The Minus Partial Order and the Shorted Matrix

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ABSTRACT

The minus partial order was defined by Hartwig [6], weakening the conditions of the star partial order of Drazin [5]. Several new properties of the minus partial order are established. The minus partial order is used to redefine the shorted matrix [15, 17] and to define the infimum $A \wedge B$ and the supremum $A \vee B$ of a pair A, B of matrices of the same order. The definition of the shorted matrix given here is similar to the Krein-Anderson-Trapp definition of the shorted positive operator.

1. INTRODUCTION AND PRELIMINARIES

Matrices are denoted by capital letters, column vectors by lowercase letters. For a matrix A, $\mathscr{M}(A)$, $\mathscr{N}(A)$, and A' denote the column span, null space, and transpose of A. $\mathscr{F}^{m\times n}$ represents the vector space of matrices of order $m \times n$ defined on a field $\mathscr{F}$. For a complex matrix A, A^* denotes its complex conjugate transpose. Two subspaces of a vector space are said to be virtually disjoint if they have only the null vector in common. $B = A \oplus (B - A)$ means $\text{Rank}\, B = \text{Rank}\, A + \text{Rank}(B - A)$ and is read as "A and $B - A$ are disjoint."

A^- denotes a generalized inverse (g-inverse) of A, that is, a solution G of the matrix equation $AGA = A$. The reflexive g-inverse A_r^- of A is a solution G of the pair of equations $AGA = A$, $GAG = G$. For a complex matrix A, a minimum norm g-inverse A_m^- is a matrix G that satisfies the pair of equations $AGA = A$, $(GA)^* = GA$. A least squares g-inverse A_l^- is similarly defined through the equations $AGA = A$, $(AG)^* = AG$. The Moore-Penrose inverse

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A^+ is the unique solution G of the simultaneous matrix equations

$$AGA = A, \quad GAG = G, \quad (AG)^* = AG, \quad (GA)^* = GA.$$

$\{A^-\}$ represents the class of all g-inverses of A. $\{A_m^-\}$ and $\{A_l^-\}$ are similarly interpreted. $\{A^- + B^-\}$ represents the class of all matrices which can be expressed as the sum of a g-inverse of A and a g-inverse of B.

Lemma 1.1 is well known. The "if" part is now folklore. The "only if" part was proved for the first time by Mitra [13].

LEMMA 1.1. *$\{B^-\} \subset \{A^-\}$ if and only if $B = A \oplus (B - A)$.*

Lemma 1.2, proved in Rao and Mitra [18, Theorem 2.4.2], is in fact a simple consequence of Lemma 1.1.

LEMMA 1.2. *The following two statements are equivalent:*

(a) $\{A^-\} = \{B^-\}$,

(b) $A = B$.

DEFINITION [18]. Matrices A and B of order $m \times n$ each are said to be parallel summable (p.s.) if $A(A+B)^-B$ is invariant under the choice of the generalized inverse $(A+B)^-$. If A and B are p.s., $A(A+B)^-B$ is called the parallel sum of A and B and denoted by the symbol $P(A, B)$. A null matrix is clearly p.s. with an arbitrary matrix of the same order.

The following lemma is proved in Rao and Mitra [18, p. 189].

LEMMA 1.3. *Nonnull matrices A and B are p.s. iff*

$$\mathcal{M}(A) \subset \mathcal{M}(A+B), \qquad \mathcal{M}(A') \subset \mathcal{M}(A'+B'),$$

or equivalently

$$\mathcal{M}(B) \subset \mathcal{M}(A+B), \qquad \mathcal{M}(B') \subset \mathcal{M}(A'+B').$$

The first part of Lemma 1.4 is easy to establish. The remaining part is proved in [1].

LEMMA 1.4. *If A and B are hermitian nonnegative definite of the same order, then A and B are p.s. Further, $P(A, B)$ is hermitian nonnegative definite, and so are $A - P(A, B)$ and $B - P(A, B)$.*

Theorem 1.1 lists some known properties of the parallel sum [1, 18].

THEOREM 1.1. *If A and B are p.s. matrices of order $m \times n$ each,*

(a) $P(A, B) = P(B, A)$;

(b) *A' and B' are p.s., and $P(A', B') = [P(A, B)]'$ (for complex matrices, A^*, B^* are also p.s. and $P(A^*, B^*) = [P(A, B)]^*$);*

(c) *for a matrix C of order $p \times m$ and rank m, CA and CB are p.s. and $P(CA, CB) = CP(A, B)$;*

(d) $\{[P(A, B)]^-\} = \{A^- + B^-\}$;

(e) $\mathcal{M}[P(A, B)] = \mathcal{M}(A) \cap \mathcal{M}(B)$;

(f) *$P[P(A, B), C] = P[A, P(B, C)]$ when all the parallel sum operations involved are permissible;*

(g) *If A and B are complex matrices and P_A and P_B are respectively the orthogonal projectors onto $\mathcal{M}(A)$ and $\mathcal{M}(B)$ under the norm induced by the inner product $(x, y) = y^*x$, then $2P(P_A, P_B)$ is the orthogonal projector onto $\mathcal{M}(A) \cap \mathcal{M}(B)$.*

In Section 2 we discuss some new properties of the star and minus partial orders. In particular it is shown that both these partial orders could be equivalently defined through the inclusions of classes of g-inverses of the matrices concerned. It was shown by Hartwig [6] that when the matrix A is star dominated by B, then $A^+ + (B - A)^+ = B^+$. We prove similar results for the minus order. In Section 3 the shorted matrix of Mitra and Puri [17] is redefined using the minus order. This definition is similar to the Krein-Anderson-Trapp definition [2, 11] of the shorted positive operator. It is shown that generalized inverses characterize the shorted matrix. When the matrix N is invertible, the duality theorem proved in this section shows how certain shorted versions of N and N^{-1} are connected to one another. In Section 4 the minus order is used to define the infimum $A \wedge B$ and the supremum $A \vee B$ of a pair of matrices A and B of the same order. Conditions governing the existence of the infimum and the supremum are studied in several theorems proved in that section.

2. PROPERTIES OF THE STAR AND MINUS PARTIAL ORDERS

In a star semigroup with a proper involution (denoted by $*$), Drazin [5] introduced the concept of a star partial order, which in the context of complex matrices of order $m \times n$ can be stated as follows: Let A and B be two such matrices. We write $A \overset{*}{<} B$ if

$$A^*A = A^*B, \qquad AA^* = BA^*. \tag{1}$$

It was shown that (1) is equivalent to the following:

$$A^+A = A^+B, \qquad AA^+ = BA^+. \tag{2}$$

Inspired by Drazin's work, Hartwig [6] introduced the plus partial order (later [9] renamed the minus partial order). We write $A \overset{-}{<} B$ if for some g-inverse A^- of A,

$$A^-A = A^-B, \qquad AA^- = BA^-. \tag{3}$$

Note that the minus partial order could be defined for matrices of the same order on any field, not necessarily real or complex.

The equivalence of (a) and (b) in Theorem 2.1 is due to Hartwig [6]. The other part is fairly straightforward. See for example the proof of Lemma 1.2 in [16].

