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Alternative quantization of the Hamiltonian in loop quantum cosmology

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ABSTRACT

Since there are quantization ambiguities in constructing the Hamiltonian constraint operator in isotropic loop quantum cosmology, it is crucial to check whether the key features of loop quantum cosmology are robust against the ambiguities. In this Letter, we quantize the Lorentz term of the gravitational Hamiltonian constraint in the spatially flat FRW model by two approaches different from that of the Euclidean term. One of the approaches is very similar to the treatment of the Lorentz part of Hamiltonian in loop quantum gravity and hence inherits more features from the full theory. Two symmetric Hamiltonian constraint operators are constructed respectively in the improved scheme. Both of them are shown to have the correct classical limit by the semiclassical analysis. In the loop quantum cosmological model with a massless scalar field, the effective Hamiltonians and Friedmann equations are derived. It turns out that the classical big bang is again replaced by a quantum bounce in both cases. Moreover, there are still great possibilities for the expanding universe to recollapse due to the quantum gravity effect.

1. Introduction

An important motivation of the theoretical search for a quantum theory of gravity is the expectation that the singularities predicted by classical general relativity would be resolved by the quantum gravity theory. This expectation has been confirmed by the recent study of certain isotropic models in loop quantum cosmology (LQC) [1–3], which is a simplified symmetry-reduced model of a full background-independent quantum theory of gravity [4], known as loop quantum gravity (LQG) [5–8]. In loop quantum cosmological scenario for a universe filled with a massless scalar field, the classical singularity gets replaced by a quantum bounce [3,9,10]. Moreover, It is also revealed in the effective scenarios that there are great possibilities for a spatially flat FRW expanding universe to recollapse due to the quantum gravity effect [11]. However, as in the ordinary quantization procedure, there are quantization ambiguities in constructing the Hamiltonian constraint operator. Thus it is crucial to check the robustness of the key results against the quantization ambiguities. Moreover, since LQC serves as a simple arena to test ideas and constructions induced in the full LQG, it is important to implement those treatments from the full theory to LQC as more as possible.

In this Letter we will propose alternative versions of Hamiltonian operator for isotropic LQC, which inherit more features from full LQG comparing to the conventional one so far considered in the literatures. It is well known that the Hamiltonian constraint in the full theory is composed of two terms, the so-called Euclidean and Lorentz terms. In spatially flat and homogeneous models, the two terms are proportional to each other. Thus people usually rewrite the Lorentz term in the form of the Euclidean one classically and then quantize their combination [3]. However, this treatment is impossible in the full theory, where the Lorentz term has to be quantized in a form quite different from the Euclidean one [7,12]. The issue that we are considering is what would happen in the improved dynamics setting of LQC if one kept the distinction of the two terms as in full theory rather than mixed them. Could one construct an operator corresponding to the Lorentz term in a way similar to that in full LQG? If so, could the classical big bang singularity still be replaced by a quantum bounce in the new quantum dynamics? To answer these questions, two alternative Hamiltonian constraint operators including the Lorentz terms are constructed respectively in the improved scheme in this Letter, which are both shown to have the correct classical limit by the semiclassical analysis. In the spatially flat FRW model with a massless scalar field, the effective Hamiltonians and Friedmann equations are derived in both case. It turns out that the classical big bang is again replaced by a quantum bounce. Moreover, there are still great possibilities for the expanding universe to recollapse due to the quantum gravity effect.

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In the spatially flat and isotropic model, one has to first introduce an elementary cell $\mathcal{V}$ and restrict all integrations to this cell. Fix a fiducial flat metric ${}^o q_{ab}$ and denote by V_o the volume of the elementary cell $\mathcal{V}$ in this geometry. The gravitational phase space variables—the connections A_a^i and the density-weighted triads E_i^a—can be expressed as [13]

$$A_a^i = cV_o^{-1/3}\,{}^o\omega_a^i \quad \text{and} \quad E_i^a = pV_o^{-2/3}\sqrt{{}^o q}\,{}^o e_i^a, \tag{1}$$

where $({}^o\omega_a^i, {}^o e_i^a)$ are a set of orthonormal co-triads and triads compatible with ${}^o q_{ab}$ and adapted to the edges of the elementary cell $\mathcal{V}$. In terms of p, the physical triad and cotriad are given by

$$e_i^a = \mathrm{sgn(p)}|p|^{-1/2}V_o^{1/3}\,{}^o e_i^a,$$
$$e_a^i = \mathrm{sgn(p)}|p|^{1/2}V_o^{-1/3}\,{}^o\omega_a^i. \tag{2}$$

The basic (nonvanishing) Poisson bracket is given by

$$\{c, p\} = \frac{\kappa\gamma}{3}, \tag{3}$$

where $\kappa = 8\pi G$ (G is the Newton's constant) and γ is the Barbero–Immirzi parameter.

To pass to the quantum theory, one constructs a kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}} = L^2(\mathbb{R}_{\mathrm{Bohr}}, \mathrm{d}\mu_{\mathrm{Bohr}})$, where $\mathbb{R}_{\mathrm{Bohr}}$ is the Bohr compactification of the real line and $\mathrm{d}\mu_{\mathrm{Bohr}}$ is the Haar measure on it [13]. The abstract $*$-algebra represented on the Hilbert space is based on the holonomies of the connection A_a^i. In the Hamiltonian constraint of LQG, the gravitational connection A_a^i appears through its curvature F_{ab}^i. Since there exists no operator corresponding to c, only holonomy operators are well defined. Hence one is led to express the curvature in terms of holonomies. Similarly, in the improved dynamics setting of LQC [3], to express the curvature one employed the holonomies

$$h_i^{(\bar{\mu})} := \cos\frac{\bar{\mu}c}{2}\mathbb{I} + 2\sin\frac{\bar{\mu}c}{2}\tau_i \tag{4}$$

