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A PROBLEM ON TRIPLES

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If a system H of triples (3-uniform hypergraph) on n vertices has the following property: for every 3-coloring of the vertex-set there exists a 3-colored triple, what is the minimum size ($S(n)$) of H? The first values of $S(n)$ are computed, and the asymptotic behaviour of this function is studied.

0. Introduction

A h-graph (or uniform hypergraph of rank h, see [1]) is a pair $H = (X, \mathscr{E})$, where $\mathscr{E}$ is a set of subsets of X, every subset having h elements: $\mathscr{E} \subset \mathscr{P}_h(X)$.

The elements of X are the vertices, the elements of $\mathscr{E}$ are the edges.

In [2] the following problem is introduced: what is the minimum number of edges of a h-graph H with n vertices, such that, for every coloring of X with c colours, there exists at least one edge of H in which no color appears more than once? $S(n, h, c)$ denotes this minimum number of edges.

The general case is studied in [2]. In [3], the special cases $c = h$ and $c = h = n - 2$ are studied. In the present paper, we consider the simplest (non-trivial) case: $c = h = 3$.

Definition 0.1. We call c-coloring of X any mapping g from X *onto* the set $\{1, 2, \ldots, c\}$. For every $x \in X$, $g(x)$ is the *colour* of x.

Definition 0.2. Let E be a subset of X, and g a c-coloring of X, E is *strongly colored* in g iff

$$x \in E,\ y \in E,\ g(x) = g(y) \quad \text{implies } x = y.$$

Definition 0.3. Let $\mathscr{H}(n, h, c)$ be the set of all the h-graphs H with n vertices, such that, for every c-coloring of the vertex-set, there exists at least one strongly colored edge in H. $S(n, h, c)$ is the minimum number of edges of an element of $\mathscr{H}(n, h, c)$.

For the sake of simplicity, we set

$$\mathscr{H}(n) = \mathscr{H}(n, 3, 3) \quad \text{and} \quad S(n) = S(n, 3, 3).$$

Definition 0.4. Let g be a c-coloring of X, and x a fixed element of X. We say that g belongs to $C(x)$ iff $g(y)=g(x)$ implies $y=x$ (x is the only element of X with colour $g(x)$).

Remark 0.5. Let $H=(X,\mathscr{E})$ be a h-graph, and x a vertex of H. Let Hx be the $(h-1)$-graph defined as follows:

$$\mathscr{E}x=\{E \mid E\in\mathscr{E}, x\in E\} \quad \mathscr{F}x=\{E-\{x\} \mid E\in\mathscr{E}x\} \quad Hx=(X-\{x\},\mathscr{F}x).$$

By considering the c-colorings belonging to $C(x)$, it was proved in [3] that, if $H\in\mathscr{H}(n,h,h)$, then $Hx\in\mathscr{H}(n-1,h-1,h-1)$, and hence that $|\mathscr{E}x|\geq S(n-1,h-1,h-1)$.

For $c=h=3$, this implies that if $H\in\mathscr{H}(n)$, then for every vertex x the *graph* Hx is connected; hence, every vertex x is contained in at least $n-2$ edges of H, and $S(n)\geq n(n-2)/3$.

Moreover, for every 3-graph H, (a) and (b) are equivalent:

(a) for every 3-coloring of X belonging to $C(x)$ there exists a strongly colored edge of H.

(b) Hx is a connected graph.

Remark 0.6. Let H be the 3-graph on n vertices whose edges are all the triples containing a given vertex. Then $H\in\mathscr{H}(n)$ and hence $S(n)\leq(n-1)(n-2)/2$. Hence

$$1/3\leq \liminf S(n)/n^2\leq \limsup S(n)/n^2\leq 1/2.$$

Conjecture. Lim $S(n)/n^2$ exists and is equal to 1/3.

In this direction, we prove in this paper the following results:

Theorem 0.1. *For* $3\leq n\leq 11$, $S(n)=\{n(n-2)/3\}$. ($\{\lambda\}$ *is the smallest integer* $\geq\lambda$.)

