Note

The complexity of subgraph isomorphism for classes of partial $k$-trees

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Abstract

We present a clear demarcation between classes of bounded tree-width graphs for which the subgraph isomorphism problem is NP-complete and those for which it can be solved in polynomial time. In previous work, it has been shown that this problem is solvable in polynomial time if the source graph either has bounded degree or is $k$-connected, for $k$ the tree-width of the two graphs. As well, it has also been shown that for certain specific connectivity or degree conditions, the problem becomes NP-complete. Here we give a complete characterization of the complexity of this problem on bounded tree-width graphs for all possible connectivity conditions of the two input graphs. Specifically, we show that when the source graph is not $k$-connected or has more than $k$ vertices of unbounded degree the problem is NP-complete; this answers an open question of Matoušek and Thomas.

Many of our reductions are restricted to using a subset of graphs of bounded tree-width, namely graphs of bounded path-width. As a direct result of our constructions, we also show that when the source and target graphs have connectivity less than $k$ or at least one has $k$ vertices of unbounded degree, the subgraph isomorphism problem for bounded path-width graphs is NP-complete.

1. Introduction

The subgraph isomorphism problem is known to be NP-complete for general graphs, but can be solved in polynomial time for many restricted classes of graphs. We can phrase the problem as that of trying to determine whether or not there is a subgraph of an input graph $H$ that is isomorphic to an input graph $G$. In this paper, we study this problem when $G$ and $H$ are both graphs of bounded tree-width with various...
connectivity and degree conditions, and show that there is a clear division between those cases in which the problem is polynomial time and those in which it is NP-complete.

Polynomial-time algorithms for subgraph isomorphism have been devised for restricted classes of graphs, including trees [16], two-connected outerplanar graphs [13], and two-connected series-parallel graphs [14]. These are all graphs of bounded tree-width (as defined formally in Section 2), for which the problem is NP-complete in general [21]. More generally, it has been shown that for $G$ and $H$ partial $k$-trees, if $G$ either has bounded degree or is $k$-connected then there is a polynomial time algorithm for subgraph isomorphism [15].

One natural question is that of determining the complexity of this problem if the constraints on $G$ are further relaxed. In this paper we study two scenarios, namely, allowing the connectivity of $G$ to be less than $k$ and allowing a constant number of nodes of unbounded degree. Such questions were also studied by Matoušek and Thomas [15] who showed that the problem is NP-complete when $G$ is a tree, $H$ is a graph of tree-width two, and each graph has at most one node of degree greater than three. However, they left as an open problem the case where, for example, $G$ has connectivity $\frac{k+1}{2}$ and $G$ and $H$ are both partial $k$-trees, and hypothesized that this problem may in fact be solvable in polynomial time. Our results directly show that the problem is, in fact, NP-complete.

In this paper we examine a subset of the class of graphs of bounded tree-width, the graphs of bounded path-width. We derive the complexity of the subgraph isomorphism problem for $G$ and $H$ both graphs of path-width $k$ where $G$ is $g$-connected and $H$ is $h$-connected, for both $g$ and $h$ less than $k$. Since NP-completeness results obtained for graphs of bounded path-width automatically apply to graphs of bounded tree-width, similar results are obtained for this larger class. Furthermore, we show that when $G$ and $H$ have tree-width $k$ with $H$ $k$-connected and $G$ with connectivity less than $k$, the problem is again NP-complete. We thus obtain sharp divisions for subgraph isomorphism on bounded tree-width graphs and nearly sharp divisions for bounded path-width graphs.

2. Preliminaries

All graphs in this paper will be simple. We denote the vertex and edge sets of a graph $G$ by $V(G)$ and $E(G)$, respectively. We will be working extensively with trees and paths; the reader is expected to have a basic familiarity with these types of graphs (see [7] for more background material). For $P$ a path on $\ell$ vertices, we will often write $P = v_1, \ldots, v_\ell$ by which we mean $V(P) = \{v_1, \ldots, v_\ell\}$ and $E(P) = \{(v_i, v_{i+1}) : 1 \leq i < \ell\}$ where all $v_i$'s are distinct.

In our discussions of connectivity, when we say that a graph $G$ is $g$-connected, we mean that $G$ is $g$-vertex-connected but not $(g+1)$-connected. Several lemmas will make use of the notion of a split graph [9], a graph $G$ such that $V(G)$ can be partitioned
into subsets $V_1(G)$ and $V_2(G)$ such that for all $u,v \in V_1(G)$, the edge $(u,v)$ is in $E(G)$ and for all $u,v \in V_2(G)$, the edge $(u,v)$ is not in $E(G)$. Lemma 2.1 follows from the definition above.

