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Chebyshev polynomials on symmetric matrices ☆

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ABSTRACT

In this paper we evaluate Chebyshev polynomials of the second kind on a class of symmetric integer matrices, namely on adjacency matrices of simply laced Dynkin and extended Dynkin diagrams. As an application of these results we explicitly calculate minimal projective resolutions of simple modules of symmetric algebras with radical cube zero that are of finite and tame representation type.

1. Introduction

Chebyshev polynomials are a sequence of recursively defined polynomials. They appear in many areas of mathematics such as numerical analysis, differential equations, number theory and algebra [10]. Although they have been known and studied for a long time, they continue to play an important role in recent advances in many subjects, for example in numerical integration, polynomial approximation, or spectral methods (e.g. [8]). It is interesting to note that they also play an important part in the representation theory of algebras (e.g. [4,6,7,11]). There are several closely related Chebyshev polynomials. Amongst these, the polynomials usually referred to as Chebyshev polynomials of the

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first kind, and Chebyshev polynomials of the second kind are the ones that often naturally appear; for example, they arise as solutions of special cases of the Sturm–Liouville differential equation or in dimension counting in representation theory (e.g. [1,5,9]).

For Chebyshev polynomials of the first and the second kind, the recursive definition is equivalent to a definition by a determinant formula. Symmetric integer matrices are a key to this definition. Focusing on the polynomials of the second kind, we exhibit some surprising properties of Chebyshev polynomials in relation to these symmetric matrices. In fact the symmetric matrices we consider are adjacency matrices of Dynkin diagrams and extended Dynkin diagrams. Dynkin diagrams play an import role in Lie theory, where they give a classification of root systems. However, they also appear in areas that have no obvious connection to Lie theory as for example in singularity theory where they are linked to Kleinian singularities, or, for example, in representation theory of algebras where they classify symmetric algebras of radical cube zero of finite and tame representation type [2].

The motivation for the study of the Chebyshev polynomials evaluated on matrices comes from the representation theory of the algebras classified in [2]. We begin with a detailed study of the Chebyshev polynomials evaluated on adjacency matrices of Dynkin diagrams, where we show that in the case of Dynkin diagrams, the families of polynomials are periodic and in the case of the extended Dynkin diagrams the families grow linearly. We then show as an application how the general results we obtain can be applied to the representation theory of the symmetric algebras of radical cube zero. We will see that the Chebyshev polynomials govern the minimal projective resolutions for these algebras and that they give rise to a method to calculate the constituents of minimal projective resolutions of simple modules.

We will now outline the content of this paper. In the next section we recall the definition of Chebyshev polynomials of the second kind, we define the polynomials we will be working with and we introduce the Dynkin diagrams together with a labeling of these diagrams which we will use throughout the paper. In Section 3 we evaluate Chebyshev polynomials on the adjacency matrices of Dynkin diagrams and extended Dynkin diagrams. In Section 4 a link with the representation theory of symmetric algebras of radical cube zero of finite and tame representation type is described. In particular, we show how the results of Section 3 can be used to calculate minimal projective resolutions of the simple modules. Finally in Section 5 we show a more general result on Chebyshev polynomials evaluated on positive symmetric matrices.

2. Definitions

2.1. Chebyshev polynomials

We briefly recall the definition of Chebyshev polynomials of the second kind; good references are [10,12].

The Chebyshev polynomial of the second kind $U_n(x)$ of degree n is defined by

$$U_n(x) = \sin(n+1)\theta/\sin\theta \quad \text{where } x = \cos\theta.$$

From this definition the following recurrence relations with initial conditions can be deduced

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad \text{with } U_0(x) = 1 \text{ and } U_1(x) = 2x.$$

Furthermore an easy calculation shows (see also [12] page 26) that $U_n(x) = \det(2xI_n - A_n)$ where A_n is an $n \times n$ matrix that has zeros everywhere except directly above and directly below the diagonal where all the entries are equal to one.

We will work with the version of the Chebyshev polynomial defined by

$$f_n(x) = \det(xI - A_n)$$

where A_n is defined as above, and I is the identity matrix. These polynomials are also sometimes called Dickson polynomials of the second kind. An easy calculation shows that $f_n(x)$ is defined by the recurrence relation $f_n(x) = xf_{n-1}(x) - f_{n-2}(x)$ with initial conditions $f_0(x) = 1$ and $f_1(x) = x$. Furthermore, we set $f_{-1}(x) = 0$. All matrices have entries in $\mathbb{Q}$.

2.2. Dynkin diagrams and adjacency matrices

The Dynkin diagrams and extended Dynkin diagrams we are going to consider are the ones of type A, D, E and $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{D}}$, $\widetilde{\mathrm{E}}$ as well as the diagrams of type L, $\widetilde{\mathrm{L}}$, and $\widetilde{\mathrm{DL}}$ (see Appendix).

Let G be an undirected graph with n vertices labeled by the set $\{1, 2, \ldots, n\}$. The adjacency matrix of G is a $n \times n$ matrix where the entry in position (i, j) is given by the number of edges between the vertices i and j. Modulo conjugation by a permutation matrix, the adjacency matrix is independent of the choice of labeling.

However, in what follows we work with particular adjacency matrices corresponding to a particular labeling of the graphs. We refer the reader to the Appendix for the labeling of the diagrams we have chosen. In the case of the Dynkin diagrams this labeling corresponds to the Dynkin labeling. It follows from Lemma 3.1 in the next section that our results are, up to permutation, independent of the labeling.

Our goal is to evaluate the Chebyshev polynomials of the second kind on the adjacency matrices of the Dynkin diagrams and the extended Dynkin diagrams as well as the diagrams of type L, $\widetilde{L}$, and $\widetilde{DL}$.

3. Evaluating Chebyshev polynomials

3.1. Evaluating Chebyshev polynomials on matrices – general results

In this section we list some useful facts about evaluating the Chebyshev polynomials on matrices. We begin with a lemma collecting some straightforward facts.

Lemma 3.1. *For any square matrix M, we have $Mf_k(M) = f_k(M)M$ and for any symmetric matrix S, the matrix $f_k(S)$ is symmetric for all $k \geqslant 0$. If T is an invertible $n \times n$ matrix then for all $n \times n$ matrices M we have $f_k(TMT^{-1}) = Tf_k(M)T^{-1}$.*

Definition 3.2. Let X be a square matrix, we say that the sequence of matrices $(f_k(X))_{k\geqslant 0}$ is periodic of period $\leqslant p$, if $p > 1$ and if p satisfies $f_{p-1}(X) = 0$ and $f_p(X) = I$.

Remark. Suppose that $(f_k(X))_{k\geqslant 0}$ is periodic of period $\leqslant p$. Then for any integer k, we can write $k = qp + r$ with $0 \leqslant r < p$, and $f_k(X) = f_r(X)$.

Lemma 3.3. *Assume X is a square matrix such that $f_d(X) = 0$ for some $d > 1$. Then for $0 \leqslant k \leqslant d + 1$, we have*

$$f_{d+k}(X) + f_{d-k}(X) = 0. \qquad (*)$$

Moreover, $f_{2d+1}(X) = 0,\ f_{2d+2}(X) = I$ and hence $f_{2d+3}(X) = X$. Therefore the sequence $(f_k(X))_{k\geqslant 0}$ is periodic, of period $\leqslant 2d + 2$.

Proof. The recursion for the Chebyshev polynomials can be rewritten as

$$xf_m(x) = f_{m+1}(x) + f_{m-1}(x), \quad \text{for } m \geqslant 0.$$

Consider now $x^k f_d(x)$ for $0 \leqslant k \leqslant d + 1$ and substitute $x = X$. Since $f_d(X) = 0$, induction on k will show that $X^k f_d(X) = f_{d+k}(X) + f_{d-k}(X) = 0$, for $0 \leqslant k \leqslant d + 1$.

The case $k = 0$ is clear. Assume now that the statement is true for all j where $j \leqslant k$ and suppose $k < d + 1$. Then

$$\begin{aligned} 0 = X^{k+1} f_d(X) &= X[X^k f_d(X)] \\ &= X[f_{d+k}(X) + f_{d-k}(X)] \\ &= f_{d+k+1}(X) + f_{d+k-1}(X) + f_{d-k+1}(X) + f_{d-k-1}(X) \end{aligned}$$

$$= f_{d+k+1}(X) + X^{k-1}f_d(X) + f_{d-k-1}(X)$$
$$= f_{d+k+1}(X) + 0 + f_{d-k-1}(X)$$

as required.

For the last part, let $k = d + 1$ and recall that $f_{-1}(X) = 0$, hence $f_{2d+1}(X) = 0$. Then let $k = d$, and recall that $f_0(X) = I$ which implies $f_{2d}(X) = -I$. Since $f_{2d+1}(X) = 0$ we can apply $(*)$ with $2d + 1$ instead of d. We then obtain $f_{2d+2}(X) + f_{2d}(X) = 0$ and hence $f_{2d+2}(X) = I$ and finally $f_{2d+3}(X) = X \cdot I - 0 = X$. $\square$

If for some $d \geqslant 1$ the matrices $f_d(X)$ and $f_{d+1}(X)$ are equal, then periodicity follows by Lemma 3.3.

Lemma 3.4. *Assume X is a square matrix such that $f_d(X) = f_{d+1}(X)$ for some integer $d \geqslant 1$. Then for $1 \leqslant k \leqslant d + 1$, we have*

$$f_{d+1+k}(X) = f_{d-k}(X).$$

In particular, $f_{2d+2}(X) = 0$.