THEOREM 2.1. *The following statements are equivalent:*

$$A \overset{-}{<} B, \tag{4a}$$

$$B = A \oplus (B - A), \tag{4b}$$

A and $B - A$ are p.s. and

$$P(A, B - A) = 0. \tag{4c}$$

Theorem 2.2 is a simple consequence of Theorem 2.1 and Lemma 1.1. We shall however give an independent proof.

THEOREM 2.2. *The condition* (3) *is equivalent to the condition*

$$\{B^-\} \subset \{A^-\}. \tag{5}$$

Proof. (3) $\Rightarrow$ $A = AA^-B = BA^-A$ $\Rightarrow$

$$AB^-A = AA^-BB^-BA^-A = AA^-BA^-A = AA^-A = A.$$

Conversely (5) $\Rightarrow$ $A(I - B^-B) = 0$ $\Rightarrow$ $A = AB^-A = AB^-B$ $\Rightarrow$ AB^-

$(B-A)=0 \Rightarrow B^{-}AB^{-}(B-A)=0$. Note that $B^{-}AB^{-} \in \{A^{-}\}$, and for this choice of A^{-} the first part of (3) holds. The second part of (3) is similarly established. ■

Though the conditions (3) and (5) are equivalent, (5) appears to be a more natural and transparent representation of the partial ordering concerned.

REMARK 1. In a regular ring R it was shown by Hartwig and Luh [8, Theorem 2.2(xiii)] that $a \overset{-}{<} b$ is equivalent to

$$a \in bRb, \tag{6a}$$

$$\{b^{-}\} \subset \{a^{-}\}. \tag{6b}$$

For matrices however (6a) is implied by (6b).

The following theorem on the star order corresponds to Theorem 2.2 for the minus order.

THEOREM 2.3. *$A \overset{*}{<} B$ if and only if*

$$\{B_m^{-}\} \subset \{A_m^{-}\}, \qquad \{B_l^{-}\} \subset \{A_l^{-}\}. \tag{7}$$

Proof. "Only if" part: Clearly $A \overset{*}{<} B \Rightarrow A \overset{-}{<} B \Rightarrow \{B^{-}\} \subset \{A^{-}\}$ by Theorem 2.2. Further, Equation (1) defining the star order implies that $\mathscr{M}(A)$ and $\mathscr{M}(B-A)$ are mutually orthogonal under the usual inner product $(x,y)=y^{*}x$, and so are $\mathscr{M}(A^{*})$ and $\mathscr{M}[(B-A)^{*}]$. Consider now B_m^{-}, an arbitrary minimum norm g-inverse of B. Then $B_m^{-}B=B^{+}B$ is the orthogonal projector onto $\mathscr{M}(B^{*})$, and

$$B_m^{-}B=B_m^{-}A+B_m^{-}(B-A)=A^{+}A+(B-A)^{+}(B-A). \tag{8}$$

Since $\mathscr{M}(A^{*})$ and $\mathscr{M}[(B-A)^{*}]$ are virtually disjoint (that is, have only the null vector in common),

$$(8) \quad \Rightarrow \quad B_m^{-}A=A^{+}A=(B_m^{-}A)^{*}$$

$$\Rightarrow \quad B_m^{-} \in \{A_m^{-}\}.$$

Similarly, if B_l^{-} is an arbitrary least squares inverse of B, then $BB_l^{-}=BB^{+}$ is the orthogonal projector onto $\mathcal{M}(B)$ and one concludes in a similar manner that $B_l^- \in \{A_l^-\}$.

"if" part: To avoid triviality, let us assume that A is nonnull.

Assume now that $\{B_m^-\} \subset \{A_m^-\}$ and $\{B_l^-\} \subset \{A_l^-\}$. Clearly $B^+ \in \{B_m^-\} \subset \{A_m^-\}$. Hence

$$B^+ + V(I - BB^+) \in \{B_m^-\} \subset \{A_m^-\},$$

where V is a complex matrix of appropriate order, otherwise arbitrary. $A\{B^+ + V(I - BB^+)\}A = A$ for arbitrary $V \Rightarrow AV(I - BB^+)A = 0$ for arbitrary $V \Rightarrow (I - BB^+)A = 0$. Similarly $\{B_l^-\} \subset \{A_l^-\} \Rightarrow B^+ + (I - B^+B)U \in \{B_l^-\} \subset \{A_l^-\} \Rightarrow A(I - B^+B) = 0$. Hence for arbitrary U and V, $B^+ + (I - B^+B)U + V(I - BB^+) \in \{A^-\} \Rightarrow \{B^-\} \subset \{A^-\} \Rightarrow A \overset{-}{<} B$ by Theorem 2.2. From Theorem 2.1 it is seen that A and $B - A$ are parallel summable and the parallel sum $P(A, B - A) = 0$. Now

$$\begin{aligned} P(A, B - A) = P(B - A, A) = (B - A)B^-A \\ &= (B - A)B_m^-A = (B - A)A^*(B_m^-)^* = 0 \\ &\Rightarrow \quad (B - A)A^*(B_m^-)^*A^* = (B - A)A^* = 0. \end{aligned}$$

Similarly

$$\begin{aligned} P(A, B - A) &= AB^-(B - A) = AB_l^-(B - A) = (B_l^-)^*A^*(B - A) = 0 \\ &\Rightarrow \quad A^*(B_l^-)^*A^*(B - A) = A^*(B - A) = 0. \end{aligned}$$

Thus the matrices A and B are seen to satisfy Equation (1), and we have

$$A \overset{*}{<} B. \qquad ∎$$

We refer to the following theorem due to Hartwig [6].

THEOREM 2.4. *If* $A \overset{*}{<} B$ *then*

$$B^+ = A^+ + (B - A)^+. \tag{9}$$

Conversely (9) *and* (4*b*) $\Rightarrow$ $A \overset{*}{<} B$.

Similar to Theorem 2.4, we have Theorem 2.5.

THEOREM 2.5. *If* $A \bar{<} B$ *then*

$$\{B^-\} \subset \{A^- + (B-A)^-\}. \tag{10}$$

Proof. From Theorem 2.1 it is seen that $A \bar{<} B \Rightarrow A$ and $B - A$ are parallel summable and the parallel sum $P(A, B-A) = 0$. Also, since $\{[P(A, B-A)]^-\} = \{A^- + (B-A)^-\}$ [Theorem 1.1(d)] and any arbitrary matrix of order $n \times m$ is a g-inverse of the null matrix of order $m \times n$, (10) follows as a trivial consequence. ■

The claim made in Theorem 2.6 is more nontrivial.

THEOREM 2.6. *If* $A \bar{<} B$ *then*

$$\{B_r^-\} \subset \{A_r^- + (B-A)_r^-\}. \tag{11}$$

Proof. Let $G \in \{B_r^-\}$. Then

$$G = GBG = GAG + G(B-A)G.$$

By Theorem 2.2, $A \bar{<} B \Rightarrow G \in \{A^-\}$. Hence $GAG \in \{A_r^-\}$. Similarly $G(B-A)G \in \{(B-A)_r^-\}$. ■

REMARK 2. The following counterexample shows (11) alone does not imply $A \bar{<} B$. Consider

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix},$$

and check that B, A, and $B - A$ are involutions.