along an edge parallel to the triad ${}^o e_i^a$ of length $\bar{\mu}\sqrt{|p|} \equiv D$, where D is a constant, with respect to the physical metric q_{ab}, where $\mathbb{I}$ is the identity 2×2 matrix and $\tau_i = -i\sigma_i/2$ (σ_i are the Pauli matrices). Note that there are ambiguities in assigning a constant to the length of the edge. In more general treatments proposed in Ref. [14], certain function $f(p)$ would be assigned to a constant. However, here we consider only the above improved treatment. Thus, the elementary variables could be taken as the functions $\exp(i\bar{\mu}c/2)$ and the physical volume $V = |p|^{3/2}$ of the cell, which have unambiguous operator analogs. It is convenient to work with the v-representation. In this representation, states $|v\rangle$, constituting an orthonormal basis in $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$, is more directly adapted to the volume operator $\hat{V}$ as

$$\hat{V}|v\rangle = \left(\frac{8\pi\gamma\ell_p^2}{6}\right)^{3/2}\frac{|v|}{L'}|v\rangle, \tag{5}$$

where $\ell_p^2 = G\hbar$ and

$$L' = \frac{4}{3}\sqrt{\frac{\pi\gamma\ell_p^2}{3D}}. \tag{6}$$

The action of $\widehat{\exp(i\bar{\mu}c/2)}$ is given by

$$\widehat{\exp(i\bar{\mu}c/2)}|v\rangle = |v+1\rangle. \tag{7}$$

Now let us consider the gravitational field coupled with a massless scalar field ϕ. The Hamiltonian of the matter field is given by $H_\phi = |p|^{-3/2}p_\phi^2/2$, where p_ϕ denotes the momentum of ϕ. Hence the total Hamiltonian constraint is given by $H = H_{\mathrm{grav}} + H_\phi$. The basic Poisson bracket for the matter field is given by

$$\{\phi, p_\phi\} = 1. \tag{8}$$

The Hamiltonian evolution equations for the matter field read

$$\dot{p}_\phi = \{p_\phi, H_\phi\} = 0 \Longrightarrow p_\phi = \text{constant},$$
$$\dot{\phi} = \{\phi, H_\phi\} = \frac{p_\phi}{|p|^{3/2}}, \tag{9}$$

which show that ϕ is monotonic function of the time parameter. So the scalar field can be regarded as internal time. To quantize the matter field, we can choose the standard Schrödinger representation for scalar field. The kinematical Hilbert space, $\mathcal{H}_\phi = L^2(\mathbb{R}, \mathrm{d}\phi)$, is the space of the square integrable functions on $\mathbb{R}$ endowed with the Lebesgue measure. Hence the kinematical Hilbert space of the gravitational field coupled with the scalar field is $\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}} \otimes \mathcal{H}_\phi$. The elementary operators of the scalar field are defined by:

$$(\hat{\phi}\Psi)(v,\phi) := \phi\Psi(v,\phi),$$
$$(\hat{p}_\phi\Psi)(v,\phi) := -i\hbar\frac{d}{d\phi}\Psi(v,\phi), \qquad \forall\Psi(v,\phi) \in \mathcal{H}_{\mathrm{kin}}. \tag{10}$$

In the following sections, we will construct two different Hamiltonian operators including the Lorentz term in the above quantum kinematic framework. Their classical limits are confirmed by calculating the expectation values of these new Hamiltonian operators with respect to suitable semiclassical states. By this approach we also obtain the effective descriptions of quantum dynamics in both cases. In the end we will summarize the results and discuss some of their ramifications.

2. Alternative regularized Hamiltonian constraints

Because of homogeneity, we can assume that the lapse N is constant and from now onwards set it to be one. The gravitational Hamiltonian constraint is given by

$$H_{\mathrm{grav}} = \int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}\left[\epsilon_{ijk}F_{ab}^i - 2\left(1+\gamma^2\right)K_{[a}^j K_{b]}^k\right]$$
$$\equiv H^E(1) - 2\left(1+\gamma^2\right)T(1), \tag{11}$$

where F_{ab} is the curvature (the field strength) of connection A_a^i, and K_a^i is the extrinsic curvature. The symmetry-reduced expressions read

$$H^E(1) = \int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}\epsilon_{ijk}F_{ab}^i = \frac{3}{\kappa}c^2\sqrt{|p|},$$
$$T(1) = \int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}K_{[a}^j K_{b]}^k = \frac{3}{2\kappa\gamma^2}c^2\sqrt{|p|},$$
$$H_{\mathrm{grav}} = H^E(1) - 2\left(1+\gamma^2\right)T(1) = -\frac{3}{\kappa\gamma^2}c^2\sqrt{|p|}. \tag{12}$$

In the following, we will discuss alternative regularization of the gravitational Hamiltonian constraint.

2.1. Regularization ambiguity of the field strength of the gravitational connection

From Eq. (12), it is easy to see that the Euclidean term $H^E(1)$ is classically proportional to the Lorentz term $T(1)$ in the spatially

flat and isotropic model. Thus people usually rewrite the Lorentz term in the form of the Euclidean one and then quantize their combination [3]. However, there exists a typical quantization ambiguity arising from the quantization of the field strength of the gravitational connection. We now introduce an alternative method to regularize it. Due to the spatial homogeneity and isotropy, one can fix the spin connection as $\Gamma_a^i = 0$, which leads to

$$A_a^i = \Gamma_a^i + \gamma K_a^i = \gamma K_a^i, \tag{13}$$

$$F_{ab}^i = 2\partial_{[a}A_{b]}^i + \epsilon^i{}_{jk}A_a^j A_b^k = \epsilon^i{}_{jk}A_a^j A_b^k, \tag{14}$$

where an internal gauge fixing is employed to get Eq. (14). Thus the Hamiltonian constraint can be simplified as

$$\begin{aligned} H_{\mathrm{grav}} &= \int_{\mathcal{V}} d^3x \frac{\epsilon^{ijk}E_j^a E_k^b}{2\kappa\sqrt{\det(q)}}\left[\epsilon_{ilm}A_a^l A_b^m - \frac{1+\gamma^2}{\gamma^2}\epsilon_{ilm}A_a^l A_b^m\right] \\ &= -\int_{\mathcal{V}} d^3x \frac{\epsilon^{ijk}E_j^a E_k^b}{2\kappa\gamma^2\sqrt{\det(q)}}\epsilon_{ilm}A_a^l A_b^m. \end{aligned} \tag{15}$$