Proof. Assume $f_d(X) = f_{d+1}(X)$. Then we have

$$f_{d-1}(X) = Xf_d(X) - f_{d+1}(X) \text{ and } f_{d+2}(X) = Xf_{d+1}(X) - f_d(X)$$

and hence $f_{d-1}(X) = f_{d+2}(X)$. For the inductive step, assume the statement is true for $1 \leqslant m \leqslant k$. Then

$$\begin{aligned} f_{d-k}(X) &= Xf_{d-k+1}(X) - f_{d-k+2}(X) \\ &= Xf_{d+k}(X) - f_{d+k-1}(X) \\ &= f_{d+k+1}(X). \qquad \square \end{aligned}$$

Some matrices allow a reduction, based on the following lemma, which gives a criterion to determine when a sequence of matrices has linear growth.

Lemma 3.5. *Assume X is a square matrix such that for some matrix Z and for some $q \geqslant 2$ we have $f_q(X) = f_{q-2}(X) + Z$, and where $ZX = XZ = 2X$. Then*

(a) for $1 \leqslant t \leqslant q - 1$, we have

$$f_{q+t}(X) = \begin{cases} f_{q-2-t}(X) + 2f_t(X) & t \text{ odd} \\ f_{q-2-t}(X) + 2f_t(X) + (-1)^{\frac{t}{2}+1}(2I - Z) & t \text{ even}, \end{cases}$$

(b) if $Z = 2I$, for $t \geqslant -1$, we have $f_{2q+t}(X) = 2f_{q+t}(X) - f_t(X)$,
(c) if $Z = 2I$ and $m = rq + u$ where $-1 \leqslant u \leqslant q - 2$ and $r \geqslant 2$, we have

$$f_{rq+u}(X) = rf_{q+u}(X) - (r-1)f_u(X).$$

Corollary 3.6. *Assume X is a square matrix such that for some matrix Z we have $f_q(X) = f_{q-2}(X) + Z$, for $q \geqslant 2$ and where $ZX = XZ = 2X$ and where $Z \neq 2I$. Then*

$$f_{2q}(X) - f_{2q-2}(X) = \begin{cases} 2Z - 2I & q \text{ odd} \\ 2Z - 2I + (-1)^{\frac{q-2}{2}+1}(4I - 2Z) & q \text{ even}. \end{cases}$$

Proof. By the Chebyshev recursion formula we have $f_{2q}(X) = Xf_{2q-1}(X) - f_{2q-2}(X)$ and $2q - 1 = q + (q-1)$ and $2q-2 = q + (q-2)$. Therefore we can apply Lemma 3.5(a) and the result follows. $\square$

Remark. (I) Later, the matrix Z will often be equal to $2I$. In this case (a) is equal to $f_{q+t}(X) = f_{q-2-t}(X) + 2f_t(X)$. Furthermore, an easy calculation shows that (c) becomes $f_{rq+u}(X) = rf_{q-2-u}(X) + (r+1)f_u(X)$.

(II) Observe that part (a) describes $f_k(X)$ for $q < k \leqslant 2q-1$. Since every natural number $m \geqslant 2q-1$ can be written in the form given in part (c), this gives a description of $f_k(X)$ for all $k \geqslant q$. It thus follows from the formula in part (c) that a sequence $(f_k(X))_k$ that satisfies the hypotheses of Lemma 3.5 with $Z = 2I$ has linear growth.

Proof of Lemma 3.5. (a) We use induction on t. For $t = 1$, we have

$$\begin{aligned} f_{q+1}(X) &= Xf_q(X) - f_{q-1}(X) \\ &= Xf_{q-2}(X) + XZ - f_{q-1}(X) \\ &= Xf_{q-2}(X) + 2X - f_{q-1}(X) \\ &= f_{q-3}(X) + 2f_1(X) \end{aligned}$$

if we recall that $f_1(X) = X$.

For $t = 2$, we have

$$\begin{aligned} f_{q+2}(X) = Xf_{q+1}(X) - f_q(X) &= X(f_{q-3}(X) + 2f_1(X)) - f_q(X) \\ &= Xf_{q-3}(X) + 2Xf_1(X) - (f_{q-2}(X) + Z) \\ &= f_{q-4}(X) + 2f_2(X) + 2I - Z. \end{aligned}$$

For the inductive step suppose first that t is odd. Then we have

$$\begin{aligned} f_{q+t+1}(X) &= Xf_{q+t}(X) - f_{q+t-1}(X) \\ &= X[f_{q-2-t}(X) + 2f_t(X)] - [f_{q-2-(t-1)}(X) + 2f_{t-1}(X) + (-1)^{\frac{t-1}{2}+1}(2I - Z)] \\ &= f_{q-2-(t+1)}(X) + 2f_{t+1}(X) + (-1)^{\frac{t+1}{2}+1}(2I - Z). \end{aligned}$$

Now suppose that t is even. Then we have

$$\begin{aligned} f_{q+t+1}(X) &= Xf_{q+t}(X) - f_{q+t-1}(X) \\ &= X[f_{q-2-t}(X) + 2f_t(X) + (-1)^{\frac{t}{2}+1}(2I - Z)] - [f_{q-2-(t-1)}(X) + 2f_{t-1}(X)] \\ &= f_{q-2-(t+1)}(X) + 2f_{t+1}(X) + (-1)^{\frac{t}{2}+1}(2X - ZX) \\ &= f_{q-2-(t+1)}(X) + 2f_{t+1}(X) \end{aligned}$$

since $ZX = 2X$.

(b) The case $t = -1$ follows from part (a). Let $t = 0$, then

$$\begin{aligned} f_{2q}(X) &= Xf_{2q-1}(X) - f_{2q-2}(X) \\ &= X[2f_{q-1}(X)] - [2f_{q-2}(X) + f_0(X)] \\ &= 2f_q(X) - I \end{aligned}$$

where the equality $f_{2q-2}(X) = f_0(X) + 2f_{q-2}(X)$ follows from part (a). Let $t \geqslant 1$, and assume the equation holds for $t-1$ and $t-2$, then

$$\begin{aligned} f_{2q+t}(X) &= Xf_{2q+(t-1)}(X) - f_{2q+(t-2)}(X) \\ &= X[2f_{q+t-1}(X) - f_{t-1}(X)] - [2f_{q+t-2}(X) - f_{t-2}(X)] \\ &= 2f_{q+t}(X) - f_t(X). \end{aligned}$$

(c) The case $r = 2$ follows from part (b). Assume now $r \geqslant 3$ and write $rq + u = 2q + t$ where $t = (r-2)q + u$. Then by part (b) we have

$$(*) \qquad f_{rq+u}(X) = f_{2q+t}(X) = 2f_{q+t}(X) - f_t(X).$$

If $r = 3$ then $2q + u = q + t$ and using part (b) again we have $f_{q+t}(X) = 2f_{q+u}(X) - f_u(X)$. Substituting this into equation $(*)$ gives

$$f_{3q+u}(X) = 2[2f_{q+u}(X) - f_u(X)] - f_{q+u}(X) = 3f_{q+u}(X) - 2f_u(X)$$

as required.

Now let $r > 3$. We write $(r-1)q + u = q + t$ and $(r-2)q + u = t$. Then equation $(*)$ gives

$$\begin{aligned} f_{rq+u}(X) &= 2f_{q+t}(X) - f_t(X) \\ &= 2[f_{(r-1)q+u}(X)] - f_{(r-2)q+u}(X) \\ &= 2[(r-1)f_{q+u}(X) - (r-2)f_u(X)] - [(r-2)f_{q+u}(X) - (r-3)f_u(X)] \\ &= rf_{q+u}(X) - (r-1)f_u(X) \end{aligned}$$

and this completes the proof. $\square$

Definition 3.7. Given some $n \times n$ matrix X, we define X^0 to be the matrix obtained from X by reversing the entries in each row. In particular, in I^0 the $i, n-i+1$-th entries are equal to 1 for all $1 \leqslant i \leqslant n$ and all other entries equal to zero.

Note that $(X^0)^0 = X$, and that $X^0 = X \cdot I^0$. Furthermore, we write I_n^0 if we need to specify the size of the matrix.

With this we have the following variation of Lemma 3.5. The proof is a straightforward modification of that of Lemma 3.5 and we leave the details to the reader.

Lemma 3.8. *Assume X is a square matrix such that for some matrix Z and for some $q \geqslant 2$ we have $f_q(X) = f_{q-2}(X) + Z$, and where $XZ = 2X^0$. Then*

(a) for $1 \leqslant t \leqslant q-1$, we have

$$f_{q+t}(X) = f_{q-2-t}(X) + 2f_t(X)^0.$$

In particular, $f_{2q-2}(X) = 2f_{q-2}(X)^0 + I$ and $f_{2q-1}(X) = 2f_{q-1}(X)^0$,

(b) for $t \geqslant -1$ we have $f_{2q+t}(X) = 2f_{q+t}(X)^0 - f_t(X)$,

(c) if $m = rq + u$ where $-1 \leqslant u \leqslant q-2$ and $r \geqslant 2$, we have

$$f_{rq+u}(X) = \begin{cases} rf_{q+u}(X)^0 - (r-1)f_u(X) & r \text{ even} \\ rf_{q+u}(X) - (r-1)f_u(X)^0 & r \text{ odd.} \end{cases}$$

Remark. Note if Z is equal to $2I^0$ then (c) can be rewritten as

$$f_{rq+u}(X) = \begin{cases} rf_{q-u-2}(X)^0 + (r+1)f_u(X) & r \text{ even} \\ rf_{q-u-2}(X) + (r+1)f_u(X)^0 & r \text{ odd.} \end{cases}$$

3.2. Substituting type A

In this paragraph we evaluate the Chebyshev polynomials on the adjacency matrix of a Dynkin diagram of type A. This will be the basis for the calculation of almost all the other finite types as well as all the extended types.