Theorem 0.2. Lim inf $S(n)/n^2\leq 97/242=0.4008\ldots$.

Theorem 0.3. Lim sup $S(n)/n^2<0.5$.

1. Proof of Theorem 0.1

By Remark 0.5, Theorem 0.1 is a consequence of the following propositions:

Proposition 1.1. $S(3)\leq 1$. (*Immediate.*)

Proposition 1.2. $S(4)\leq 3$. (*More generally*, $S(n,h,n-1)=\{(n-1)/(n-h)\}$, *see* [2].)

Proposition 1.3. $S(5) \leq 5$.

Proof. Let $X = Z/5Z$, and $\mathscr{E} = \{(i, i+1, i+2) \mid i \in Z/5Z\}$. Then, $H = (X, \mathscr{E})$ belongs to $\mathscr{H}(5)$. More generally, it is proved in [3] that $S(n, n-2, n-2) = n(n-1)/2 - f(n)$, where $f(n)$ is the maximum number of edges of a graph of order n and girth 5.

Proposition 1.4. $S(6) \leq 8$.

Proof. Let $X = Z/6Z$, and

$$\mathscr{E} = \{(i, i+1, i+4), (i, i+4, i+2), (i, i+2, i+3), (i, i+3, i+5) \mid i \in X\}$$

and

$$H = (X, \mathscr{E}).$$

H has exactly 8 edges, and, for every vertex x, Hx is a connected graph. Hence, it follows from Remark 0.5 that, in order to prove that $H \in \mathscr{H}(6)$, it is enough to check that, for every 3-coloring g such that every color appears twice, there exists a strongly colored edge.

Let g be such a coloring. If, for every i, $g(i) = g(i+3)$, then the edge $(0, 2, 4)$ is strongly colored. Otherwise, by the cyclic symmetry of H, we can assume that $g(0) = 1$ and $g(3) = 2$. Then, if either $g(2)$ or $g(5)$ equals 3, then $(0, 2, 3)$ or $(0, 3, 5)$ is strongly colored, and if not, $g(1) = g(4) = 3$ and $(1, 2, 5)$ is strongly colored.

Proposition 1.5. $S(7) \leq 12$.

Proof. Let $X = \{1, 2, \ldots, 7\}$, and (omitting the commas in the triples): $\mathscr{E}' = \{(126), (147), (153), (234), (257), (367), (456)\}$ ($\mathscr{E}'$ is a projective plane on 7 points).

$$\mathscr{E}'' = \{(124), (157), (237), (356), (146)\}, \quad \text{and } H = (X, \mathscr{E}' \cup \mathscr{E}'').$$

For every vertex x, Hx is connected. Hence, for proving $H \in \mathscr{H}(7)$, we can only consider the 3-colorings where two colours appear exactly twice, and the last colour appears thrice (for example colour 1). Let g be such a coloring, then at least one edge of $\mathscr{E}'$ is strongly colored. Indeed, let a, b, c, d be four distinct points such that $g(a) = g(b) = 2$ and $g(c) = g(d) = 3$. Either these four points are independent (no three of them are collinear) or three of them are collinear. In either case, the existence of a strongly colored line is easily checked.

Proposition 1.6. $S(8) \leq 16$.

Proof. Let $X = Z/8Z$. Let $H = (X, \mathscr{E})$ be the 3-graph defined as follows: $\mathscr{F}_0 = \{(6, 1), (1, 5), (5, 7), (7, 4), (4, 3), (3, 2)\}$

