Highest weight modules over pre-exp-polynomial Lie algebras

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\begin{abstract}
In this paper, we introduce pre-exp-polynomial Lie algebras which include loop algebras, Virasoro-like algebras and some quantum torus Lie algebras. We study “highest weight” representations of these \(\mathbb{Z}_{n+1}\)-graded Lie algebras. More precisely, we show that non-graded and graded irreducible highest weight modules with the same highest weight simultaneously have all finite-dimensional weight spaces or not, and they have all finite-dimensional weight spaces if and only if the highest weight is an exp-polynomial “highest weight”. We also show that non-graded and graded highest weight Verma modules with the same highest weight are simultaneously irreducible or not, and we determine necessary and sufficient conditions for a Verma module to be irreducible.
\end{abstract}

\section{Introduction}

Representations of many infinite-dimensional Lie algebras have many important applications in mathematics and physics. Various highest weight modules play a crucial role in this aspect. The relation to physics is well established in the book on conformal field theory [11]. Recently, developing representation theory for higher rank infinite-dimensional Lie algebras has attracted extensive atten-
tion of many mathematicians and physicists. These Lie algebras include extended affine Lie algebras, loop algebras and quantum torus Lie algebras.

Extended affine Lie algebras are natural multi-variable generalizations of affine Kac–Moody algebras. The theory of affine Lie algebras is rich and beautiful, having connections with diverse areas of mathematics and physics. Extended affine Lie algebras have also been proven themselves to be useful for the applications. The papers [3,4,18] constructed useful weight representations for extended affine Lie algebras.

Frenkel, Jing and Wang [16] used representations of loop Lie algebras to construct a new form of the McKay correspondence. Inami et al., studied toroidal symmetry in the context of a 4-dimensional conformal field theory [20,21]. There are also applications of toroidal Lie algebras to soliton theory [9,22]. We refer readers to [1,2,5–7,12–14,17] for various representations for loop algebras and for toroidal Lie algebras. In particular, a class of very interesting irreducible modules with finite-dimensional weight spaces over full toroidal Lie algebras were constructed in [6].

Quantum torus Lie algebras is another class of interesting Lie algebras closely related to extended affine Lie algebras. Recently, various interesting weight representations were constructed in [10,15,18,19,26], and certain level of classifications were discussed in [23].

For research on a same problem for different algebras, different techniques and different methods have been used in the above papers. Most of the above mentioned Lie algebras are extragraded exp-polynomial Lie algebras defined in [8]. So the concept of exp-polynomial Lie algebras is very wide. The new “highest weight” modules constructed in [8] have been used in several places, for example, classification of irreducible Harish-Chandra modules [25], some affine Kac–Moody algebra modules [29].

From the above remark we see that it is hard to find a uniform method to study a problem for all exp-polynomial Lie algebras. In this paper we introduce pre-exp-polynomial Lie algebras in which a class of Lie algebras are exp-polynomial Lie algebras. There are some interesting examples for pre-exp-polynomial Lie algebras, including loop algebras (Example 1), some quantum torus Lie algebras (Example 2) and Virasoro-like algebras (Example 3).

Higher rank infinite-dimensional Lie algebras do not possess the ordinary triangular decomposition (as defined in [28]). Several mathematicians have developed different ways for finding irreducible weight modules for such Lie algebras. One of the classical problems is to find irreducible weight modules with finite-dimensional homogeneous spaces. In the present paper, we have considered graded and non-graded weight modules with the same highest weight for pre-exp-polynomial Lie algebras (including several known Lie algebras) and shown that under some natural conditions they both have all finite-dimensional weight spaces or not at the same time. Also we have found necessary and sufficient conditions for a Z-graded Verma G-module to be irreducible. We have shown the non-graded and graded Verma G-modules are irreducible or not at the same time.

Let us first recall the construction of highest weight modules over \( \mathbb{Z}^{n+1} \)-graded Lie algebras constructed in [8]. Let us denote the set of integers, non-negative integers, positive integers, the complex numbers by \( \mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{C} \) respectively.

Let \( G \) be a \( \mathbb{Z} \times \mathbb{Z}^n \)-graded Lie algebra and let \( G = G^- \oplus G^{(0)} \oplus G^+ \) be a decomposition of \( G \) relative to the \( \mathbb{Z} \)-grading. The subalgebra \( G^{(0)} \) is a \( \mathbb{Z}^n \)-graded infinite-dimensional Lie algebra which can be noncommutative. We take some natural module \( V \) for \( G^{(0)} \) (usually \( V \) is either \( \mathbb{Z}^n \)-graded or finite-dimensional). Parallel to the construction of a usual highest weight module, we let \( G^+ \) act on \( V \) trivially, and introduce the induced module

\[
\tilde{M}(V) = \text{Ind}_{G^{(0)} + G^+}^{G^-} V \simeq U(G^-) \otimes_{\mathbb{C}} V.
\]

If \( V \) is \( \mathbb{Z}^n \)-graded then \( \tilde{M}(V) \) inherits a \( \mathbb{Z} \times \mathbb{Z}^n \)-grading and we call it a graded module; if \( V \) is finite-dimensional and not \( \mathbb{Z}^n \)-graded, then we call \( \tilde{M}(V) \) a non-graded module. We denote by \( \tilde{M}(V) \) graded Verma module and non-graded Verma module corresponding to whether \( V \) is \( \mathbb{Z}^n \)-graded.

The Verma module \( M(V) \) always has a quotient module \( M(V) \) which does not have any proper submodule trivially intersecting \( V \). In many cases, \( M(V) \) will have finite-dimensional homogeneous components. In [8], it was shown that \( M(V) \) has finite-dimensional homogeneous components when
$G$ is an exp-polynomial Lie algebra and $V$ is an exp-polynomial module. The converse of this result is generally not true for a distinguished spanning set, see [14, Theorem 2.2]. But it is true if we suitably choose a distinguished spanning set for some exp-polynomial algebras. Part of the present paper is to study this converse.

The paper is organized as follows. In Section 2 we recall from [8] the definitions for exp-polynomial Lie algebras, exp-polynomial modules, and define our pre-exp-polynomial Lie algebras. Then we give the statement of our main results (Theorems 2.9, 2.10, 2.11 and 2.12). We provide examples of such Lie algebras and modules as well as give examples of applications of the main theorems. In Section 3 we give the proof of Theorems 2.9 and 2.10, and in Section 4 we prove Theorems 2.11 and 2.12.

2. Pre-exp-polynomial Lie algebras

In this section, we first recall some definitions from [8], then introduce pre-exp-polynomial Lie algebras, give some examples, state the main results in this paper and present some applications.

We assume that all Lie algebras and vector spaces are over $\mathbb{C}$ in this paper, although $\mathbb{C}$ can be replaced by any field of characteristic 0.

**Definition 2.1.** The algebra of exp-polynomial functions in $r$ variables, $n_1, \ldots, n_r$, is the algebra of functions $f(n_1, \ldots, n_r): \mathbb{Z}^r \to \mathbb{C}$ generated as an algebra by functions $n_j$ and $a^n_j$, where various $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, r$.

An exp-polynomial function may be written as a finite sum

$$f(n_1, \ldots, n_r) = \sum_{k \in \mathbb{Z}_+^r} \sum_{a \in (\mathbb{C}^*)^r} c_{k,a} n_{k_1}^{a_{1}} \cdots n_{k_r}^{a_{r}},$$

where $c_{k,a} \in \mathbb{C}$, $k = (k_1, \ldots, k_r)$ with $k_j \geq 0$, and $a = (a_1, \ldots, a_r)$.

**Definition 2.2.** Let $G = \bigoplus_{\alpha \in \mathbb{Z}^n} G_{\alpha}$ be a $\mathbb{Z}^n$-graded Lie algebra and $K$ be an index set. Then $G$ is said to be an exp-polynomial Lie algebra if $G$ has a homogeneous spanning set $\{g_k(\alpha) \mid k \in K, \alpha \in \mathbb{Z}^n\}$ with $g_k(\alpha) \in G_{\alpha}$, and there exists a family of exp-polynomial functions $\{f_{k,r}^{s}(\alpha, \beta) \mid k, r, s \in K\}$ in the $2n$ variables $\alpha_j, \beta_j$ and where for each $k, r$ the set $\{s \mid f_{k,r}^{s}(\alpha, \beta) \neq 0\}$ is finite, such that

$$[g_k(\alpha), g_r(\beta)] = \sum_{s \in K} f_{k,r}^{s}(\alpha, \beta) g_s(\alpha + \beta), \quad \text{for } k, r \in K, \alpha, \beta \in \mathbb{Z}^n.$$ (2.1)

This homogeneous spanning set $\{g_k(\alpha) \mid k \in K, \alpha \in \mathbb{Z}^n\}$ is called a distinguished spanning set.