Assume A is the adjacency matrix of a Dynkin diagram of A_n, $n \geqslant 2$. Then by the Cayley–Hamilton Theorem we know that $f_n(A) = 0$ and by Lemma 3.3 the sequence $(f_k(A))_{k \geqslant 0}$ is periodic of period $\leqslant 2n+2$.

Let Θ_k be the subset of $\Theta = \{(i,j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$ given by

$$\Theta_k = \{(i,j) : k+2 \leqslant i+j \leqslant 2n-k,\ \ i+j \equiv k (\mathrm{mod}\ 2), -k \leqslant j-i \leqslant k\}.$$

We think of this as a subset of $\mathbb{N} \times \mathbb{N}$. Note that Θ_k consists of the points in Θ of parity $i+j \equiv k(\mathrm{mod}\ 2)$ which lie in the rectangle given by the lines

$$x+y = k+2,\ x+y = 2n-k,\ y-x = k,\ y-x = -k. \tag{1}$$

This rectangle has corners $(1, k+1)$, $(k+1, 1)$ and $(n-k, n)$, $(n, n-k)$. In particular, $\Theta_n = \emptyset$ and $\Theta_{-1} = \emptyset$.

Write E_{ij} for the usual matrix unit.

Proposition 3.9. *For $-1 \leqslant k \leqslant n$,*

$$f_k(A) = \sum_{(i,j) \in \Theta_k} E_{ij}. \tag{2}$$

In particular, $f_n(A) = 0$.

Corollary 3.10. *The family $(f_k(A))_k$, $k \geqslant -1$ is periodic of period $\leqslant 2n+2$.*

Proof. It follows from Lemma 3.3 that for $0 \leqslant k \leqslant n+1$, $f_{n+k}(A) + f_{n-k}(A) = 0$. and that the sequence $(f_k(A))_{k \geqslant 0}$ is periodic of period at most $2n+2$. □

Furthermore, $f_k(A) = 0$ if and only if $k = mn + m - 1 = m(n+1) - 1$ for $m \geqslant 0$ and the entries of $f_k(A)$ are known, for all $k \geqslant 0$.

Proof of Proposition 3.9. Let $-1 \leqslant k \leqslant n$ and let W_k be the expression on the right hand side of (2). To prove (2) it is enough to show that

$$AW_k = W_{k+1} + W_{k-1}.$$

We have $(\sum_{l=1}^{n-1} E_{l,l+1})E_{ij} = E_{i-1,j}$, except in the case $i = 1$ where $(\sum_{l=1}^{n-1} E_{l,l+1})E_{1j} = 0$. Note, however, that this only occurs for $(1, k+1)$, that is in the 'top corner' of Θ_k. Similarly, $(\sum_{l=1}^{n-1} E_{l+1,l})E_{ij} = E_{i+1,j}$ except in the case $i = n$ where $(\sum_{l=1}^{n-1} E_{l+1,l})E_{nj} = 0$. This occurs only for $(n, n-k)$, that is the 'bottom corner' of Θ_k. Therefore $AE_{ij} = E_{i-1,j} + E_{i+1,j}$ for all $(i,j) \in \Theta_k$, with the two exceptions as described above.

If we visualize the E_{ij} occurring in W_k as grid points in the rectangle defined by Θ_k then the terms of AW_k are obtained by replacing each (i,j) in this rectangle by the two points below and above, namely $(i-1,j)$ and $(i+1,j)$ (with the exceptions of the top corner where $(i+1,j)$ is missing, and the bottom corner where $(i-1,j)$ is missing).

Following this process we obtain only the points (r,s) with $r+s \equiv k-1(\mathrm{mod}\ 2)$ and each point (r,s) inside the rectangle defined by (1) is obtained twice. Additionally we get all points (r,s) lying on the lines

$$x+y = k+1,\ x+y = 2n-k+1,\ \text{and}\ y-x = k+1,\ y-x = -k-1$$

exactly once.

Thus we obtain precisely the points corresponding to W_{k+1} and to W_{k-1}, while the points in the intersection appear twice. □

3.3. *Substituting type L*

In this paragraph we evaluate the Chebyshev polynomials on the adjacency matrix of a diagram of type L. Let L be the adjacency matrix of a diagram of type L_n. Then we can express L in terms of A such that $L = A + E_{11}$ where A is the matrix of type A_n as in the previous section. We will now express the matrices $f_k(L)$ in terms of the matrices $f_k(A)$. For $k = 1, 2, \ldots, n-1$ we define

$$T_k := \sum_{1 \leqslant i,j,i+j \leqslant k+1} E_{ij}.$$

That is, each entry in the upper left corner, up to the line $i + j = k + 1$, is equal to 1 and all other entries are zero.

Proposition 3.11. *(a) For $k = 1, 2, \ldots, n-1$ we have $f_k(L) = f_k(A) + T_k$.*
(b) $f_n(L) = f_{n-1}(L)$.
(c) For $1 \leqslant k \leqslant n+1$ we have $f_{n-1+k}(L) = f_{n-1-k}(L)$. In particular $f_{2n-1}(L) = I$ and $f_{2n}(L) = 0$.
(d) We have $f_{4n+1}(L) = 0$ and $f_{4n+2}(L) = I$.

Corollary 3.12. *The sequence $(f_k(L))_k$ is periodic of period $\leqslant 4n + 2$.*

Proof. (a) The proof is by induction. Clearly, $f_1(L) = f_1(A) + T_1$. Suppose that $k < n - 1$ and that $f_k(L) = f_k(A) + T_k$ holds. Then

$$\begin{aligned}
f_{k+1}(L) &= Lf_k(L) - f_{k-1}(L)\\
&= (A + E_{11})(f_k(A) + \sum_{i=1}^{k}\sum_{j=1}^{k+1-i} E_{ij}) - f_{k-1}(L)\\
&= Af_k(A) + \sum_{i=1}^{k}\sum_{j=1}^{k+1-i} (E_{i+1j} + E_{i-1j}) + E_{11}f_k(A) + E_{11}\sum_{i=1}^{k}\sum_{j=1}^{k+1-i} E_{ij} - f_{k-1}(L)\\
&= Af_k(A) + \sum_{i=1}^{k-1}\sum_{j=1}^{k-i} E_{ij} + \sum_{i=2}^{k+1}\sum_{j=1}^{k+2-i} E_{ij} + \sum_{j=1}^{k+1} E_{1j} - f_{k-1}(L)\\
&= Af_k(A) - f_{k-1}(A) + \sum_{i=1}^{k-1}\sum_{j=1}^{k-i} E_{ij} + \sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i} E_{ij} - \sum_{i=1}^{k-1}\sum_{j=1}^{k-i} E_{ij}\\
&= f_{k+1}(A) + \sum_{i=1}^{k+1}\sum_{j=1}^{k+2-i} E_{ij}.
\end{aligned}$$

(b) The calculation in part (a) also holds when $k = n - 1$ and therefore

$$\begin{aligned}
f_n(L) &= Lf_{n-1}(L) - f_{n-2}(L)\\
&= f_n(A) + \sum_{i=1}^{n}\sum_{j=1}^{k+1-i} E_{ij}\\
&= 0 + f_{n-1}(L_n).
\end{aligned}$$

Part (c) and (d) follow from Lemma 3.4. □

Remark 3.13. (I) We keep the record that the calculation in part (a) shows that for $k < n-1$

$$E_{11}T_k + AT_k - T_{k-1} = T_{k+1}.$$

(II) Later we have to use the matrices of type L but where the labeling is reversed. Then we have the description as in the above proposition, where the only change is that T_k is replaced by $B_k := I^0 T_k$. This matrix is obtained from B_k by reflecting in the line $i + j = n + 1$. For this, the calculation in part (a) shows that for $k < n-1$

$$E_{nn}B_k + AB_k - B_{k-1} = B_{k+1}.$$

3.4. Substituting extended type L

Fix $n \geqslant 2$ and let $\widetilde{L} = \widetilde{L}_n$ be the adjacency matrix of a diagram of type $\widetilde{\mathrm{L}}_{\mathrm{n}}$ as described in Section 2.2. In this paragraph we evaluated the Chebyshev polynomial on $\widetilde{L}$ and show that the family $f_k(\widetilde{L})$ is of linear growth.

We note that $\widetilde{L}$ can be expressed in terms of the adjacency matrix A of A_n. Namely, $\widetilde{L} = A + E_{11} + E_{nn}$. The next proposition shows that the terms of $f_k(\widetilde{L}_n)$ are a sum of $f_k(A)$ and an upper and a lower triangular matrix whose entries are all equal to 1. Recall the definition of T_k and B_k from Section 3.3.

Proposition 3.14. *For $1 \leqslant k \leqslant n-1$, we have*

$$f_k(\widetilde{L}) = f_k(A) + T_k + B_k.$$

Proof. We proof the result by induction. The result holds for $k = 1$ and a direct calculation shows that it also holds for $k = 2$. Suppose it holds for all $l \leqslant k$. Then by definition we have $f_{k+1}(\widetilde{L}) = Xf_k(\widetilde{L}) - f_{k-1}(\widetilde{L})$ and by induction hypothesis this is equal to

$$(A + E_{11} + E_{nn})(f_k(A) + T_k + B_k) - f_{k-1}(A) - T_{k-1} - B_{k-1}.$$

To prove the stated formula we need

$$(E_{11} + E_{nn})(f_k(A) + T_k + B_k) + A(T_k + B_k) - T_{k-1} - B_{k-1} = T_{k+1} + B_{k+1}.$$

This is true by the record kept in Remark 3.13. $\square$

Let U be the $n \times n$ matrix all of whose entries are equal to 1. Recall also the definition of I^0 and X^0, and note that $2XI^0 = 2X^0$. As special cases of the previous, and by applying Lemma 3.8, we have therefore

Corollary 3.15. *We have that $f_{n-2}(\widetilde{L}) = U - I^0$, and $f_{n-1}(\widetilde{L}) = U$, and $f_n(\widetilde{L}) = U + I^0$. Hence*

$$f_n(\widetilde{L}) = f_{n-2}(\widetilde{L}) + 2I^0$$

and the sequence $(f_k(\widetilde{L}))_{k \geqslant 0}$ has linear growth.

