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A six-dimensional view on twistors

Annamaria Sinkovics, Erik Verlinde

Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract

We study the twistor formulation of the classical $\mathcal{N}=4$ super-Yang–Mills theory on the quadric submanifold of $\mathbb{CP}^{3|3}\times\mathbb{CP}^{3|3}$. We reformulate the twistor equations in six dimension, where the superconformal symmetry is manifest, and find a connection to complexified AdS_5.

1. Introduction

Recently there has been a renewed interest in twistor formulations of gauge theories. By studying the structure of maximally helicity violating amplitudes Witten has constructed a new formulation of $\mathcal{N}=4$ gauge theory [1]. The structure of these amplitudes becomes apparent when transforming from momentum to twistor space, where the amplitudes are supported on holomorphic curves. Follow-up works extended this result for non-maximally helicity violating amplitudes [2], for "googly amplitudes" (i.e., amplitudes in the opposite helicity description) [3], for analyzing loop amplitudes [4] and gravity amplitudes [5]. For further extension of the formalism and for advances on the diagrammatics of amplitude computations see [6].

These results can be interpreted [1] by formulating $\mathcal{N}=4$ gauge theory as a topological string theory, the B-model with target space of the supermanifold $\mathbb{CP}^{3|4}$. In this way the structure of perturbative Yang–Mills amplitudes arises by including the contribution of D-instantons. Alternative string formulations for describing the perturbative $\mathcal{N}=4$ twistor space amplitudes has also been proposed in [7], see also [8]. In this note we are interested in a different formulation of the $\mathcal{N}=4$ SYM, proposed by Witten [9]. According to [9], the full classical Yang–Mills field equations, not just the self-dual or anti-self-dual part, can be constructed in terms of a vector bundle on a quadric submanifold $Q\in\mathbb{CP}^{3|3}\times\mathbb{CP}^{3|3}$. This formulation generalizes Ward's construction [10] of (anti-)self-dual gauge fields from vector bundles on a single $\mathbb{CP}^3$. A concise summary of Ward's formulation is given in the appendix of [1]. Unlike the formulation on $\mathbb{CP}^{3|4}$, the construction from $Q\in\mathbb{CP}^{3|3}\times\mathbb{CP}^{3|3}$ is parity symmetric, a helicity flip exchanges the two $\mathbb{CP}^{3|3}$s.

E-mail address: sinkovic@science.uva.nl (A. Sinkovics).

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The connection between Yang–Mills theory and the quadric Q has gained further interest through the recent duality conjectures in topological string theory. First, in [11] it was argued that an S-duality relates the B-model on $\mathbb{CP}^{3|4}$ to the A-model on the same supermanifold, see also [12]. Secondly, it was conjectured that by mirror symmetry the A-model on $\mathbb{CP}^{3|4}$ becomes the B-model on the quadric Q in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. A proof of this mirror map was presented in [13], see also [14]. The D-instanton contributions of the original B-model are mapped first through the S-duality to perturbative A-model amplitudes. After the additional mirror symmetry we arrive again at the B-model, but without the D-instanton contributions. This means the Yang–Mills equations could be directly related to the B-model on the quadric, and may thus be formulated in terms of a holomorphic Chern–Simons theory on this space. This possibility was already mentioned in [1], but for a concrete realization of this idea one needs to overcome the obstacle of finding a proper measure on the quadric.

It is interesting to note that, while the B-model on $\mathbb{CP}^{3|4}$ is related to weakly coupled $\mathcal{N} = 4$ SYM, the B-model on the quadric should again be give a strong coupled formulation, since it follows from the conjectured S-duality for topological strings and mirror symmetry [11,12]. Furthermore, the target space Q_5 of the B-model is a complex five-dimensional supermanifold and is symmetric under the superconformal group $SU(4|3)$. It is therefore natural to ask whether there is a connection with the $AdS_5 \times S^5$, which is also dual to the same theory in the same regime. In this Letter we will make a step in this direction by reformulating the (super)twistor equations in a 6-dimensional notation that makes the superconformal invariance manifest. In our formulation 4d Minkowski space will be identified with the lightcone embedded in the six-dimensional flat space modded out by rescalings. This projective version of 6d space has also other components, one of which can be identified with AdS_5. The twistor equations rewritten in the 6d notation take a particularly simple form.