**Lemma 2.1.** A split graph with minimum degree $k$ is $k$-connected.

The graphs constructed in our reductions will often consist of taking the union of a number of different graphs.

**Definition.** Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the union of $G_1$ and $G_2$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. If a node $u \in V_1$ is the same as the node $v \in V_2$, we say that $u$ is identified with $v$ in the union, or that the union is formed by identifying $u$ and $v$.

Notice that unions preserve simple graphs: when we take the union of the edge set we do not allow multiple edges to be formed.

The next definitions and lemmas give characterizations of partial $k$-trees and partial $k$-paths.

**Definition.** A $k$-tree [19] is defined inductively as follows:
1. $K_k$ (a $k$-clique, the complete graph on $k$ nodes) is a $k$-tree.
2. Let $G$ be a $k$-tree on $n$ nodes and let $K$ be a $k$-clique in $G$. Then the $n + 1$ node graph $G'$ formed from $G$ by adding a new node $v$ adjacent to all nodes of $K$ is a $k$-tree.
A partial $k$-tree is a subgraph of a $k$-tree.

**Definition.** A tree-decomposition [20] of a graph $G$ is a pair $(T, \chi)$, where $T$ is a tree and $\chi : V(T) \rightarrow \{\text{subsets of } V(G)\}$ satisfying:
1. for every $a \in V(G)$, there is an $x \in V(T)$ such that $a \in \chi(x)$;
2. for every $e = (a,b) \in E(G)$, there is an $x \in V(T)$ such that $a,b \in \chi(x)$; and
3. for $x,y,z \in V(T)$, if $y$ is on the path from $x$ to $z$ in $T$ then $\chi(x) \cap \chi(z) \subseteq \chi(y)$.

The width of a tree-decomposition $(T, \chi)$ is $\Delta = \max\{|\chi(x)| - 1 : x \in V(T)\}$ and the tree-width of a graph $G$ is the minimum width over all its tree-decompositions.

**Lemma 2.2** (van Leeuwen [22]). $G$ is a partial $k$-tree if and only if $G$ has tree-width at most $k$.

**Definition.** A path-decomposition of a graph $G$ is a tree-decomposition $(P, \chi)$ in which $P$ is a path.

The definitions of the width of a path-decomposition and the path-width of $G$ are defined analogously. We can also modify the definition of partial $k$-trees to yield a different characterization of path-width $k$ graphs. In our algorithms, we make use of
the recursive construction of \( k \)-paths, defined below. Elsewhere a \( k \)-path is known as a \( k \)-caterpillar \([2, 18]\) and defined as a \( k \)-tree that is an interval graph \([9]\).

**Definition.** A \( k \)-path is defined inductively as follows:

1. \( K_k \) is a \( k \)-path.
2. \( K_{k+1} \) is a \( k \)-path with one node designated as the distinguished node.
3. Let \( G \) be a \( k \)-path on \( n > k \) nodes with distinguished node \( v \). Let \( K \) be either the \( k \)-clique to which \( v \) is adjacent or a \( k \)-clique involving \( v \). Then the \( n + 1 \) node graph \( G' \) formed from \( G \) by adding a new node \( w \) adjacent to all nodes of \( K \) is a \( k \)-path with \( w \) the distinguished node.

A partial \( k \)-path is a subgraph of a \( k \)-path.

We will be working extensively with the \( k \)-path definition; the following further definitions will be useful. We can view the formation of a \( k \)-path as a step-by-step procedure, consisting first of the creation of a clique of size \( k \) and then the addition, in steps 1 through \( \ell \), of the \( (k + 1) \)st through \( (k + \ell) \)th nodes in the \( k \)-path. The first \( k \) nodes will be called the original clique. The distinguished node added at step \( i \) will be denoted \( u_i \), and the \( k \)-clique to which it is attached will be called the attachment clique, \( C_i \). Thus, \( C_i \) will either be identical to \( C_{i-1} \) or will contain \( u_{i-1} \).

**Lemma 2.3** (Arnborg et al. \([2]\)). \( G \) is a partial \( k \)-path if and only if \( G \) has path-width at most \( k \).