3.5. General setup for the remaining infinite families

All remaining infinite families of Dynkin and extended Dynkin types are based on the above calculations and fit into a more general set-up described in this section. Namely, in this section we substitute a symmetric block matrix X of the form

$$X = \begin{pmatrix} 0 & S \\ S^t & W \end{pmatrix},$$

where we assume that $SS^tS = 2S$, and hence that S^tS and SS^t are projections. Assume further that S has rank one. Then for any matrix M of the appropriate size, SMS^t is a scalar multiple of SS^t.

The following lemma is a straightforward calculation.

Lemma 3.16. *Write $f_k(X)$ as block matrix, $f_k(X) = \begin{pmatrix} H_k & S_k \\ (S_k)^t & W_k \end{pmatrix}$. Then we have*

$$H_{k+1} = SS_k^t - H_{k-1}$$
$$S_{k+1} = SW_k - S_{k-1}$$
$$W_{k+1} = S^tS_k + WW_k - W_{k-1}.$$

We can express the adjacency matrices of the remaining infinite families in question in terms of S and W and we will apply Lemma 3.16 as follows. Let ε_1 be the row vector of length n whose first entry is equal to 1 and whose other entries are all equal to 0.

(1) For type D_m, $m \geqslant 3$, of size m, the adjacency matrix of type D_m is given by choosing W to be of type A_n with $n = m - 2$, and S to be the matrix with two rows, each row equal to ε_1.
(2) For type $\widetilde{D}_{m+1}$, $m \geqslant 4$, of size $m + 2$, we take S as in (1), and for W we take the matrix D of type D_m but with the labeling reversed, we will call this matrix V, that is $V = I^0DI^0$.
(3) For type $\widetilde{A}_n$, $n \geqslant 3$, of size $n + 1$, we take $W = A_n$ and S to be the matrix with one row equal to ε_1.
(4) For type $\widetilde{DL}_m$, $m \geqslant 3$, of size m, we take $W = I^0L_nI^0$ with $n = m - 2$, that is W is L_n but with reversed order, and we take S as in (1) so that S is a two-row matrix with both rows equal to ε_1.

Take X as above, and S_k, W_k and H_k as in the recursion in Lemma 3.16. We will now calculate the first few terms explicitly.

Definition 3.17. We define invariants c and λ_c of X, to be the first integer $c > 1$, and the scalar λ_c, such that $Sf_c(W)S^t = \lambda_cSS^t$ is non-zero.

Proposition 3.18. *Let c be as above and let $1 \leqslant k \leqslant c + 2$. Then*

(a) $S_k = S\left(\sum_{i\geqslant 0} f_{k-1-2i}(W)\right)$, $\quad W_k = f_k(W) + \sum_{i\geqslant 0} \psi_{k-2-2i}$

where $\psi_x = \sum_{0\leqslant r\leqslant x} f_r(W)S^tSf_{x-r}(W)$.

(b) *For $k < c + 2$*

$$H_k = \begin{cases} 0 & k \text{ odd} \\ I & k \equiv 0 \bmod 4 \\ SS^t - I & k \equiv 2 \bmod 4. \end{cases}$$

(c) $$H_{c+2} = \begin{cases} (1+\lambda_c)SS^t - H_c & c \text{ even} \\ \lambda_cSS^t & c \text{ odd}. \end{cases}$$

We make the convention that $\psi_x = 0$ and $f_x(W) = 0$ if $x < 0$.

Proof. (a) Induction on k. The cases $k = 1$ and $k = 2$ are clear. So assume true for all j with $1 \leqslant j \leqslant k$, and suppose $k < c + 2$, then

$$S_{k+1} = SW_k - S_{k-1} = S\left(f_k(W) + \sum_{i\geqslant 0} \psi_{k-2-2i}\right) - S\left(\sum_{j\geqslant 0} f_{k-2-2j}(W)\right).$$

Let $x \leqslant k-2$, since $k < c+2$ we have $x < c$, and therefore $Sf_{x-y}(W)S^t = 0$ for all $y \geqslant 0$. This implies that almost all terms of $S\psi_x$ are zero, and

$$S\psi_x = SS^tSf_x(W) = 2Sf_x(W).$$

Substituting this gives

$$S_{k+1} = Sf_k(W) + 2S\left(\sum_{i\geqslant 0} f_{k-2-2i}(W)\right) - S\left(\sum_{j\geqslant 0} f_{k-2-2j}(W)\right)$$

which proves the claim.

Next, consider W_{k+1}, substituting the terms using the induction hypothesis we get

$$W_{k+1} = S^tS\left[\sum_{i\geqslant 0} f_{k-1-2i}(W)\right] + Wf_k(W) + \sum_{i\geqslant 0} W\psi_{k-2-2i} - f_{k-1}(W) - \sum_{i\geqslant 0}\psi_{k-3-2i}.$$

By the recursion, $Wf_k(W) - f_{k-1}(W) = f_{k+1}(W)$. Moreover, it also follows from the recursion that, for $x \geqslant 0$,

$$W\psi_x - \psi_{x-1} = \psi_{x+1} - S^tSf_{x+1}(W).$$

Substituting these, and noting that the terms $-S^tSf_{x+1}(W)$ cancel for all $x = k-2-2i$ gives the claim.

(b) and (c) The cases $k = 1$ and $k = 2$ are clear. For the inductive step, we have if $k < c+2$ that

$$H_{k+1} = S(S_k^t) - H_{k-1} = S\left(\sum_{i\geqslant 0} f_{k-1-2i}(W)\right)S^t - H_{k-1}.$$

With the assumption, $k-1-2i \leqslant c$ for all $i \geqslant 0$, and the only way to get this equal to c is for $k = c+1$ and $i = 0$. Hence

$$S\sum_{i\geqslant 0} f_{k-1-2i}(W)S^t = \begin{cases} Sf_c(W)S^t & k-1 = c \text{ odd} \\ Sf_c(W)S^t + 2SS^t & k-1 = c \text{ even} \\ SS^t & k-1 < c,\ k-1 \text{ even} \\ 0 & \text{else} \end{cases}$$

and recalling that $Sf_c(W)S^t = \lambda_c SS^t$, this gives the claim. $\square$

3.6. *Substituting type D*

In this paragraph we evaluate the Chebyshev polynomials on the adjacency matrices of Dynkin diagrams of type D. Let X be the matrix associated to type D_m, such that S and $W = A_n$ for $m = n+2$ are as described in Section 3.5. According to Proposition 3.9 and the remark on Θ_k preceding it, the parameters defined in Definition 3.17 are $c = 2n$ and $\lambda_c = 1$. Therefore we obtain the expressions of $f_k(D_m)$ for $k \leqslant 2n+2$ from Proposition 3.18.

Lemma 3.19. *For* $1 \leqslant k < 2n+1$, $f_k(D_m) \neq 0$ *and* $f_{2n+1}(D_m) = 0$.

Proof. First, we observe that

$$\sum_{i\geqslant 0} f_{2n-2i}(W) = 0, \quad \sum_{i\geqslant 0} f_{2n-1-2i}(W) = 0. \tag{3}$$

Namely, each of these is a sum of terms of the form $f_{n+t}(W) + f_{n-t}(W)$ for some $0 \leqslant t \leqslant n+1$, and by Lemma 3.3 this sum is equal to zero. Furthermore, if $1 \leqslant r < 2n-1$ then the first row of $\sum_{i\geqslant 0} f_{r-2i}(W)$ is non-zero: for example, the entries of the first row of $f_0(W)$ (or $f_1(W)$) do not cancel.

We start by showing that for $1 \leqslant k < 2n+1$, $f_k(D_m) \neq 0$. Suppose $1 \leqslant k < 2n+1$. Then by Proposition 3.18 we have

$$S_k = S\left[\sum_{i\geqslant 0} f_{k-1-2i}(W)\right].$$

This is the two-row matrix where both rows are equal to the first row of $\sum_{i\geqslant 0} f_{k-1-2i}(W)$. By the above this is non-zero except when $k = 2n$.

Thus for $k < 2n$ we have $S_k \neq 0$ and therefore $f_k(D_m) \neq 0$. If $k = 2n$, then k is even and by Proposition 3.18 $H_k \neq 0$, and so $f_{2n}(D_{2m}) \neq 0$.

Finally consider $f_{2n+1}(D_m)$. It follows from (3) that $S_{2n+1} = 0$, and Proposition 3.18 implies $H_{2n+1} = 0$. Recall that $f_{2n+1}(W) = 0$, and thus

$$W_{2n+1} = \sum_{i\geqslant 0} \psi_{2n-1-2i} = \sum f_r(W)SS^t f_s(W) \tag{4}$$

where the sum is over all $r, s \geqslant 0$ with $r+s \leqslant 2n-1$ and $r+s$ odd.

Given such r, s, define r' and s' by $r+r' = 2n$ and $s'+s = 2n$. Then for $0 \leqslant k \leqslant n+1$, $r = n-k$ implies $r' = n+k$ and similarly $s = n-k$ implies $s' = n+k$ or vice versa. This implies that $f_r(W) + f_{r'}(W) = 0$ and $f_s(W) + f_{s'}(W) = 0$.