2. Twistor construction of the Yang–Mills equations

Let us first fix conventions. We work in signature $\eta^{\mu\nu} = \{-, +, +, +\}$, and use complexified Minkowski space $\mathbb{M}_4$ so that the coordinate $x \in \mathbb{M}_4$ are complex. In this Letter we will write most equations explicitly in coordinates that are defined for non-compact Minkowski space, but our results can be extended to its compactified version, which we also denote by $\mathbb{M}_4$. Undotted and dotted indices denote spinors transforming in the $(1/2, 0)$ and $(0, 1/2)$ representations, and can be raised and lowered with the two-index antisymmetric tensor ϵ. In spinor notation

$$x_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} x_\mu = x_0 \delta_{a\dot{a}} + \vec{x}\vec{\sigma}_{a\dot{a}}.$$

The bosonic twistor equation is written as [1]

$$V_{\dot{a}} + x_{a\dot{a}} V^a = 0, \quad \dot{a} = 1, 2. \tag{2.1}$$

It can be viewed in two ways: given x, it determines a curve in the space $\mathbb{CP}^3$, which is parametrized by the homogeneous coordinates λ and μ. The space $\mathbb{CP}^3$ is called twistor space. The curve itself is a copy of $\mathbb{CP}^1$, since the equation can be solved for V^a of $V_{\dot{a}}$, or the reverse. In the analysis of the scattering amplitudes, this curve arises after Fourier transforming the amplitudes from momentum to twistor space. After the transformation, the amplitudes are localized on the curve given by the twistor equation. From another point of view, given V^a and $V_{\dot{a}}$, the twistor equation determines a two-dimensional subspace in $\mathbb{M}_4$, called alpha-plane. The twistor equation is naturally connected to the (anti-)self-dual Yang–Mills equation via Ward's construction. The basic idea of this construction is that the information of (anti-)self-dual gauge fields can be encoded in the structure of complex vector bundles. Consider a complex vector bundle over $\mathbb{M}_4$ with a connection on it. In general, parallel transport with this connection is not integrable. However, according to Ward's construction, we have integrability when restricting to the alpha-planes, if and only if the gauge field is anti-self-dual. The set of all alpha-planes is the twistor space $\mathbb{CP}^3$. There is of course analogous construction for a self-dual gauge field, where the complex 2-planes of integrability are now called beta-planes. Thus by Ward's theorem, an (anti-)self-dual gauge field corresponds to a vector bundle on the twistor space $\mathbb{CP}^3$, and this vector bundle is trivial when restricted to a 2-dimensional subspace defined by the twistor equation. It is natural to try to extend this construction to the full Yang–Mills equations. While the self-dual gauge field equations are algebraic, the full Yang–Mills equations are a differential equation. In [9] Witten achieves the extension by embedding Minkowski space in a bigger space, $\mathbb{M}_4 \times \mathbb{M}_4$. By writing $x \in \mathbb{M}_4$ as $x = \frac{1}{2}(y+z)$ with

$$(y, z) \in \mathbb{M}_4 \times \mathbb{M}_4,$$

one can split the 4d connection ∇_x as

$$\nabla_x = \nabla_y + \nabla_z. \tag{2.2}$$

The original Minkowski space thus corresponds to the diagonal $y = z$ inside $\mathbb{M}_4 \times \mathbb{M}_4$. The connection ∇_x satisfies the Yang–Mills field equations if ∇_y is anti-self-dual, ∇_z is self-dual and both connections mutually commute. Thus we have

$$\begin{aligned} &[\nabla_y, \nabla_y] + [\nabla_y, \nabla_y]^* = 0, \\ &[\nabla_z, \nabla_z] - [\nabla_z, \nabla_z]^* = 0 \end{aligned} \tag{2.3}$$

and

$$[\nabla_y, \nabla_z] = 0.$$

From these relations it follows that

$$[\nabla_x, \nabla_x] = [\nabla_y, \nabla_y] + [\nabla_z, \nabla_z]$$

and finally the Jacobi identity implies

$$\begin{aligned} &\big[\nabla_x, [\nabla_x, \nabla_x]^*\big] \\ &\quad = \big[\nabla_y + \nabla_z, -[\nabla_y, \nabla_y] + [\nabla_z, \nabla_z]\big] = 0. \end{aligned} \tag{2.4}$$

One of the main points of [9] is that a gauge connection can only be split in this way if it corresponds to a vector bundle on $\mathbb{CP}^3 \times \mathbb{CP}^3$, again trivial on each $\mathbb{CP}^1 \times \mathbb{CP}^1$. This is a very strong requirement that is not satisfied by a general solution of the Yang–Mills field equations. Every gauge field on $\mathbb{M}_4$, not necessary satisfying any equation, is equivalent to a vector bundle on the manifold

$$Q_5 = \left\{ (U, V) \in \mathbb{CP}^3 \times \mathbb{CP}^3 \;\middle|\; \sum_{\alpha=1}^{4} u_\alpha v^\alpha = 0 \right\}. \tag{2.5}$$