DEFINITION 1. A pair of matrices A and C are said to satisfy condition α if A and C are parallel summable and so are at least one of the following pairs: (1) A and $C_0 = C - P(A, C)$, (2) C and $A_0 = A - P(A, C)$.

THEOREM 2.7. *If A and $B-A$ satisfy condition α and*

$$B^{-}=A^{-}+(B-A)^{-} \tag{12}$$

for some choice of the g inverses involved, then

$$A \bar{<} B.$$

Proof. Since A and $B-A$ are parallel summable, (12) $\Rightarrow$

$$\{B^{-}\}\subset\{A^{-}+(B-A)^{-}\}=\{[P(A,B-A)]^{-}\}.$$

Hence by Lemma 1.1, $P(A,B-A)$ and $B-P(A,B-A)$ are disjoint matrices. However, if A and $(B-A)_0=B-A-P(A,B-A)$ are parallel summable, then

$$\mathcal{M}(A)\subset\mathcal{M}[B-P(A,B-A)].$$

Since by Theorem 1.1(e)

$$\mathcal{M}[P(A,B-A)]=\mathcal{M}(A)\cap\mathcal{M}(B-A),$$

this implies

$$\mathcal{M}[P(A,B-A)]\cap\mathcal{M}[B-P(A,B-A)]=\mathcal{M}[P(A,B-A)]=\{0\}$$

$$\Rightarrow\quad P(A,B-A)=0\quad\Rightarrow\quad A\bar{<}B.$$

The argument is similar for the case where $B-A$ and $A-P(A,B-A)$ are parallel summable. ∎

From Lemma 1.4 it is clear that a pair of hermitian nonnegative definite matrices of the same order satisfy condition (α). We have thus the following corollary to Theorem 2.7.

COROLLARY 2.1. *When A and $B-A$ are hermitian nonnegative definite,* (11) $\Rightarrow$ $A\bar{<}B$.

DEFINITION 2. A pair of matrices A and C satisfy condition β if for every reflexive g-inverse $(A+C)_r^-$ of $A+C$ there exist reflexive g-inverses A_r^- and C_r^- of A and C respectively such that

$$A_r^- \overset{-}{<} (A+C)_r^-, \qquad C_r^- \overset{-}{<} (A+C)_r^-. \tag{13}$$

Since the rank of a reflexive g-inverse is the same as the rank of the original matrix, it is seen from Theorem 2.6 that

$$A \overset{-}{<} B \quad \Rightarrow \quad A \text{ and } B-A \text{ satisfy condition } \beta. \tag{14}$$

The following example shows that the converse of (14) is not true in general, but note that condition β does imply parallel summability (Theorem 2.8).

EXAMPLE. Consider

$$B=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \qquad A=\begin{pmatrix}1&0&-1\\0&1&0\\1&0&-1\end{pmatrix}$$

and the reflexive g-inverses

$$A_r^- = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \qquad (B-A)_r^- = \begin{pmatrix}2&1&-1\\-1&0&1\\1&1&0\end{pmatrix},$$

both of which are idempotent matrices and hence dominated by the identity matrix under the minus partial order.

THEOREM 2.8. *If A and $B-A$ satisfy condition β, then A and $B-A$ are parallel summable.*

Proof. If A and $B-A$ are not parallel summable, by Lemma 1.3 at least one of the inclusion statements

$$\mathcal{M}(A) \subset \mathcal{M}(B), \qquad \mathcal{M}(A') \subset \mathcal{M}(B')$$

is not true. Suppose $\mathcal{M}(A) \not\subset \mathcal{M}(B)$.

Then there exists a m-tuple x such that

$$x \in \mathscr{M}(A) \quad \text{but} \quad x \notin \mathscr{M}(B).$$

Let the columns of the matrix B_1 form a basis of $\mathscr{M}(B)$, and $(K : y)'$ be a left inverse of $(B_1 : x)$. Clearly

$$K'B_1 = I, \qquad K'x = 0. \tag{15}$$

Choose $W = (K'B)^- K' \in \{B_r^-\}$. If there exists a reflexive g-inverse A_r of A such that

$$A_r^- \bar{<} W,$$

such a g-inverse clearly satisfies the condition

$$\mathscr{M}(A_r^-) \subset \mathscr{M}(W), \qquad \mathscr{M}([A_r^-]') \subset \mathscr{M}(W'),$$

which implies $A_r^- = WJW$ for some matrix J. We have $A = AA_r^- AA_r^- A = AWJWAWJWA$, from which we conclude

$$\text{Rank}(WAW) = \text{Rank}\, A.$$

(See Theorem 2.1 of Mitra [12] in this connection.) This is impossible on account of (15). One arrives at a similar contradiction if $\mathscr{M}(A') \not\subset \mathscr{M}(B')$. ■

Note that if $A \bar{<} B$, it does not follow that for each reflexive g-inverse A_r^- of A there exists a reflexive g-inverse $(B - A)_r^-$ of $B - A$ such that

$$A_r^- + (B - A)_r^- \in \{B_r^-\}. \tag{16}$$

To convince oneself about this one only has to consider the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B - A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the choice

$$A_r^- = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The following theorem gives necessary and sufficient conditions on the reflexive g-inverse of A which ensure that the condition (16) is satisfied.

THEOREM 2.9. *Let* $A \bar{<} B$ *and* X *be a reflexive g-inverse of* A. *For a reflexive g-inverse* $(B-A)_r^-$ *of* $B-A$ *to exist such that* (16) *is true, it is necessary and sufficient that*

$$(B-A)X(B-A)=0, \tag{17a}$$

$$X(B-A)X=0. \tag{17b}$$

Proof. The necessity part is straightforward. For the sufficiency part note that since $A \bar{<} B$, one can without any loss of generality write

$$B=L_1'L_2, \qquad A=L_1'\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}L_2,$$

where L_1 and L_2 are matrices of full row rank. Since $X \in \{A_r^-\}$, this can be expressed as

$$X=R_2ER_1',$$

where R_1 and R_2 are right inverses of L_1 and L_2 respectively and E is a reflexive g-inverse of

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

E is therefore of the form

$$\begin{pmatrix} I & T \\ W & WT \end{pmatrix}.$$

(17a) and (17b) imply respectively $WT=0$ and $TW=0$. Observe that

$$Y=R_2\begin{pmatrix} 0 & -T \\ -W & I \end{pmatrix}R_1' \in \{(B-A)_r^-\} \tag{18}$$

and $X+Y=R_2R_1' \in \{B_r^-\}$.