It is clear that both $r+s'$ and $s+r'$ are odd. We now claim that precisely one of $r+s'$ and $r'+s$ is strictly less than $2n$.

Assume for a contradiction that $r'+s \geqslant 2n$ and $r+s' \geqslant 2n$. Then because both expressions are odd, they are both strictly larger than $2n$ and we have $r'+s > 2n = r'+r$ and thus $s > r$. But $r+s' > 2n = s+s'$ implies $r > s$, a contradiction.

We get a similar contradiction if we assume $r'+s < 2n$ and $r+s' < 2n$ and therefore exactly one of $r+s'$ and $r'+s$ is strictly less than $2n$.

Suppose now that $r'+s < 2n$. Then the terms in (3) where labels of the form r, r', s, s' occur are precisely

$$f_r(W)S^tSf_s(W) + f_{r'}(W)S^tSf_s(W).$$

But this expression is zero since $f_r(W) + f_{r'}(W) = 0$. $\square$

The following two lemmas give more precise information about particular entries of $f_k(D_m)$.

Lemma 3.20. *We have*

$$f_{2n}(D_m) = \begin{cases} I & n \text{ even} \\ \begin{pmatrix} I_2^0 & 0 \\ 0 & I_n \end{pmatrix} & n \text{ odd}. \end{cases}$$

Proof. Consider $W_{2n} = f_{2n}(W) + \sum_i \psi_{2n-2-2i}$. We know $f_{2n}(W) = I$ and so we must show that $\sum \psi_{2n-2-2i} = 0$. However this follows directly from an argument similar to the one in the previous Lemma. Similarly one shows that $S_{2n} = 0$. The result then follows from Proposition 3.18. $\square$

Lemma 3.21. *Assume $1 \leqslant k \leqslant 2n$. Then the last row of $f_k(D_m)$ is equal to*

$$\begin{array}{ll} \begin{pmatrix} 0 & 0 & \varepsilon_{n-k} \end{pmatrix} & 1 \leqslant k < n \\ \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \end{pmatrix} & k = n \\ \begin{pmatrix} 0 & 0 & \varepsilon_{k-n} \end{pmatrix} & n < k \leqslant 2n. \end{array}$$

Hence the last row of $f_k(D_m)$ reversed is

$$\begin{array}{ll} \begin{pmatrix}\varepsilon_{k+1} & 0 & 0\end{pmatrix} & 1 \leqslant k < n \\ \begin{pmatrix}0 & \ldots & 0 & 1 & 1\end{pmatrix} & k = n \\ \begin{pmatrix}\varepsilon_{n-(k-n)+1} & 0 & 0\end{pmatrix} & n < k \leqslant 2n. \end{array}$$

Proof. (1) We need the last column of S_k, the transpose of this gives the first two entries for the required last row.

Recall $S_k = S[\sum_i f_{k-1-2i}(W)]$. This is the 2-row matrix where each row is equal to the first row of $\sum_i f_{k-1-2i}(W)$. We need only the $1n$ entry of this sum.

The $1n$ entry of $f_x(W)$ is 1 for $x = n-1$ and -1 for $x = n+1$ and is zero for any other $x \leqslant 2n$.

Case 1. Assume n is odd. Then $f_x(W)_{1n} = 0$ for all odd $x \leqslant 2n$. Set $x = k-1-2i$, so for k even, the last column of S_k is zero.

Now let k be odd, and consider the $x = k-1-2i$. We have $f_x(W)_{1n} = 1$ for $x = n-1$ and $= -1$ for $x = n+1$ and is zero otherwise. It follows that the 1n entry of $\sum_i f_{k-1-2i}(W)$ in this case is equal to 1 if $k-1 = n-1$ and is zero otherwise, by cancelation.

This shows that the last column of S_k is zero unless $k = n$ and then it is of the form $\binom{1}{1}$.

Case 2. Assume n is even. Then for k odd (all x even), as before the last column of S_k is zero. Assume k is even, consider $x = k-1-2i$. We have $f_x(W)_{1n}$ as before. It follows that the $1n$ entry of $\sum_i f_{k-1-2i}(W)$ is equal to 1 if $k-1 = n-1$ and zero otherwise. Again, the last column of S_k is zero unless $k = n$ and then it is of the form $\binom{1}{1}$.

(2) Now we determine the last row of W_k, recall from Proposition 3.18 that $W_k = f_k(W) + \sum_{i\geqslant 0} \psi_{k-2-2i}$.

Assume first that $1 \leqslant k \leqslant n$. We claim that then the last row of $\sum_i \psi_{k-2-2i}$ is zero, hence the last row of W_k is equal to ε_{n-k} for $k < n$, and is zero for $k = n$.

Consider $(Sf_a(W))^t(Sf_b(W))$. If this has last row non-zero then we must have that the first row of $f_a(W)$ has non-zero $1n$ entry. This occurs only for $a = n-1$ or $a = n+1$ but here we have only $a + b \leqslant k-2 \leqslant n-2$. So this has last row equal to zero. This implies the claim.

Now consider $k = n + r$ where $1 \leqslant r \leqslant n$. Then $f_{n+r}(W) = -f_{n-r}(W)$ and this has last row equal to $-\varepsilon_r$. We claim that $\sum_i \psi_{n+r-2-2i} = 2\varepsilon_r$. (This will imply the statement.)

We use induction on r. Assume first that $r = 1$.

Then $\sum_i \psi_{n-1-2i} = \psi_{n-1} + \sum_{i>0} \psi_{n-1-2i}$. In the sum, the last row is zero (by the argument in the previous case). The last row of ψ_{n-1} has non-zero contribution only from $f_{n-1}(W)S^t$ and this is $2\varepsilon_1$.

For the inductive step, write

$$\sum_i \psi_{n+r-2-2i} = \psi_{n+r-2} + \sum_{i\geqslant 0} \psi_{n+r-4-2i}.$$

By the inductive hypothesis the sum is equal to $2\varepsilon_{r-2}$. Now consider ψ_{n+r-2}. This has only two terms with non-zero last row, and the last row of ψ_{n+r-2} is equal to the last row of

$$(Sf_{n-1}(W))^t Sf_{r-1} + (Sf_{n+1}(W))^t Sf_{r-3}.$$

This is equal to $2\varepsilon_r - 2\varepsilon_{r-2}$. In total we get the stated answer. $\square$

Using 4.3 with $d = 2n+1$, the previous three Lemmas imply that $f_{2m-3-k}(D_m) = f_{2m-3+k}(D_m)$ for $0 \leqslant k \leqslant 2m-3$; and the periodicity of $(f_k(D_m))_{k\geqslant 0}$ follows.

Corollary 3.22. *The family $(f_k(D_m))$ is periodic of period $\leqslant 2(2m-2)$.*

3.7. Substituting type $\widetilde{D}$

Let X be of type $\widetilde{D}_{m+1}$ of size $m+2=n+4$. That is, $X=\begin{pmatrix} 0 & S \\ S^t & V \end{pmatrix}$, where V is equal to D_{n+2} with reversed order (explicitly $V=I^0D_{n+2}I^0$), and where $S=\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$ of size $2\times m$. Recall from 3.17 the definition of the invariants c and λ_c.

Lemma 3.23. *For X as described above, we have $c=2n$ and $\lambda_c=1$.*

Proof. The parameter λ_c is the 11 entry of $f_c(V)$, which is equal to the nn entry of $f_c(D_m)$ when this is non-zero the first time. The nn entry of $f_k(D_m)$ is the nn entry of the matrix W_k occurring in the recursion for type D_m, with the notation of Section 3.6.

Let $k\leqslant 2n$. We know from Proposition 3.18 that $W_k=f_k(W)+\sum_{i\geqslant 0}\psi_{k-2-2i}$. We notice that $(f_k(W))_{nn}\neq 0$ only if $f_k(W)=\pm I$ and this occurs for the first time when $k=2n$.

Now consider the nn entry of ψ_x for $x\leqslant 2n-2$. For this we need the nn entry of $(Sf_r(W))^t(Sf_s(W)$ for $r+s=x\leqslant 2n-2$. This is equal to

$$2(f_r(W))_{1n}\cdot f_s(W)_{1n}. \qquad (*)$$

The $1n$ entry of any $f_t(W)$ for $t\leqslant 2n-2$ is only non-zero for $t=n-1$ or $n+1$. So the number $(*)$ is zero for $r+s<2n-2$. If $r+s=2n-2$ then it is only non-zero for $r=s=n-1$ and then it is equal to 1. $\square$

By Proposition 3.18 we now obtain the expressions for the matrices constituting $f_k(X)$ for $1\leqslant k\leqslant 2n+2$, and this is enough to prove linear growth. Recall from Definition 3.7 the definition of I_2^0.

Proposition 3.24. *We have*

$$f_{2n+2}(X)-f_{2n}(X)=\begin{cases} 2I & n \text{ odd} \\ 2\widetilde{I} & n \text{ even} \end{cases}$$

where

$$\widetilde{I}=\begin{pmatrix} I_2^0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_2^0 \end{pmatrix}.$$

Hence the sequence $(f_k(X))_k$ has linear growth.

Proof. By 3.18, using $c=2n$ and $\lambda_c=1$, we have $H_{2n+2}-H_{2n}=2SS^t-2H_{2n}=2I_2$ for n odd, and $H_{2n+2}-H_{2n}=2(SS^t-I_2)$ if n is even.