Here $\alpha = (a, \dot{a})$ is a four-component spinor index. The space Q_5 has complex dimension 5 and can be viewed as the space of all lightlike lines in $\mathbb{M}_4$. The lightlike lines through a given point in $\mathbb{M}_4$ form a $\mathbb{CP}^1 \times \mathbb{CP}^1$ inside Q_5, with one $\mathbb{CP}^1$ in each factor of $\mathbb{CP}^3 \times \mathbb{CP}^3$. A vector bundle associated with a gauge field on $\mathbb{M}_4$ is trivial on every such $\mathbb{CP}^1 \times \mathbb{CP}^1$. Gauge fields that satisfy the Yang–Mills equation $D^* F = 0$ corresponds to

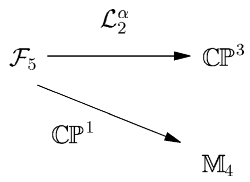

Fig. 1. Twistor space $\mathbb{CP}^3$ and Minkowski space $\mathbb{M}_4$ can be regarded as the base of a fiber bundle with total space $\mathcal{F}_5$. The corresponding fibers are the alpha-plane $\mathcal{L}_2^\alpha$ and $\mathbb{CP}^1$.

a vector bundle on Q_5 that can be extended to a small local neighborhood of Q_5 inside $\mathbb{CP}^3 \times \mathbb{CP}^3$. To be precise, it is necessary and sufficient to extend the vector bundle from Q_5 up to third order in a local Taylor expansion. This means that the vector bundle actually lives on a quadric given by $(U_\alpha V^\alpha)^4 = 0$, which can then be taken as the actual defining equation of Q_5. The extension of the connection to third order away from Q_5 also implies that the Yang–Mills gauge field can be extended to $\mathbb{M}_4 \times \mathbb{M}_4$ away from the diagonal up to third order in the $w = y - z$. Ward's construction relates a connection on Q_5 to an anti-self-dual connection ∇_y and self-dual connection ∇_z, but to get the Yang–Mills equations these connections also have to commute in the neighborhood of the diagonal. This is what leads to the above mentioned requirements.

It is useful to compare this twistor construction of the Yang–Mills equations with the usual one for the (anti-)self-dual equation in terms of a schematic diagram, as indicated in Fig. 1. Complexified Minkowski $\mathbb{M}_4$ and the usual twistor space $\mathbb{CP}^3$ can both be seen as projections of the same five-dimensional space denoted by $\mathcal{F}_5$. The $\mathbb{CP}^1$ fiber over $\mathbb{M}_4$ corresponds to the set of all alpha-planes through a given point. The two-dimensional fiber $\mathcal{L}_2^\alpha$ over $\mathbb{CP}^3$ is just the alpha-plane itself. Similarly we can construct a fiber bundle over Q_5 by taking the lightray $\mathcal{L}_1 \sim \mathbb{CP}^1$ parametrized by a point in Q_5 as the fiber. The resulting total space $\mathcal{F}_6$ is also a fiber bundle over $\mathbb{M}_4$ with fiber equal to $\mathbb{CP}^1 \times \mathbb{CP}^1$, which is the space of all lightrays through a given point on $\mathbb{M}_4$. This is shown in Fig. 2.

2.1. Supersymmetric extension

As in the bosonic space, we can examine if the supersymmetric Yang–Mills equations are equivalent to an integrability conditions along lightlike lines through a given point in the superspace. It turns out

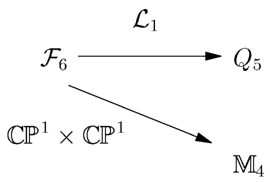

Fig. 2. The total space $\mathcal{F}_6$ is fibered over the space of lightrays Q_5 with the lightray $\mathcal{L}_1$ itself as fiber. Points in Minkowski space $\mathbb{M}_4$ lift to $\mathbb{CP}^1 \times \mathbb{CP}^1$ fibers in $\mathcal{F}_6$.

that this is indeed the case for $\mathcal{N} = 3$ or $\mathcal{N} = 4$ supersymmetry (these two theories are basically equivalent). The supertwistor equations for the alpha-plane are [1]

$$V_{\dot{a}} + x^R_{a\dot{a}} V^a = 0, \quad \dot{a} = 1, 2,$$
$$\psi_I + \theta_{aI} V^a = 0, \quad I = 1, \dots, \mathcal{N}. \tag{2.6}$$

Here we introduced anti-commuting coordinates θ_{aI}, $I = 1, \dots, \mathcal{N}$ for $\mathcal{N}$ supersymmetries. The supertwistor space $\mathbb{CP}^{3|3}$ is thus parametrized by $(V^a, V_{\dot{a}}, \psi^I)$, where the ψ^I are spinless anti-commuting variables. Analogously, the beta-plane equations are given as

$$U_a + x^L_{a\dot{a}} U^{\dot{a}} = 0,$$
$$\eta^I + \theta^I_{\dot{a}} U^{\dot{a}} = 0. \tag{2.7}$$

The pair of Eqs. (2.6) for the alpha-planes can, as in the bosonic case, be found by a partial Fourier transformation of the MHV amplitudes to twistor space. Similarly, the beta-plane equations (2.7) arise by a Fourier transformation of the MHV amplitudes with opposite helicity. It turns out, however, that the x-coordinate that appears in these equations is different for the left-handed and right-handed helicities, hence the superscript.