$$\mathscr{E} = \{(i, a+i, b+i) \mid i \in Z/8Z, (a, b) \in \mathscr{F}_0\}.$$

H has exactly 16 edges, and, for every vertex x, Hx is connected. Hence, it remains to consider the 3-colorings for which every colour appears at least twice. For such a coloring g, one of the colours (for example 1) appears exactly twice. By the cyclic symmetry of H, we can assume that $g(0)=1$. If $g(6)$, or $g(1)$, or $g(3)$, or $g(2)$ equals 1, there exist two consecutive elements a and b in the sequence $(6,1,5,7,4,3,2)$ such that $g(a)=2$ and $g(b)=3$, and hence $(0,a,b)$ is a strongly colored edge of H. Otherwise, we can assume $g(6)=2$, and either $g(0)=g(5)=1$, $g(6)=g(1)=2$, $g(7)=g(4)=g(3)=g(2)=3$ or $g(0)=g(7)=1$, $g(6)=g(1)=g(5)=2$, $g(4)=g(3)=g(2)=3$ or $g(0)=g(4)=1$, $g(6)=g(1)=g(5)=g(7)=2$, $g(3)=g(2)=3$, then either $(1,4,5)$ or $(1,2,7)$ or $(1,3,4)$ is a strongly colored edge of H.

Proposition 1.7. $S(9)\leq 21$.

Proof. Let $X=Z/9Z$,

$$\mathcal{F}_0=\{(1,6),(6,3),(3,4),(4,2),(2,7),(7,5),(5,8)\}$$

and

$$\mathscr{E}=\{(i,a+i,b+i)\,|\,i\in X,(a,b)\in\mathcal{F}_0\},\quad H=(X,\mathscr{E}).$$

Then, H has exactly 21 edges, and $H\in\mathscr{H}(9)$: as above, we consider only the 3-colorings for which every colour appears at least twice:

(1) 3-colorings in which every colour appears 3 times: we can assume $g(0)=1$. Then either there exists a strongly colored edge in $\mathscr{E}_0$ (see Remark 0.5), or the vector $V=(g(1),g(6),g(3),g(4),g(2),g(7),g(5),g(8))$ takes essentially one of the following values:

$$\begin{array}{ll}(1,2,2,2,1,3,3,3), & (2,1,2,2,1,3,3,3)\\ (2,2,2,1,3,1,3,3), & (2,1,3,3,3,1,2,2)\\ (2,2,2,1,3,3,1,3), & (2,2,1,2,1,3,3,3)\\ (2,2,2,1,3,3,3,1), & (2,2,1,3,3,3,1,2)\\ (2,2,2,1,1,3,3,3), & \end{array}$$

in each case, either $(2,3,8)$ or $(3,5,7)$ or $(3,6,7)$ is a strongly colored edge of H.

(2) 3-colorings in which one of the colours appears exactly twice (for example colour 1).

By the cyclic symmetry of H, we can assume $g(0)=1$. Then either there exists a strongly coloured edge in $\mathscr{E}_0$, or the vector V takes essentially one of the following values:

$$\begin{array}{ll}(2,2,1,3,3,3,3,3), & (2,2,2,1,3,3,3,3)\\ (2,2,2,2,1,3,3,3), & (2,2,2,2,2,1,3,3)\end{array}$$

in each case either $(3,6,7)$ or $(1,4,7)$ or $(1,2,7)$ or $(3,5,7)$ is a strongly colored edge of H.

Proposition 1.8. $S(10) \leq 27$.

Proof. Let $H = (X, \mathscr{E})$ with $X = \{0, 1, 2, \ldots, 9\}$, and (omitting the commas in the triples)

$$\mathscr{E} = \{(012), (023), (034), (045), (056), (067), (078), (089), (129), (137), (146), (157), (158), (169), (189), (238), (248), (256), (268), (279), (346), (349), (359), (367), (457), (489), (579)\}.$$

Then H has 27 edges, and we have to prove that $H \in \mathscr{H}(10)$. For every vertex x, Hx is connected. Hence, we consider:

(1) The 3-colorings for which one of the colours (e.g. colour 1) appears exactly twice: let $g(a) = g(b) = 1$.

We remark that if there exists such a coloring g for which no edge of H is strongly colored, then the graph

$$Ga, b = (X - \{a, b\}, \mathscr{F}a, b)$$

where

$$\mathscr{F}a, b = \{F \mid F \in \mathscr{F}a, b \notin F\} \cup \{F \mid F \in \mathscr{F}b, a \notin F\}$$

is not connected.

Hence, we have to check that the 45 graphs Ga, b are connected. We omit here this tedious operation.