Most of the work on graphs of bounded path-width has focused on determining path-width of graphs. This problem is NP-complete in general \([1]\), but can be solved in polynomial time for classes such as graphs of bounded tree-width \([4]\), permutation graphs \([5]\), cocomparability graphs of bounded dimension \([11]\), cotriangulated graphs \([9]\), cographs \([6]\), splitgraphs \([10]\), and interval graphs. Tree-width and path-width have been shown to be equal for cographs \([6]\), as well as for the more general class of asteroidal-triple-free graphs \([17]\). More generally, it has been shown that for any graph on \( n \) nodes, the path-width is in \( O(\log n \cdot \text{tree-width}) \) \([3, 12]\).

### 3. General properties of the reductions

In the remainder of this paper, we establish the complexity of subgraph isomorphism for \( G \) and \( H \) both partial \( k \)-paths of varying connectivities. The subgraph isomorphism problem is that of determining if there is a subgraph of \( H \) isomorphic to \( G \), or more formally:

**Instance:** A source graph \( G \) and a target graph \( H \).

**Question:** Is there a one-to-one function (a **subgraph isomorphism from \( G \) to \( H \)**) \( \phi : V(G) \to V(H) \) such that \((u, v) \in E(G) \) implies that \( (\phi(u), \phi(v)) \in E(H) \)?

We prove the following theorem:
Theorem 3.1. **SUBGRAPH ISOMORPHISM** on partial k-trees can be solved in polynomial time when G is g-connected, H is h-connected, and g = k, and is NP-complete otherwise.

The polynomial-time algorithm is due to Matoušek and Thomas [15]. The NP-completeness results follow from Lemmas 4.6 and 5.3. As a byproduct of our constructions in the proofs of these lemmas, we also obtain the following result.

Theorem 3.2. **SUBGRAPH ISOMORPHISM** on partial k-trees is NP-complete when G and H each have all but k nodes of degree at most k + 2.

It is not difficult to see that the subgraph isomorphism problem is in NP; we only need prove that the appropriate problems are NP-complete. Our reductions will make use of the NP-complete problem 3-Partition [8]:

**Instance:** A finite set \( \mathcal{I} = \{I_1, I_2, \ldots, I_{3m}\} \) of positive integers and a positive integer \( B \) such that \( \sum_j I_j = mB \) and for each \( j \), \( B/4 < I_j < B/2 \).

**Question:** Can \( \mathcal{I} \) be partitioned into \( m \) disjoint subsets \( \{C_1, C_2, \ldots, C_m\} \) (of 3 elements each) such that for \( 1 \leq i \leq m \), \( \sum_{I \in C_i} I = B \)?

Recall that 3-Partition is strongly NP-complete so we can assume that the values \( I_j \) for a specific instance are specified in unary. This fact will be crucial to our reductions as the graphs G and H we will construct will each have size \( O(mB) \). For the remainder of this section, we will specify an instance of 3-Partition by a tuple \( (I_1, \ldots, I_{3m}; B) \).

Without loss of generality we will also assume that \( I_j > 1 \) for all \( j \).

We now delineate classes of graphs which will be used in our reductions.

**Definition.** A **j-path spiral of length \( \ell \)** is a graph consisting of the following:

1. a total of \( (j-1)+\ell \) nodes, where \( c_1, \ldots, c_{j-1} \) are center nodes and \( b_1, \ldots, b_{\ell} \) are exterior nodes (in order), with \( b_1 \) and \( b_{\ell} \) the first and last exterior nodes, respectively;
2. edges between each pair of center nodes, forming a clique of center edges;
3. edges between each exterior node and all the center nodes, forming the set of radial edges; and
4. edges between \( b_i \) and \( b_{i-1} \) for \( i > 1 \), forming the set of exterior edges.

We can consider the \( j-1 \) center nodes to be \( j-1 \) of the \( j \) nodes in the original clique in the construction of a \( j \)-path, and the exterior nodes the added nodes (with the \( j \)th node in the original clique being either the first exterior node or a node removed later to form a partial \( j \)-path). Lemma 3.3 can be verified for a path-decomposition \((P, \chi)\) where \( P = w_0, w_1, \ldots, w_{\ell} \) and \( \chi(w_0) = \{c_1, \ldots, c_{j-1}\} \), \( \chi(w_1) = \{c_1, \ldots, c_{j-1}, b_1\} \), and for \( i > 1 \), \( \chi(w_i) = \{c_1, \ldots, c_{j-1}, b_{i-1}, b_i\} \). The connectivity follows from the \( j-1 \) paths between each pair of nodes due to Lemma 2.1 (where the center nodes are viewed as \( V_1(G) \)) and the fact that the exterior edges can be used to form a \( j \)th path between any two nodes in the graph.
Lemma 3.3. A $j$-path spiral $\mathcal{S}$ of length $\ell$ is a $j$-connected partial $j$-path with a width-$j$ path-decomposition $(P, \chi)$ such that one endpoint of $P$ is labeled by the center nodes of $\mathcal{S}$.