It is easy to check the following identity [7,12],

$$\frac{\epsilon_{ijk}E^{aj}E^{bk}}{\sqrt{\det(q)}} = \frac{2}{\kappa\gamma}\tilde{\epsilon}^{abc}\{A_c^i, V\}, \tag{16}$$

where $\tilde{\epsilon}^{abc}$ is the Levi-Civita density. Putting Eq. (16) into (15), we obtain

$$H_{\mathrm{grav}} = \int_{\mathcal{V}} d^3x \frac{4}{\kappa^2\gamma^3}\tilde{\epsilon}^{abc}\,\mathrm{Tr}\big(A_a A_b\{A_c, V\}\big). \tag{17}$$

Now in order to quantize H_{grav}, we have to replace the expression with the elementary variables. Taking account of the definition (4) of holonomies, we have the identities

$$c\tau_i = \lim_{\bar{\mu}\to 0}\frac{1}{2\bar{\mu}}\big[h_i^{(\bar{\mu})} - h_i^{(\bar{\mu})^{-1}}\big], \tag{18}$$

and

$$\{c\tau_k, V\} = -\frac{1}{\bar{\mu}}h_k^{(\bar{\mu})}\{h_k^{(\bar{\mu})^{-1}}, V\}. \tag{19}$$

Eqs. (18) and (19) imply that the Hamiltonian constraint (17) can be written as

$$H_{\mathrm{grav}} = \lim_{\bar{\mu}\to 0} H^{\mathrm{R}(\bar{\mu})}_{\mathrm{grav}}, \tag{20}$$

where

$$\begin{aligned} H^{\mathrm{R}(\bar{\mu})}_{\mathrm{grav}} = -\frac{\mathrm{sgn}(p)}{\kappa^2\gamma^3\bar{\mu}^3}\epsilon^{ijk}\,\mathrm{Tr}\big[&\big(h_i^{(\bar{\mu})} - h_i^{(\bar{\mu})^{-1}}\big) \\ &\times \big(h_j^{(\bar{\mu})} - h_j^{(\bar{\mu})^{-1}}\big)h_k^{(\bar{\mu})}\{h_k^{(\bar{\mu})^{-1}}, V\}\big], \end{aligned} \tag{21}$$

from which an alternative quantization of the Hamiltonian constraint can be straitforwardly implemented [15].

2.2. Regularization ambiguities of the Hamiltonian constraint

In the following, we will regularize the two terms of the gravitational Hamiltonian constraint in Eq. (12) respectively. From Eq. (13) one obtains the classically equivalent expression of Eq. (11) as

$$H^S_{\mathrm{grav}} = H^E(1) - 2\big(1+\gamma^2\big)T_S(1), \tag{22}$$

where

$$T_S(1) = \frac{1}{\gamma^2}\int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}A_{[a}^j A_{b]}^k. \tag{23}$$

Since the expressions (11) and (22) inherit more features of the Hamiltonian constraint in the full theory, we will take them separately as the starting-points of our quantization. To this aim, the first step is to give their regularized expressions which would be suitable for quantization. Note that the improved quantum operator representing the Euclidean Hamiltonian constraint $H^E(1)$ was first introduced in [3], and its regularized formulation reads

$$\begin{aligned} H^{E,\bar{\mu}}(1) = \frac{2\,\mathrm{sgn}(p)}{\kappa^2\gamma\bar{\mu}^3}\epsilon^{ijk}\,\mathrm{Tr}\big(&h_i^{(\bar{\mu})}h_j^{(\bar{\mu})}h_i^{(\bar{\mu})^{-1}}h_j^{(\bar{\mu})^{-1}} \\ &\times h_k^{(\bar{\mu})}\{h_i^{(\bar{\mu})^{-1}}, V\}\big). \end{aligned} \tag{24}$$

Now our task is to give the regularized formulations of the Lorentz terms $T_S(1)$ and $T(1)$ in Eqs. (22) and (11) respectively. Let us first deal with the symmetry-reduced Lorentzian term $T_S(1)$. Using Eq. (16), Eq. (23) can be written as

$$\begin{aligned} T_S(1) &= -\frac{2}{\kappa^2\gamma^3}\int_{\mathcal{V}} d^3x\,\tilde{\epsilon}^{abc}\,\mathrm{Tr}\big(A_a A_b\{A_c, V\}\big) \\ &= -\frac{2\,\mathrm{sgn}(p)}{\kappa^2\gamma^3}\epsilon^{ijk}\,\mathrm{Tr}\big(c\tau_i c\tau_j\{c\tau_k, V\}\big). \end{aligned} \tag{25}$$

By the identities (18) and (19), Eq. (25) can be written as

$$T_S(1) = \lim_{\bar{\mu}\to 0} T_S^{\bar{\mu}}(1), \tag{26}$$

where

$$\begin{aligned} T_S^{\bar{\mu}}(1) = \frac{\mathrm{sgn}(p)}{2\kappa^2\gamma^3\bar{\mu}^3}\epsilon^{ijk}\,\mathrm{Tr}\big(&\big[h_i^{(\bar{\mu})} - h_i^{(\bar{\mu})^{-1}}\big] \\ &\times \big[h_j^{(\bar{\mu})} - h_j^{(\bar{\mu})^{-1}}\big]h_k^{(\bar{\mu})}\{h_k^{(\bar{\mu})^{-1}}, V\}\big) \end{aligned} \tag{27}$$

Putting Eqs. (24) and (27) together, we obtain the regularized Hamiltonian constraint corresponding to (22) as

$$H^{S,\bar{\mu}}_{\mathrm{grav}} = H^{E,\bar{\mu}}(1) - 2\big(1+\gamma^2\big)T_S^{\bar{\mu}}(1). \tag{28}$$

Let us now turn to the original Lorentz term in Eq. (11):

$$T(1) = \int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}K_{[a}^j K_{b]}^k. \tag{29}$$

Though this term was considered in some early literature [16,17], here we will treat it in the new improved quantization framework [3]. As in the full theory [7,12], the extrinsic curvature can be written as

$$K_a^i = \frac{1}{\kappa\gamma}\{A_a^i, K\} = \frac{1}{\kappa\gamma^3}\{A_a^i, \{H^E(1), V\}\}, \tag{30}$$

where

$$K = \int_{\mathcal{V}} d^3x K_a^i E_i^a = \frac{3}{\gamma}cp \tag{31}$$

is the integrated trace of K_a^i. Hence Eq. (29) can be reexpressed as

$$\begin{aligned} T(1) &= -\frac{2}{\kappa^4\gamma^3}\int_{\mathcal{V}} d^3x\,\tilde{\epsilon}^{abc}\,\mathrm{Tr}\big(\{A_a, K\}\{A_b, K\}\{A_c, V\}\big) \\ &= -\frac{2\,\mathrm{sgn}(p)}{\kappa^4\gamma^3}\epsilon^{ijk}\,\mathrm{Tr}\big(\{c\tau_i, K\}\{c\tau_j, K\}\{c\tau_k, V\}\big). \end{aligned} \tag{32}$$