**Definition 2.3.** Let $G = \bigoplus_{\alpha \in \mathbb{Z}^n} G_{\alpha}$ be an exp-polynomial Lie algebra. A $G$ module $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_{\alpha}$ is called a $\mathbb{Z}^n$-graded exp-polynomial module if $V$ has a basis $\{v_j(\alpha)\}_{j \in J, \alpha \in \mathbb{Z}^n}$, and there exists a family of exp-polynomial functions $h_{k,j}^{s}(\alpha, \beta)$ for $k \in K, j, s \in J$ such that

$$g_k(\alpha)v_j(\beta) = \sum_{s \in J} h_{k,j}^{s}(\alpha, \beta) v_s(\alpha + \beta),$$

where $(g_k(\alpha))_{k \in K}$ is the distinguished spanning set for $G$, and for each $k, j$ the set $\{s \mid h_{k,j}^{s}(\alpha, \beta) \neq 0\}$ is finite. The homogeneous components of the $\mathbb{Z}^n$-grading on $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_{\alpha}$ are given by $V_{\alpha} = \text{Span} \{v_j(\alpha)\}_{j \in J}.$

**Definition 2.4.** Let $G$ be a $\mathbb{Z}^n$-graded exp-polynomial Lie algebra. We call this algebra $\mathbb{Z}^n$-extragraded if $G$ has another $\mathbb{Z}$-gradation.
G = ∐_{i ∈ Z} G^{(i)}, \quad (2.2)

and the set \( K \) is a disjoint union of finite subsets \( K_i \),

\[ K = \bigcup_{i ∈ Z} K_i, \]

such that the elements of the homogeneous spanning set \( \{ g_k(α) \mid k ∈ K_i, \ α ∈ ℤ^n \} \) are homogeneous of degree \( i \) under this new \( ℤ \)-gradation and span \( G^{(i)} \).

Similar to the concept of local Lie algebras as in [27], we first introduce local exp-polynomial Lie algebras.

**Definition 2.5.** Let \( K \) be a nonempty finite set and \( n ∈ ℤ \). A \( ℤ^{n+1} \)-graded Lie algebra \( G = \bigoplus_{i ∈ ℤ} \bigoplus_{α ∈ ℤ^n} G^{(i)}(α) \), is called a local exp-polynomial Lie algebra if

1. (P1) \( G \) is generated by \( G^{(1)} \) and \( G^{(-1)} \) as a Lie algebra, where \( G^{(i)} = \bigoplus_{α ∈ ℤ^n} G^{(i)}(α) \).
2. (P2) There exist nonzero-vector generating sets \( \{ g_k^{(±1)}(α) \mid k ∈ K \} \) for \( G^{(1)}(α) \), \( G^{(-1)}(α) \) respectively, \( \{ g_k^{(0)}(α) \mid k ∈ K \} \) for \( G^{(0)}(α) \) and exp-polynomial functions \( f_{i,j}^{(k,0)}(α, β) \), \( f_{i,j}^{(k,1)}(α, β) \) and \( f_{i,j}^{(k,-1)}(α, β) \) for any \( i, j, k ∈ K \) such that

\[
\begin{align*}
\left[ g_i^{(1)}(α), g_j^{(-1)}(β) \right] &= \sum_{k ∈ K} f_{i,j}^{(k,0)}(α, β) g_k^{(0)}(α + β), \\
\left[ g_i^{(0)}(α), g_j^{(1)}(β) \right] &= \sum_{k ∈ K} f_{i,j}^{(k,1)}(α, β) g_k^{(1)}(α + β), \\
\left[ g_i^{(0)}(α), g_j^{(-1)}(β) \right] &= \sum_{k ∈ K} f_{i,j}^{(k,-1)}(α, β) g_k^{(-1)}(α + β).
\end{align*}
\]

Now we introduce our pre-exp-polynomial Lie algebras which we shall study in the present paper.

**Definition 2.6.** A local exp-polynomial Lie algebra \( G \) as defined in Definition 2.5 is called a pre-exp-polynomial Lie algebra if

1. (P3) \( G^{(0)} \) is abelian.
2. (P4) \( f_{i,j}^{(k,0)}(α, β) = δ_i,j δ_{i,k} f_i(α) \).
3. (P5) \( \{ g_k^{(±1)}(α) \mid k ∈ K \} \) are bases for \( G^{(1)}(α) \) and \( G^{(-1)}(α) \) respectively.

**Remark 1.** The set \( \{ g_k^{(0)}(α) \mid k ∈ K \} \) is not necessarily basis for \( G^{(0)}(α) \).

**Remark 2.** Condition (P3) is like to have the Cartan subalgebra for a semisimple Lie algebra. This condition indeed excludes a lot of exp-polynomial algebras, for example Witt algebras and generalized Virasoro algebras with rank \( n + 1 \) which do not have an abelian subalgebra with index group \( ℤ^n \). Still there are some interesting Lie algebras left, see examples below. Conditions (P1) and (P2) are natural. Even we can add the natural condition that \( G \) does not have any nonzero ideal contained in \( G^+ + G^- \). We have conditions (P4) and (P5) in the definition because of some technical reasons.
Remark 3. There are pre-exp-polynomial Lie algebras which are not exp-polynomial Lie algebras. Let us consider the case $n = 1$. Let $g^{(±1)}(α)$ be a basis for $G^{(±1)}(α)$ respectively, and $g^{(0)}(α)$ be a basis for $G^{(0)}(α)$. Let $G$ be the free Lie algebra generated by $\{g^{(±1)}(α), g^{(0)}(α) \mid α ∈ \mathbb{Z}\}$ and let $\tilde{G}$ be the quotient algebra of $G$ modulo the relations $[g^{(1)}(α), g^{(−1)}(β)] = g^{(0)}(α + β)$, $[g^{(0)}(α), g^{(1)}(β)] = g^{(1)}(α + β)$ and $[g^{(0)}(α), g^{(−1)}(β)] = −g^{−1}(α + β)$ for all $α, β ∈ \mathbb{Z}$. One can show that $G$ is a pre-exp-polynomial Lie algebra with $K = \{1\}$, but not an exp-polynomial Lie algebra since $[g^{(1)}(α + β), g^{(−1)}(β)]$ are linearly independent for all $α, β ∈ \mathbb{Z}$, thus one cannot find a finite set $\{g^{(2)}_k(α) \mid k ∈ K(2)\}$, where $K(2)$ is an index set, and functions $f^{(k)}(α + β, −β)$ such that

$$[g^{(1)}(α + β), g^{(1)}(−β)] = ∑_{k ∈ K2} f^{(k)}(α + β, −β)g^{(2)}_k(α).$$

For convenience, let $A = \mathbb{C}[t_1^{±1}, t_2^{±1}, . . . , t^n]^{2}$ be the Laurent polynomial algebra. We denote $W_A = W ⊗ A$ for any vector space $W$. Now let us look at some examples.

Example 1. Let $g$ be a Kac–Moody algebra of rank $r$, $h$ a Cartan subalgebra of $g$, $Π = \{α_1, α_2, . . . , α_r\} ⊆ h^*$ the root basis, and $e_1, e_2, . . . , e_r, h_1, h_2, . . . , h_r, f_1, f_2, . . . , f_r$ the Chevalley generators of $g$. Denote by $Δ, Δ_+$ and $Δ_−$ the sets of all roots, positive roots and negative roots respectively. Then we have the following root space decomposition of $g$ with respect to $h$

$$g = ∑_{α ∈ Δ_−} g_α ⊕ h ⊕ ∑_{α ∈ Δ_+} g_−α.$$

Consider the principal $\mathbb{Z}$-gradation of derived algebra $g' = ∇_{i ∈ \mathbb{Z}} g_i$ where $g_i$ is spanned by all root space $g_α$ where $α$ has height $i$.

It is easy to see that the $\mathbb{Z}^{n+1}$-graded loop algebra $g'_A = ∇_{i ∈ \mathbb{Z}} g_i ⊗ A$ becomes a pre-exp-polynomial algebra, where we take $K = \{1, 2, . . . , r\}$, $g^{(1)}_k(α) = e_k ⊗ t^α$, $g^{(−1)}_k(α) = f_k ⊗ t^α$, $g^{(0)}_k(α) = h_k ⊗ t^α$ for $k ∈ K$.

Example 2. Consider an associative quantum torus

$$C_q = ∇_{i ∈ \mathbb{Z}, α ∈ \mathbb{Z}^n} C t_i^α$$

generated by the variables $t_0^±, t_1^±, . . . , t_n^±$, where $t_1, . . . , t_n$ commute with each other and $t_0$ does not commute with any nonconstant element in $C[t_1, . . . , t_n]$, but satisfies the relations: $t_j t_0 = q_j t_0 t_j$ for some $q_1, . . . , q_n ∈ \mathbb{C}^*$. From the associative algebra $C_q$ we define the Lie algebra $Ω_q$ with the brackets

$$[t_0^α(α), t_0^β(β)] = (q^{iα} − q^{−iβ} + δ(α, (−i, −β))q^{−iα}) t_0^{i+1}(α + β),$$

where $q^α = q_1^α q_2^α . . . q_n^α$. We can easily verify that $Ω_q$ is a pre-exp-polynomial Lie algebra with $K = \{1\}$, $g^{(1)}_k(α) = t_0^{iα}$, $g^{(−1)}_k(α) = t_0^{−iα}$, $g^{(0)}_k(α) = (1 − q^α + δ_{α, 0}) t^α$, and with coefficient functions $f_i(α) = q^α$, $f^{(1)}_j(α, β) = −(1 − q^{−2}) f^{(−1)}_j(α, β) = (1 − q^{−2}) (q^{−α} − 1)$. The Lie algebra $Ω_q$ is actually an exp-polynomial algebra, see [23] for details.