Now consider $S_{2n+2}-S_{2n}$. Here most of terms of the sum cancel and we obtain

$$S_{2n+2}-S_{2n}=Sf_{2n+1}(V)=S\cdot 0=0.$$

Finally, for $V_{2n+2}-V_{2n}$ most of the terms of the sum cancel as well and we are left with $V_{2n+2}-V_{2n}=f_{2n+2}(V)-f_{2n}(V)+\psi_{2n}$. Furthermore, we have

$$f_{2n+2}(V)-f_{2n}(V)=-2f_{2n}(V)=\begin{cases} -2I_{n+2} & n \text{ even} \\ -2U & n \text{ odd} \end{cases}$$

where U is the matrix $f_{2n}(D_m)$ reversed.

It remains to calculate ψ_{2n}. For that we note that

$$(Sf_0(V))^t(Sf_{2n}(V)) = 2E_{11} = (Sf_{2n}(V))^t(Sf_0(V))$$
$$(Sf_1(V))^t(Sf_{2n-1}(V)) = 2E_{22} = (Sf_{2n-1}(V))^t(Sf_1(V))$$
$$\dots\dots$$
$$(Sf_{n-1}(V))^t(Sf_{n+1}(V)) = 2E_{nn} = (Sf_{n+1}(V))^t(Sf_{n-1}(V))$$
$$(Sf_n(V))^t(Sf_n(V)) = 2(E_{n+1,n+1} + E_{n+1,n+2} + E_{n+2,n+1} + E_{n+2,n+2}).$$

Thus $\psi_{2n} = \begin{pmatrix} 4I_n & 0 \\ 0 & 2SS^t \end{pmatrix}$, and

$$V_{2n+2} - V_{2n} = \begin{cases} 2I_{n+2} & n \text{ odd} \\ \begin{pmatrix} 2I_n & 0 \\ 0 & 2(SS^t - I_2) \end{pmatrix} & n \text{ even} \end{cases}$$

The conclusion follows from Lemma 3.5 with $Z = 2I$ for n odd. Assume n is even, then applying Corollary 3.6 twice gives $f_{8n+8}(X) - f_{8n+6}(X) = 2I$, and linear growth follows from Lemma 3.5. □

3.8. *Substituting extended type $\widetilde{A}$*

Let now X be the matrix of type $\widetilde{A}_n$, that is X is an $n+1 \times n+1$ matrix of the form $X = \begin{pmatrix} 0 & S \\ S^t & W \end{pmatrix}$, with

$$W = A_n \text{ and } S = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

where S is a $1 \times n$ matrix. One checks that the invariants c and λ_c of X as defined in 3.17 are as follows.

Lemma 3.25. *Let $1 \leqslant c$ be minimal such that $Sf_c(W)S^t = \lambda_c SS^t \neq 0$. Then $c = n-1$, and $\lambda_c = 1$.*

Therefore, by Proposition 3.18 we get the matrices $f_k(X)$ for $1 \leqslant k \leqslant n+1$.

Proposition 3.26. *We have*

$$f_{n+1}(X) - f_{n-1}(X) = 2I.$$

Hence the sequence $(f_k(X))_k$ has linear growth.

Proof. We have $H_{n+1} - H_{n-1} = 2$. Furthermore, $S_{n+1} - S_{n-1} = Sf_n(W) = 0$. Consider $W_{n+1} - W_{n-1}$, this is equal to

$$f_{n+1}(W) - f_{n-1}(W) + \psi_{n-1}$$

(since most of the terms of the sums cancel).

We know from type A that $f_{n+1}(W) - f_{n-1}(W) = -2I^0$. A straightforward calculation shows that

$$\psi_{n-1} = 2(I + I^0).$$

Combining these gives the first statement. We get linear growth from Lemma 3.5. □

3.9. *Substituting type $\widetilde{\mathrm{DL}}$*

Let X be the adjacency matrix of type $\widetilde{DL}_m$ with m vertices, and $m = n+2$. With the notation of Section 3.5 X has the following blocks: $S = \left(\begin{smallmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{smallmatrix}\right)$, and $W = V$ where $V = I^0 L I^0$ is the adjacency matrix of type L_n reversed.

Recall that $f_{2n}(V) = 0$ and $f_k(V) \neq 0$ for $1 \leqslant k < 2n$. Furthermore $f_{2n-1}(V) = I$. The invariants c and λ_c of X, given by $\lambda_c SS^t = Sf_c(V)S^t \neq 0$, and $Sf_k(V)S^t = 0$ for $1 < k < c$ are as follows.

Lemma 3.27. *We have $c = 2n - 1$ and $\lambda_c = 1$.*

Proof. The parameter $Sf_k(V)S^t$ is the $(1, 1)$ entry of $f_k(V)$ which is equal to the (n, n) entry of $f_k(L_n)$. It follows from Section 3.3 that this entry is equal to 1 for $c = 2n-1$ and equal to 0 for $1 < k < 2n-1$. □

We can now apply Proposition 3.18:

Proposition 3.28. *We have*

$$f_{2n+1}(X) - f_{2n-1}(X) = Z$$

where

$$Z = \begin{pmatrix} SS^t & 0 \\ 0 & 2I_n \end{pmatrix}$$

and $XZ = ZX = 2X$. Hence the sequence $(f_k(X))_k$ has linear growth.

Proof. Using the formulae from Proposition 3.18 with $c = 2n - 1$, we have $H_{2n+1} - H_{2n-1} = Sf_{2n-1}(V)S^t = SS^t$. Next, consider $S_{2n+1} - S_{2n-1}$. Here most of the sum cancels and the only expression remaining is

$$Sf_{2n}(V) = S \cdot 0 = 0.$$

Consider now $V_{2n+1} - V_{2n-1}$. This is equal to

$$f_{2n+1}(V) - f_{2n-1}(V) + \psi_{2n-1}.$$

By Proposition 3.11(c) combined with Lemma 3.3, we have $f_{2n+1}(V) - f_{2n-1}(V) = -2I$. We now calculate ψ_{2n-1}, this is equal to

$$\psi_{2n-1} = 2\left[\sum_{r=0}^{n-1} (Sf_r)^t \cdot (Sf_{2n-1-r})\right].$$

Similarly to the proof of Proposition 9.2 we find that $(Sf_r)^t \cdot (Sf_{2n-1-r}) = 2E_{r+1,r+1}$, for $0 \leqslant r \leqslant n-1$. Hence $V_{2n+1} - V_{2n-1} = -2I + 4I = 2I$, as required. This proves that the difference $f_{2n+1}(X) - f_{2n-1}(X)$ is as stated, and one checks that $XZ = 2X$. Now by applying Corollary 3.6 first with $q' = 2q$ we obtain $f_{q'}(X) - f_{q'-2}(X) = Z'$ where $Z' = \begin{pmatrix} I_2^0 & 0 \\ 0 & 2I_n \end{pmatrix}$ and where $Z'X = XZ' = 2X$. Then we set $q'' = 2q' = 4q$ and by Corollary 3.6 we obtain $f_{q''}(X) - f_{q''-2}(X) = 2I$ and then by Lemma 3.5 linear growth follows. □

3.10. Substituting type E

Similar results can be shown for type E. However, all we need for the application to representation theory that we have in mind, are some simple facts that an easy calculation by hand can provide. Namely, we need to know when $f_k(E_i)$ is equal to zero for the first time when $i = 6, 7$ or 8 where E_i for $i = 6, 7, 8$ is the adjacency matrix of the Dynkin diagram of type E as given in Section 2.2. This happens exactly for $f_{11}(E_6)$, $f_{17}(E_7)$ and $f_{29}(E_8)$.

Furthermore we need the explicit expressions of the matrices preceding this first zero matrix. They are given by

$$f_{10}(E_6) = (\varepsilon_6, \varepsilon_5, \varepsilon_3, \varepsilon_4, \varepsilon_2, \varepsilon_1)^T,$$

$$f_{16}(E_7) = I_7,$$
$$f_{10}(E_8) = I_8,$$

where ε_i is as before, that is, it is the row vector that has a one in place i and zero elsewhere.

3.11. Summary

In the preceding sections we have calculated explicitly the Chebyshev polynomials evaluated on the adjacency matrices of the Dynkin diagrams of types A, D, E, of the extended Dynkin diagrams $\widetilde{A}$, $\widetilde{D}$, as well as on the diagrams of type L, $\widetilde{L}$ and $\widetilde{DL}$. In particular we have shown that the Dynkin diagrams and the type L diagram give rise to periodic families and that the extended Dynkin diagrams and the diagrams of types $\widetilde{L}$ and $\widetilde{DL}$, give rise to families that have linear growth.

Theorem 3.29. *Let $\mathcal{D}$ be the adjacency matrix of a Dynkin diagram of type A, D, E or a diagrams of type L. Then for $r \geqslant 1$, we have $f_k(\mathcal{D}) = 0$ if and only if $k = rh - 1$ where h denotes the Coxeter number of the associated diagram.*

Theorem 3.30. *Let $\mathcal{D}$ be the adjacency matrix of an extended Dynkin diagram or of a diagram of type $\widetilde{L}$ or $\widetilde{DL}$. Then for $m = rq + u$ where $-1 \leqslant u \leqslant q - 2$ and $2 \leqslant r$ we have the recurrence relation*

$$f_{rq+u}(\mathcal{D}) = rf_{q+u}(\mathcal{D}) - (r-1)f_u(\mathcal{D})$$

(a) if $\mathcal{D} = \widetilde{A}_n$ then $q = n + 1$;
(b) if $\mathcal{D} = \widetilde{D}_n$ then $q = 2n - 4$ for n even and $q = 8n - 16$ for n odd;
(c) if $\mathcal{D} = \widetilde{L}_n$ then $q = n$;
(d) if $\mathcal{D} = \widetilde{DL}_n$ then $q = 8n - 4$;
(e) if $\mathcal{D} = \widetilde{E}_6$ then $q = 12$, if $\mathcal{D} = \widetilde{E}_7$ then $q = 72$, and if $\mathcal{D} = \widetilde{E}_8$ then $q = 60$.