Moreover, we have the following identity

$$\{c\tau_i, K\} = -\frac{2}{3\bar{\mu}} h_i^{(\bar{\mu})} \{h_i^{(\bar{\mu})^{-1}}, K\}. \tag{33}$$

Using the identities (19) and (33), Eq. (32) can be written as

$$T(1) = \lim_{\bar{\mu}\to 0} T_F^{\bar{\mu}}(1), \tag{34}$$

where

$$\begin{aligned} T_F^{\bar{\mu}}(1) = {} & \frac{8\,\mathrm{sgn}(p)}{9\kappa^4\gamma^7\bar{\mu}^3}\epsilon^{ijk}\,\mathrm{Tr}\big(h_i^{(\bar{\mu})}\{h_i^{(\bar{\mu})^{-1}}, \{H^{E,\bar{\mu}}(1), V\}\} \\ & \times h_j^{(\bar{\mu})}\{h_j^{(\bar{\mu})^{-1}}, \{H^{E,\bar{\mu}}(1), V\}\} \\ & \times h_k^{(\bar{\mu})}\{h_k^{(\bar{\mu})^{-1}}, V\}\big). \end{aligned} \tag{35}$$

Putting Eqs. (24) and (35) together, we obtain the regularized Hamiltonian constraint corresponding to (11) as

$$H^{F,\bar{\mu}}_{\mathrm{grav}} = H^{E,\bar{\mu}}(1) - 2\big(1+\gamma^2\big)T_F^{\bar{\mu}}(1). \tag{36}$$

3. Alternative Hamiltonian constraint operators

Since both regularized Hamiltonian constraints (28) and (36) are now expressed in terms of elementary variables and their Poisson brackets, which have unambiguous quantum analogs, it is straightforward to write down the quantum operators $\hat{H}^{S,\bar{\mu}}_{\mathrm{grav}}$ and $\hat{H}^{F,\bar{\mu}}_{\mathrm{grav}}$. However, the limit $\bar{\mu}\to 0$ of these operators do not exist, not only for the Euclidean term $\hat{H}^{E,\bar{\mu}}(1)$, but also for the Lorentzian term $\hat{T}_S^{\bar{\mu}}(1)$ or $\hat{T}_F^{\bar{\mu}}(1)$. In fact, even in the full theory, there are no local operators representing connections and curvatures. To get unambiguous operators, one should have recourse to the area gap as in the improved scheme [3], where $\bar{\mu}$ is given by

$$\bar{\mu}^2|p| = \Delta, \tag{37}$$

here $\Delta = 4\sqrt{3}\pi\gamma\ell_p^2$ is a minimum nonzero eigenvalue of the area operator [10]. The Euclidean Hamiltonian constraint operator $\hat{H}^E(1)$ corresponding to (24) is given in the improved scheme by [3]

$$\begin{aligned} \hat{H}^E(1) = {} & -\frac{\gamma^2}{2\kappa}\sin(\bar{\mu}c)\Bigg[\frac{24i\,\mathrm{sgn}(v)}{\kappa\hbar\gamma^3\bar{\mu}^3}\Big(\sin\Big(\frac{\bar{\mu}c}{2}\Big)\hat{V}\cos\Big(\frac{\bar{\mu}c}{2}\Big) \\ & - \cos\Big(\frac{\bar{\mu}c}{2}\Big)\hat{V}\sin\Big(\frac{\bar{\mu}c}{2}\Big)\Big)\Bigg]\sin(\bar{\mu}c), \end{aligned} \tag{38}$$

where, for clarity, we have suppressed hats over the operators $\sin(\bar{\mu}c/2)$, $\cos(\bar{\mu}c/2)$ and $\mathrm{sgn}(v)/\bar{\mu}^3$. In the v-representation where

$$v := \frac{\mathrm{sgn}(p)|p|^{3/2}}{2\pi\gamma\ell_p^2\sqrt{\Delta}}, \tag{39}$$

$\hat{H}^E(1)$ acts on the basis $|v\rangle$ of $\mathcal{H}^{\mathrm{grav}}_{\mathrm{kin}}$ as

$$\begin{aligned} & \hat{H}^E(1)|v\rangle \\ & = -\frac{\gamma^2}{2\kappa}\big[f_+(v)|v+4\rangle + f_o(v)|v\rangle + f_-(v)|v-4\rangle\big], \end{aligned} \tag{40}$$

where

$$\begin{aligned} & f_+(v) = \frac{27}{16}\sqrt{\frac{8\pi}{6}}\frac{L\ell_p}{\gamma^{3/2}}\big(|v+3| - |v+1|\big)(v+2), \\ & f_-(v) = f_+(v-4), \qquad f_o(v) = -f_+(v) - f_-(v), \end{aligned} \tag{41}$$

here

$$L = \frac{4}{3}\sqrt{\frac{\pi\gamma\ell_p^2}{3\Delta}}. \tag{42}$$

Now we turn to the Lorentz part. The regularized expression (27) enables us to define a self-adjoint operator on $\mathcal{H}^{\mathrm{grav}}_{\mathrm{kin}}$ in the improved scheme corresponding to (23) as

$$\begin{aligned} \hat{T}_S(1) = {} & -\sin\frac{\bar{\mu}c}{2}\Bigg[\frac{24i\,\mathrm{sgn}(v)}{\kappa^2\hbar\gamma^3\bar{\mu}^3}\Big(\sin\frac{\bar{\mu}c}{2}\hat{V}\cos\frac{\bar{\mu}c}{2} \\ & - \cos\frac{\bar{\mu}c}{2}\hat{V}\sin\frac{\bar{\mu}c}{2}\Big)\Bigg]\sin\frac{\bar{\mu}c}{2}. \end{aligned} \tag{43}$$