Example 3. Let $B$ be an additive subgroup of $C$ with rank $n$. The Virasoro-like algebra $L(B)$ over $C$ is the Lie algebra with a $C$-basis $\{d_x \mid x ∈ \mathbb{Z} ⊕ B\}$ and subject to the following commutator relations
Lemma 2.7

where \( x = (x^{(1)}, x^{(2)}) \), \( y = (y^{(1)}, y^{(2)}) \), \( \{y^{(1)}_1, y^{(2)}_1\} = \det \left( \begin{array}{cc} y^{(1)}_1 & y^{(2)}_1 \\ x^{(1)} & x^{(2)} \end{array} \right) \) \( d_{x+y} \), \( \forall x, y \in \mathbb{Z} \oplus B \).

From now on we assume that \( G \) is such a pre-exp-polynomial Lie algebra. We denote

\[
g_k^{(i)}(\alpha) = g_k^{(i)} \otimes t^\alpha. \quad (2.3)
\]

For example, \( g_k^{(1)} \otimes (3t^\alpha + 5t^\beta) = 3g_k^{(1)}(\alpha) + 5g_k^{(1)}(\beta) \) for \( \alpha, \beta \in \mathbb{Z}^n \), \( k \in K \). The symbol \( g_k^{(i)} \otimes t^\alpha \) is not actually a tensor product, just for the purpose of later use.

Assume that \( \psi \in (G^{(0)})^* \), the dual space of \( G^{(0)} \). Then we have the 1-dimensional \( G^{(0)} \)-module \( V_\psi = C\nu \) via \( \nu v = \psi(\chi)v \) for any \( \chi \in G^{(0)} \). At the same time \( V_\psi \otimes A \) becomes a \( \mathbb{Z}^n \)-graded \( G^{(0)} \)-module via \( g_k^{(0)}(\alpha)(v \otimes t^\beta) = \psi(g_k^{(0)}(\alpha)) (v \otimes t^{\alpha+\beta}) \). We shall simply write \( v \otimes t^\beta = t^\beta \). We denote the \( \mathbb{Z}^n \)-graded \( G^{(0)} \)-submodule generated by 1 by \( V_\psi \). Let \( \Gamma_\psi = \{ \alpha \in \mathbb{Z}^n \mid t^\alpha \in V_\psi \} \).

The first two parts in following lemma are well known, see [12].

**Lemma 2.7.**

(a) \( V_\psi \) is a graded-simple \( G^{(0)} \)-module if and only if \( \Gamma_\psi \) is a subgroup of \( \mathbb{Z}^n \).

(b) If \( \Gamma_\psi \) is a subgroup of \( \mathbb{Z}^n \), let \( \tilde{\Gamma}_\psi = \mathbb{Z}^n / \Gamma_\psi \). Then \( V_\psi \otimes A = \bigoplus_{m \in \tilde{\Gamma}_\psi} U(G^{(0)})(v \otimes t^m) \) as \( G^{(0)} \)-modules where by \( m \in \tilde{\Gamma} \psi \) we mean one representative element in \( \tilde{\Gamma}_\psi \).

(c) As \( G^{(0)} \)-modules, \( U(G^{(0)})(v \otimes t^m) \) is isomorphic to \( V_\psi \) for any \( \alpha \in \mathbb{Z}^n \).

One can easily obtain (c) by the following \( G^{(0)} \)-module isomorphism \( V_\psi \otimes A \to V_\psi \otimes A, v \otimes t^\beta \mapsto v \otimes t^{\beta - \alpha} \).

Let \( G^+ = \bigoplus_{i \geq 1} G^{(i)} \), \( G^- = \bigoplus_{i \leq -1} G^{(i)} \). Then we have the decomposition

\[
G = G^- \oplus G^{(0)} \oplus G^+. \quad (2.4)
\]

We can define the action of \( G^+ \) on \( V_\psi \) by \( G^+ V_\psi = 0 \) and then consider the induced module

\[
\tilde{M}(V_\psi) = \text{Ind}_{G^{(0)}+G^+}^{G^{(0)}+G} V_\psi \simeq U(G^-) \otimes_C V_\psi. \quad (2.5)
\]

It is clear that \( \tilde{M}(V_\psi) \) is a \( \mathbb{Z}^{n+1} \)-graded module over \( G \) and

\[
\tilde{M}(V_\psi)^{(i)} = \bigoplus_{i \leq 0, \alpha \in \mathbb{Z}^n} \tilde{M}(V_\psi)^{(i)}_\alpha, \quad (2.6)
\]

where \( \tilde{M}(V_\psi)^{(i)}_\alpha \) is naturally defined, for example, \( \tilde{M}(V_\psi)^{(0)}_\alpha = (V_\psi)_\alpha \). In general, the homogeneous components \( \tilde{M}(V_\psi)^{(i)}_\alpha \) with \( i < 0 \) are infinite-dimensional.

It is easy to see that \( \tilde{M}(V_\psi) \) has a unique maximal proper \( \mathbb{Z}^{n+1} \)-graded submodule \( \tilde{M}^{\text{red}} \) which intersects trivially with \( V_\psi \). Let

\[
[d_x, d_y] = \det \left( \begin{array}{cc} y^{(1)}_1 & y^{(2)}_1 \\ x^{(1)} & x^{(2)} \end{array} \right) d_{x+y}, \quad \forall x, y \in \mathbb{Z} \oplus B,
\]
Lemma 2.8. Assume that $G$ is a pre-exp-polynomial Lie algebra defined in Definition 2.6. Then we have the induced $\mathbb{Z}^{n+1}$-gradation

$$M(V_{\bar{\psi}}) = \tilde{M}(V_{\bar{\psi}})/\tilde{M}^{\rad}.$$  

(2.7)

If we replace $V_{\bar{\psi}}$ with $V_{\psi}$ in the previous construction, as in (2.5) and (2.7) we define a $\mathbb{Z}$-graded $G$ module $\tilde{M}(V_{\bar{\psi}})$ and its $\mathbb{Z}$-graded factor-module $M(V_{\bar{\psi}}) = \tilde{M}(V_{\bar{\psi}})/\tilde{M}^{\rad}$.

For any non-graded $G$-module $W$, we can make $W_A$ into a $\mathbb{Z}^n$-graded $G$-module as follows:

$$g^{(i)}(\alpha)(w \otimes t^\beta) = (g^{(i)}(\alpha)w) \otimes t^{a_i+\beta}$$

where $w \in W$ and $g^{(i)}(\alpha) \in G^{(i)}(\alpha)$.

Lemma 2.8. Assume that $\Gamma_{\bar{\psi}}$ is a group.

(a) As $G$-modules, $M(V_{\bar{\psi}}) \otimes A = \bigoplus_{\alpha \in \mathbb{Z}^n} U(G)(v \otimes t^m)$.

(b) As $G$-modules, $U(G)(v \otimes t^\alpha)$ is isomorphic to $M(V_{\bar{\psi}})$ for any $\alpha \in \mathbb{Z}^n$.

Proof. (a) follows from $U(G)(v \otimes t^m) = U(G^-)U(G^{(0)})(v \otimes t^m) = U(G^-)U(G^{(0)})(v \otimes t^m) \simeq U(G^-)V_{\bar{\psi}}$ which is irreducible.

(b) follows Lemma 2.7(b) and the fact that the above module $U(G^-)V_{\bar{\psi}} \simeq M(V_{\bar{\psi}})$.

One can easily see that Lemma 2.8 also holds for Verma module $\tilde{M}(V_{\bar{\psi}})$.

The main results of this paper are the following theorems.

Theorem 2.9. Assume that $G$ is a pre-exp-polynomial Lie algebra defined in Definition 2.6 and suppose that $\psi \in (G^{(0)})^+$ and $G$ does not have any nonzero ideal contained in $G^+$. Then the following statements are equivalent:

(a) $G$ module $M(V_{\bar{\psi}})$ has all finite-dimensional homogeneous spaces.

(b) There exists a nonzero polynomial $P(X) \in \mathbb{C}[X]$ such that $\psi$ satisfies the following condition

$$\psi(g^{(0)}_l \otimes AP(t_j)) = 0.$$ 

for every $l, j$.

(c) There exists exp-polynomial function $p_k(\alpha)$ for $k \in K$ such that $\psi(g^{(0)}(\alpha)) = p_k(\alpha)$ for all $k \in K$ and all $\alpha \in \mathbb{Z}^n$.

Theorem 2.10. Assume that $G$ is a pre-exp-polynomial Lie algebra defined in Definition 2.6. Suppose that $\psi \in (G^{(0)})^+$ and $G$ does not have any nonzero ideal contained in $G^+$ and suppose that $\Gamma_{\bar{\psi}}$ is a subgroup of $\mathbb{Z}^n$. Then the $\mathbb{Z}^n$-graded $G$ module $M(V_{\bar{\psi}})$ has all finite-dimensional homogeneous spaces if and only if $M(V_{\bar{\psi}})$ has all finite-dimensional homogeneous spaces.

Theorem 2.11. Let $G$ be a pre-exp-polynomial Lie algebra defined in Definition 2.6. $\psi \in (G^{(0)})^+$. Then the $\mathbb{Z}$-graded $G$ module $\tilde{M}(V_{\bar{\psi}})$ is not irreducible if and only if there is a $\mathbb{Z}^{n+1}$-graded nonzero ideal of $G$ contained in $G^-$ or there exists $k \in K$, distinct $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n$ and $a_1, a_2, \ldots, a_r \in \mathbb{C}$ such that $f_k(\alpha)\psi(\sum_i a_i g^{(0)}_k(\alpha + \alpha_i)) = 0$ for all $\alpha \in \mathbb{Z}^n$.

