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A note on the free degrees of homeomorphisms on genus 2 orientable compact surfaces

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ABSTRACT

For a compact surface F, the free degree $\mathfrak{fr}(F)$ of homeomorphisms on F is defined as the maximum of least periods among all periodic points of self-homeomorphisms on F. We show that $\max_b \mathfrak{fr}(F_{2,b}) = 12$.

1. Background

Let $f : F \to F$ be a self-homeomorphism on a compact surface F. The free degree $\mathfrak{fr}(f)$ of f is a positive integer n such that $f^1, f^2, \ldots, f^{n-1}$ have no fixed points while f^n has at least a fixed point.

We denote the free degree of all homeomorphisms on F by $\mathfrak{fr}(F) = \max\{\mathfrak{fr}(f) \mid f \in Homeo(F)\}$, the free degree of all orientation preserving homeomorphisms by $\mathfrak{fr}^+(F) = \max\{\mathfrak{fr}(f) \mid f \in Homeo^+(F)\}$ and the free degree of all orientation reversing homeomorphisms by $\mathfrak{fr}^-(F) = \max\{\mathfrak{fr}(f) \mid f \in Homeo^-(F)\}$.

In nineteen forties, J. Nielsen [4] proved that for a genus g orientable closed surface F_g,

$$\mathfrak{fr}^+(F_g) = \begin{cases} 2 \text{ or } 3, & \text{if } g = 2, \\ 2g - 2, & \text{if } g > 2. \end{cases}$$

The exact value of $\mathfrak{fr}^+(F_2) = 2$ was determined by W. Dicks and J. Llibre [2] in 1996.

S. Wang [7] extended those results to all homeomorphisms case and to non-orientable closed surfaces case. Especially, he showed that $\mathfrak{fr}(F_2) = 4$.

J. Wu and X. Zhao [8] studied the free degrees of homeomorphisms on compact surfaces with boundaries. For a genus g with b holes orientable surface $F_{g,b}$, they got an upper bound $24g - 24$.

In this paper, for genus 2 orientable compact surfaces $F_{2,b}$, we have

Theorem 1.1. $\max_b \mathfrak{fr}(F_{2,b}) = 12$.

Remark. In his/her report the referee has informed us that Theorem 1.1 is a special case of the results contained in Moira Chas's 1998 thesis [1] (recently available in the arXiv.org). In fact the results in [1] have also covered the main results in [8].

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We prove Theorem 1.1 in Sections 2 and 3. We list some important results of M. Chas in an additional remark at the end of the paper.

2. The upper bound of $\mathfrak{fr}(F_{2,b})$

Theorem 2.1. *If f is a self-homeomorphism on $F_{2,b}$, then there exists a positive integer $n \leqslant 12$ such that the Nielsen number $N(f^n) > 0$, hence $\mathfrak{fr}(f) \leqslant 12$.*

Proof. The procedure of our proof is essentially the same as [8, Lemma 7.1]. We prove the theorem by using a reduction on the number of boundary components b.

Since $\mathfrak{fr}(f) \leqslant \min\{n \mid \text{the Nielsen number } N(f^n) > 0\} \leqslant \min\{n \mid \text{the Lefschetz number } L(f^n) \neq 0\}$ and the Nielsen (Lefschetz) number of a map is a homotopic invariant, by Thurston's classification theorem of homeomorphisms of surface [5], we only have to prove $N(f^n) > 0$ or $L(f^n) \neq 0$ ($n \leqslant 12$) for f being periodic or pseudo-Anosov or reducible. Furthermore, we may assume that f is in standard form introduced by B. Jiang and J. Guo [3] for simplifying the estimate of the Nielsen (Lefschetz) number.

Case 1: $b = 0$. We already knew $L(f^n) \neq 0$ for some $n \leqslant 4$ in the proof of S. Wang [7, Theorem 1].

Case 2: f is periodic. If we collapse each boundary component of $F_{2,b}$ to a point, f induces a periodic homeomorphism $\bar{f}$ on F_2. They have the same order n and $N(f^n) = N(id) = 1$. By S. Wang [6], $n \leqslant 12$.