3. THE SHORTED MATRIX

Shorted matrices were introduced by Mitra and Puri [17], extending Krein's [11] and Anderson and Trapp's [2] definition of a shorted positive operator. We shall give here a definition based on the minus partial order, which resembles the definition of the shorted positive operator. No attempt will be made to prove Theorems 3.1–3.4 given below. Their proofs are minor modifications of those of similar theorems proved in [14], [15], and [17] with the condition (20) replaced by an equivalent rank additivity condition. Mitra and Puri [17] also show how the shorted matrix is related to the concept of generalized Schur complement due to Ando [3]. For a recent survey paper on the Schur complement and the shorted matrix, the reader is referred to Carlson [4].

Let $N \in \mathscr{F}^{m\times n}$, $U \in \mathscr{F}^{m\times p}$, $V \in \mathscr{F}^{q\times n}$. Write $\mathscr{S} = \mathscr{M}(U)$, $\mathscr{T} = \mathscr{M}(V')$, and $\mathscr{C}$ for the class of matrices

$$\mathscr{C} = \left\{C : C \in \mathscr{F}^{m\times n}, C \overset{-}{<} N, \mathscr{M}(C) \subset \mathscr{S}, \mathscr{M}(C') \subset \mathscr{T}\right\}. \qquad (19)$$

Definition. If $\mathscr{C}$ has a unique maximal element, this maximal element is called the shorted matrix N relative to $\mathscr{S}$ and $\mathscr{T}$ and denoted by the symbol $S(N|\mathscr{S},\mathscr{T})$.

Let F denote the matrix

$$F = \begin{pmatrix} N & U \\ V & 0 \end{pmatrix}.$$

Theorem 3.1. *If*

$$\begin{pmatrix} N & U \\ 0 & 0 \end{pmatrix} \overset{-}{<} F, \qquad \begin{pmatrix} N & 0 \\ V & 0 \end{pmatrix} \overset{-}{<} F, \qquad (20)$$

then the class $\mathscr{C}$ has a unique maximal element under the minus partial order.

Theorem 3.2. *Unless $\mathscr{C}$ consists exclusively of the null matrix, the condition (20) is also necessary for the class $\mathscr{C}$ to have a unique maximal element.*

THEOREM 3.3. *Let the matrices N, U, V satisfy the condition* (20) *and*

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \in \{F^-\}, \tag{21}$$

where $C_1 \in \mathscr{F}^{n\times m}$. Then

$$UC_3U = U, \qquad VC_2V = V, \tag{22}$$

$$UC_3N = NC_2V = UC_4V. \tag{23}$$

The common matrix in (23) *is invariant under the choice of the g-inverse of F and is the shorted matrix $S(N|\mathscr{S}, \mathscr{T})$.*

THEOREM 3.4. *When $S(N|\mathscr{S}, \mathscr{T})$ is non-null*

(a) $\mathscr{M}[S(N|\mathscr{S}, \mathscr{T})] = \mathscr{M}(N) \cap \mathscr{S}$, $\mathscr{M}([S(N|\mathscr{S}, \mathscr{T})]') = \mathscr{M}(N') \cap \mathscr{T}$,
(b) $\{[S(N|\mathscr{S}, \mathscr{T})]^-\} = \{N^- + X\}$,

where X is an arbitrary solution of the matrix equation

$$VXU = 0. \tag{24}$$

The following theorem reminds one of the duality relationship between the minimum norm and least squares g-inverses and has been motivated by the same [18, Theorem 3.2.4].

THEOREM 3.5 (The duality theorem). *Let $\mathscr{S}_0$ stand for $\mathscr{N}(V)$, the nullspace of V, and $\mathscr{T}_0$ for $\mathscr{N}(U')$. Let $m = n$, N be invertible, and N, U, V satisfy* (20). *Then $S(N^{-1}|\mathscr{S}_0, \mathscr{T}_0)$ exists and is given by*

$$N^{-1} - N^{-1}UC_3 = N^{-1} - C_2VN^{-1}. \tag{25}$$

Proof. The equality in (25) follows from (23). Check that

$$V(N^{-1} - N^{-1}UC_3) = V(N^{-1} - C_2VN^{-1}) = VN^{-1} - VN^{-1} = 0, \tag{26}$$

$$(N^{-1} - N^{-1}UC_3)U = N^{-1}U - N^{-1}U = 0. \tag{27}$$

We now show that $\mathscr{M}(N^{-1}UC_3)$ is virtually disjoint with $\mathscr{S}_0$, that is, they have only the null vector in common. If not, let $x = N^{-1}UC_3y \in \mathcal{M}(N^{-1}UC_3) \cap \mathcal{S}_0$. Then $Vx = VN^{-1}UC_3y = VC_2VN^{-1}y = VN^{-1}y = 0$. Hence $x = N^{-1}UC_3y = C_2VN^{-1}y = 0$.

That $\mathcal{M}[(N^{-1}UC_3)'] = \mathcal{M}[(C_2VN^{-1})']$ is virtually disjoint with $\mathcal{T}_0$ is similarly established. Hence

$$N^{-1} = (N^{-1} - N^{-1}UC_3) \oplus N^{-1}UC_3. \tag{28}$$

Let $C \in \mathcal{F}^{n\times n}$, $C \bar{<} N^{-1}$, $\mathcal{M}(C) \subset \mathcal{S}_0$, $\mathcal{M}(C') \subset \mathcal{T}_0$; then

$$N^{-1} - C = N^{-1}UC_3 \oplus (N^{-1} - N^{-1}UC_3 - C).$$

However, $C \bar{<} N^{-1}$ implies

$$\begin{aligned} N^{-1} &= C \oplus (N^{-1} - C) \\ &= C \oplus [N^{-1}UC_3 \oplus (N^{-1} - N^{-1}UC_3 - C)], \end{aligned} \tag{29}$$

which in turn implies

$$N^{-1} - N^{-1}UC_3 = C \oplus (N^{-1} - N^{-1}UC_3 - C).$$

Hence

$$C \bar{<} N^{-1} - N^{-1}UC_3. \qquad \blacksquare$$

THEOREM 3.6. *Let $N \in \mathcal{F}^{m\times n}$, $U \in \mathcal{F}^{m\times p}$, $V \in \mathcal{F}^{q\times n}$, and let there exist a matrix C_0 such that*

$$\{C_0^-\} = \{N^- + X\}, \tag{30}$$

where N^- is an arbitrary g-inverse of N and X an arbitrary solution of the matrix equation

$$VXU = 0. \tag{31}$$

Then the shorted matrix $S(N|\mathcal{S}, \mathcal{T})$ exists, where $\mathcal{S} = \mathcal{M}(U)$, $\mathcal{T} = \mathcal{M}(V')$, and

$$C_0 = S(N|\mathcal{S}, \mathcal{T}).$$

[Note that the crucial part of the proof is to establish the existence of the shorted matrix. Because of the one to one correspondence between a matrix and its class of generalized inverses (Lemma 1.2) and Theorem 3.4, the rest of the theorem then follows as a simple consequence.]