Theorem 2.12. Let $G$ be a pre-exp-polynomial Lie algebra defined in Definition 2.6. $\psi \in (G^{(0)})^+$. Assume that $\Gamma_{\bar{\psi}}$ is a subgroup. Then the graded $G$ module $\tilde{M}(V_{\bar{\psi}})$ is irreducible if and only if the non-graded $G$ module $\tilde{M}(V_{\bar{\psi}})$ is irreducible.
Remark 4. If \( \Gamma_\psi \) is not a subgroup, Theorems 2.10 and 2.12 are not true in general. Consider the case \( n = 1 \) and take \( \psi(g_k^{(0)}(i)) = 0 \) if \( i \geq 0 \) and \( \psi(g_k^{(0)}(i)) = 1 \) if \( i < 0 \). Then we know that \( \bar{M}(V_\psi) = M(V_\psi) \) is irreducible and has infinite-dimensional weight spaces, but \( \tilde{M}(V_\psi) \) is not irreducible and \( M(V_\psi) \) is 1-dimensional if we redefine \( M(V_\psi) \) as the irreducible quotient of \( \bar{M}(V_\psi) \).

Applying Theorems 2.9 and 2.11 we see

**Corollary 2.13.** Let \( G \) be a pre-exp-polynomial Lie algebra defined in Definition 2.6 with \( n = 1, \psi \in (G^{(0)})^* \). Assume \( G \) does not have a nonzero ideal contained in \( G^+ + G^- \). Then \( \mathbb{Z} \)-graded \( G \) module \( M(V_\psi) \) is not irreducible if and only if \( M(V_\psi) \) has all finite-dimensional homogeneous spaces.

For convenience we denote \( \varepsilon_i = (\delta_{1,i}, \ldots, \delta_{n,i}) \in \mathbb{Z}^n \).

In [23,24], the authors proved that, for exactly two variable case \( (n = 1) \) with nonzero central charge, \( \mathbb{Z}^2 \)-graded simple modules with all finite-dimensional weight spaces are some of highest weight modules \( M(V_\psi) \). Our Theorem 2.9 answers which \( M(V_\psi) \) should be in the class.

**Example 4.** Let us consider the algebra in Example 1. In this case Theorems 2.9 and 2.10 give Lemmas 3.7 and 3.6 in [11]. Suppose \( \psi \in (G^{(0)})^* = (\mathfrak{h}_1')^* \) where \( \mathfrak{h}_1' \) is the Cartan subalgebra of \( g' \) and suppose \( \Gamma_\psi \) is a group. From Theorems 2.11 and 2.12 we know that Verma modules \( \bar{M}(V_\psi) \) and \( M(V_\psi) \) are not irreducible if and only if there exists \( k \in K \), distinct \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n \) and nonzero \( a_1, a_2, \ldots, a_r \in \mathbb{C} \) such that \( \psi(\sum_{i=1}^{r} a_i h_k(\alpha + \alpha_i)) = 0 \) for all \( \alpha \in \mathbb{Z}^n \).

**Example 5.** Let us consider the algebra in Example 2. In this case when \( n = 1 \), Theorem 2.9 gives Theorem 2.1 in [14]. Suppose \( \psi \in (G^{(0)})^* = (\mathbb{C}[t_1^\pm 1, \ldots, t_n^\pm 1])^* \) and suppose \( \Gamma_\psi \) is a group. From Theorems 2.9 and 2.10, we know that modules \( M(V_\psi) \) and \( \bar{M}(V_\psi) \) have all finite-dimensional homogeneous spaces if and only if \( \psi \) is an exp-polynomial function on the basis \( \{g_1^{(0)}(\alpha) = (1 - q^\alpha + \delta_{\alpha,0})^\alpha | \alpha \in \mathbb{Z}^n \} \).

From Theorems 2.11 and 2.12 we know that Verma modules \( \bar{M}(V_\psi) \) and \( M(V_\psi) \) are not irreducible if and only if there exists distinct \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n \) and nonzero \( a_1, a_2, \ldots, a_r \in \mathbb{C} \) such that \( \psi(\sum_{i=1}^{r} a_i (1 - q^{\alpha_1 + \delta_{\alpha_1,0}}) \alpha^{\alpha + \alpha_i}) = 0 \) for all \( \alpha \in \mathbb{Z}^n \).

**Example 6.** Let us consider the algebra in Example 3. Suppose \( \psi \in (G^{(0)})^* \) and suppose \( \Gamma_\psi \) is a group. From Theorems 2.9 and 2.10, we know that modules \( M(V_\psi) \) and \( \bar{M}(V_\psi) \) have all finite-dimensional homogeneous spaces if and only if \( \psi \) is an exp-polynomial function on the basis \( \{g_1^{(0)}(\alpha) = (-\alpha + \delta_{\alpha,0}) d_{0,\alpha} | \alpha \in B \} \).

From Theorems 2.11 and 2.12 we know that Verma modules \( \bar{M}(V_\psi) \) and \( M(V_\psi) \) are not irreducible if and only if there exists distinct \( \alpha_1, \alpha_2, \ldots, \alpha_r \in B \) and nonzero \( a_1, a_2, \ldots, a_r \in \mathbb{C} \) such that \( \psi(\sum_{i=1}^{r} a_i (-\alpha_i + \delta_{\alpha_i,0}) d_{0,\alpha + \alpha_i}) = 0 \) for all \( \alpha \in B \).

3. **Proofs of Theorems 2.9 and 2.10**

In this section we shall prove Theorems 2.9, and 2.10. Before starting off, we first state the following lemma which can follow from Lemma 2.1 in [8].

**Lemma 3.1.** Suppose \( f(\alpha) \) is an exp-polynomial function for \( \alpha \in \mathbb{Z}^n \). If there is a subgroup \( \Gamma_0 \subset \mathbb{Z}^n \) and \( \gamma \in \mathbb{Z}^n \) such that \( \mathbb{Z}^n = \Gamma_0 \oplus \mathbb{Z} \gamma \), and if there exists \( k_0 \in \mathbb{N} \) such that \( f(\Gamma_0 + k \gamma) = 0 \) for all \( k > k_0 \), then \( f = 0 \).

**Proof of Theorem 2.9.** If \( f_k(\alpha) = 0 \) for a fixed \( \alpha \in \mathbb{Z}^n \) and a fixed \( k \in K \), we have

\[
\left[ g_k^{(1)}(\alpha), g_l^{(-1)}(\beta) \right] = \delta_{k,l} f_k(\alpha) g_k^{(0)}(\alpha + \beta) = 0
\]
for all \( \beta \in \mathbb{Z}^n \) and any \( l \in K \). Then we see that \( g^{(1)}_k(\alpha) \) can generate an ideal of \( G \) contained in \( G^+ \), which contradicts the assumption. Thus \( f_\alpha(\alpha) \neq 0 \) for any \( k \in K \) and any \( \alpha \in \mathbb{Z}^n \).

(a) \( \Rightarrow \) (b). Since \( \text{dim} \ M_{-1} \) is finite, for each fixed \( k \in K \) and \( j = 1, \ldots, n \), there exists a nonzero polynomial \( p_{k,j}(t_j) = \sum_{i=0}^{n_j} a_i t_j^i \in \mathbb{C}[t_j] \) where \( a_i \in \mathbb{C} \) with \( a_0 a_{n_j} \neq 0 \) such that

\[
\left( g^{(-1)}_k \otimes p_{k,j}(t_j) \right) v = 0.
\]

Applying \( g^{(1)}_k(\beta) \) to the above equation, we obtain that

\[
0 = g^{(1)}_k(\beta) \left( g^{(-1)}_k \otimes p_{k,j}(t_j) \right) v = \left( f_k(\beta) g^{(0)}_k \otimes t^\beta p_{k,j}(t_j) \right) v.
\]

So we have

\[
0 = f_k(\beta) \left( g^{(0)}_k \otimes t^\beta p_{k,j}(t_j) \right) v.
\]

We see that

\[
0 = \left( g^{(0)}_k \otimes t^\beta p_{k,j}(t_j) \right) v = \psi \left( g^{(0)}_k \otimes A p_{k,j}(t_j) \right) v
\]

to give

\[
\psi \left( g^{(0)}_k \otimes A p_{k,j}(t_j) \right) = 0, \text{ for every } k, \text{ and for every } j.
\]

Let \( P(t) = \prod_{k,j} p_{k,j}(t) \), then we have

\[
\psi \left( g^{(0)}_k \otimes A p(t) \right) = 0, \text{ for every } k \in K.
\]

(b) \( \Rightarrow \) (a). Suppose there exists a nonzero \( P(X) \in \mathbb{C}[X] \) such that

\[
\psi \left( g^{(0)}_k \otimes t^\beta p(t_j) \right) = 0, \text{ for every } k, j \text{ and for every } \beta \in \mathbb{Z}^n.
\]

We know that

\[
\left( g^{(0)}_k \otimes t^\beta p(t_j) \right) M_0 = 0, \text{ for every } k, j \text{ and for every } \beta \in \mathbb{Z}^n. \tag{3.1}
\]