Case 3: f is pseudo-Anosov. Denote the stable singular foliation of f by $\mathcal{F}^s$. Let l be the minimal length of orbits of f-action on the set of all 1-prong boundary components.

Subcase 3.1: $l = 0$. Suppose $q : F_{2,b} \to F_2$ is a map collapsing each boundary component of $F_{2,b}$ to one point. Then f induces a map $\bar{f}$ on F_2 satisfying the following commutative diagram

$$\begin{array}{ccc} F_{2,b} & \xrightarrow{\ f\ } & F_{2,b} \\ \downarrow q & & \downarrow q \\ F_2 & \xrightarrow{\ \bar{f}\ } & F_2 \end{array} \tag{2.1}$$

where $q(\mathcal{F}^s)$ is the stable singular foliation of the pseudo-Anosov map $\bar{f}$. By S. Wang [7], $p = \mathfrak{fr}(\bar{f}) \leqslant 4$. There exists a point $x \in F_2$ which is a fixed point of $\bar{f}^p$. Since $\bar{f}^p$ is also a pseudo-Anosov map on a closed surface, x is an m-prong ($m > 1$) singularity and consists a fixed point class of $\bar{f}^p$. We have two sub-subcases.

(3.1.1) $\bar{f}^p$ is orientation preserving. Then the index $ind(\bar{f}^p, x)$ is non-zero.

If $q^{-1}(x)$ is a singleton, it is an isolated fixed point of f^p with non-zero index. f^p is in standard form and $q^{-1}(x)$ is also an essential fixed point class of f^p. We have $N(f^p) > 0$.

If $q^{-1}(x)$ is not a singleton, it is an m-prong boundary circle of $\mathcal{F}^s$ and is invariant under f^p. Then f^{pm} maps each singularity on $q^{-1}(x)$ to itself. By B. Jiang and J. Guo [3, Lemma 2.1], we have $ind(f^{pm}, q^{-1}(x)) = -m \neq 0$ and $N(f^{pm}) > 0$. It is sufficient to prove $pm \leqslant 12$.

(1) If $m = 2$ or 3, then $pm \leqslant 12$.

(2) If $m \geqslant 4$, since $q(\mathcal{F}^s)$ has no 1-prong singularities, we have

$$-2 = \chi(F_2) = \sum_{k=2}^{\infty}\left(1 - \frac{k}{2}\right)P_k\big(q(\mathcal{F}^s)\big) \leqslant \left(1 - \frac{m}{2}\right)P_m\big(q(\mathcal{F}^s)\big)$$

where $P_k(q(\mathcal{F}^s))$ is the number of k-prong singularities of $q(\mathcal{F}^s)$. It follows that

$$mP_m\big(q(\mathcal{F}^s)\big) \leqslant 4\left(1 + \frac{2}{m-2}\right) \leqslant 8.$$

Since f permutes the m-prong boundary components of $F_{2,b}$, we have that

$$p \leqslant P_m^{bd}(\mathcal{F}^s) \leqslant P_m^{bd}(\mathcal{F}^s) + P_m^{int}(\mathcal{F}^s) = P_m\big(q(\mathcal{F}^s)\big)$$

where $P_m^{bd}(\mathcal{F}^s)$ is the number of m-prong boundary components of $F_{2,b}$ and $P_m^{int}(\mathcal{F}^s)$ is number of m-prong singularities in the interior of $F_{2,b}$. Hence $pm \leqslant 8$.

(3.1.2) $\bar{f}^p$ is orientation reversing. Then $p = 1$ or 3 and the index $ind(\bar{f}^{2p}, x) = 1 - m \neq 0$.

If $q^{-1}(x)$ is a singleton, it is an isolated fixed point of f^{2p} with non-zero index. f^{2p} is in standard form and $q^{-1}(x)$ is also an essential fixed point class of f^{2p}. We have $N(f^{2p}) > 0$.