(2) The 3-colorings where two colours appear exactly thrice, and the other one four times. Let g be such a coloring, and let us assume, in order to get a contradiction that no edge of H is strongly colored for g.

Case I. $g(2) \neq g(8)$, e.g. $g(2) = 1$ and $g(8) = 2$. Then $g(3)$, $g(4)$, $g(6)$ have colour 1 or 2.

Subcase I.I: $g(9) = 1$. Then $g(0)$ and $g(1)$ equal 1 or 2, and we get a contradiction, since only two vertices are left for colour 3.

Subcase I.II. $g(9) = 2$. Then $g(1)$ and $g(7)$ equal 1 or 2, and the contradiction follows as above.

Subcase I.III. $g(9) = 3$. Then, we can deduce successively: $g(1)$ and $g(7)$ equal 1 or 3, $g(0)$, $g(1)$, $g(4)$ equal 2 or 3, $g(1) = 3$, $g(4) = 2$, $g(6) = 2$ or 3, $g(6) = 2$, $g(3) = 2$, $g(0) = 3$, and the contradiction: (023) is strongly colored.

Case II. $g(2) = g(8)$

Subcase II.I. $g(2) = g(8) = g(0)$. Let V be the vector $(g(0), g(1), \ldots, g(9))$. Then, the only essential values of V for which no edge of $\mathscr{E}_0$ is strongly colored are

(1, 2, 1, 2, 1, 3, 3, 3, 1, 2), (1, 2, 1, 2, 2, 1, 3, 3, 1, 3), (1, 3, 1, 2, 2, 1, 3, 3, 1, 2) and (1, 3, 1, 2, 2, 2, 1, 3, 1, 3). But then, the edges (3, 4, 6) or (4, 5, 7) or (4, 8, 9) are strongly colored.

Subcase II.II. $g(0) = 1$, $g(2) = g(8) = 2$. Then, the only essential values of V for which no edge of $\mathscr{E}_0$ is strongly colored are (1, 2, 2, 1, 3, 3, 3, 1, 2, 2), (1, 1, 2, 1, 3, 3, 3, 1, 2, 2), (1, 2, 2, 1, 3, 3, 3, 1, 2, 1), for which (3, 4, 9) or (5, 7, 9) or (4, 8, 9) are strongly colored.

Proposition 1.9. $S(11) \leq 33$.

Proof. Let $X = Z/11Z$ and $H = (X, \mathscr{E})$ the 3-graph such that

$$\mathscr{F}_0 = \{(2, 10), (10, 5), (5, 6), (6, 1), (1, 3), (3, 7), (7, 4), (4, 8), (8, 9)\}$$

and

$$\mathscr{E} = \{(i, a + i, b + i) \mid i \in X, (a, b) \in \mathscr{F}_0\}.$$

H has exactly 33 edges and, for every vertex x, Hx is connected. Hence, in order to prove $H \in \mathscr{H}(11)$, we consider:

(1) The 3-colorings where at least one colour appears exactly twice (e.g. colour 1). Let g be such a coloring; we can assume, by the cyclic symmetry of H, that $g(0) = 1$. Let V be the vector $V = (g(2), g(10), g(5), g(6), g(1), g(3), g(7), g(4), g(8), g(9))$. Then, it is not difficult to make a census of the 6 essential values of V for which no edge of $\mathscr{E}_0$ is strongly colored. But then, for these 6 values, (4, 5, 10) or (5, 6, 8) or (1, 6, 7) or (3, 7, 10) or (4, 5, 7) or (1, 4, 8) are strongly colored.

(2) The 3-colorings where all the colours appear at least thrice. Leg g be such a coloring. We can assume that colour 1 appears exactly thrice, and, by the cyclic symmetry of H, that $g(0) = 1$ and at least one of the colours $g(1)$, $g(2)$, $g(3)$ is equal to 1. Then, it is not difficult to make a census of the 22 essential values of the vector V for which no edge of $\mathscr{E}_0$ is strongly colored. But then, either (1, 6, 7) or (5, 6, 8) or (2, 7, 8) or (3, 4, 6) or (4, 5, 7) is strongly colored in these colorings.