From \( g^{(1)}_k(\beta) \left( g^{(-1)}_k \otimes t^\alpha p(t_j) \right) v = \delta_{k,k} \left( g^{(0)}_k \otimes t^{\alpha + \beta} p(t_j) \right) v = 0 \), the irreducibility of \( M(V_\psi) \) and (P1), we see that

\[
\left( g^{(-1)}_k \otimes t^\alpha p(t_j) \right) v = 0, \text{ for every } k, j \text{ and for every } \alpha \in \mathbb{Z}^n. \tag{3.2}
\]

Since \( G^- \) is generated by \( G^{(-1)} \) as a Lie algebra, we know that

\[
G^{(-1)} M_{-i} = M_{-i-1}, \text{ for every } i \in \mathbb{Z}^+, \text{ and if } v \in M_{-i}, \text{ where } i > 0, \text{ satisfies } G^{(1)} v = 0 \text{ then } v = 0.
\]
Next, by induction on \( s \), we show that

**Claim.** For any \( s \in \mathbb{Z}^+ \), we have

\[
(g_k^{(0)} \otimes r^\alpha P(t_j))_{M-s} = 0, \quad \text{for every } k, j \text{ and for every } \alpha \in \mathbb{Z}^n.
\]

and

\[
(g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s} = 0, \quad \text{for every } k, j \text{ and for every } \alpha \in \mathbb{Z}^n.
\]

Formulae (3.1) and (3.2) ensure the claim for \( s = 0 \). Suppose the claim holds for \( s \). Now let us consider the claim for \( s + 1 \). For every \( j, k, l \) and for every \( \alpha, \beta \in \mathbb{Z}^n \), we have

\[
(g_l^{(0)} \otimes r^\beta P(t_j))(g_k^{(-1)} \otimes r^\alpha M_{-s})
\]

\[
= (g_k^{(-1)} \otimes r^\alpha)(g_l^{(0)} \otimes r^\beta P(t_j))_{M-s} + \left[ g_l^{(0)} \otimes r^\beta P(t_j), g_k^{(-1)} \otimes r^\alpha M_{-s} \right]
\]

\[
= (g_k^{(-1)} \otimes r^\alpha)(g_l^{(0)} \otimes r^\beta P(t_j))_{M-s} + \sum_{r \in K} f_{l,k}^{(r,-1)}(\beta, \alpha)(g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s}.
\]

By induction, we have \((g_l^{(0)} \otimes r^\beta P(t_j))_{M-s} = 0\) and \((g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s} = 0\). Thus \((g_l^{(0)} \otimes r^\beta P(t_j))(g_k^{(-1)} \otimes r^\alpha M_{-s}) = 0\) for every \( k, j \) and for every \( \alpha, \beta \in \mathbb{Z}^n \). Since \( G^{(-1)}M_{-s} = M_{-s-1} \), we have proved the first formula in claim for \( s + 1 \).

Using this newly established formula, for any \( k, j \) and for every \( \alpha \in \mathbb{Z}^n \), and using induction for \( s \), i.e.,

\[
(g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s} = 0,
\]

we have

\[
(g_l^{(1)} \otimes r^\beta)(g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s-1}
\]

\[
= (g_k^{(-1)} \otimes r^\alpha P(t_j))(g_l^{(1)} \otimes r^\beta)_{M-s-1} + \left[ g_l^{(1)} \otimes r^\beta, g_k^{(-1)} \otimes r^\alpha P(t_j) \right]_{M-s-1}
\]

\[
= (g_k^{(-1)} \otimes r^\alpha P(t_j))(g_l^{(1)} \otimes r^\beta)_{M-s-1} + \delta_{l,k} f_k(\beta)(g_k^{(0)} \otimes r^\alpha P(t_j))_{M-s-1}.
\]

We notice that \((g_l^{(1)} \otimes r^\beta)_{M-s-1} \in M_{-s-1}\), and \((g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s} = 0\), also \((g_k^{(0)} \otimes r^\alpha P(t_j))_{M-s-1} = 0\) by the established formula. So \((g_l^{(1)} \otimes r^\beta)(g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s-1} = 0\) for all \( k \) and \( \alpha \). Since \( G^{(1)} \) generates \( G^+ \), then \((g_k^{(-1)} \otimes r^\alpha P(t_j))_{M-s-1} = 0\) for every \( k, j \) and for every \( \alpha \in \mathbb{Z}^n \). Thus claim is true.

From the second formula of claim, we see that

\[
\dim M_{-s-1} \leq |K| \deg P(t_j) \dim M_{-s}, \quad \text{for every } s \in \mathbb{Z}^+,
\]

where \(|K|\) is the number of the elements in \( K \). Then (a) follows.

From [26, Lemma 2.7], we know that (b) and (c) are equivalent. \( \square \)
From the above proof (a) ⇒ (b), we know

**Corollary 3.2.** G module $M(V_\psi)$ has at least one infinite-dimensional homogeneous space if and only if there exist $j$ and $k$ such that the vectors:

$$g_k^{(-1)}(\alpha + i\varepsilon_j)v; \quad i \in \mathbb{Z}$$

are linearly independent for any fixed $\alpha \in \mathbb{Z}^n$.

**Proof of Theorem 2.10.** We break the proof into two cases.

**Case 1.** The rank of $\Gamma_\psi$ is $n$, then the index $r$ of $\Gamma_\psi$ in $\mathbb{Z}^n$ is finite. As $\mathbb{Z}^n$-graded $G$ modules, $M(V_\psi) \otimes A \cong \bigoplus_r \text{copies} \ M(V_\bar{\psi})$. It is clear that $M(V_\bar{\psi})$ has all finite-dimensional homogeneous spaces if and only if $M(V_\psi)$ has all finite-dimensional homogeneous spaces.

**Case 2.** The rank of $\Gamma_\psi$ is less than $n$.

As $\mathbb{Z}^n$-graded $G$ modules, again from the fact that $M(V_\psi) \otimes A \cong \bigoplus_r \text{copies} \ M(V_\bar{\psi})$, maybe infinitely many copies, we see that $M(V_\bar{\psi})$ has all finite-dimensional homogeneous spaces if $M(V_\psi)$ has all finite-dimensional homogeneous spaces.

Now we assume that $M(V_\psi)$ has at least one infinite-dimensional homogeneous space. Then we know that $\psi \neq 0$. There exist a subgroup $\Gamma_0 \subset \mathbb{Z}^n$ and $\gamma \in \mathbb{Z}^n$ such that $\Gamma_\psi \subset \Gamma_0$ and $\mathbb{Z}^n = \Gamma_0 \oplus \mathbb{Z}\gamma$.

Without loss of generality we may assume that $\gamma = \varepsilon_1$ and $\Gamma_0 = \mathbb{Z}\varepsilon_2 + \cdots + \mathbb{Z}\varepsilon_n$.

From Corollary 3.2 we know there exist $j$ and $k$ such that the vectors:

$$g_k^{(-1)}(\alpha + i\varepsilon_j)v, \quad i \in \mathbb{Z}$$

are linearly independent for any fixed $\alpha \in \mathbb{Z}^n$, where $a_1 = 0$ and $a_j = \varepsilon_j$. We may assume that $j = k = 1$, i.e.,

$$g_1^{(-1)}(\alpha + i\varepsilon_1)v, \quad i \in \mathbb{Z}$$

(3.3)

are linearly independent for any fixed $\alpha \in \mathbb{Z}^n$.

Then there exists $\gamma_1 \in \mathbb{Z}^n$ such that (take $\alpha = 0$)

$$0 \neq g_1^{(1)}(\gamma_1)(g_1^{(-1)}(-i\varepsilon_1))v = f_1(\gamma_1)\psi(g_1^{(0)}(\gamma_1 - i\varepsilon_1))v.$$  

(3.4)

Then we know that $\gamma_1 - i\varepsilon_1 \in \Gamma_\psi$ and $\gamma_1 - j\varepsilon_1 \notin \Gamma_\psi$ if $j \neq i$. Noting that $\psi(g_1^{(0)}(\gamma')) = 0$ if $\gamma' \notin \Gamma_\psi$, we see that

$$g_1^{(1)}(\gamma_1)(g_1^{(-1)}(-j\varepsilon_1))v = \delta_{i,j}f_1(\gamma_1)\psi(g_1^{(0)}(\gamma_1 - i\varepsilon_1))v.$$  

(3.5)

Let us prove that in $M(V_\bar{\psi})_{-2}$, the vectors:

$$S = \{g_1^{(-1)}(i\varepsilon_1)g_1^{(-1)}(-i\varepsilon_1)v \mid i \in \mathbb{N}\}$$

are linearly independent. Otherwise we assume that
where all $a_i \neq 0$. We denote the left-hand side of the above expression by $X$.