If $q^{-1}(x)$ is not a singleton, it is an m-prong boundary circle of $\mathcal{F}^s$ and is invariant under f^p. Since $\bar{f}^p$ is orientation reversing, f^p is also orientation reversing. Since f^p is in standard form, $f^p|_{q^{-1}(x)}$ is a reflection. So $f^{2p}|_{q^{-1}(x)} = id$. By B. Jiang and J. Guo [3, Lemma 2.1], we have $ind(f^{2p}, q^{-1}(x)) = -m \neq 0$ and $N(f^{2p}) > 0$, where $2p \leqslant 6$.

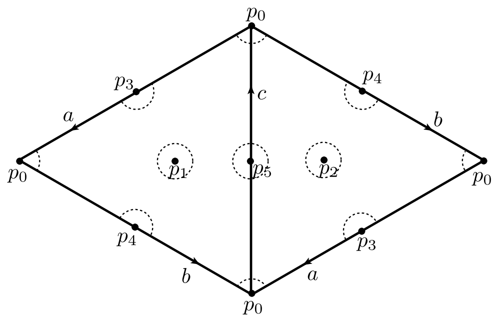

Fig. 1. An order 6 periodic homeomorphism on a torus.

Subcase 3.2: $0 < l \leqslant 12$. Let C be a 1-prong boundary component such that $f^l(C) = C$. By B. Jiang and J. Guo [3, Lemma 3.6], if f^l is orientation preserving, then C is the fixed point class of f^l with index -1. So $N(f^l) > 0$. If f^l is orientation reversing, then there is a fixed point $x \in C$ with index 1. We also have $N(f^l) > 0$.

Subcase 3.3: $l > 12$. Let $q : F_{g,b} \to F_{g,b'}$ be a map collapsing each 1-prong boundary component of $F_{2,b}$ to a point. The pseudo-Anosov map f induces a homeomorphism $\bar{f}$ on $F_{2,b'}$. We have the following commutative diagram

$$\begin{array}{ccc} F_{2,b} & \xrightarrow{f} & F_{2,b} \\ \downarrow q & & \downarrow q \\ F_{2,b'} & \xrightarrow{\bar{f}} & F_{2,b'}. \end{array}$$

Of course, $b' < b$ and $\bar{f}$ is not in standard form. By the inductive assumption, $N(\bar{f}^m) > 0$ for some $m \leqslant 12 < l$. Since q is surjective and only collapses 1-prong boundary components of $F_{2,b}$, by the definition of l, we know any essential fixed point class of $\bar{f}^m$ contains at lease one essential fixed point classes of f^m. We have $N(f^m) > 0$.

Case 4: f is reducible. Let P be a reduced piece with the biggest genus among all pieces. Assume that $P \cong F_{g,b'}$. Thus, either $g < 2$ or $g = 2$ and $b' < b$.

Subcase 4.1: $g = 2$. Clearly, $f|_P$ is a homeomorphism on P. By assumption of reduction, $N(f^n|_P) > 0$ for some $n \leqslant 12$. So $N(f^n) > 0$.

Subcase 4.2: $g = 0$ or 1. Consider the quotient map $q : F_{2,b} \to F_2$ and the induced homeomorphism satisfying the commutative diagram (2.1). Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be the cutting system for f. We assume that $q(\gamma_j)$ is essential in F_2 for $j = 1, 2, \ldots, t'$, and inessential for $j = t' + 1, \ldots, k$. We write $\Gamma' = \{\gamma_1, \gamma_2, \ldots, \gamma_{t'}\}$. Then each component of $F_2 - q(\Gamma')$ is a union of one component of $F_2 - q(\Gamma)$ and other components of $F_2 - q(\Gamma)$ are disks. This implies that $t' > 0$ and the maximal genus of the components of $F_2 - q(\Gamma')$ is still g. Since each curve $q(\gamma_j)$ in $q(\Gamma')$ is essential in F_2, the Euler characteristic number of each component of $F_2 - q(\Gamma')$ is negative.