The alpha- and beta-planes are invariant under supersymmetry and superconformal variations. The supersymmetry variations which leave the set of alpha-plane equations invariant are

$$\delta x^R_{a\dot{a}} = -\epsilon^I_{\dot{a}} \theta_{aI}, \qquad \delta\theta_{aI} = \epsilon_{Ia},$$
$$\delta V_{\dot{a}} = -\epsilon^I_{\dot{a}} \psi_I, \qquad \delta\psi_I = -\epsilon_{aI} V^a.$$

The alpha-planes are also invariant under the superconformal variations

$$\delta x^R_{a\dot{a}} = x^R_{b\dot{a}} \tilde{\epsilon}^{bI} \theta_{aI}, \qquad \delta\theta_{aI} = \tilde{\epsilon}^{\dot{a}}_I x^R_{a\dot{a}} - \tilde{\epsilon}^{bJ} \theta_{bJ} \theta_{aI},$$
$$\delta V^a = \tilde{\epsilon}^{aI} \psi_I, \qquad \delta\psi_I = \tilde{\epsilon}_{\dot{a}I} V^{\dot{a}}.$$

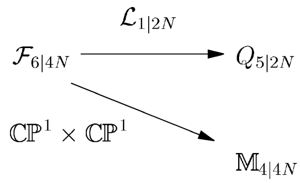

Fig. 3. The total space $\mathcal{F}_{6,4N}$ is fibered over the space of supersymmetric lightrays $Q_{5|2N}$ as well as over supersymmetric Minkowski $\mathbb{M}_{4|4N}$. The corresponding fibers are the superlightray $\mathcal{L}_{1|N}$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

The supersymmetry and superconformal variations which leave the beta-planes invariant are completely analogous. By comparing the transformation rules we find the chiral and anti-chiral coordinates are related as

$$x^R_{a\dot{a}} = -x^L_{a\dot{a}} + \theta_{aI} \theta^I_{\dot{a}}.$$

Super lightrays are obtained by intersecting the alpha- and beta-planes. Imposing both the alpha-plane equations (2.6) as well as those for the beta-planes (2.7) leads to a condition on the supertwistors. Namely, one only obtains a non-trivial solution provided that

$$U_a V^a + U_{\dot{a}} V^{\dot{a}} + \psi_I \eta^I = 0. \tag{2.8}$$

This defines the generalization of the manifold Q_5 to the supersymmetric situation

$$Q_{5|2N} = \big\{(U, \eta, V, \psi) \in \mathbb{CP}^{3|N} \times \mathbb{CP}^{3|N} \mid U_\alpha V^\alpha + \psi_I \eta^I = 0\big\}. \tag{2.9}$$

The quadric submanifold $Q_{5|2N}$ is the space of all supersymmetric lightlike lines [15]. Just as in the bosonic case one can define a fibre bundle over it with total space $\mathcal{F}_{6|4N}$, which projects down on $\mathbb{M}_{4|4N}$ along $\mathbb{CP}^1 \times \mathbb{CP}^1$ fibers, see Fig. 3.

The supersymmetric lightlike lines $\mathcal{L}_{1|2N}$, unlike in the bosonic case, are not one-dimensional, thus integrability along them is not any more a trivial condition. According to [9], for $\mathcal{N} = 3$ supersymmetry the integrability on the quadric corresponds to the $\mathcal{N} = 3$ supersymmetric equations of motion. For $\mathcal{N} = 4$ supersymmetry an additional condition is necessary, see [9]. Henceforth, in the rest paper we will take $\mathcal{N} = 3$. Ward's construction relates a vector bundle on $Q_{5|6}$ to a supergauge field on supersymmetric Minkowski space $\mathbb{M}_{4|12}$. It is an interesting open problem whether $N = 3$ SYM can be reformulated as the holomorphic Chern–Simons theory corresponding to a B-model topological string on $Q_{5|6}$. The quadric $Q_{5|6}$ is a Calabi–Yau supermanifold, thus the B-model in principle can be constructed on it, and as mentioned before, the existence of the B-model on the quadric is also supported by a conjectured duality chain. The main problem in formulating the holomorphic Chern–Simons theory appears to be the definition of the appropriate measure. This question will not be addressed in this Letter. Our main concern is the reformulation of the twistor equations in a manifestly superconformal fashion, and in this way clarify the role of the quadric in super-Yang–Mills.