Now suppose $k_0 = \max\{\|k_i\| : i = 1, 2, \ldots, r\}$. For any $k > k_0$, any $\alpha \in \mathbb{Z}^n$ with the first component to be 0, noting that $\alpha + (k \pm k_i)\varepsilon_1 \notin \mathcal{F}_\psi$ and $\psi(\alpha + (k \pm k_i)\varepsilon_1) = 0$ for all $i$, we have the following computations

\[
0 = g_1^{(1)}(\alpha + k\varepsilon_1)X = \sum_{i=1}^{r} a_i \left[ g_1^{(1)}(\alpha + k\varepsilon_1), g_1^{(-1)}(k_i\varepsilon_1) \right] g_1^{(-1)}(-k_i\varepsilon_1) \nu
\]

\[
= f_1(\alpha + k\varepsilon_1) \sum_{i=1}^{r} a_i g_1^{(0)}(\alpha + k\varepsilon_1 + k_i\varepsilon_1) g_1^{(-1)}(-k_i\varepsilon_1) \nu
\]

\[
= f_1(\alpha + k\varepsilon_1) \sum_{i=1}^{r} a_i \sum_{j \in K} f_{1,1}^{(j,-1)}(\alpha + k\varepsilon_1 + k_i\varepsilon_1, -k_i\varepsilon_1) g_j^{(-1)}(\alpha + k\varepsilon_1) \nu.
\]

Since $g_j^{(-1)}(\alpha + k\varepsilon_1) \nu$ are linearly independent,

\[
f_1(\alpha + k\varepsilon_1) \sum_{i=1}^{r} a_i f_{1,1}^{(j,-1)}(\alpha + k\varepsilon_1 + k_i\varepsilon_1, -k_i\varepsilon_1) = 0
\]

for any $j \in K, k > k_0$, and all $\alpha \in \mathbb{Z}^n$ with the first component being 0. By Lemma 3.1, we see that

\[
f_1(\alpha) \sum_{i=1}^{r} a_i f_{1,1}^{(j,-1)}(\alpha + k\varepsilon_1, -k_i\varepsilon_1) = 0 \quad (3.6)
\]

for any $l \in \mathbb{Z}, \alpha \in \mathbb{Z}^n$ and $i \in \{1, 2, \ldots, r\}$.

Now using (3.3)-(3.6), we compute

\[
0 = g_1^{(1)}(\gamma_k)X
\]

\[
= \sum_{i=1}^{r} a_i \left[ g_1^{(1)}(\gamma_k), g_1^{(-1)}(k_i\varepsilon_1) \right] g_1^{(-1)}(-k_i\varepsilon_1) + g_1^{(-1)}(k_i\varepsilon_1) \left[ g_1^{(1)}(\gamma_k), g_1^{(-1)}(-k_i\varepsilon_1) \right] \nu
\]

\[
= f_1(\gamma_k) \sum_{i=1}^{r} a_i g_1^{(0)}(\gamma_k + k_i\varepsilon_1) g_1^{(-1)}(-k_i\varepsilon_1) \nu + a_1 f_1(\gamma_k) \psi\left( g_1^{(0)}(\gamma_k - k_1\varepsilon_1) \right) g_1^{(-1)}(k_1\varepsilon_1) \nu
\]

\[
= \left( f_1(\gamma_k) \sum_{i=1}^{r} a_i \sum_{j \in K} f_{1,1}^{(j,-1)}(\gamma_k + k_i\varepsilon_1, -k_i\varepsilon_1) \right) g_j^{(-1)}(\gamma_k) \nu
\]

\[
+ a_1 f_1(\gamma_k) \psi\left( g_1^{(0)}(\gamma_k - k_1\varepsilon_1) \right) g_1^{(-1)}(k_1\varepsilon_1) \nu
\]

\[
= a_1 f_1(\gamma_k) \psi\left( g_1^{(0)}(\gamma_k - k_1\varepsilon_1) \right) g_1^{(-1)}(k_1\varepsilon_1) \nu \neq 0 \quad \text{(using (3.4))},
\]

which is a contradiction. Thus $S$ is linearly independent, i.e., $M(V_\psi)$ has an infinite-dimensional homogeneous space. The theorem follows. □
4. Proofs of Theorems 2.11 and 2.12

Before starting the proof of Theorem 2.11, we need to define a total ordering on a PBW basis for the universal enveloping algebra $U = U(G^-) = \bigoplus_{i \in \mathbb{Z}^+} U(G^-)_i$ of $G^-$. We will use the obvious gradation $U = \bigoplus_{i \in \mathbb{Z}^+} U_i$.

For $m \in \mathbb{N}$ and $x \in \mathbb{Z}^n$, we fix a basis $B_{-m}(x) = \{g_k^{(i)}(x) | k \in K_{m}(x)\}$ for $g_{-m}(x)$ where $K_{m}(x)$ is an index set, maybe infinite. Note that $K_{-1}(x) = K$. Then we fix a total ordering $>$ on each $B_{-m}(x)$.

We define $g_k^{(i)}(x) > g_l^{(j)}(x)$ if $i > j$, or $i = j$ and $x > y$, or $i = j$ and $y = x$ but $g_k^{(i)}(x) > g_l^{(j)}(y)$, where the order $x > y$ is the lexicographical order on $\mathbb{Z}^n$, for example $(2,1,\ldots,1) > (1,2,\ldots,2)$.

We have the obvious meaning for $\geq, \leq$ and $<$. We fix a PBW basis $\mathcal{B}$ for $U(G^-)$ consisting of the following elements:

$$g_k^{(i_1)}(\alpha_1)g_k^{(i_2)}(\alpha_2) \cdots g_k^{(i_r)}(\alpha_r),$$

where $g_k^{(i_1)}(\alpha_1) \succ g_k^{(i_2)}(\alpha_2) \succ \cdots \succ g_k^{(i_r)}(\alpha_r)$. We call this $r$ the height of the element $g_k^{(i_1)}(\alpha_1)g_k^{(i_2)}(\alpha_2) \cdots g_k^{(i_r)}(\alpha_r)$, which is denoted by $\text{ht}(g_k^{(i_1)}(\alpha_1)g_k^{(i_2)}(\alpha_2) \cdots g_k^{(i_r)}(\alpha_r))$. We now define a term ordering on $\mathcal{B}$ as follows:

$$g_k^{(i_1)}(\beta_1)g_k^{(i_2)}(\beta_2) \cdots g_k^{(i_s)}(\beta_s) > g_k^{(i_1)}(\alpha_1)g_k^{(i_2)}(\alpha_2) \cdots g_k^{(i_r)}(\alpha_r)$$

if $s > r$, or $r = s$ and $(\beta_1, \ldots, \beta_s) > (i_1, \ldots, i_s, \alpha_1, \ldots, \alpha_s)$ in the lexicographical order. Then $>$ is a total ordering on $\mathcal{B}$.

For any nonzero $x \in U(G^-)$, we can uniquely write it as a linear combination of elements in $\mathcal{B}$: $x = \sum_{i=1}^{m} a_i X_i$, where $0 \neq a_i \in \mathbb{C}$, $X_i \in \mathcal{B}$, and $X_1 > X_2 > \cdots > X_m$. We define the height of $x$ as $\text{ht}(X_1)$, and the highest term $\text{hm}(x)$ of $x$ as $a_1 X_1$. For convenience, we define $\text{ht}(0) = -1$ and $\text{hm}(0) = 0$.

It is clear that $\mathcal{B} v := \{Xv | X \in \mathcal{B}\}$ is a basis for the Verma module $M = M(V_\psi)$ where $\psi$ is again the highest weight vector of $M$. We define

$$\text{ht}(Xv) := \text{ht}(X), \quad \text{hm}(Xv) := \text{hm}(X)v, \quad \forall X \in U(G^-).$$

At last we need to define the notation

$$U_{=r} = \{x \in U(G^-) | \text{ht}(X) \leq r\}.$$

It is easy to see that $U_{=r} \subseteq U_{s} \subseteq U_{s+1}$ for all $r, s, s' \in \mathbb{N}$.

Also we note $g_k^{(i_1)}(\alpha)U_{=r} \subseteq U_{=s}^r$ for all $r, s \in \mathbb{Z}$ and $k \in K$ (where we have regarded $U_{=1} = 0$ for $l > 0$) and $g_k^{(i_1)}(\alpha_1)g_k^{(i_2)}(\alpha_2) \cdots g_k^{(i_r)}(\alpha_r) \in U_{=1}^r \cdots U_{=1}^{i_1} \cdots U_{=1}^{i_r}$ for any $i_1, i_2, \ldots, i_r \in \mathbb{N}$.

Now we are ready to give

**Proof of Theorem 2.11.** ($\Leftarrow$). If there is a $\mathbb{Z}^{n+1}$-graded nonzero ideal $I$ of $G$ properly contained in $G^-$ then $U(G^-)^Z_I v$ is a nonzero proper submodule of $M(V_\psi)$. Thus $M(V_\psi)$ is reducible.

Now suppose there exist distinct $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n$ and $a_1, a_2, \ldots, a_r \in \mathbb{C}$ such that

$$f_1(\alpha)\psi \left( \sum_{i=1}^{r} a_i g_1^{(0)}(\alpha + \alpha_i) \right) = 0$$

for all $x \in \mathbb{Z}^n$. For any $x \in \mathbb{Z}^n$ and any $k \in K$, we have
\[ g_k^{(1)}(\alpha) \left( \sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i) \right) v = \delta_{k,1} f_1(\alpha) \sum_{i=1}^{r} a_i g_1^{(0)}(\alpha + \alpha_i) v \]

\[ = \delta_{k,1} f_1(\alpha) \psi \left( \sum_{i=1}^{r} a_i g_1^{(0)}(\alpha + \alpha_i) \right) v = 0. \]

Thus \((\sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i))\) can generate a nonzero proper submodule in \(\tilde{M}(V, \psi)\). Again \(\tilde{M}(V, \psi)\) is reducible.  