Its action on the basis $|v\rangle$ reads

$$\hat{T}_S(1)|v\rangle = S_+(v)|v+2\rangle + S_o(v)|v\rangle + S_-(v)|v-2\rangle, \tag{44}$$

where

$$\begin{aligned} & S_+(v) = -\frac{27}{16}\sqrt{\frac{8\pi}{6}}\frac{L\ell_p}{\kappa\gamma^{3/2}}(v+1)\big(|v+2| - |v|\big), \\ & S_-(v) = S_+(v-2), \qquad S_o(v) = -S_+(v) - S_-(v). \end{aligned} \tag{45}$$

Hence we obtain the action of the Hamiltonian constraint operator corresponding to (22) on $|v\rangle$ as

$$\begin{aligned} \hat{H}^S_{\mathrm{grav}}|v\rangle = {} & \hat{H}^E_{\mathrm{grav}}(1)|v\rangle - 2\big(1+\gamma^2\big)\hat{T}_S(1)|v\rangle \\ = {} & f'_+(v)|v+4\rangle + S'_+(v)|v+2\rangle \\ & + \big[f'_o(v) + S'_o(v)\big]|v\rangle \\ & + S'_-(v)|v-2\rangle + f'_-(v)|v-4\rangle, \end{aligned} \tag{46}$$

where $f'_*(v) = -\frac{\gamma^2}{2\kappa}f_*(v)$, $S'_*(v) := -2(1+\gamma^2)S_*(v)$, here $* = +, -, o$.

On the other hand, the regularized expression (35) enables us to define the other operator on $\mathcal{H}^{\mathrm{grav}}_{\mathrm{kin}}$ corresponding to (29) as

$$\begin{aligned} \hat{T}_F(1) = {} & -\frac{96i}{9\kappa^4\gamma^7\hbar^5}\Big(\sin\frac{\bar{\mu}c}{2}\hat{B}\cos\frac{\bar{\mu}c}{2} - \cos\frac{\bar{\mu}c}{2}\hat{B}\sin\frac{\bar{\mu}c}{2}\Big) \\ & \times\Bigg[\frac{\mathrm{sgn}(v)}{\bar{\mu}^3}\Big(\sin\frac{\bar{\mu}c}{2}\hat{V}\cos\frac{\bar{\mu}c}{2} - \cos\frac{\bar{\mu}c}{2}\hat{V}\sin\frac{\bar{\mu}c}{2}\Big)\Bigg] \\ & \times\Big(\sin\frac{\bar{\mu}c}{2}\hat{B}\cos\frac{\bar{\mu}c}{2} - \cos\frac{\bar{\mu}c}{2}\hat{B}\sin\frac{\bar{\mu}c}{2}\Big), \end{aligned} \tag{47}$$

where

$$\hat{B} \equiv \big[\hat{H}^E(1), \hat{V}\big]. \tag{48}$$

It is easy to see from Eq. (47) that $\hat{T}_F(1)$ is a symmetric operator. Its action on $|v\rangle$ reads

$$\begin{aligned} \hat{T}_F(1)|v\rangle = {} & \frac{\sqrt{6}}{2^8\times 3^3}\frac{\gamma^{3/2}}{\kappa^{3/2}\hbar^{1/2}}\frac{1}{L}\big(F_+(v)|v+8\rangle + F_o(v)|v\rangle \\ & + F_-(v)|v-8\rangle\big), \end{aligned} \tag{49}$$

where

$$\begin{aligned} F_+(v) = {} & \big[M_v(1,5)f_+(v+1) - M_v(-1,3)f_+(v-1)\big] \\ & \times(v+4)M_v(3,5) \\ & \times\big[M_v(5,9)f_+(v+5) - M_v(3,7)f_+(v+3)\big], \\ F_-(v) = {} & \big[M_v(1,-3)f_-(v+1) - M_v(-1,-5)f_-(v-1)\big] \\ & \times(v-4)M_v(-5,-3) \\ & \times\big[M_v(-3,-7)f_-(v-3) - M_v(-5,-9)f_-(v-5)\big], \end{aligned}$$

$$
\begin{aligned}
F_o(v) = {} & \big[M_v(1,5) f_+(v+1) - M_v(-1,3) f_+(v-1)\big] \\
& \times (v+4) M_v(3,5) \\
& \times \big[M_v(5,1) f_-(v+5) - M_v(3,-1) f_-(v+3)\big] \\
& + \big[M_v(1,-3) f_-(v+1) - M_v(-1,-5) f_-(v-1)\big] \\
& \times (v-4) M_v(-5,-3) \\
& \times \big[M_v(-3,1) f_+(v-3) - M_v(-5,-1) f_+(v-5)\big],
\end{aligned} \tag{50}
$$

here

$$
M_v(a,b) := |v+a| - |v+b|. \tag{51}
$$

Hence the action of the Hamiltonian constraint operator corresponding to (11) on $|v\rangle$ is given by

$$
\begin{aligned}
\hat{H}^F_{\rm grav}|v\rangle &= \hat{H}^E_{\rm grav}(1)|v\rangle - 2\big(1+\gamma^2\big)\hat{T}_F(1)|v\rangle \\
&= F'_+(v)|v+8\rangle + f'_+(v)|v+4\rangle \\
&\quad + \big[F'_o(v) + f'_o(v)\big]|v\rangle \\
&\quad + f'_-(v)|v-4\rangle + F'_-(v)|v-8\rangle,
\end{aligned} \tag{52}
$$

where $F'_*(v) := -2(1+\gamma^2)\frac{\sqrt{6}}{2^8\times 3^3}\frac{\gamma^{3/2}}{\kappa^{3/2}\hbar^{1/2}}\frac{1}{L}F_*(v)$, here $* = +, -, o$.

Thus, both of the new proposed Hamiltonian constraint operators in Eqs. (46) and (52) are also difference operators with constant steps in eigenvalues of the volume operator $\hat{V}$. But they contain more terms with steps of different size comparing to the operator (38).