3. The twistor equations in six-dimensional notation

The 4d Minkowski space $\mathbb{M}$ can be identified with the set of all lightlike directions in a 6d flat space–time $\mathbb{M}_{4,2}$ with signature $--++++$ and metric

$$ds^2 = -dX^+ dX^- + dX_\mu dX^\mu, \quad \mu = 1, \ldots, 4. \tag{3.1}$$

Specifically, a lightlike direction in $\mathbb{M}_{4,2}$ can be parametrized as

$$\left(X^+, X^-, X_\mu\right) \sim \left(1, x^2, x_\mu\right), \tag{3.2}$$

where x_μ are the coordinates for a point in $\mathbb{M}$ and x^2 its length.The 6d isometry group $SO(4,2)$ acts as the conformal group on $\mathbb{M}$. Twistors are naturally formulated in this six-dimensional language. In this way, a twistor corresponds to a chiral spinor

$$V^\alpha = \begin{pmatrix} V^{\dot a} \\ V^a \end{pmatrix} \tag{3.3}$$

transforming in the **4** of $SU(2,2)$. For every point in the 6d space–time we can define the anti-symmetric matrix

$$X_{\alpha\beta} \equiv \begin{pmatrix} X^+\epsilon_{\dot a\dot b} & X_{b\dot a} \\ -X_{a\dot b} & -X^-\epsilon_{ab} \end{pmatrix}, \tag{3.4}$$

where we used the spinor notation introduced before, so $X_{a\dot b} = \sigma^\mu_{a\dot b} X_\mu$. The twistor equation then takes the simple form

$$X_{\alpha\beta} V^\beta = 0. \tag{3.5}$$

The first component of this equation gives

$$X^+ V_{\dot a} + X_{a\dot a} V^a = 0. \tag{3.6}$$

By using the scale invariance to define $x_{a\dot a} = X_{a\dot a}/X_+$, one recognizes the twistor equation (2.1). The second component gives an additional equation, which is equivalent to the twistor equation provided

$$-X^+X^- + X_\mu X^\mu = 0.$$

This is the lightcone condition on the six-dimensional embedding space. We thus recover the standard twistor equation describing the anti-self-dual alpha-plane in $\mathbb{M}$. In the six-dimensional space $\mathbb{M}_{4,2}$ where rescalings are not modded out, the twistor equation defines an anti-self-dual null 3-plane through the origin. We will also call this alpha-plane. Similarly there are beta-planes that are self-dual and that can be described via a similar twistor equation but with the dotted and undotted indices interchanged. Specifically, we can raise indices as

$$X^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} X_{\gamma\delta}, \tag{3.7}$$

where $\epsilon^{a\dot b\dot a b} = \epsilon^{\dot a\dot b}\epsilon^{ab}$ (other entries follow by permutation). The twistor equation for beta-planes then takes the form

$$X^{\alpha\beta} U_\beta = 0. \tag{3.8}$$

Imposing both type of twistor equations amounts to intersecting the alpha- and beta-planes. Generically these only intersect in the origin of the 6d space. In Minkowski space this means there is no intersection at all. To get a non-zero intersection one should impose that

$$U_\alpha V^\alpha = 0. \tag{3.9}$$

This can be seen most easily in components: $X^+ U_a V^a = -U^{\dot b} X_{a\dot b} V^a = -X^+ U^{\dot b} V_{\dot b}$. The intersection of an alpha- and beta-plane yields in this case a null two-plane through the origin in 6d, and corresponds to a lightray in 4d.

3.1. Superconformal invariance

We now proceed with the supersymmetric extension of the alpha- and beta-planes on the quadric. The starting point is that the set of supertwistor equations has to be invariant under the superconformal group in six dimension, $SU(4|3)$. The superconformal group acts linearly on (U_α, η_I) and on (V^α, ψ^I), which transform in the conjugate representations $(4|3)$ and $(\bar{4}|\bar{3})$. Since the R-symmetry group is $U(3)$, the η_I and ψ^I are in different representations. Taking the product of the $(4|3)$ with itself we look for the anti-symmetric combinations. The anti-symmetric 6 is identified with the $X_{\alpha\beta}$, while the anti-symmetric odd 12 are the $\Theta_\alpha{}^I$. The representation is completed by taking the symmetric combination 6 of (3×3), let us call this Φ_{IJ}. The symmetric field Φ_{IJ} can be thought of as a metric, which can be used to raise and lower the $U(3)$ indices.