2. Asymptotic results

Theorem 2.1. *For every $n \geq 3$, $S(2n) \leq 4S(n) + 3(n - 1)$.*

Proof. Let $H = (X, \mathscr{E})$ be a 3-graph such that $H \in \mathscr{H}(n)$ and $|\mathscr{E}| = S(n)$. We shall construct a 3-graph $H' = (X', \mathscr{E}')$ such that $H' \in \mathscr{H}(2n)$ and $|\mathscr{E}'| = 4S(n) + 3(n - 1)$, hence proving the theorem.

Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ with $X \cap Y = \emptyset$ and $X' = X \cup Y$.

Let $\mathscr{E}'' = \{x_i, y_j, y_k) \mid (x_i, x_j, x_k) \in \mathscr{E}\}$ then $|\mathscr{E}''| = 3S(n)$.

Let $\mathscr{G}_1 = \{(x_i, x_{i+1}, y_{i+1}) \mid 1 \leq i \leq n - 1\}$ and $\mathscr{G}_2 = \{(x_i, y_i, y_{i+1}) \mid 1 \leq i \leq n - 1\} \cup \{(x_{i+1}, y_i, y_{i+1}) \mid 1 \leq i \leq n - 1\}$.

Finally, let $\mathscr{E}' = \mathscr{E} \cup \mathscr{E}'' \cup \mathscr{G}_1 \cup \mathscr{G}_2$. Then, $H' = (X', \mathscr{E}') \in \mathscr{H}(2n)$: indeed, let g be a 3-coloring of X':

Case I. The three colours appear in X. Then, since $H \in \mathscr{H}(n)$, there exists a strongly colored edge in $\mathscr{E} \subset \mathscr{E}'$.

Case II. Exactly one colour appears in X (e.g. colour 1). Let $Y_m = \{i \mid g(y_i) = m\}$ $(m = 1,2,3)$. Then $Y_2 \neq \emptyset$ and $Y_3 \neq \emptyset$.

Subcase II.I. $Y_1 = \emptyset$. Then, there exist i and j such that $|i - j| = 1$, $i \in Y_2$, $j \in Y_3$. Then $(x_i, y_i, y_j) \in \mathscr{G}_2$ is strongly colored in g.

Subcase II.II. $Y_1 \neq \emptyset$. Let h be the 3-coloring of X defined by: $h(x_i) = m$, for $i \in Y_m$, $m = 1,2,3$. There exists $(x_i, x_j, x_k) \in \mathscr{E}$, such that $h(x_i) = 1$, $h(x_j) = 2$, $h(x_k) = 3$. Then, $(x_i, y_j, y_k) \in \mathscr{E}''$ is strongly colored in the coloring g.

Case III. Exactly two colours appear in X (e.g. colours 1 and 2). Let

$$X_p = \{i \mid g(x_i) = p\} \qquad (p = 1,2)$$

$$Y_m = \{i \mid g(x_i) = m\} \qquad (m = 1,2,3)$$

$$A = X_1 \cap Y_1, \qquad B = X_1 \cap Y_2, \qquad C = X_1 \cap Y_3$$

$$D = X_2 \cap Y_1, \qquad E = X_2 \cap Y_2, \qquad F = X_2 \cap Y_3.$$

Since $Y_3 \neq \emptyset$, we can assume for instance that $F \neq \emptyset$.

Subcase III.I: $E \neq \emptyset$. Let h be the 3-coloring of X defined by $h(x_i) = 2$ for $i \in E$, $h(x_i) = 3$ for $i \in F$, and $h(x_i) = 1$ for $i \in X_1 \cup D$. Then there exists $(x_i, x_j, x_k) \in \mathscr{E}$, such that $h(x_i) = 1$, $h(x_j) = 2$, $h(x_k) = 3$.

If $i \in X_1$, then $(x_i, y_j, y_k) \in \mathscr{E}''$ is strongly colored in g.

If $i \in D$, then $(y_i, x_j, y_k) \in \mathscr{E}''$ is strongly colored for g.

Subcase III.II. $E = \emptyset$.