(4.2.1) $g = 1$. Let Q be a component of $F_2 - q(\Gamma')$ with genus 1. Let k be the orbit length of Q under the action of $\bar{f}$ and m be number of boundary components of Q. Since each component of $F_2 - q(\Gamma')$ has non-positive Euler characteristic number. We have $k\chi(Q) = -km \geqslant \chi(F_2) = -2$. It follows that $k = 1$ and $m \leqslant 2$ or $k = 2$ and $m = 1$. In either case, Q and its boundary components are invariant under $\bar{f}^2$. So $f^2|_P$ keeps at least a boundary component of the closure P of $q^{-1}(Q)$ invariant. The genus of P is 1. By [8, Lemma 5.2], $L((f^2|_P)^p) \neq 0$ for some $p \leqslant 6$ because f^2 is orientation preserving. It follows that $N(f^n|_P) > 0$ for some $n \leqslant 12$. So $N(f^n) > 0$.

(4.2.2) $g = 0$. Each component of $F_2 - q(\Gamma')$ is a disk with holes. They all have non-positive Euler characteristic number. From the additivity of Euler characteristic numbers, there must be a component Q with $\chi(Q) < 0$. Let k be the orbit length of Q under the action of $\bar{f}$ and m be number of boundary components of Q. Then we have $k\chi(Q) = k(2 - m) \geqslant \chi(F_2) = -2$. This implies that $km \leqslant 2 \cdot \frac{m}{m-2} \leqslant 6$ because $\chi(Q) < 0$ i.e. $m \geqslant 3$. Since $\bar{f}^k$ permutes the boundary components of Q, then there exists a positive integer $m' \leqslant m$ such that $\bar{f}^{km'}$ fixes set-wisely at least three boundary components. The closure P of $q^{-1}(Q)$ is a disk with holes and $f^{km'}|_P$ also keeps at least three boundary components invariant. Since $(f^{km'}|_P)^2$ is orientation preserving, by [8, Lemma 5.1], $L((f^{km'}|_P)^2) \neq 0$. So $N(f^n) > 0$ for some $n \leqslant 2km' \leqslant 2km \leqslant 12$. □

3. Example: a free degree 12 periodic homeomorphism on $F_{2,10}$

The construction is motivated by S. Wang [6, Lemma 5]. In fact, we just cut the neighborhoods of some periodic points on a surface shown by the example in that lemma.

Step 1: An order 6 *periodic homeomorphism on a torus.* Consider a torus F_1 as the union of two regular 3-gon's in the plane shown in Fig. 1. The oriented edges marked by the same letter are identified. All corner points are identified to a point on F_1.

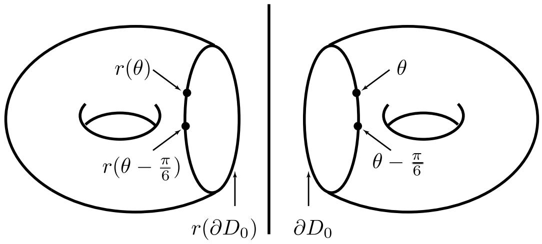

Fig. 2. A free degree 12 homeomorphism on $F_{2,10}$.

ϕ is a map who restricted on each 3-gon a rotation of angle $\frac{2}{3}\pi$. Let ρ be a rotation of angle π in the plane switching two regular 3-gon's. Then $\rho \circ \phi$ induces a periodic homeomorphism f_1 of order 6 on F_1. The image of the corner points p_0 is a fixed point. The images of the centers of two 3-gon's p_1 and p_2 are periodic points of order 2. The images of the centers of the oriented edges p_3, p_4 and p_5 are periodic points of order 3. The images of all the other points are of order 6.