The Hamiltonian constraint of the scalar field has been quantized in the literatures as [3,13]

$$
\hat{H}_\phi = \frac{1}{2}\widehat{|p|^{-3/2}}\,\widehat{p^2_\phi}, \tag{53}
$$

where the action of $\widehat{|p|^{-3/2}}$ on $|v\rangle$ reads

$$
\begin{aligned}
\widehat{|p|^{-3/2}}|v\rangle = {} & \left(\frac{3}{2}\right)^3\left(\frac{6}{\kappa\hbar\gamma}\right)^{3/2} L|v| \\
& \times \big||v+1|^{1/3} - |v-1|^{1/3}\big|^3|v\rangle.
\end{aligned} \tag{54}
$$

Thus we obtain alternative total Hamiltonian operators respectively as

$$
\hat{H}_S = \hat{H}^S_{\rm grav} + \hat{H}_\phi, \tag{55}
$$

$$
\hat{H}_F = \hat{H}^F_{\rm grav} + \hat{H}_\phi. \tag{56}
$$

Note that in both quantum dynamics the scalar field ϕ can be used as emergent time. But the different expressions of gravitational Hamiltonian operators may lead to different physics, which may be examined in some aspects. We will consider their classical limit and effective dynamics in next section.

4. Classical limit and effective dynamics

It has been shown in [11,18] that the improved Hamiltonian constraint operator constructed in [3] has the correct classical limit. In this section, we will show that the two Hamiltonian constraint operators constructed in this Letter also have the correct classical limit. Moreover, the effective Hamiltonian incorporating higher order quantum corrections can also be obtained. In order to do the semiclassical analysis, it is convenient to introduce a new variable:

$$
b := \frac{\sqrt{\Delta}}{2}\frac{c}{\sqrt{|p|}} \tag{57}
$$

conjugate to v with the Poisson bracket $\{b, v\} = 1/\hbar$. Then the classical Hamiltonian constraint in FRW model can be written as

$$
\begin{aligned}
H &= H_{\rm grav} + H_\phi \\
&= -\frac{3^2\sqrt{6}}{2}\frac{\hbar^{1/2}}{\gamma^{3/2}\kappa^{1/2}} L|v|b^2 + \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|v|}{L}\rho,
\end{aligned} \tag{58}
$$

where

$$
\rho = \frac{1}{2}\left(\frac{6}{\kappa\gamma\hbar}\right)^3\left(\frac{L}{|v|}\right)^2 p^2_\phi \tag{59}
$$

is the energy density of the scalar field. Let us first consider the gravitational part. Since there are uncountable basis vectors, the natural Gaussian semiclassical states live in the algebraic dual space of some dense set in $\mathcal{H}^{\rm grav}_{\rm kin}$. A semiclassical state $(\Psi_{(b_o,v_o)}|$ peaked at a point (b_o, v_o) of the gravitational classical phase space reads:

$$
(\Psi_{(b_o,v_o)}| = \sum_{v\in\mathbb{R}} e^{-[(v-v_o)^2/2d^2]} e^{ib_o(v-v_o)}(v|, \tag{60}
$$

where d is the characteristic "width" of the coherent state. For practical calculations, we use the shadow of the semiclassical state $(\Psi_{(b_o,v_o)}|$ on the regular lattice with spacing 1 [19], which is given by

$$
|\Psi\rangle = \sum_{n\in\mathbb{Z}}\big[e^{-(\epsilon^2/2)(n-N)^2} e^{-i(n-N)b_o}\big]|n\rangle, \tag{61}
$$

where $\epsilon = 1/d$ and we choose $v_o = N \in \mathbb{Z}$. Since we consider large volumes and late times, the relative quantum fluctuations in the volume of the universe must be very small. Therefore we have the restrictions: $1/N \ll \epsilon \ll 1$ and $b_o \ll 1$. One can check that the state (60) is sharply peaked at (b_o, v_o) and the fluctuations are within specified tolerance [11,18]. The semiclassical state of matter part is given by the standard coherent state

$$
(\Psi_{(\phi_o,p_\phi)}| = \int {\rm d}\phi\, e^{-[(\phi-\phi_o)^2/2\sigma^2]} e^{ip_\phi(\phi-\phi_o)/\hbar}(\phi|, \tag{62}
$$

where σ is the width of the Gaussian. Thus the whole semiclassical state reads $(\Psi_{(b_o,v_o)}| \otimes (\Psi_{(\phi_o,p_\phi)}|$.

The task is to use this semiclassical state to calculate the expectation values of the two Hamiltonian operators in Eqs. (55) and (56) to a certain accuracy. In the calculation, one gets the expression with the absolute values, which is not analytical. To overcome the difficulty we separate the expression into a sum of two terms: one is analytical and hence can be calculated straightforwardly, while the other becomes exponentially decayed out. We first see the expectation value of the Hamiltonian operator $\hat{H}_F$, which inherits more features of the full theory. The expectation values of the two terms of $\hat{H}^F_{\rm grav}$ in Eq. (52) are respectively calculated as

$$
\begin{aligned}
\big\langle\hat{H}^E(1)\big\rangle &= \frac{3^2\sqrt{6}}{2^3}\frac{\gamma^{1/2}\hbar^{1/2}}{\kappa^{1/2}} L|v_o|\left[e^{-4\epsilon^2}\sin^2(2b_o)\right. \\
&\quad \left. + \frac{1}{2}\big(1 - e^{-4\epsilon^2}\big)\right] + O\big(e^{-N^2\epsilon^2}\big), \\
\big\langle\hat{T}_F(1)\big\rangle &= \frac{3^2\sqrt{6}}{2^6}\frac{\hbar^{1/2}}{\gamma^{3/2}\kappa^{1/2}} L|v_o|\left[e^{-16\epsilon^2}\sin^2(4b_o)\right. \\
&\quad \left. + \frac{1}{2}\big(1 - e^{-16\epsilon^2}\big)\right] + O\big(e^{-N^2\epsilon^2}\big).
\end{aligned} \tag{63}
$$

In the calculation of $\langle\hat{H}_\phi\rangle$, one has to calculate the expectation value of the operator $\widehat{|p|^{-3/2}}$, which is given by [11]:

$$\langle\widehat{|p|^{-3/2}}\rangle = \left(\frac{6}{8\pi\gamma\ell_p^2}\right)^{3/2}\frac{L}{N}\left[1+\frac{1}{2N^2\epsilon^2}+\frac{5}{9N^2}+O\left(1/N^4\epsilon^4\right)\right] + O\left(e^{-N^2\epsilon^2}\right)+O\left(e^{-\pi^2/\epsilon^2}\right).$$