Remark. It is easily checked that for $\mathcal{D}$ one of the extended types $\widetilde{E}$ with q given as above we have $f_q(\mathcal{D}) - f_{q-2}(\mathcal{D}) = 2I$.

4. Application to representation theory

In this section we show how the previous results can be used to calculate the minimal projective resolutions of the simple modules of a class of symmetric algebras – namely those that are of radical cube zero and of tame or of finite representation type. Our method gives the indecomposable projective components in each degree of the projective resolutions through a description of the radical layers of the syzygies.

Let K be a field and let Λ be a finite dimensional K-algebra. Then Λ is symmetric if there exists a linear map $\nu : \Lambda \to k$ such that for all $a, b \in \Lambda$, $\nu(ab) = \nu(ba)$ such that $\mathrm{Ker}(\nu)$ contains no non-zero left or right ideal. Recall that the Jacobson radical $J(\Lambda)$ of the algebra is the smallest ideal of Λ such that the quotient is semisimple.

Let Λ be a finite dimensional symmetric K-algebra such that $J^3(\Lambda) = 0$. We assume that Λ is indecomposable, and that $J^2(\Lambda) \neq 0$. These algebras are classified in [2] according to the minimal projective resolution of non-projective finitely generated Λ-modules. Namely, if $S_1, S_2, \ldots, S_n$ are the simple Λ-modules, let $\mathcal{E}_n = (e_{ij})_{i,j=1,\ldots,n}$ where $e_{ij} = \dim\ \mathrm{Ext}^1_\Lambda(S_i, S_j)$, which is a symmetric matrix. Then if the largest eigenvalue λ of $\mathcal{E}_n$ is > 2, the dimensions in the minimal projective resolutions of the non-projective finite-dimensional Λ-modules are unbounded, and the algebra is of wild representation type. If $\lambda = 2$, the algebra is of tame representation type and the minimal projective resolutions either are bounded or grow linearly, and if $\lambda < 2$ they are bounded. In the latter two cases the algebras are classified by Dynkin diagrams A, D, E or the graph L for the finite representation type, and by the extended Dynkin diagrams $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{D}}$, $\widetilde{\mathrm{E}}$ or the graphs $\widetilde{\mathrm{L}}$ and $\widetilde{\mathrm{DL}}$ for the tame representation type.

To each of these diagrams $\mathcal{D}$ we associate the quiver $Q(\mathcal{D})$, which is obtained by replacing each edge of the diagram by a pair of arrows pointing in opposite directions. Let K be an algebraically closed field and let $KQ(\mathcal{D})$ be the path algebra of $Q(\mathcal{D})$. Then one has an ideal I of $KQ(\mathcal{D})$ such that the corresponding quotient algebra $\Lambda = KQ(\mathcal{D})/I$ is a symmetric algebra such that $J^3(\Lambda) = 0$.

We recall some properties of indecomposable Λ-modules as stated in [2]. The Loewy length of a Λ-module is at most 3. If P is a projective indecomposable Λ-module, then $\mathrm{rad}^2(P) = \mathrm{soc}(P) \cong \mathrm{hd}(P)$. If M is indecomposable but not simple or projective, then the Loewy length of M is equal to 2 and we denote by $d(M)$ the column vector $\begin{pmatrix} \alpha(M) \\ \beta(M) \end{pmatrix}$ where $\alpha(M) = \underline{\dim}\ \mathrm{hd}(M)$ and $\beta(M) = \underline{\dim}\ \mathrm{rad}(M)$ where $\underline{\dim} V$ denotes the dimension vector of the finite dimensional Λ-module V. This is the column vector of length n whose ith entry corresponds to the multiplicity of the simple module S_i as a composition factor in V. If S is a simple Λ-module then $\alpha(S) = \underline{\dim}\ S$ and $\beta(S) = 0$. And if $P(S)$ is a projective cover of S then the dimension vector of its three radical layers are described by the vector $\begin{pmatrix} \alpha(S) \\ \mathcal{E}_n\alpha(S) \\ \alpha(S) \end{pmatrix}$. More generally, the projective cover $P(M)$ of a non-projective indecomposable Λ-module M has in its radical layers the dimension vectors $\alpha(M)$, $\mathcal{E}_n\alpha(M)$ and $\alpha(M)$. Therefore if $\Omega(M)$ is non-simple it has in its radical layers the dimension vectors $\mathcal{E}_n\alpha(M) - \beta(M)$ and $\alpha(M)$. Furthermore, $d(\Omega(M)) = Bd(M)$ with $B = \begin{pmatrix} \mathcal{E}_n & -I_n \\ I_n & 0 \end{pmatrix}$ where I_n is the identity $n \times n$ matrix. Therefore for M an indecomposable non-projective Λ-module we get the recurrence relation

$$d(\Omega^k(M)) = B^k d(M)$$

if none of the $\Omega^j(M)$ for $j = 0, \dots, k$ are simple Λ-modules. Furthermore, we observe that the entries of the matrix B^k can be given in terms of Chebyshev polynomials.

Lemma 4.1. *Let M be an $n \times n$ matrix and let $B = \begin{pmatrix} M & -I_n \\ I_n & 0 \end{pmatrix}$. Then for $i \geqslant 0$,*

$$B^i = \begin{pmatrix} f_i(M) & -f_{i-1}(M) \\ f_{i-1}(M) & -f_{i-2}(M) \end{pmatrix}$$

where $f_i(x)$ is the Chebyshev polynomial of the second kind defined by the recurrence relation $f_i(x) = xf_{i-1}(x) - f_{i-2}(x)$ and with initial conditions $f_0(x) = 1$ and $f_1(x) = x$.

Proof. By definition of B we have $f_0(M) = I_n$ and $f_1(M) = M$ and we pose $f_{-1}(M) = 0$. Suppose now the result holds for i. Then

$$B^iB = \begin{pmatrix} f_i(M) & -f_{i-1}(M) \\ f_{i-1}(M) & -f_{i-2}(M) \end{pmatrix} \begin{pmatrix} M & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} f_{i+1}(M) & -f_i(M) \\ f_i(M) & -f_{i-1}(M) \end{pmatrix}. \qquad \square$$

If $\Omega(M)$ is simple then $\Omega(M) = soc(P(M))$ and $d(\Omega(M)) = (\alpha(M), 0)^T$, however $Bd(M) = (0, \alpha(M))^T$. Thus we know that $\Omega^k(M)$ is simple if the first n entries of the vector $B^kd(M)$ are zero and if there is only one non-zero entry in the $n+1$ to $2n$ components of $B^kd(M)$.

Let Λ be one of the algebras defined above. We say that Λ is of type A, D, E, L, $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{D}}$ etc. (or $\mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{L}_n, \widetilde{\mathrm{A}}_n, \widetilde{\mathrm{D}}_n$ etc. if we need to specify the number of vertices of the diagram) if the underlying diagram of the quiver of Λ is of that type.

The following two theorems are an application of the results of Section 3 . In the case of Theorem 4.2 this gives a proof of a result that can be deduced from Theorem 2.1 in [3].

Theorem 4.2. *The minimal projective resolutions of the simple Λ-modules are periodic*

(a) of period $2n$ if Λ is of type A_n,
(b) of period $2n$ if Λ is of type L_n,
(c) of period $2n-3$ if Λ is of type D_n for n even and of period $2(2n-3)$ for n odd,
(d) of period 22 if Λ is of type E_6, of period 17 if Λ is of type E_7 and of period 29 if Λ is of type E_8.

Definition 4.3. We call a projective resolution $\mathbf{R}^\bullet$ of a Λ-module M of *linear growth of factor p* if given the first p terms

$$R_{p-1} \to \cdots \to R_1 \to R_0 \to M \to 0$$

of $\mathbf{R}^\bullet$, all other terms of $\mathbf{R}^\bullet$ have the form

$$R_{kp+l} = N_l^{\oplus k} \oplus R_l \quad \text{for} \quad 0 \leqslant l \leqslant p-1$$

where N_l is a direct sum of components R_i for $0 \leqslant i \leqslant p$.

Let q be as given in Theorem 3.30.

Theorem 4.4. *Let Λ be of extended Dynkin type or of type $\widetilde{DL}$. Then the minimal projective resolutions of the simple Λ-modules are of linear growth of factor q.*

Theorem 4.2 follows directly from the calculation of the syzygies in the next proposition.

Proposition 4.5. *(a) Let Λ be of type A_n. Then for all simple Λ-modules S_i we have $\Omega^j(S_i)$ is not simple for $j < n$ and $\Omega^n(S_i) = S_{n-i+1}$.*

(b) Let Λ be of type L_n. Then for all simple Λ-modules S_i, we have $\Omega^j(S_i)$ is not simple for $j < 2n$ and $\Omega^{2n}(S_i) = S_i$.