The superconformal generators satisfy

$$\{Q^I_\alpha, Q^\beta_J\} = M_\alpha{}^\beta \delta^I_J + \delta_\alpha{}^\beta M^I{}_J$$

where $M_\alpha{}^\beta$ are the Lorentz generators, here written as a matrix in the fundamental representation of $SU(4)$, and the $M_I{}^J$ the $SU(3)$ R-symmetry generators. The superconformal generators contain both the supersymmetry and additional conformal generators, which in the four-dimensional notation appeared separately. For the invariance of the twistor equations under the superconformal group, we put the equations in the smallest possible representation $(4|3)$ of the superconformal group $SU(4|3)$. We obtain the four even and three odd equations

$$X_{\alpha\beta} V^\beta + \Theta_\alpha{}^J \psi_J = 0, \tag{3.10}$$

$$\Theta_\alpha{}^I V^\alpha + \Phi^{IJ} \psi_J = 0. \tag{3.11}$$

This set of equations describes the super alpha-planes, and are invariant under the action of the superconformal generators. Indeed, examining the action of the generators on the set of twistor equations, we find that (3.10) is annihilated by Q^I_α, and is mapped into (3.11) under Q^β_J. The odd equations (3.11) are mapped to the even equations (3.10) under Q^I_α, and are annihilated by the Q^β_J. The beta-plane equations can be derived by analogous reasoning and read

$$U_\alpha X^{\alpha\beta} + \eta^J \Theta^\beta{}_J = 0,$$
$$U_\alpha \Theta^\alpha{}_I + \eta^J \Phi_{JI} = 0. \tag{3.12}$$

The relation between the alpha- and beta-plane equations will be further clarified in the following section.

We end this section by explaining how to get the supertwistor equations in the 4d notation of Section 2. We only consider the equations for the alpha-planes. To reduce (3.10) and (3.11) to the 4d twistor equations (2.6) one first converts the four component indices α and β to two-component notation. In addition, one has to identify the 4d coordinates $x^R_{a\dot{a}}$ and $\theta_{\dot{a}}$ in terms of the 6d ones. This turns out to be trickier than one might have expected at first: $x^R_{a\dot{a}}$ and $\theta_{\dot{a}}$ are non-linearly defined in terms of the 6d coordinates. With hindsight this is not surprising because the 4d superconformal symmetries act non-linearly on the coordinates, while the action on the 6d coordinates is linear. After a bit of straightforward but tedious algebra one finds that the correct identifications are

$$x^R_{a\dot{a}} = \left(X_{a\dot{a}} - \Theta_a{}^I \Theta_{\dot{a}I}\right) \Big/ \left(X_+ - \frac{1}{2}\Theta_{\dot{a}}{}^I \Theta^{\dot{a}}{}_I\right),$$
$$\theta^I_{\dot{a}} = \Theta_{\dot{a}}{}^I - x^R_{a\dot{a}} \Theta^{aI}. \tag{3.13}$$

The coordinate Φ^{IJ} was used to raise and lower indices, and disappears in the 4d picture. The reduction of the beta-plane equations (3.12) to four-dimensional notation proceeds analogously.

4. Combining alpha- and beta-planes: connection with AdS_5

The twistor equations for the alpha- and beta-planes are scale invariant in 6 dimensions. One can add a point at infinity by introducing another coordinate chart obtained by the inversion map

$$X_{\alpha\beta} \to -4\zeta^2 \frac{X_{\alpha\beta}}{X^2}, \qquad X^2 = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} X_{\alpha\beta} X_{\gamma\delta}, \tag{4.1}$$

where ζ is an arbitrary scale. Inversion sends the lightcone $X^2 = 0$ to the point at infinity. We will argue that it also exchanges the alpha- and beta-planes. We first consider the bosonic twistor equations. We use the parameter ζ to modify the bosonic twistor equation as follows

$$X_{\alpha\beta} V^\beta = \zeta U_\alpha. \tag{4.2}$$

Further, instead of the lightcone we consider the five-dimensional submanifold

$$X^2 = -4\zeta^2. \tag{4.3}$$

This equation describes a complexified (anti-)de Sitter space. What we have achieved by introducing the parameter ζ is that on the submanifold (4.3) the modified alpha-plane equation (4.2) is equivalent to the analogous equation for the beta-plane

$$X^{\alpha\beta}U_\beta = \zeta V_\alpha. \tag{4.4}$$

This follows from the identity

$$X^{\alpha\beta}X_{\beta\gamma} = \zeta^2\delta^\alpha{}_\gamma,$$

where we made use of (4.3). When $\zeta \neq 0$ we have the freedom to rescale X and put $\zeta = 1$. In this case the submanifold (4.3) is the fixed locus of the inversion map. Furthermore, in the limit $\zeta \to 0$ one recovers the "old" equations (3.5) and (3.8), and the AdS-manifold reduces again to the lightcone. The above procedure is analogous to the way in which the massive Dirac equation reduces to two decoupled Weyl equation for the left- and right-handed components of a spinor. In this analogy the parameter ζ is as a "mass", and V and U the left-handed and right-handed Weyl spinors.