For clarity, we will suppress the label o in the following. Collecting these results we can obtain an effective Hamiltonian with the relevant quantum corrections of order $\epsilon^2, 1/\nu^2\epsilon^2, \hbar^2/\sigma^2 p_\phi^2$ as:

$$\mathcal{H}^F_{\rm eff} = -\frac{3^2\sqrt{6}}{2^3}\frac{\hbar^{1/2}}{\gamma^{3/2}\kappa^{1/2}}L|\nu| \times\left[\sin^2(2b)\left[1-\left(1+\gamma^2\right)\sin^2(2b)\right]+2\epsilon^2\right] +\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|\nu|}{L}\rho\left(1+\frac{1}{2|\nu|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right). \quad (64)$$

Hence the classical constraint (58) is reproduced up to small quantum corrections and therefore the Hamiltonian operator $\hat{H}_F$ has correct classical limit. We can further obtain the Hamiltonian evolution equation of ν by taking its Poisson bracket with $\mathcal{H}^F_{\rm eff}$ as

$$\dot{\nu}_F = 3|\nu|\sqrt{\frac{\kappa}{3}\rho_c}\sin(2b)\cos(2b)\left[1-2\left(1+\gamma^2\right)\sin^2(2b)\right], \quad (65)$$

where $\rho_c = 3/(\kappa\gamma^2\Delta)$. The vanishing of the effective Hamiltonian constraint (64) gives

$$\sin^2(2b)\left[1-\left(1+\gamma^2\right)\sin^2(2b)\right] = \frac{\rho}{\rho_c}\left(1+\frac{1}{2|\nu|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right)-2\epsilon^2. \quad (66)$$

For the classical region, $b\ll 1$ and $\rho\ll\rho_c$, we have from Eq. (66)

$$\sin^2(2b) = \frac{1-\sqrt{1-\chi_F}}{2(1+\gamma^2)}, \quad (67)$$

where

$$\chi_F = 4\left(1+\gamma^2\right)\left[\frac{\rho}{\rho_c}\left(1+\frac{1}{2|\nu|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right)-2\epsilon^2\right]. \quad (68)$$

The modified Friedmann equation can then be derived as

$$H_F^2 = \left(\frac{\dot{\nu}_F}{3\nu}\right)^2 = \frac{\kappa}{3}\frac{\rho_c}{4(1+\gamma^2)^2}\left(1-\sqrt{1-\chi_F}\right)\left(1+2\gamma^2+\sqrt{1-\chi_F}\right) \times(1-\chi_F). \quad (69)$$

It is easy to see that if one neglects the small quantum corrections in the classical region, $\chi_F\ll 1$ for $\rho\ll\rho_c$, one gets

$$H_F^2 \approx \frac{\kappa}{3}\frac{\rho_c}{4(1+\gamma^2)^2}\frac{1}{2}\chi_F 2\left(1+\gamma^2\right)\approx\frac{\kappa}{3}\rho, \quad (70)$$

which reduces to the standard Friedmann equation. However, quantum geometry effects lead to a modification of the Friedmann equation especially at the scales when ρ becomes comparable to ρ_c. Remarkable changes to the classical theory happen when the Hubble parameter in Eq. (69) vanishes by

$$1-\chi_F = 0. \quad (71)$$

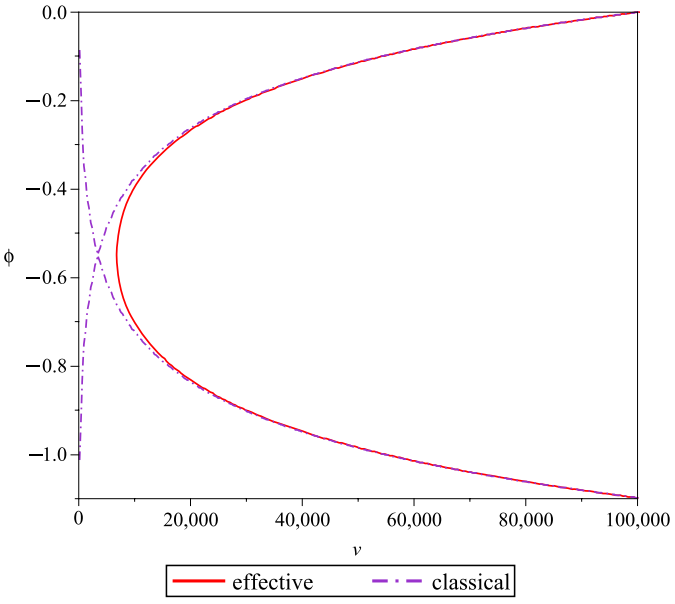

Fig. 1. The effective dynamics represented by the observable $\nu|_\phi$ are compared to classical trajectories. In this simulation, the parameters were: $G=\hbar=1$, $p_\phi = 10\,000$, $\epsilon = 0.001, \sigma = 0.01$ with initial data $\nu_o = 100\,000$.

If we consider only the leading order contribution in Eq. (68), this can happen when

$$\rho = \rho_c/4\left(1+\gamma^2\right). \quad (72)$$

Thus, when energy density of the scalar field reaches to the leading order critical energy density $\rho_c^F = \rho_c/4(1+\gamma^2)$, the universe bounces from the contracting branch to the expanding branch. The quantum bounce implied by (69) is shown in Fig. 1.