Proof. Let C_1 be a matrix of full column rank such that

$$\mathscr{M}(C_1)=\mathscr{M}(N)\cap\mathscr{M}(U),$$

and D_1' a matrix of full column rank such that

$$\mathscr{M}(D_1')=\mathscr{M}(N')\cap\mathscr{M}(V').$$

Let $(N^-)_0$ be a particular choice of N^-. A typical member of $\{N^- + X\}$ is therefore

$$(N^-)_0+Y+X=(N^-)_0+Z,$$

where X, Y, and Z are arbitrary solutions of

$$VXU=0,\quad NYN=0,\quad D_1ZC_1=0,\qquad \text{respectively,} \tag{32}$$

while if (30) holds, a typical member of $\{C_0^-\}$ is

$$(N^-)_0+Z_0,$$

where Z_0 is an arbitrary solution of

$$C_0Z_0C_0=0. \tag{33}$$

Then (30), (32), and (33) $\Leftrightarrow$

$$\mathscr{M}(C_0)=\mathscr{M}(C_1)=\mathscr{M}(N)\cap\mathscr{M}(U),$$
$$\mathscr{M}(C_0')=\mathscr{M}(D_1')=\mathscr{M}(N')\cap\mathscr{M}(V'),$$
$$\dim[\mathscr{M}(N)\cap\mathscr{M}(U)]=\dim[\mathscr{M}(N')\cap\mathscr{M}(V')]=r\quad(\text{say}).$$

Let N be of rank s, and C_2 and D_2' be matrices of $s-r$ columns each such that

$$\mathscr{M}(N)=\mathscr{M}(C_1:C_2),\qquad \mathscr{M}(N')=\mathscr{M}(D_1':D_2').$$

Then

$$N=(C_1:C_2)F_n\begin{pmatrix}D_1\\D_2\end{pmatrix},$$

where

$$F_n=\begin{pmatrix}U_{11}&U_{12}\\U_{21}&U_{22}\end{pmatrix}$$

is an invertible matrix of order $s\times s$. Let

$$F_n^{-1}=\begin{pmatrix}U^{11}&U^{12}\\U^{21}&U^{22}\end{pmatrix}.$$

Then

$$\begin{pmatrix}D_1\\D_2\end{pmatrix}_R^{-1}F_n^{-1}(C_1:C_2)_L^{-1}\in\{N^-\},$$

and $C_0N^-C_0=C_0\Rightarrow$

$$\begin{aligned}D_1\begin{pmatrix}D_1\\D_2\end{pmatrix}_R^{-1}F_n^{-1}(C_1:C_2)_L^{-1}C_1&=(I:0)F_n^{-1}\begin{pmatrix}I\\0\end{pmatrix}\\&=U^{11}\end{aligned}$$

is nonsingular.

We show that this implies the existence of the shorted matrix. If U^{11} is nonsingular, so is $U^{22}-U^{21}(U^{11})^{-1}U^{12}$. Then, noting that

$$\begin{pmatrix}U_{11}&U_{12}\\U_{21}&U_{22}\end{pmatrix}=\begin{pmatrix}U^{11}&U^{12}\\U^{21}&U^{22}\end{pmatrix}^{-1},$$

we have

$$U_{11}=(U^{11})^{-1}+(U^{11})^{-1}U^{12}\left(U^{22}-U^{21}(U^{11})^{-1}U^{12}\right)^{-1}U^{21}(U^{11})^{-1},$$

$$U_{12}=-(U^{11})^{-1}U^{12}\left(U^{22}-U^{21}(U^{11})^{-1}U^{12}\right)^{-1},$$

$$U_{21}=-\left(U^{22}-U^{21}(U^{11})^{-1}U^{12}\right)^{-1}U^{21}(U^{11})^{-1},$$

$$U_{22}=\left(U^{22}-U^{21}(U^{11})^{-1}U^{12}\right)^{-1}.$$

Note that U_{22} is nonsingular. We now show that $N_0 = C_1(U_{11} - U_{12}U_{22}^{-1}U_{21})D_1$ is in fact the shorted matrix $S(N|\mathscr{S}, \mathscr{T})$.

Write $N_1 = (C_1U_{12} + C_2U_{22})U_{22}^{-1}(U_{21}D_1 + U_{22}D_2)$, and observe that

$$N = N_0 + N_1$$

and that since U_{22} is nonsingular, $\mathscr{M}(N_1)$ is virtually disjoint with $\mathscr{M}(C_1) = \mathscr{M}(N) \cap \mathscr{S}$, and $\mathscr{M}(N_1')$ with $\mathscr{M}(D_1') = \mathscr{M}(N') \cap \mathscr{T}$. Let $C \in \mathscr{F}^{m \times n}$, $C \bar{<} N$, $\mathscr{M}(C) \subset \mathscr{S}$, $\mathscr{M}(C') \subset \mathscr{T}$. Then clearly

$$\mathscr{M}(C) \subset \mathscr{M}(N) \cap \mathscr{S}, \qquad \mathscr{M}(C') \subset \mathscr{M}(N') \cap \mathscr{T}.$$

Also $C \bar{<} N \quad \Rightarrow$

$$\begin{aligned} N &= C \oplus (N - C) \\ &= C \oplus [(N_0 - C) \oplus N_1]. \end{aligned}$$

Hence $N_0 = C \oplus (N_0 - C) \;\Rightarrow\; C \bar{<} N_0$.

The matrix N_0 thus satisfies the definition of the shorted matrix $S(N|\mathscr{S}, \mathscr{T})$. ■

4. SUPREMUM AND INFIMUM OF A PAIR OF MATRICES

Let $A, B \in \mathscr{F}^{m \times n}$. Consider the classes of matrices in $\mathscr{F}^{m \times n}$ defined as follows:

$$\overline{\mathscr{C}} = \left\{ C : A \bar{<} C,\ B \bar{<} C \right\}, \tag{34}$$

$$\underline{\mathscr{C}} = \left\{ C : C \bar{<} A,\ C \bar{<} B \right\}. \tag{35}$$

If $\overline{\mathscr{C}}$ is nonempty, the supremum of matrices A and B, denoted by $A \vee B$, is the unique minimal element in $\overline{\mathscr{C}}$ if one exists. The infimum of A and B is similarly the unique maximal element in $\underline{\mathscr{C}}$ if one exists, and is denoted by $A \wedge B$. We note that the null matrix is a member of $\underline{\mathscr{C}}$. Hence $\underline{\mathscr{C}}$ is always nonempty, but $\overline{\mathscr{C}}$ is not necessarily so.