(\(\Rightarrow\)). Now we assume there is not a \(\mathbb{Z}^{n+1}\)-graded nonzero ideal of \(G\) contained in \(G^-\) and there does not exist \(k \in K\), distinct \(\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n\) and \(a_1, a_2, \ldots, a_r \in \mathbb{C}\) such that \(f_1(\alpha)\psi(\sum_{i=1}^{r} a_i g_k^{(0)}(\alpha + \alpha_i)) = 0\) for all \(\alpha \in \mathbb{Z}^n\). We shall use the decompositions \(M = \tilde{M}(V, \psi) = \bigoplus_{i \in \mathbb{Z}^+} \tilde{M}_{-i}\) and \(M = M(V, \psi) = \bigoplus_{i \in \mathbb{Z}^+} M_{-i}\). We need to show that \(\tilde{M}(V, \psi) = M(V, \psi)\), i.e., \(M_{-n} = M_{-n}\) for all \(n \in \mathbb{Z}^+\). We shall do this by induction on \(n\). The result for \(n = 0\) is trivial and the result for \(n = 1\) follows from a similar proof to that of Theorem 2.9 ((a) \(\Rightarrow\) (b)).

Now we consider the case \(n > 1\) and suppose that \(M_{-k} = M_{-k}\) for all \(0 \leq k < n\).

It suffices to prove that \(X v \neq 0\) in \(\tilde{M}(V, \psi)\) for any nonzero \(X \in U(G^-)_{-n}\). We write \(X = \sum_{i=1}^{m} a_i X_i\) where \(0 \neq a_i \in \mathbb{C}, X_i \in M, X_1 > X_2 > \cdots > X_m\). We are going to show that \(X v \neq 0\).

We break up the proof into two different cases.

**Case 1.** \(\text{ht}(X) < n\).

Suppose \(X_1 = g_k^{(-i_1)}(\alpha_1) g_{k_2}^{(-i_2)}(\alpha_2) \cdots g_{k_r}^{(-i_r)}(\alpha_r) s_{k_{r+1}}^{(-1)}(\alpha_{r+1}) \cdots s_{k_{r+s}}^{(-1)}(\alpha_{r+s}) \in M_{-1}\) with \(r > 0\), and \(i_r \geq 2\). By the assumption there exist \(k \in K\) and \(\alpha \in \mathbb{Z}^n\) such that \([g_k^{(1)}(\alpha), g_{k_r}^{(-i_r)}(\alpha_r)] \neq 0\). Since \(g_k^{(1)}(\alpha)(X v) \in \tilde{M}(-n-1) = M(-n-1)\), by the inductive hypothesis we know that

\[ \text{hm}(g_k^{(1)}(\alpha)(X v)) = \text{hm}([g_k^{(1)}(\alpha), a_1 X_1] v). \]

which appears in

\[-la_1 g_{k_1}^{(-i_1)}(\alpha_1) g_{k_2}^{(-i_2)}(\alpha_2) \cdots [g_k^{(1)}(\alpha), g_{k_r}^{(-i_r)}(\alpha_r)] g_{k_{r+1}}^{(-1)}(\alpha_{r+1}) \cdots g_{k_{r+s}}^{(-1)}(\alpha_{r+s}) v \neq 0, \]

where \(l\) is the number of \(q\) such that \(g_{k_q}^{(-i_q)}(\alpha_q) = g_{k_r}^{(-i_r)}(\alpha_r)\). Then \(g_k^{(1)}(\alpha)(X v) \neq 0\) in \(\tilde{M}\), yielding \(X v \neq 0\) in \(\tilde{M}\).

**Case 2.** \(\text{ht}(X) = n\).

To the contrary in this case, we assume that \(X v = 0\) in \(M(V, \psi)\). There exist \(r, s\) such that \(\text{ht}(X_i) = n, 1 \leq i \leq r, r + 1 \leq i \leq s\) and \(\text{ht}(X_i) \leq n - 2, s + 1 \leq i \leq m\).

For \(1 \leq i \leq r\), each \(X_i\) is of the form \(X_i = g_{k_{i,1}}^{(-1)}(\alpha_{i,1}) g_{k_{i,2}}^{(-1)}(\alpha_{i,2}) \cdots g_{k_{i,n}}^{(-1)}(\alpha_{i,n})\). For \(k \in K\) and \(\alpha > \alpha_{i,j}\) for all \(i, j\) (for example the first component of \(\alpha\) is sufficiently large) we compute

\[ g_k^{(1)}(\alpha)(X_i v) = [g_k^{(1)}(\alpha), g_{k_{i,1}}^{(-1)}(\alpha_{i,1}) g_{k_{i,2}}^{(-1)}(\alpha_{i,2}) \cdots g_{k_{i,n}}^{(-1)}(\alpha_{i,n})] v \]

\[ = f_k(\alpha) \sum_{p=1}^{n} \delta_{k,k_{i,p}} g_{k_{i,1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i,p}}^{(0)}(\alpha + \alpha_{i,p}) \cdots g_{k_{i,n}}^{(-1)}(\alpha_{i,n}) v \]

\[ = f_k(\alpha) \sum_{p=1}^{n} \delta_{k,k_{i,p}} g_{k_{i,1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i,p}}^{(-1)}(\alpha_{i,p}) \cdots g_{k_{i,n}}^{(-1)}(\alpha_{i,n}) g_k^{(0)}(\alpha + \alpha_{i,p}) v \]
\begin{align*}
&+ f_k(\alpha) \sum_{p=1}^{n} \sum_{q=p+1}^{n} \delta_{k,k_i,p} g_{k_i,1}^{(-1)}(\alpha_{i,1}) \cdots g_k^{(-1)}(\alpha_p) \\
&\cdots \left( \sum_{l \in K} f_{k,k_i,q}^{(-1)}(\alpha + \alpha_{i,p}, \alpha_{i,q}) g_{l}^{(-1)}(\alpha + \alpha_{i,p} + \alpha_{i,q}) \right) \cdots g_{k_i,n}^{(-1)}(\alpha_{i,n}) v \\
&= f_k(\alpha) \sum_{p=1}^{n} \delta_{k,k_i,p} \psi \left( g_k^{(0)}(\alpha + \alpha_{i,p}) \right) g_{k_i,1}^{(-1)}(\alpha_{i,1}) \cdots g_k^{(-1)}(\alpha_{i,p}) \cdots g_{k_i,n}^{(-1)}(\alpha_{i,n}) v \\
&+ f_k(\alpha) \sum_{p=1}^{n} \sum_{q=p+1}^{n} \sum_{l \in K} \delta_{k,k_i,p} f_{k,k_i,q}^{(-1)}(\alpha + \alpha_{i,p}, \alpha_{i,q}) \\
&\times g_{k_i,1}^{(-1)}(\alpha_{i,1}) \cdots g_k^{(-1)}(\alpha_{i,p}) \cdots g_{k_i,n}^{(-1)}(\alpha_{i,n}) v \\
&= f_k(\alpha) \sum_{p=1}^{n} \delta_{k,k_i,p} \psi \left( g_k^{(0)}(\alpha + \alpha_{i,p}) \right) g_{k_i,1}^{(-1)}(\alpha_{i,1}) \cdots g_k^{(-1)}(\alpha_{i,p}) \cdots g_{k_i,n}^{(-1)}(\alpha_{i,n}) v \\
&+ f_k(\alpha) \sum_{p=1}^{n} \sum_{q=p+1}^{n} \sum_{l \in K} \delta_{k,k_i,p} h_{k,k_i,q}^{(l)}(\alpha + \alpha_{i,p}, \alpha_{i,q}) g_{l}^{(-1)}(\alpha + \alpha_{i,p} + \alpha_{i,q}) g_{k_i,1}^{(-1)}(\alpha_{i,1}) \\
&\cdots g_{k_i,n}^{(-1)}(\alpha_{i,n}) v \mod(U^{(n-2)}_{-n+1} v), \quad (4.1)
\end{align*}

where \( \hat{\cdot} \) means the factor is missing.

For \( r + 1 \leq i \leq s \), each \( X_i \) is of the form \( X_i = g_{k_1}^{(-2)}(\alpha_{i,1}) g_{k_2}^{(-1)}(\alpha_{i,2}) \cdots g_{k_{n+1}}^{(-1)}(\alpha_{i,n+1}) \). For \( k \in K \) and \( \alpha > \alpha_i \) for all \( i, j \) we compute

\begin{align*}
g_k^{(1)}(\alpha)(X_i v) &= \left[ g_k^{(1)}(\alpha), g_{k_1}^{(-2)}(\alpha_{i,1}) g_{k_2}^{(-1)}(\alpha_{i,2}) \cdots g_{k_{n+1}}^{(-1)}(\alpha_{i,n}) \right] v \\
&= \sum_{p \in K} y_{k,k_i,1}^{(p)}(\alpha, \alpha_{i,1}) g_p^{(-1)}(\alpha + \alpha_{i,1}) g_{k_2}^{(-1)}(\alpha_{i,2}) \\
&\cdots g_{k_{n+1}}^{(-1)}(\alpha_{i,n+1}) v \mod(U^{(n-2)}_{-n+1} v), \quad (4.2)
\end{align*}

where \( y_{k,k_i,1}^{(p)}(\alpha, \alpha_{i,1}) \) are exp-polynomial functions on \( \alpha \). Indeed, this comes from the following computation

\begin{align*}
&\left[ g_k^{(1)}(\alpha), \left[ g_k^{(-1)}(\beta), g_j^{(-1)}(\gamma) \right] \right] \\
&= f_k(\alpha) \left( g_k^{(0)}(\alpha + \beta), g_j^{(-1)}(\gamma) \right) + \delta_{k,j} \left[ g_k^{(-1)}(\beta), g_k^{(0)}(\alpha + \gamma) \right] \\
&= f_k(\alpha) \sum_{i \in K} \left( f_{k,j}^{(i)}(\alpha + \beta, \gamma) - \delta_{k,j} f_{k,k}^{(i)}(\beta, \alpha + \gamma) \right) g_i^{(-1)}(\alpha + \beta + \gamma),
\end{align*}

where on the right-hand side all functions are exp-polynomial functions on \( \alpha \).