The constructions in our reductions will involve forming unions of $j$-paths and $i$-stars, or star-unions, defined below. Lemma 3.4 is straightforward to verify.

Definition. A $j$-star is a graph consisting of the following:
1. a total of $j + 3$ nodes where $d_1, \ldots, d_j$ are the clique nodes and $p_1, p_2, p_3$ are the pendant nodes;
2. clique edges between each pair of clique nodes forming a clique of size $j$; and
3. pendant edges between each pendant node and all clique nodes.

Lemma 3.4. A $j$-star is a $j$-connected $j$-path with a width-$j$ path-decomposition such that the label of one endpoint contains the clique nodes.

Definition. Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be $(j+1)$-path spirals of varying lengths. Let $\mathcal{X}$ be an $i$-star for $i \geq j$, and let $j$ of the clique nodes of $\mathcal{X}$ be called identified nodes. Then, the star-union of $\mathcal{X}$ and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ is the graph formed by taking the union of $\mathcal{S}_1, \ldots, \mathcal{S}_r$ and $\mathcal{X}$ in which the center nodes of each $\mathcal{S}_i$ are identified with the identified nodes of $\mathcal{X}$.

4. Reduction for $H$ not $k$-connected

In this section we begin by focusing on the proof of Lemma 4.6 for the case in which $G$ has connectivity no greater than that of $H$. We then give a construction for handling the case in which $G$ has greater connectivity than $H$. We start with some technical lemmas.

Lemma 4.1 can be proved using Lemma 2.1 by viewing the $i$ clique nodes of the $i$-star as the clique part (i.e. $V_i(G)$) of a split graph.

Lemma 4.1. The star-union of a $(j+1)$-path spiral and an $i$-star, $i \geq j$, is $j$-connected.

The next set of lemmas show that a star-union has the correct width. The proof of Lemma 4.2 is straightforward and hence omitted.

Lemma 4.2. Let $G$ be a partial $k$-path and $v_1, \ldots, v_k$ be nodes of $G$ such that there is a path-decomposition $(P, \chi)$ of $G$ of width $k$ with one endpoint of $P$ labeled by $\{v_1, \ldots, v_k\}$. Let $\mathcal{S}$ be a $(j+1)$-path spiral ($k > j$) and $G'$ be the union of $G$ and $\mathcal{S}$ formed by identifying the $j$ center nodes with any $j$ nodes from among $v_1, \ldots, v_k$, say $v_1, \ldots, v_j$. Then $G'$ has a width-$k$ path-decomposition such that the label of one endpoint contains $\{v_1, \ldots, v_j\}$. 
In the proof of Lemma 4.2, the path-decomposition constructed has the property that the label of one endpoint of the path contains all the identified nodes \( \{v_1, \ldots, v_l\} \) (and consequently all of the center nodes of the spiral). Thus, by iterating this lemma with Lemma 3.3, we obtain the following corollary:

**Corollary 4.3.** The star-union of a \( k \)-star and a set of \((j + 1)\)-path spirals, \( k > j \), is a \( j \)-connected partial \( k \)-path.

In the reductions, each of \( G \) and \( H \) will consist of star-unions. We will show that the clique nodes in \( G \) must map to the clique nodes in \( H \), and that the ways in which the spirals in \( G \) map to the spirals in \( H \) dictate the partition of the items. Here we make a general observation which is the key to our reductions.