Let $\mathscr{S}_a$, $\mathscr{T}_a$, $\mathscr{S}_b$, and $\mathscr{T}_b$ be defined as follows:

$$\mathscr{S}_a = \mathscr{M}(A), \quad \mathscr{S}_b = \mathscr{M}(B), \quad \mathscr{T}_a = \mathscr{M}(A'), \quad \mathscr{T}_b = \mathscr{M}(B').$$

We shall prove the following theorem:

THEOREM 4.1. *If* $S(A|\mathscr{S}_b, \mathscr{T}_b)$ *and* $S(B|\mathscr{S}_a, \mathscr{T}_a)$ *exist, then* $\bar{\mathscr{C}}$ *is nonempty iff*

$$S(A|\mathscr{S}_b, \mathscr{T}_b) = S(B|\mathscr{S}_a, \mathscr{T}_a). \tag{36}$$

Proof. Let $C \in \bar{\mathscr{C}}$. Then

$$\begin{aligned} S(A|\mathscr{S}_b, \mathscr{T}_b) &= S[S(C|\mathscr{S}_a, \mathscr{T}_a)|\mathscr{S}_b, \mathscr{T}_b] = S(C|\mathscr{S}_b \cap \mathscr{S}_a, \mathscr{T}_b \cap \mathscr{T}_a) \\ &= S(C|\mathscr{S}_a \cap \mathscr{S}_b, \mathscr{T}_a \cap \mathscr{T}_b) = S[S(C|\mathscr{S}_b, \mathscr{T}_b)|\mathscr{S}_a, \mathscr{T}_a] \\ &= S(B|\mathscr{S}_a, \mathscr{T}_a). \end{aligned}$$

This establishes the necessity of (36). Conversely if (36) holds,

$$A + B - S(A|\mathscr{S}_b, \mathscr{T}_b) \in \bar{\mathscr{C}}. \qquad \blacksquare$$

REMARK 3. $\bar{\mathscr{C}}$ can be nonempty when $S(A|\mathscr{S}_b, \mathscr{T}_b)$ and/or $S(B|\mathscr{S}_a, \mathscr{T}_a)$ is not defined. Consider for example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and note that $A \overset{-}{<} I$, $B \overset{-}{<} I$, since both A and B are idempotent. In fact for $c \neq 0$, a, b, c arbitrary otherwise,

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}$$

dominates both A and B under the minus partial order. The shorted matrices concerned do not exist, since

$$\dim[\mathscr{M}(A) \cap \mathscr{M}(B)] = 2, \qquad \dim[\mathscr{M}(A') \cap \mathscr{M}(B')] = 1.$$

THEOREM 4.2. *If* (36) *holds,* $A \vee B$ *exists iff at least one of the matrices* $A - S(A|\mathscr{S}_b, \mathscr{T}_b)$ *and* $B - S(B|\mathscr{S}_a, \mathscr{T}_a)$ *is a null matrix.*

Proof. Let us write

$$A - S(A|\mathscr{S}_b, \mathscr{T}_b) = S_a,$$

$$B - S(B|\mathscr{S}_a, \mathscr{T}_a) = S_b,$$

$$S(A|\mathscr{S}_b, \mathscr{T}_b) = S(B|\mathscr{S}_a, \mathscr{T}_a) = S.$$

We have seen in the proof of Theorem 4.1 that $A + B - S = S_a + S_b + S \in \overline{\mathscr{C}}$. We show that it is in fact a minimal element in $\overline{\mathscr{C}}$. This follows from the fact that if $C \in \overline{\mathscr{C}}$,

$$A \bar{<} C \quad \Rightarrow \quad \mathscr{M}(A) \subset \mathscr{M}(C),$$

$$B \bar{<} C \quad \Rightarrow \quad \mathscr{M}(B) \subset \mathscr{M}(C).$$

Hence $\mathscr{M}(A:B) \subset \mathscr{M}(C) \Rightarrow \operatorname{Rank} C \geqslant \dim \mathscr{M}(A) + \dim \mathscr{M}(B) - \dim \mathscr{M}(A) \cap \mathscr{M}(B) = \operatorname{Rank} S_a + \operatorname{Rank} S_b + \operatorname{Rank} S$.

That $S_a + S_b + S$ is not the unique minimal element unless at least one of S_a and S_b is a null matrix follows from the fact that if both S_a and S_b are nonnull, $S_a + S_b + S + S_b K S_a$ for arbitrary K is also a minimal element in $\overline{\mathscr{C}}$. This is seen as follows:

$S_a + S$ and $S_b + S_b K S_a$ are disjoint matrices.

Clearly $\mathscr{M}(S_a + S)$ and $\mathscr{M}[S_b(I + KS_a)]$ are virtually disjoint. If for some vectors u and v,

$$u'(S_a + S) = v'(S_b + S_b K S_a),$$

then $w'S_a + u'S = v'S_b$, where $w' = u' - v'S_b K$. This implies

$$v'S_b = 0' \Rightarrow v'(S_b + S_b K S_a) = 0',$$

since $\mathscr{M}(S_b')$ is virtually disjoint with $\mathscr{M}(S_a' : S')$. This shows $\mathscr{M}(S_a' + S')$ and $\mathscr{M}(I + KS_a)'S_b']$ are also virtually disjoint. Thus

$$A = S_a + S \bar{<} S_a + S_b + S + S_b K S_a.$$

Similarly $B = S_b + S \bar{<} S_a + S_b + S + S_b K S_a \Rightarrow S_a + S_b + S + S_b K S_a \in \bar{\mathscr{C}}$. However,

$$\operatorname{Rank}(S_a + S_b + S + S_b K S_a) \leqslant \operatorname{Rank} S_a + \operatorname{Rank}[S_b(I + KS_a)] + \operatorname{Rank} S$$
$$\leqslant \operatorname{Rank} S_a + \operatorname{Rank} S_b + \operatorname{Rank} S.$$

This shows $S_a + S_b + S + S_b K S_0$ is also a minimal element in $\bar{\mathscr{C}}$. Hence $A \vee B$ does not exist. If $A - S(A|\mathscr{S}_b, \mathscr{T}_b) = 0$, then $A = S(A|\mathscr{S}_b, \mathscr{T}_b) = S(B|\mathscr{S}_a, \mathscr{T}_a)$ $\bar{<} B$. Clearly here $A = A \wedge B$, $B = A \vee B$. Similarly, when $B - S(B|\mathscr{S}_a, \mathscr{T}_a)$ $= 0$ we have $B = A \wedge B$, $A = A \vee B$. ■

It was shown in Theorems 4.1 and 4.2 that when both $S(A|\mathscr{S}_b, \mathscr{T}_b)$ and $S(B|\mathscr{S}_a, \mathscr{T}_a)$ exist, $A \vee B$ does not exist unless one of the matrices A and B is dominated by the other under the minus partial order. The following theorem shows that this is the only instance when $A \vee B$ exists.

THEOREM 4.3. *Nonexistence of even one of* $S(A|\mathscr{S}_b, \mathscr{T}_b)$ *and* $S(B|\mathscr{S}_a, \mathscr{T}_a)$ *implies the nonexistence of* $A \vee B$.