This construction can be generalized to the supertwistor equations. The modified form of the supersymmetric alpha-plane equations read

$$X_{\alpha\beta}V^\beta + \Theta^I{}_\alpha\psi_I = \zeta U_\alpha,$$
$$\Theta^I{}_\alpha V^\alpha + \Phi^{IJ}\psi_J = \zeta\eta^I, \tag{4.5}$$

where we added the supertwistors for the beta-plane equations on the right-hand side, multiplied with an auxiliary parameter ζ. Note that the modification with $\zeta \neq 0$ is consistent with the $SU(4|3)$ superconformal symmetries. All of the variables appearing here are projective coordinates: one has the freedom to rescale (U, η) and (V, ψ) by arbitrary and independent complex variables. We can also rescale the coordinates (X, Θ, Φ) simultaneously with ζ. In principle this allows us to put the parameter ζ to an arbitrary value.

Eqs. (4.5) directly imply the quadric relation

$$U_\alpha V^\alpha + \psi_I\eta^I = 0 \tag{4.6}$$

as can be seen by replacing U_β and η^I by the expressions on the l.h.s. We can modify the supersymmetric beta-plane equations in a similar way

$$U_\beta X^{\beta\alpha} + \Theta^\alpha{}_I\eta^I = \zeta V^\alpha,$$
$$U_\alpha\Theta^\alpha{}_I + \Phi_{IJ}\eta^J = \zeta\psi_I. \tag{4.7}$$

We now require that these equations are consistent with (4.5). This leads to a number of relations between the X, Θ and Φ coordinates with upper and lower indices. Let us first take $\zeta \neq 0$. Note that both the alpha- and beta-plane equations can be written in matrix form by combining the coordinates as follows

$$\zeta^{-1}\begin{pmatrix} X & \Theta \\ \Theta & \Phi \end{pmatrix}. \tag{4.8}$$

The only difference between the alpha- and beta-planes is that in one case the indices are up and in the other case down, and more importantly that (U, η) and (V, ψ) are interchanged. It is now easy to see that the two sets of equations are consistent for $\zeta \neq 0$ if and only if the matrix (4.8) is invertible, and its inverse is simply obtained by replacing upper with lower indices. This leads to the relations

$$X^{\alpha\gamma}X_{\gamma\beta} + \Theta^\alpha{}_I\Theta_\beta{}^I = \zeta^2\delta^\alpha{}_\beta,$$
$$\Theta^\alpha{}_I\Theta_\alpha{}^J + \Phi_{IK}\Phi^{KJ} = \zeta^2\delta_I{}^J$$

and

$$X^{\alpha\beta}\Theta_\beta{}^I + \Theta^\alpha{}_J\Phi^{JI} = 0,$$
$$\Theta^\beta{}_I X_{\beta\alpha} + \Phi_{IJ}\Theta_\alpha{}^J = 0.$$

With these relations (4.5) is equivalent to (4.7), and hence it suffices to keep only one or the other set of equations.

To show the equivalence we assumed that $\zeta \neq 0$. But now we can take the limit $\zeta \to 0$ and obtain both the alpha- and the beta-plane equations. In this limit

$$X^{\alpha\beta}X_{\alpha\beta} - \Theta^\alpha{}_I\Theta_\alpha{}^I = 0. \tag{4.9}$$

This describes the superlightcone in 6d. But when $\zeta \neq 0$ we find

$$X^{\alpha\beta}X_{\alpha\beta} - \Theta^\alpha{}_I\Theta_\alpha{}^I = -4\zeta^2. \tag{4.10}$$

Here we recognize a complexified supersymmetric version of AdS_5. The appearance of AdS_5 is not entirely surprising in view of the symmetries of the equations and the use of a 6d notation. What about the S^5? Could this be described by the Φ coordinates? In our description the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ has been broken to (complexified) $SU(3)$. This suggest that one should not expect to find an S^5 because it is not consistent with the symmetries. But it is interesting to note that Φ can be identified with the (complexified) space of symmetric $SU(3)$ matrices, which is isomorphic to $SU(3)/SO(3)$ and is indeed 5-dimensional.