**Lemma 4.4.** An instance \((I_1, \ldots, I_{3m}; B)\) of 3-Partition has a solution if and only if \( G \) is isomorphic to a subgraph of \( H \), where

\[ G \text{ is the star-union of a } k \text{-star } K \text{ and a set of } (g + 1)\text{-path spirals of lengths } I_1, \ldots, I_{3m} \text{ for some } 0 \leq g < k; \text{ and} \]

\[ H \text{ is the star-union of a } k' \text{-star } K' \text{ and a set of } m \text{ }(h + 1)\text{-path spirals each of length } B, \text{ for some } g < h < k. \]

**Proof.** Suppose that \( G \) is isomorphic to a subgraph of \( H \) and let \( f : V(G) \to V(H) \) be the subgraph isomorphism. Clearly \( f \) is a bijection since \( |V(G)| = |V(H)| \); we first show that \( f \) maps \( X \) to \( X' \) such that identified nodes of \( X \) map to (a possible subset of the) identified nodes of \( X' \). All clique nodes in both \( G \) and \( H \) have degree at least \( k + 2 \) whereas all other nodes have degree at most \( k + 1 \), so the clique nodes of \( G \) must map to the clique nodes of \( H \). As well, the identified nodes have degree \( k - 1 + 3 + mB > k + 2 \) in both \( G \) and \( H \), so among the clique nodes, the identified nodes must map to identified nodes. Finally, consider a pendant node \( p \) of \( G \). If it maps to some exterior node \( v \) of \( H \) then an exterior neighbour \( w \) of \( v \) must have a pre-image which is adjacent to \( p \); such a neighbour exists since \( I_j > 1 \) for all \( j \). But \( p \) only has neighbours that are clique nodes, not exterior nodes.

Thus, we have shown that the exterior nodes in the spirals of \( G \) must map into exterior nodes in the spirals of \( H \). We now show that the instance \((I_1, \ldots, I_{3m}; B)\) of 3-Partition has a solution. Since exterior edges must map to exterior edges, the images of the exterior nodes in a particular spiral in \( G \) must form a consecutive set of exterior nodes in \( H \). Consequently, each of the spirals \( S_i \) must be contained entirely in some \( S_j \) for some \( j \). Since \( f \) is a bijection, this forms a partition of \( \{I_1, \ldots, I_{3m}\} \) into sets of size \( B \) each specified by the \( S_j \), as required. The proof of the other direction is straightforward. □

Corollary 4.5 follows from Lemma 4.4 and Corollary 4.3, for \( G \) the star-union of a \( k \)-star and \((g + 1)\)-path spirals \( S_1, \ldots, S_{3m} \) of lengths \( I_1, \ldots, I_{3m} \), respectively, and \( H \)
the star-union of a $k$-star and $(h + 1)$-path spirals $T_1, \ldots, T_m$ of length $B$. From the construction of $G$ and $H$ it is clear that all but $g$ nodes of $G$ and $h$ nodes of $H$ have degree at most $k + 2$, as needed for Theorem 3.2.

**Corollary 4.5.** *SUBGRAPH ISOMORPHISM* on partial $k$-paths is NP-complete when $G$ is $g$-connected, $H$ is $h$-connected, and $g \leq h < k$.

We now show how our previous reductions can be modified to allow $H$ to have lower connectivity than $G$, to obtain the more general result:

**Lemma 4.6.** *SUBGRAPH ISOMORPHISM* on partial $k$-paths is NP-complete when $G$ is $g$-connected, $H$ is $h$-connected, and $g$ and $h$ are both less than $k$.

**Proof.** The case in which $g \leq h < k$ is handled in Corollary 4.5, in the remainder of this proof we assume $h < g < k$. For an instance $\mathcal{F} = \{I_1, \ldots, I_{3m}; B\}$ of 3-Partition, we begin by constructing graphs $G$ and $H'$ both having connectivity $g$ as in Corollary 4.5.

We form a split graph $D_h$ by attaching $h$ nodes of degree one, $w_1, \ldots, w_h$, to distinct nodes of a complete graph $K_h$. Notice that any $h$ of the $g$ identified nodes $\{d_1, \ldots, d_g\}$ in the star-union $H'$ form an $h$-clique. We construct $H$ to be the union of $H'$ and $D_h$ formed by identifying any $h$ nodes of $\{d_1, \ldots, d_g\}$ with $w_1, \ldots, w_h$.

Next, we prove that $H$ is an $h$-connected partial $k$-path. The graph $H$ has connectivity $h$ since each of $H'$ and $K_h$ have connectivity at least $h$ and $\{w_1, \ldots, w_h\}$ forms a cutset in $H$. As well, since by Lemma 4.2 and Corollary 4.3 there is a width $k$ path-decomposition $(P, \chi)$ of $H'$ such that the label of one endpoint of $P$ contains $\{d_1, \ldots, d_g\}$, this path-decomposition can easily be extended to a width $k$ path-decomposition of $H$. Hence $H$ is an $h$-connected partial $k$-path.