Proof. Assume without any loss of generality that $S(A|\mathscr{S}_b, \mathscr{T}_b)$ does not exist, and let $C = A \vee B$. Then $B \bar{<} C \Rightarrow C = B \oplus (C - B)$. One can therefore write $C = L_2' L_1$ and $B = L_2' D L_1$, where L_1 and L_2 are of full row rank, and

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$A \bar{<} C \Rightarrow A = L_2' H L_1$, where

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

is idempotent. Let λ be a scalar $\neq 0, 1$ such that

$$\det[I + (\lambda - 1)(I - H_{22})] \neq 0.$$

If $H_{22} = I$, choose

$$T = \begin{pmatrix} I & -(\lambda - 1)H_{12} \\ 0 & I + (\lambda - 1)(I - H_{22}) \end{pmatrix} = T_1$$

or

$$T = \begin{pmatrix} I & 0 \\ -(\lambda-1)H_{21} & I+(\lambda-1)(I-H_{22}) \end{pmatrix} = T_2$$

according as H_{12} or H_{21} is nonnull. If $H_{22} \neq I$, either choice is permissible. Clearly $\det T \neq 0$, which implies Rank $L_2'TL_1 = \text{Rank}\, C$. We now show $C_0 = L_2'TL_1$ dominates both A and B under the minus partial order and is in fact a minimal element in $\bar{\mathscr{C}}$. That $B \bar{<} C_0$ is trivial. We have

$$T_1 - H = \begin{pmatrix} I-H_{11} & -\lambda H_{12} \\ -H_{21} & \lambda(I-H_{22}) \end{pmatrix},$$

which has the same column span as $I - H$. Hence Rank$(T_1 - H) = $ Rank$(I - H)$ and we have

$$T_1 = H \oplus (T_1 - H).$$

Similarly it is seen that $T_2 = H \oplus (T_2 - H)$. In either case $A \bar{<} C_0$, and since C and C_0 have same rank, C_0 is seen to be another distinct minimal element in $\bar{\mathscr{C}}$. This is in conflict with our assumption that $C = A \vee B$. The excluded case, namely $H_{22} = I$ and $H_{12} = 0$, $H_{21} = 0$, is inconsistent with the assumption that $S(A|\mathscr{S}_b, \mathscr{T}_b)$ does not exist, since here

$$L_2'\begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} L_1$$

is in fact seen to be the shorted matrix $S(A|\mathscr{S}_b, \mathscr{T}_b)$. ■

THEOREM 4.4. *If* (36) *is satisfied, then* $P(A, B)$ *exists and*

$$A \wedge B = 2P(A, B) \tag{37}$$

is the unique maximal element in $\underline{\mathscr{C}}$.

Proof. We need Lemma 4.1 which is easily established using Theorems 1.1(d) and 3.4(b) of this paper and Theorem 2.1 of [16].

LEMMA 4.1. *If* $S(A|\mathscr{S}_b, \mathscr{T}_b)$ *and* $S(B|\mathscr{S}_a, \mathscr{T}_a)$ *are both defined, then*

$$P[S(A|\mathscr{S}_b, \mathscr{T}_b), S(B|\mathscr{S}_a, \mathscr{T}_a)] = P(A, B).$$

in the sense that, if either side is defined, so is the other, and they are equal.

Here $S(A|\mathscr{S}_b, \mathscr{T}_b) = S(B|\mathscr{S}_a, \mathscr{T}_a) = S$ *(say). Hence*

$$P[S(A|\mathscr{S}_b, \mathscr{T}_b), S(B|\mathscr{S}_a, \mathscr{T}_a)] = \frac{S}{2} = P(A, B).$$

If $C \in \underline{\mathscr{C}}$, *then* $C \overset{-}{<} A,\ C \overset{-}{<} B \Rightarrow C \overset{-}{<} A,\ \mathscr{M}(C) \subset \mathscr{S}_b,\ \mathscr{M}(C') \subset \mathscr{T}_b \Rightarrow C \overset{-}{<} S(A|\mathscr{S}_b, \mathscr{T}_b) = S$. *Also* $S(A|\mathscr{S}_b, \mathscr{T}_b) = S(B|\mathscr{S}_a, \mathscr{T}_a) = S \Rightarrow S \overset{-}{<} A$, $S \overset{-}{<} B \Rightarrow S \in \underline{\mathscr{C}}$. *Hence*

$$A \wedge B = S = 2P(A, B). \qquad \blacksquare$$

REMARK 4. The formula (37) is strikingly similar to the expression for the infimum of two orthogonal projections in a finite dimensional complex vector space given by Anderson and Duffin [1] and reproduced in Theorem 1.1(g).

Let $A \overset{*}{\vee} B$ and $A \overset{*}{\wedge} B$ represent the star supremum and the star infimum of a pair of matrices A and B with the star order replacing the minus order in the definitions of $\overline{\mathscr{C}}$ and $\underline{\mathscr{C}}$ given in (34) and (35) respectively. It was shown by Holladay [10, Lemma 7.34] that

$$A \overset{*}{\wedge} B \overset{*}{<} 2A(A+B)^{+}B.$$

Clearly $A \overset{*}{\wedge} B = B \overset{*}{\wedge} A \overset{*}{<} 2B(A+B)^{+}A$, and $A(A+B)^{+}B$ is not necessarily equal to $B(A+B)^{+}A$, since A and B need not be p.s. It was shown further that [10, Lemma 7.35] when $A \overset{*}{\vee} B$ exists,

$$A \overset{*}{\wedge} B = 2A(A+B)^{+}B = 2B(A+B)^{+}A.$$

REMARK 5. $A \wedge B$ may not always exist. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and note that for arbitrary a, b, both

$$\begin{pmatrix} a & 1 \\ a & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b/2 & b \\ (2-b)/2 & 2-b \end{pmatrix}$$

are dominated by A and B under the minus partial order. It was shown by Hartwig and Drazin [7] that $\mathscr{C}^{m\times n}$, the vector space of complex matrices of order $m \times n$, is a lower semilattice under the * order, that is, for a pair of arbitrary matrices $A, B \in \mathscr{C}^{m\times n}$, $\underline{\mathscr{C}}$ has a unique maximal element if the minus partial order is replaced by * order in the definition of $\underline{\mathscr{C}}$. The above example shows that $\mathscr{C}^{m\times n}$ loses this property under the minus partial order.

REMARK 6. $A \wedge B$ may exist even if (36) is not true. Let A be nonnull matrix and $B = 2A$. Clearly $A = S(A|\mathscr{S}_b, \mathscr{T}_b) \neq B = S(B|\mathscr{S}_a, \mathscr{T}_a)$. If C is a nonnull matrix and $C \stackrel{-}{<} A$, then $A = C \oplus (A - C)$. However, $B - C = 2(A - C) \oplus C$ and C are not disjoint. Hence $C \stackrel{-}{\nless} B$. Thus the null matrix is the only member of $\underline{\mathscr{C}}$.

REMARK 7. The example discussed in Remark 6 may lead one to speculate that if $A = A_0 \oplus A_1$ and $B = A_0 \oplus kA_1$ $(k \neq 1)$, then $A \wedge B = A_0$. This however need not be true. Consider for example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then it is seen that

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is dominated by both A and B. However, $C \neq A \wedge B$, since the matrix

$$C_0 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

is also dominated by both A and B under the minus partial order. In case

$A \wedge B$ exists, it can be of rank at most 2, has to dominate both C and C_0, and has to be itself an idempotent matrix, since

$$A \wedge B \bar{<} A, \quad C \bar{<} A \wedge B, \quad C_0 \bar{<} A \wedge B$$

$$\Rightarrow \quad \mathcal{M}(A \wedge B) = \mathcal{M}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}[(A \wedge B)'] = \mathcal{M}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

However, no such idempotent matrix can exist to satisfy all the demands that have just been made, since $(0,1,1)'$ belongs to both the range and the null space of $A \wedge B$.