In a similar way, we can calculate the expectation value of the other Hamiltonian operator (55) as well. The effective Hamiltonian corresponding to $\hat{H}_S$ with the relevant quantum corrections is obtained as

$$\mathcal{H}^S_{\rm eff} = -\frac{3^2\sqrt{6}}{2}\frac{\hbar^{1/2}}{\gamma^{3/2}\kappa^{1/2}}L|\nu| \times\left[\sin^2 b\left(1+\gamma^2\sin^2 b\right)+\frac{1}{2}\epsilon^2\right] +\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|\nu|}{L}\rho\left(1+\frac{1}{2|\nu|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right). \quad (73)$$

Hence the classical constraint (58) is again reproduced up to small quantum corrections. The corresponding modified Friedmann equation can then be derived as

$$H_S^2 = \frac{\kappa}{3}\frac{\rho_c}{\gamma^4}\left(-1+\sqrt{1+\chi_S}\right)\left(1+2\gamma^2-\sqrt{1+\chi_S}\right)\left(1+\chi_S\right), \quad (74)$$

where

$$\chi_S = \gamma^2\left[\frac{\rho}{\rho_c}\left(1+\frac{1}{2|\nu|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right)-\frac{1}{2}\epsilon^2\right]. \quad (75)$$

The Hubble parameter in Eq. (74) can also vanish when

$$1+2\gamma^2-\sqrt{1+\chi_S} = 0. \quad (76)$$

Thus the quantum dynamics given by $\hat{H}_S$ has qualitatively similar feature of that given by $\hat{H}_F$. However, there are quantitative differences between them. For the leading order effective theory of

$\hat{H}_S$, when energy density of the scalar field reaches to the critical energy density $\rho_c^S = 4(1+\gamma^2)\rho_c$, the universe bounces from the contracting branch to the expanding branch.

5. Discussion

We have successfully constructed two versions of Hamiltonian operator for isotropic LQC in the improved scheme, where the Lorentz term is quantized in two approaches different from the Euclidean one. The treatments of the Lorentz term of Hamiltonian in the full LQG can be properly implemented in one of our constructions, which inherits more features of the full theory. One of the Hamiltonian operators is self-adjoint and the other is symmetric. Both of them are shown to have the correct classical limit by the semiclassical analysis. Hence the alternative Hamiltonian operators that we proposed can provide good arenas to test the ideas and constructions of the quantum dynamics in full LQG. In the spatially flat FRW model with a massless scalar field, the effective Hamiltonians and Friedmann equations are derived in both case. Although there are quantitative differences between the two versions of quantum dynamics, qualitatively they have the same dynamical features. Especially, the classical big bang is again replaced by a quantum bounce in both cases. For instance, in the leading order effective theory of $\hat{H}_F$, the universe would bounce from the contracting branch to the expanding branch when the energy density of scalar field reaches to the critical $\rho_c^F = \rho_c/4(1+\gamma^2)$. Therefore, the key feature of LQC for the resolution of the big bang singularity is still maintained for the new quantum dynamics inheriting more features of the full theory. Recall that the quantum bounce happens at ρ_c for the quantum dynamics in [3]. Thus the new quantum dynamics here lead to some quantitatively different critical energy density for the bounce.

On the other hand, the discussion in [11] can be carried out similarly. It is easy to see from Eqs. (69) and (74) that the Hubble parameter in both cases may also vanish by the vanishing of χ_F and χ_S respectively, whence the asymptotic behavior of the quantum geometric fluctuations plays a key role for the fate of the universe. By the ansatz $\epsilon = \lambda(r)v^{-r(\phi)}$ with $0 \leqslant r(\phi) \leqslant 1$, Eqs. (68) and (75) imply that there are great possibilities for the expanding universe to undergo a recollape in the future. The recollape can happen provided $0 \leqslant r < 1$ in the large scale limit. Suppose that the semiclassicality of our coherent state is maintained asymtotically so that the quantum fluctuation $1/\epsilon$ of v cannot increase as v unboundedly as v approaches infinity. Thus the recollape is in all probability as viewed from the parameter space of $r(\phi)$. Taking the effective dynamics of $\hat{H}_F$ as an example, in the scenario when $r = 0$ asymtotically, besides the quantum bounce when the matter density ρ increases to the Planck scale, the universe would also undergo a recollapse when ρ decreases to $\rho_{\rm coll}^F \approx 8(1+\gamma^2)\epsilon^2\rho_c^F$. Therefore, the quantum fluctuations also lead to a cyclic universe in this case. The cyclic universe in this effective scenario is illustrated in Fig. 2. This amazing possibility that quantum gravity manifests herself in the large scale cosmology was first revealed in [11]. Nevertheless, the condition that the semiclassicality is maintained in the large scale limit has not been confirmed. Hence further numerical and analytic investigations to the properties of dynamical semiclassical states in the alternative quantum dynamics are still desirable. It should be noted that in some simplified completely solvable models of LQC (see [9] and [20]), the dynamical coherent states could be obtained, where $r(\phi)$ approaches 1 in the large scale limit. While those treatments lead to the quantum dynamics different from ours, they raise caveats to the conjectured recollapse.

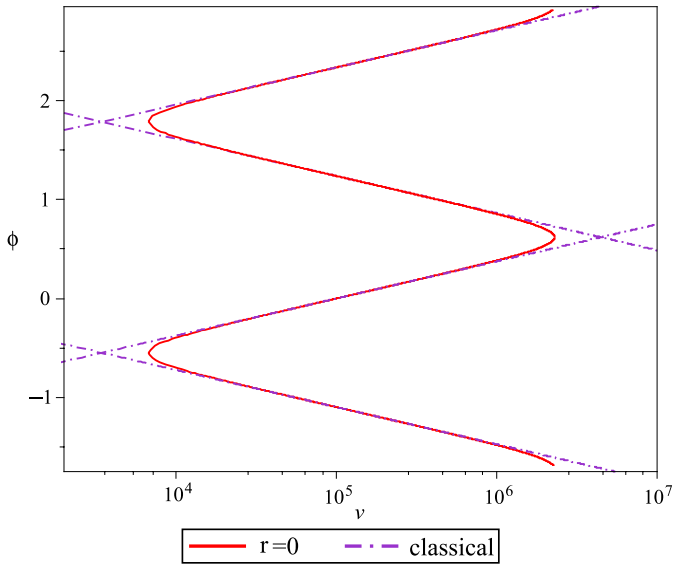

Fig. 2. The cyclic model is compared with expanding and contracting classical trajectories. In this simulation, the parameters were: $G = \hbar = 1$, $p_\phi = 10\,000, \epsilon = 0.001, \sigma = 0.01$ with initial data $v_o = 100\,000$.

To summarize, the quantum dynamics of LQC in the improved scheme is extended in order to inherit more features from the full LQG. The key features of LQC in this model, that the big bang singularity is replaced by a quantum bounce and there are great possibilities for an expanding universe to recollapse, are robust against the quantization ambiguities with the extensions. This result further supports the expectation that the above features of LQC originate not only in the symmetric model but also from the fundamental LQG.

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