Using (4.1), (4.2) and the fact that \( g_k^{(1)}(\alpha)(X_i v) \in U^{(n-2)}_{-n+1} \) for all \( s + 1 \leq i \leq m \), we obtain that, for all \( \alpha \in \mathbb{Z}^n \) with the first component sufficiently large.
\[
0 = g_k^{(1)}(\alpha)(Xv) = \left[ g_k^{(1)}(\alpha), X \right]v = \sum_{i=1}^m a_i \left[ g_k^{(1)}(\alpha), X_i \right]v
\]

\[
= f_k(\alpha) \sum_{p=1}^n \delta_{k,i_p} \psi \left( g_k^{(0)}(\alpha + \alpha_{i,p}) \right) g_{k_{i_1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) v
\]

\[
+ \sum_{\beta, k} y_{\beta,k}^{(-1)}(\alpha + \beta_1) g_{k_1}^{(-1)}(\beta_2) \cdots g_{k_{n-1}}^{(-1)}(\beta_n) v \mod (U^{n-2}v),
\]

where the last summation is over finitely many fixed \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n-1}) \) and \( k = (k_1, k_2, \ldots, k_{n-1}) \) and \( y_{\beta,k}(\alpha) \) are exp-polynomial functions in \( \alpha \). Denote

\[
x_1 = f_k(\alpha) \sum_{p=1}^n \delta_{k,i_p} \psi \left( g_k^{(0)}(\alpha + \alpha_{i,p}) \right) g_{k_{i_1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) v,
\]

\[
x_2 = \sum_{\beta, k} y_{\beta,k}^{(-1)}(\alpha + \beta_1) g_{k_1}^{(-1)}(\beta_2) \cdots g_{k_{n-1}}^{(-1)}(\beta_n) v.
\]

For all \( \alpha \in \mathbb{Z}^n \) with the first component sufficiently large, since the vectors in \( \mathcal{B} \) appearing in the above expressions of \( x_1 \) and \( x_2 \) are linearly independent we must have \( x_1 = x_2 = 0 \) for these \( \alpha \). Particularly, from \( x_2 = 0 \) we deduce from Lemma 3.1 that \( y_{\beta,k}(\alpha) = 0 \) for all \( \alpha \in \mathbb{Z}^n \) and all \( \beta, k \). Thus \( x_2 = 0 \) for all \( \alpha \in \mathbb{Z}^n \) and all \( \beta, k \). Consequently \( x_1 = 0 \) for all \( \alpha \in \mathbb{Z}^n \).

In the expression of \( x_1 \), if we take \( k = k_{1,n} \), then the highest terms are

\[
f_k(\alpha) \sum_{i \in I} p_i \psi \left( g_k^{(0)}(\alpha + \alpha_{i,n}) \right) g_{k_{i_1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) v,
\]

where \( I = \{1 \leq i \leq r \mid g_{k_{i_1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) = g_{k_{i_1}}^{(-1)}(\alpha_{i,n-1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) \} \), \( k_{i,n} = k_{1,n} \), \( p_i \) is the number of \( q \) such that \( g_{k_{i,q}}^{(-1)}(\alpha_{i,q}) = g_{k_{i,n}}^{(-1)}(\alpha_{i,n}) \) in the expression \( g_{k_{i_1}}^{(-1)}(\alpha_{i,1}) \cdots g_{k_{i_{n-1}}}^{(-1)}(\alpha_{i,n}) \) for \( i \in I \). Since \( x_1 = 0 \), \( f_k(\alpha) \sum_{i \in I} p_i \psi \left( g_k^{(0)}(\alpha + \alpha_{i,n}) \right) = 0 \).

If we denote \( P = \sum_{i \in I} p_i \alpha_i \), then \( P \neq 0 \). Consequently, \( f_k(\alpha) \psi \left( g_k^{(0)} \otimes r^{eP} \right) = 0 \) for all \( \alpha \in \mathbb{Z}^n \). Since \( g_{k_i}^{(-1)}(\alpha_{i,n}) \in \mathcal{B} \), \( g_k^{(-1)} \otimes P \neq 0 \). This contradicts the assumption in the theorem. Therefore \( Xv \neq 0 \) in this case. This completes the proof of the theorem. \( \square \)

Now we can give

**Proof of Theorem 2.12.** As \( \mathbb{Z}^{n+1} \)-graded \( G \) modules, from the fact that \( \tilde{M}(V_{\psi}) \otimes A \cong \bigoplus_{\alpha} \tilde{M}(V_{\psi}) \), maybe infinitely many copies, we see that \( \tilde{M}(V_{\psi}) \) is irreducible if \( \tilde{M}(V_{\psi}) \) is irreducible.

Now we assume that \( \tilde{M}(V_{\psi}) \) is not irreducible. We need to show that \( \tilde{M}(V_{\psi}) \) is not irreducible either. From Theorem 2.11, there is a \( \mathbb{Z}^{n+1} \)-graded nonzero ideal \( I \) of \( G \) properly contained in \( G^- \) or there exists \( k \in K \), distinct \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n \) and \( a_1, a_2, \ldots, a_r \in \mathbb{C} \) such that

\[
f_k(\alpha)(\sum_{i=1}^r a_i g_k^{(0)}(\alpha + \alpha_i)) = 0 \ \text{for all} \ \alpha \in \mathbb{Z}^n.
\]

If such an ideal \( I \) exists, then \( U(G^-)I V_{\psi} \) is a nonzero proper submodule of \( \tilde{M}(V_{\psi}) \). Thus \( \tilde{M}(V_{\psi}) \) is reducible.

Now assume that there exists \( k \in K \), distinct \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}^n \) and \( a_1, a_2, \ldots, a_r \in \mathbb{C} \) such that

\[
f_k(\alpha)(\sum_{i=1}^r a_i g_k^{(0)}(\alpha + \alpha_i)) = 0 \ \text{for all} \ \alpha \in \mathbb{Z}^n.
\]

We may assume that \( k = 1 \) and \( r \) is minimal. From a similar proof of the first paragraph in "(b) \Rightarrow (a)" of Theorem 2.9 we know that
\[
\sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i) v = 0, \quad \text{in } M(V_{\psi}). \tag{4.4}
\]

Note that since \( r \) is minimal then \( a_i \neq 0 \) for all \( i \).

**Claim.** All \( \alpha_i \) in (4.4) are in the same coset of \( \Gamma_{\psi} \).

Otherwise we may assume that \( \alpha_1, \alpha_2, \ldots, \alpha_{r_1} \) are in the same coset \( z + \Gamma_{\psi} \), all other \( \alpha_i \) are not in \( z + \Gamma_{\psi} \), and \( r_1 < r \).

Since \( r \) is minimal and \( r_1 < r \), there exists \( \beta \in \mathbb{Z}^n \) such that
\[
0 \neq g_1^{(1)}(\beta) \sum_{i=1}^{r_1} a_i g_1^{(-1)}(\alpha_i) v = f_1(\beta) \sum_{i=1}^{r_1} a_i g_1^{(0)}(\beta + \alpha_i) v. \tag{4.5}
\]

Then we know that \( \beta + \alpha_i \in \Gamma_{\psi} \) for \( i = 1, 2, \ldots, r_1 \). Clearly, \( \beta + \alpha_j \notin \Gamma_{\psi} \) for \( j > r_1 \). So
\[
g_1^{(1)}(\beta) g_1^{(-1)}(\alpha_j) v = f_1(\beta) \psi(g_1^{(0)}(\beta + \alpha_j)) v = 0
\]
for \( j > r_1 \). Thus
\[
0 = g_1^{(1)}(\beta) \sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i) v = f_1(\beta) \sum_{i=1}^{r_1} a_i g_1^{(0)}(\beta + \alpha_i) v,
\]
contrary to (4.5). The claim holds.

From the claim, there exist \( x_i \in U(G^{(0)}_{\alpha_1 - \alpha_i}) \) with respect to the \( \mathbb{Z}^n \)-gradation such that \( x_i v = v \) in \( \tilde{M}(V_{\psi}) \). Then
\[
\sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i) x_i v = 0, \quad \text{in } \tilde{M}(V_{\psi}). \tag{4.6}
\]

In \( \tilde{M}(V_{\psi}) \otimes A \), we know that \( x_i(v \otimes 1) = v \otimes t^{\alpha_1 - \alpha_i} \), and
\[
\sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i)(v \otimes t^{\alpha_1 - \alpha_i}) = \sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i)x_i(v \otimes 1)
\]
\[
= \left( \sum_{i=1}^{r} a_i g_1^{(-1)}(\alpha_i)v \right) \otimes t^{\alpha_1} = 0 \otimes t^{\alpha_1} = 0.
\]

Thus \( \tilde{M}(V_{\psi}) \) is not irreducible. \( \square \)

**Acknowledgments**

The last author wishes to thank Prof. Y. Billig and S. Eswara Rao for valuable suggestions to improve the paper.
References