SScase III.II.I: $A \neq \emptyset$ and $C \neq \emptyset$. This case is similar to the case III.I.
SScase III.II.II: $A = \emptyset$ and $C \neq \emptyset$ and $B \cup D \neq \emptyset$. Then there exist i and j such that:

$$|i - j| = 1, \qquad i \in C \cup F \quad \text{and} \quad j \in B \cup D.$$

Then $(x_j, y_i, y_j) \in \mathscr{G}_2$ is strongly colored in g.
SScase III.II.III: $A = \emptyset$ and $C \neq \emptyset$ and $B \cup D = \emptyset$. This means that only colour 3 appears in Y. Then, there exist i and j such that: $|i - j| = 1$, $i \in X_1$ and $j \in X_2$. Then, (x_i, x_j, y_k), with $k = \max(i, j)$, belongs to $\mathscr{G}_1$ and is strongly colored in g.
SScase III.II.IV: $C = \emptyset$. There exists i and j such that: $|i - j| = 1$, $i \in F$, and $j \in A \cup B \cup D$. If $j \in B \cup D$, then $(x_j, y_i, y_j) \in \mathscr{G}_2$ is strongly colored in g. If $j \in A$, then $(x_i, y_i, y_j) \in \mathscr{G}_2$ is strongly colored in g.

From Theorem 2.1. follows immediately:

Corollary 2.2. *Let n_0 be a fixed integer ≥ 3, and γ such that $S(n_0) = \gamma n_0^2 - 3n_0/2 + 1$. Then $S(n) \leq \gamma n^2 - 3n/2 + 1$, for all the values of n of the form $n = 2^k n_0$.*

Taking $n_0 = 11$ and $S(n_0) = 33$ (by Theorem 0.1), we get the following result which implies Theorem 0.2:

Corollary 2.3. *$S(n) \leq (97/242)n^2 - 3n/2 + 1$, for all the values of n of the form $n = 11 \cdot 2^k$ (k positive integer).*

Lemma 2.4. $S(n) \leq S(n-1) + n - 2$, $n \geq 4$.

Proof. Let $H = (X, \mathscr{E})$ with $H \in \mathscr{H}(n-1)$ and $|\mathscr{E}| = n - 1$. Let $x \notin X$, $X' = X \cup \{x\}$, and $\mathscr{G}x$ be a set of $n-2$ triples containing x and such that $Gx = (X, \{E - \{x\} \mid E \in \mathscr{G}x\})$ is a connected graph. Then $H' = (X', \mathscr{E} \cup \mathscr{E}x)$ belongs to $\mathscr{H}(n)$ and has $S(n-1) + n - 2$ edges, hence proving the lemma.

From Lemma 2.4 (which is a special case of a result of [2]), we can deduce an upper bound for $S(n)$, valid for all values of n:

Corollary 2.5. $S(n) \leq \binom{n-1}{2} - (3/121)(n+1)^2$, $n \geq 11$.

Proof. Let k be the integer such that

$$m = 11 \cdot 2^k \leq n \leq 11 \cdot 2^{k+1} - 1 = 2m - 1.$$

By repeated use of Lemma 2.4, we have:

$$S(n) \leq S(m) + (n-2) + (n-3) + \cdots + (m-1).$$

Hence, by Corollary 2.3,

$$S(n) \leq (97/242)m^2 - 3m/2 + 1 + \binom{n-1}{2} - \binom{m-1}{2},$$

and the result follows, using $m \geq (n+1)/2$. From Corollary 2.5, we get Theorem 0.3: $\limsup S(n)/n^2 \leq 0.5 - (3/121)$.

Note added in proof. The conjective in Section 0 was recently proved by the author.

References

[1] C. Berge, Graphes et Hypergraphes (Dunod, Paris, 1970).
[2] F. Sterboul, A new combinatorial parameter, Colloquia Mathematica Societatis J. Bolyai, 10, Infinite and finite sets (Keszthely, Hungary, 1973) 1387–1404.
[3] F. Sterboul, Un problème extrémal pour les graphes et les hypergraphes, Discrete Math. 11 (1975) 71–78.