(c) Let Λ be of type D_n. Then for all simple Λ-modules S_i, we have $\Omega^j(S_i)$ is not simple for $j < 2n-3$ and

$$\Omega^{2n-3}(S_i) = \begin{cases} S_2 & \text{if } i = 1 \text{ and } n \text{ is odd} \\ S_1 & \text{if } i = 2 \text{ and } n \text{ is odd} \\ S_i & \text{otherwise.} \end{cases}$$

(d) Let Λ be of type E_6 , E_7 or E_8. Then for all simple Λ-modules S_i, we have

$$\Omega^j(S_i) = \begin{cases} S_i & \text{if } n = 6 \text{ and } j = 11 \text{ and for } i = 3, 4 \\ S_{n-i+1} & \text{if } n = 6 \text{ and } j = 11 \text{ and for } i = 1, 2, 5, 6 \\ S_i & \text{if } n = 7 \text{ and } j = 17 \text{ and for all } i \\ S_i & \text{if } n = 8 \text{ and } j = 29 \text{ and for all } i. \end{cases}$$

Proof. The Proposition follows directly from Lemma 4.1 and the results of Section 3. □

Proof of Theorem 4.4. It is enough to establish the recurrence formula for the Chebyshev polynomials evaluated on the adjacency matrices in question, since by Lemma 4.1 it is then straightforward to determine the projective resolutions. Namely, if the algebra has n simple modules, then the indecomposable projectives constituting the rth term in a projective resolution of the simple S_i are exactly given by the first n entries of the ith row of the matrix B^r.

Let $\mathcal{D}$ be the adjacency matrix of the graph underlying Λ (recall that $\mathcal{D}$ is not of type $\widetilde{L}$). Then it follows from Theorem 3.30 in combination with Lemma 3.5 that given $f_0(\mathcal{D}), \ldots, f_q(\mathcal{D})$ we have for $q+1 \leqslant n \leqslant 2q-1$, where $n = q+l$ for $1 \leqslant l \leqslant q-1$, that $f_{q+l}(\mathcal{D}) = (f_{q-2-l}(\mathcal{D}) + f_l(\mathcal{D})) + f_l(\mathcal{D})$ since in this case $Z = 2I$. So if we calculate a minimal projective resolution of the simple S_i, then the components of $q+l$th term, for $1 \leqslant l \leqslant q-1$, are $R_{q+l} = N_l \oplus R_l$ where R_l is determined by the ith row of $f_l(\mathcal{D})$ and $N_l = R_{q-2-l} \oplus R_l$ is determined by the ith row of $f_{q-2-l}(\mathcal{D}) + f_l(\mathcal{D})$. For $n \geqslant 2q-1$ we have the following: Write $n = rq + l$ with $-1 \leqslant l \leqslant q-2$. Furthermore, note that here $Z = 2I = 2f_0(\mathcal{D})$. Then by Lemma 3.5 and the remark that follows we have, for $-1 \leqslant l \leqslant q-2$, that $R_{rq+l} = N_l^{\oplus r} \oplus R_l$ where $N_l = R_{q-2-l} \oplus R_l$. $\square$

Remark. Suppose Λ is of type $\widetilde{L}$. Then the projective resolution of the simple Λ-modules is almost of linear growth. More precisely in that case we have by Corollary 3.15 that $f_q(X) - f_{q-2}(X) = Z$ where $Z = 2I^0 = 2f_0(\mathcal{D})^0$. Given the first q terms in a projective resolution of a simple Λ-module S we then obtain by Lemma 3.8 and the remark that follows the nth term in the projective resolution of S, for $n \geqslant 2q-1$, in the following way: if we write $n = rq + l$ with $-1 \leqslant l \leqslant q-2$, then

$$N_{rq+l} = \begin{cases} (N_l^0)^{\oplus r} + R_l & r \text{ even} \\ N_l^{\oplus r} + R_l^0 & r \text{ odd} \end{cases}$$

where N_l is as defined in the proof above and the components in N_l^0 are given by the ith row of $I^0(f_{q-2-l}(\mathcal{D}) + f_l(\mathcal{D}))I^0$ and R_l^0 is given by the ith row of $I^0 f_l(\mathcal{D}) I^0$.

5. Evaluating Chebyshev polynomials on positive symmetric matrices

We conclude by a general statement on evaluating the Chebyshev polynomials $(f_k(x))_k$ on positive symmetric matrices.

Theorem 5.1. *Assume X is a symmetric matrix, with entries in $\mathbb{Z}_{\geqslant 0}$ and assume X is indecomposable. Then*

(a) $f_d(X) = 0$ for some $d \geqslant 1$ if and only if X is the adjacency matrix of a diagram of type A, D, E or L.
(b) The family $(f_k(X))_k$ grows linearly if and only if X is the adjacency matrix of a diagram of type $\widetilde{A}, \widetilde{D}, \widetilde{E}, \widetilde{L}$ or $\widetilde{DL}$.

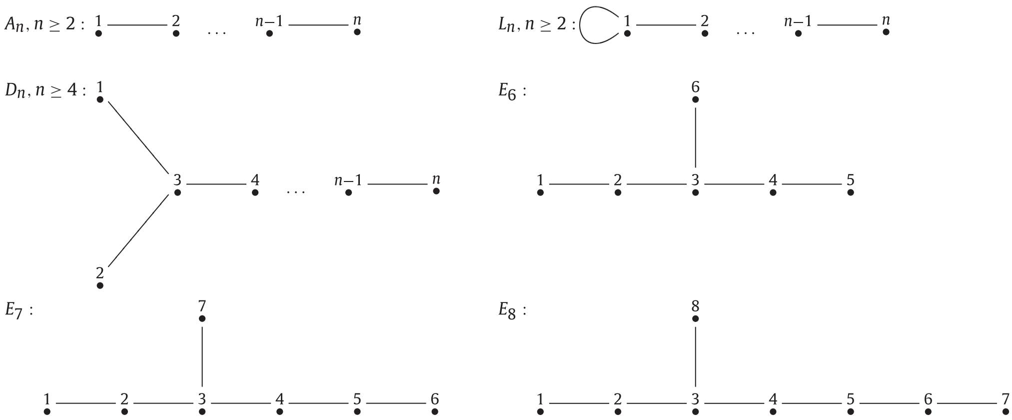
Proof. (a) If X is the adjacency matrix of a Dynkin diagram or of a graph of type L, then the result follows from Section 3. Suppose that $f_d(X) = 0$ for some n, then by Lemma 3.3 X annihilates the sequence of polynomials $f_k(x)$ periodically. Since X is a symmetric integer matrix with entries in $\mathbb{Z}_{\geqslant 0}$, then it is the adjacency matrix of a finite connected graph. To this graph we can associate a unique symmetric algebra Λ with radical cube zero, such that the sequence of polynomials $(f_k(X))_{k\geqslant 0}$ describes the growth of a minimal projective resolution of the simple modules, as explained at the beginning of Section 4. To construct this algebra, we replace each edge by a pair of arrows pointing in opposite directions. Then the algebra is the path algebra modulo the ideal generated by quadratic relations in the arrows, and there is a unique choice of such relations making the algebra symmetric with radical cube zero.

Then it follows from Section 4 that for such an algebra the minimal projective resolutions of the simple modules are periodic and thus following [2, 1.1] Λ is of type A, D, E or L.

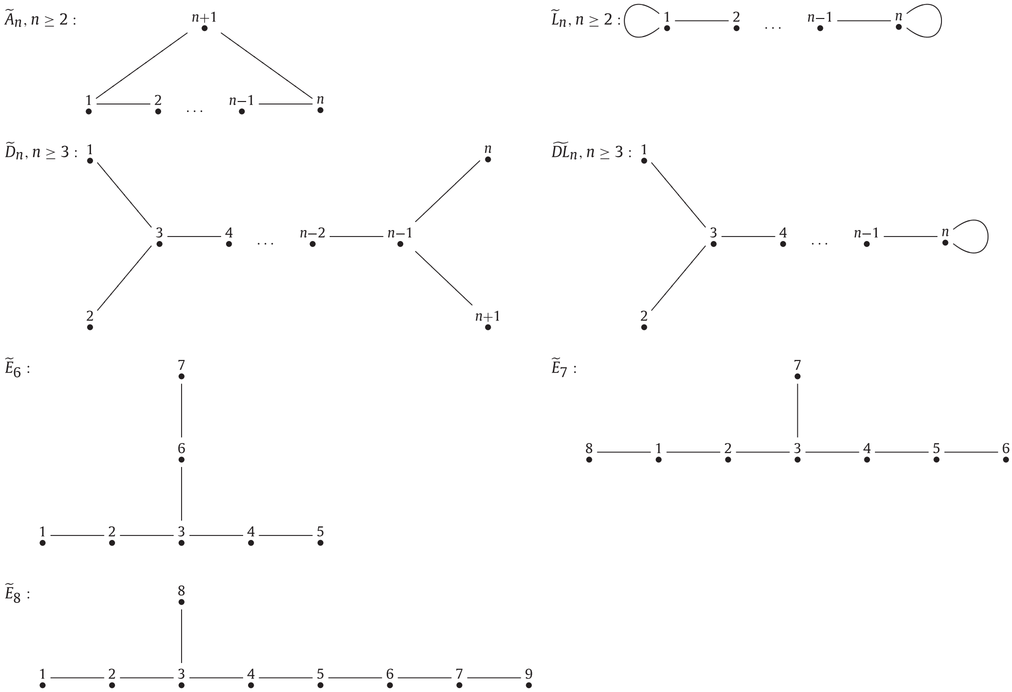
(b) If X is of extended Dynkin type or of types $\widetilde{L}$ or $\widetilde{DL}$ then by Section 3, $(f_k(X))_k$ grows linearly. Conversely, suppose that $(f_k(X))_k$ grows linearly. As in (a), X gives rise to a symmetric algebra Λ of radical cube zero. By Section 4 this implies that the minimal projective resolutions of the simple Λ-modules grow linearly. Following [2, 1.1] Λ is of type $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{D}}$, $\widetilde{\mathrm{E}}$, $\widetilde{\mathrm{L}}$, or $\widetilde{\mathrm{DL}}$. $\square$

A. Appendix

Dynkin diagrams with labels:



Extended Dynkin diagrams with labels:



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