THEOREM 4.5. *Let* $S(A|\mathcal{S}_b, \mathcal{T}_b)$ *and* $S(B|\mathcal{S}_a, \mathcal{T}_a)$ *both exist and be unequal. Then either* $A \wedge B$ *is a null matrix or* $A \wedge B$ *does not exist.*

Proof. When $S(A|\mathcal{S}_b, \mathcal{T}_b)$ and $S(B|\mathcal{S}_a, \mathcal{T}_a)$ both exist, clearly

$$A \wedge B = S(A|\mathcal{S}_b, \mathcal{T}_b) \wedge S(B|\mathcal{S}_a, \mathcal{T}_a).$$

Let $S(A|\mathcal{S}_b, \mathcal{T}_b) = L_2'L_1$ be a rank factorization of $S(A|\mathcal{S}_b, \mathcal{T}_b)$. Since $S(A|\mathcal{S}_b, \mathcal{T}_b)$ and $S(B|\mathcal{S}_a, \mathcal{T}_a)$ have the same row and column spans,

$$S(B|\mathcal{S}_a, \mathcal{T}_a) = L_2'B_0L_1$$

for some nonsingular matrix B_0. This shows that without any loss of generality one may consider the case $A = I$ and B nonsingular. Let $C \bar{<} I$, $C \bar{<} B$, and C be nonnull. We shall consider two cases separately.

Case 1. $BC \neq CB$. Here $C_0 = BCB^{-1} \neq C$ has the same rank as C, and it is easily seen that $C_0 \bar{<} I$, $C_0 \bar{<} B$. Thus C cannot be the unique maximal element in $\underline{\mathcal{C}}$.

Case 2. $BC = CB$. Here choose and fix a matrix X such that $(I - C)XC \neq 0$ and $\det[I + (I - C)X] \neq 0$. Put $C_0 = C + (I - C)XC$ and observe that C_0 has the same rank as C. Further, $C_0^2 = C_0 \Rightarrow C_0 \bar{<} I$. Also, $C_0B^{-1}C_0 = C_0 \Rightarrow C_0 \bar{<} B$ (by Theorem 2.2). Thus here also C cannot be the unique maximal element in $\underline{\mathcal{C}}$. Hence $A \wedge B$ does not exist unless the null matrix is the only member in $\underline{\mathcal{C}}$. In the example discussed under Remark 6 we have seen that $A \wedge B$ could be a null matrix. ■

REMARK 8. One may at this stage speculate that for an arbitrary pair of matrices A and B of the same order, either $A \wedge B = 0$ or $A \wedge B = 2P(A, B)$ or $A \wedge B$ may not exist. This however is not true. Consider for example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $\dim[\mathscr{M}(A) \cap \mathscr{M}(B)] = 2$, $\dim[\mathscr{M}(A') \cap \mathscr{M}(B')] = 1$. Hence $S(A|\mathscr{S}_b, \mathscr{T}_b)$, $S(B|\mathscr{S}_a, \mathscr{T}_a)$, and $P(A, B)$ do not exist.

For a matrix C to be dominated by A under the minus order, C has to be a 2×2 idempotent matrix bordered by a null row and a null column (as in A). To be dominated by B the first row of any such matrix has to be $(2,2,0)$, as any other choice would imply $\text{Rank}(B - C) = 2$. This shows that

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the only nonnull member of $\underline{C}$ and is thus equal to $A \wedge B$.

THEOREM 4.6. *Let $C \in \overline{\mathscr{C}}$ and be such that $(C - A) \wedge (C - B)$ exists. Then*

(1) $C_0 = C - (C - A) \wedge (C - B) \in \overline{\mathscr{C}}$,

(2) $C_0 \overset{-}{<} C$,

(3) $(C_0 - A) \wedge (C_0 - B) = 0$.

Proof. (1): $C_0 - A = (C - A) - (C - A) \wedge (C - B) \overset{-}{<} C - A$. Hence A and $C_0 - A$ are disjoint matrices, and

$$C_0 = A \oplus (C_0 - A) \quad \Rightarrow \quad A \overset{-}{<} C_0.$$

Similarly $B \overset{-}{<} C_0$. Hence $C_0 \in \overline{\mathscr{C}}$.

(2):

$$\begin{aligned} C = A \oplus (C - A) = A \oplus [\{C - A - (C - A) \wedge (C - B)\} \\ \oplus \{(C - A) \wedge (C - B)\}] \\ = C_0 \oplus \{(C - A) \wedge (C - B)\} \quad \Rightarrow \quad C_0 \overset{-}{<} C. \end{aligned}$$

(3): Let E be a nonnull matrix and $E \overset{-}{<} C_0 - A$, $E \overset{-}{<} C_0 - B$. Then

$E \overset{-}{<} C - A$, $E \overset{-}{<} C - B$. Further, E and $(C-A)\wedge(C-B)$ are disjoint matrices, which contradicts the assumption that $(C-A)\wedge(C-B)$ is the unique maximal element in the corresponding $\mathscr{C}$, since

$$\begin{aligned} C - A &= \{E \oplus (C_0 - A - E)\} \oplus \{(C-A)\wedge(C-B)\} \\ &= \{C_0 - A - E\} \oplus [E \oplus \{(C-A)\wedge(C-B)\}] \end{aligned}$$

and

$$\begin{aligned} C - B &= \{E \oplus (C_0 - B - E)\} \oplus \{(C-A)\wedge(C-B)\} \\ &= \{C_0 - B - E\} \oplus [E \oplus \{(C-A)\wedge(C-B)\}]. \end{aligned}$$

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Note added in proof. The author is grateful to Dr. David H. Carlson for pointing out an error in an earlier version of the statement of Theorem 3.2 and for motivating the discussions that follow. It is to be noted that the present definition of the shorted matrix is not strictly equivalent to the 1982 definition [14, 15 ,17] though they are very nearly so. If a unique shorted matrix exists under the 1982 definition, then from Theorem 2.1 in [14] it is seen that this is also the unique maximal element in $\mathscr{C}$. On the other hand if both $\mathscr{S}$ and $\mathscr{T}$ are of positive dimensions, if precisely one of the following is true

$$\mathscr{S} \cap \mathscr{M}(N) = \{0\}, \mathscr{T} \cap \mathscr{M}(N') = \{0\} \tag{38}$$

(even is (20) be untrue), then the null matrix is the only member of $\mathscr{C}$ and thus its maximal element. However Theorem 2.1 [14] implies that here the unique shorted matrix under the 1982 definition does not exist. The following example constructed on the lines of Remark 7 illustrates a similar situation with both parts of (38) being untrue. Let $N = I$ be a matrix of order 3×3, $\mathcal{S} = \mathcal{M}([0,1,1]')$, $\mathcal{T} = \mathcal{M}([0,1,-1]')$. Both these examples also point out the necessity of the exclusion clause in Theorem 3.4.

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