Step 2: A free degree 12 *homeomorphism on* $F_{2,10}$. We cut the neighborhoods D_i of p_i $(i = 0, \ldots, 5)$ to get a genus 1 surface with 6 holes $F_{1,6}$. f_1 induces a periodic homeomorphism f_2 of order 6 on $F_{1,6}$. The restriction of f_2 on ∂D_0 is a rotation of $\frac{\pi}{3}$. Put $F_{1,6}$ in E^3 and let r be a reflection about a plane (mirror). Now pick a copy $F_{1,6}$ and its mirror image $r(F_{1,6})$. Define a map f' on $F_{1,6} \cup r(F_{1,6})$ as below: for $x \in F_{1,6}$, let $f'(x) = r(x)$; for $x \in r(F_{1,6})$, let $f'(x) = f_2 \circ r(x)$. Glue $F_{1,6}$ with its mirror image $r(F_{1,6})$ by identifying θ on ∂D_0 with $r(\theta - \frac{\pi}{6})$ on $r(\partial D_0)$ and get $F_{2,10}$ (see Fig. 2). Now f' induces a periodic homeomorphism f of order 12 on $F_{2,10}$. Clearly $\mathfrak{fr}(f) = 12$.

Additional remark. In her PhD thesis [1] Moira Chas has obtained quite general results on determining the free degrees of orientable compact surfaces with boundaries. The results in [1] have determined $\mathfrak{fr}^+(F_{g,b})$ and $\mathfrak{fr}^-(F_{g,b})$ for almost all the pairs (g, b). Below we list the results in [1] which are related to the problems considered in this paper and [8].

Theorem 3.1. *([1, Theorem F (3)]) The free degrees of all orientation preserving homeomorphisms on $F_{2,b}$ are listed in the following table.*

b	1	2	3	4	5	6	7	8	9	10	11
$\mathfrak{fr}^+(F_{2,b})$	3	4	5	6	3	8	4	10	5	6	6
b	12	13	14	15	16	17	18	19	20	21	$\geqslant 22$
$\mathfrak{fr}^+(F_{2,b})$	6	7	8	8	8	9	10	10	10	10	10

Theorem 3.2. *([1, Theorem G (3)]) The free degrees of all orientation reversing homeomorphisms on $F_{2,b}$ are listed in the following table.*

b	1	2	3	4	5	6	7	8	9	10	11
$\mathfrak{fr}^-(F_{2,b})$	1	4	3	6	4	8	4	4	5	12	6
b	12	13	14	15	16	17	18	19	20	21	$\geqslant 22$
$\mathfrak{fr}^-(F_{2,b})$	6	7	8	8	8	8	8	9	10	11	12

Theorem 3.3. *([1, Theorem H]) If $g \geqslant 2$ then $\mathfrak{fr}^+(F_{g,b}) \leqslant 4g + 2$. Moreover, if $b \geqslant 6g + 6$, then equality holds.*

Theorem 3.4. *([1, Theorem I]) Let $g \geqslant 2$. Then $\mathfrak{fr}^-(F_{2,b}) \leqslant 4g + (-1)^g 4$ and equality holds if $b \geqslant 6g + 2 + (-1)^g 8$.*

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References

[1] M. Chas, Minimum periods of homeomorphisms of orientable surfaces, PhD thesis, Universitat Autonoma de Barcelona, 1998, 141+iii pp., see http://arxiv.org/abs/1204.0023.
[2] W. Dicks, J. Llibre, Orientation-preserving self-homeomorphisms of the surface of genus two have points of period at most two, Proc. Amer. Math. Soc. 124 (5) (1996) 1583–1591.
[3] B. Jiang, J. Guo, Fixed points of surface diffeomorphisms, Pacific J. Math. 160 (1) (1993) 67–89.
[4] J. Nielsen, Fixpunktfrie Afbildninger, Mat. Tidsskr. B (1942) 25–41.
[5] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.) 19 (2) (1988) 417–431.
[6] S. Wang, Maximum orders of periodic maps on closed surfaces, Topology Appl. 41 (3) (1991) 255–262.
[7] S. Wang, Free degrees of homeomorphisms and periodic maps on closed surfaces, Topology Appl. 46 (1) (1992) 81–87.
[8] J. Wu, X. Zhao, Free degrees of homeomorphisms on compact surfaces, Algebr. Geom. Topol. 11 (4) (2011) 2437–2452.