2m+3)}\left(1 + o\left(t^{-(a-c)}\right)\right)\\
&\quad = 2^{n+1}(n+2m+3)t^{2an+(2a+b)(2m+3)-(a-c)}\left(1 + o\left(t^0\right)\right).
\end{aligned}$$

Further,

$$P_4(y_1, y_2, y_3, y_4)^k + P_4(y_1, y_3, y_2, y_4)^k + P_4(y_1, y_4, y_2, y_3)^k = \varepsilon t^{4ak}\left(1 + o\left(t^0\right)\right),$$

where $\varepsilon = 1$ if $k > 0$, and $\varepsilon = 3$ if $k = 0$. Therefore,

$$Q^{n,m,k} = \varepsilon 2^{n+1}(n+2m+3)t^{(2(n+2m+2k)+5)a+(2m+3)b+c}\bigl(1+o\bigl(t^0\bigr)\bigr).$$

This implies that the only possible linear relations between the $Q^{n,m,k}$'s are between those having the same value of $(n+2m+2k, m)$.

Thus, the only linear relations that could exist between $Q^{n,m,k}$'s with fixed $n+2k+3m=d$ are between those having the same value $m+k$ (from the first substitution) and the same values of m (from the second substitution). In other words,

$$\operatorname{span}\bigl\{Q^{n,m,k} \mid n+2k+3m=d\bigr\} = \bigoplus_{n+2k+3m=d} \operatorname{span}\bigl\{Q^{n,m,k}\bigr\}.$$

It follows that the $Q^{n,m,k}$'s are indeed linearly independent, as required. $\square$

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