5. Comments on the application to Yang–Mills theory

In this note we studied the twistor construction of classical $\mathcal{N}=3$ super-Yang–Mills theory on the quadric submanifold $Q_{5|6}$ of $\mathbb{CP}^{3|3}\times\mathbb{CP}^{3|3}$. We gave a reformulation of the twistor equations in six dimension, and described the (anti-)self-dual alpha- and beta-planes in a manifest superconformal invariant notation. The superconformal symmetry naturally allows a modification of the twistor equations leading to an interesting connection with AdS_5 and its supersymmetric extension. An important question is what this implies for the Yang–Mills theory, and whether our construction can be applied to the AdS/CFT correspondence. In this concluding section we will present some comments regarding this questions for the purely bosonic twistor equations. A more complete investigation is left for future work.

Our six-dimensional view on the twistor equations can be used to extend a 4d gauge field to a 6d gauge field as follows. First one uses the fact that any 4d gauge field can be represented as a vector bundle over Q_5. Then by applying a generalization of Ward's construction to our 6d bosonic twistor equations one obtains a Yang–Mills field in six dimensions. Indeed, according to Siegel [16], a 4d Yang–Mills field, not obeying any equations, can be mapped on to a 6d gauge field satisfying

$$X^A F_{ABC}=0 \tag{5.1}$$

with

$$F_{ABC}=X_{[A}F_{BC]}, \tag{5.2}$$

where $A=1,\dots,6$ are the 6d space–time indices. Here we followed the notation of [16]. This equation expresses the integrability of the gauge field along alpha- and beta-planes. Thus a vector bundle on Q_5 is related through a generalized twistor construction to a solution of (5.1). The Yang–Mills field equations are in this notation

$$\nabla^A F_{ABC}=0. \tag{5.3}$$

To get a solution equation (5.3) one has again to locally extend the vector bundle to $\mathbb{CP}^3\times\mathbb{CP}^3$. An interesting observation in this context is that the anti-de Sitter space that we found can be related to the coordinates $(y,z)\in\mathbb{M}_4\times\mathbb{M}_4$ introduced in [9] and described in Section 2. There we wrote the Minkowski coordinate as $x=\frac{1}{2}(y+z)$. By applying the same line of thought to our AdS description we write

$$X_{\alpha\beta}=\frac{1}{2}(Y_{\alpha\beta}+Z_{\alpha\beta})\quad\text{with } Y_{\alpha\beta}V^\beta=0,\ U_\alpha Z^{\alpha\beta}=0$$

with $Y^2=Z^2=0$. Hence up to rescaling we have $Y=(1,y^2,y)$ and $Z=(1,z^2,z)$ with $(y,z)\in\mathbb{M}_4\times\mathbb{M}_4$. The modified twistor equations for $X_{\alpha\beta}$ imply

$$Y^{\alpha\beta}U_\beta=2\zeta V^\alpha,\qquad Z_{\alpha\beta}V^\beta=2\zeta U_\alpha. \tag{5.4}$$

These equations are consistent provided that $Y^{\alpha\beta}Z_{\alpha\beta}=-8\zeta^2$. In terms of z and y this gives $(z-y)^2=4\zeta^2$. In other words, the parameter ζ can be interpreted as the distance $w^2=(z-y)^2$ between the two points y and z. This observation suggests that the gauge field on the AdS submanifold can be obtained from the connections ∇_y and ∇_z in a point $(y,z)\in\mathbb{M}_4\times\mathbb{M}_4$. However, a slightly confusing point is the following. In Section 2 it was noted that an extension away from the diagonal in $\mathbb{M}_4\times\mathbb{M}_4$ requires the Yang–Mills equations to be satisfied. But here we just argued that we can extend any gauge field to 6 dimensions. We believe the resolution is that for the construction of the 6d gauge field it is not necessary that ∇_y and ∇_z mutually commute.

We end with some final comments. All these equations have presumably a supersymmetric extension. In that case one does also obtain the super-Yang–Mills equations: integrability along the superalpha- and beta-planes gives the familiar constraints of $N=3$ supersymmetry which imply the equations of motion. It would be interesting to work this out in detail in our formalism. We leave this for future work. Our work may be helpful in making a connection to the topological B-model on the quadric, and represent the Yang–Mills theory as a holomorphic Chern–Simons theory on the quadric.

Finally, it would be interesting to examine if the $\mathcal{N}=4$ gauge theory amplitudes can be formulated in terms of the twistor space $Q_{5|6}$. Although the amplitudes are in the weak coupling region, it is possible that one can find a sign of the quadric by performing a kind of Fourier transformation. Such a formulation would have the advantage of being symmetric in both helicities.

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