Since $H'$ is a subgraph of $H$, it follows immediately from Lemma 4.4 that if there is a solution to $\mathcal{F}$, then $H$ (or more specifically, $H'$) contains a subgraph isomorphic to $G$. It remains to be shown that if $H$ contains a subgraph isomorphic to $G$ then there is solution to the instance $\mathcal{F}$. To use Lemma 4.4, it will suffice to show that no subgraph of $H$ containing unidentified nodes of $D_h$ can be isomorphic to $G$. Suppose this was not the case and that $G'$ was a subgraph of $H$ containing unidentified nodes of $D_h$ isomorphic to $G$. Clearly $G$ would have a vertex of degree at most $h$, and hence connectivity at most $h < g$, yielding a contradiction. 

5. Reduction for $H$ $k$-connected

To handle the case in which $H$ is $k$-connected, we use constructions similar to those appearing in the previous sections. However, when we form the union of a $k$-star with $k$-path spirals, the resulting graph has tree-width $k$, not path-width $k$.

We begin by stating a number of technical lemmas similar to those found in Section 4. The following definition is a generalization of star-unions but yields slightly higher connectivity.
**Definition.** Let $\mathcal{K}$ be a $k$-star with clique nodes $d_1, \ldots, d_k$. Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be $j$-path spirals for $j \leq k$. Let $b_{1,i}$ be the first exterior node of $\mathcal{S}_i$. Let $G$ be the star-union of $\mathcal{K}$ and $\mathcal{S}_1, \ldots, \mathcal{S}_r$. Without loss of generality, assume the nodes $d_1, \ldots, d_{j-1}$ are identified with the center nodes of the spirals in $G$. Then the enhanced star-union of $\mathcal{K}$ and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ is the graph $G$ with an additional edge from each $b_{1,i}$ to the node $d_j$.

**Lemma 5.1.** Let $G$ be the enhanced star-union of a $k$-star $\mathcal{K}$ and a $j$-path spiral $\mathcal{S}$ of length $\ell$ for $j \leq k$. Then $G$ is a $j$-connected partial $k$-path and there is a path-decomposition of $G$ with one endpoint labeled by the clique nodes of $\mathcal{K}$.

**Proof.** To find $j$ node-disjoint paths between nodes both in $\mathcal{K}$ is trivial. For $d_i$ an exterior node in $\mathcal{S}$ and $u$ any other node in $G$, there are $j-1$ paths between $d_i$ and $u$ due to Lemma 2.1, where we view the clique nodes of the $k$-star as $V_1(G)$. A $j$th path can be formed using exterior edges, for $u \notin V(\mathcal{K})$, and exterior edges and $d_j$, for $u \in V(\mathcal{K})$.

To show that $G$ is a partial $k$-path, we construct a path-decomposition $(P, \chi)$. We define the path $P$ to be $w_0, w_1, w'_2, w'_3, w_4, \ldots, w_\ell$, and set $\chi(w_0) = \{d_1, \ldots, d_k\}$, $\chi(w_1) = \{d_1, \ldots, d_k, p_1\}$, $\chi(w'_2) = \{d_1, \ldots, d_k, p_2\}$, $\chi(w'_3) = \{d_1, \ldots, d_k, p_3\}$, $\chi(w_4) = \{d_1, \ldots, d_k, b_1\}$ and for $i > 1$, $\chi(w_i) = \{d_1, \ldots, d_{k-1}, b_{i-1}, b_i\}$. It is straightforward to verify that $(P, \chi)$ is the required path-decomposition.

We can now join together path-decompositions of enhanced star-unions to obtain tree-decompositions of enhanced star-unions when there are multiple spirals.

**Lemma 5.2.** Let $G$ be the enhanced star-union of a $k$-star $\mathcal{K}$ and $j$-path spirals $\mathcal{S}_1, \ldots, \mathcal{S}_r$, $j \leq k$. Then, $G$ is $j$-connected and has tree-width $k$.

**Proof.** Let $d_1, \ldots, d_k$ be the clique nodes in $\mathcal{K}$. Let the graph $G_i$ be the enhanced star-union of $\mathcal{K}$ with $k$-path spiral $\mathcal{S}_i$, $1 \leq i \leq r$ in which the identified nodes are $d_1, \ldots, d_{j-1}$. Then, by Lemma 5.1, we can form width-$k$ path-decompositions $(P_1, \chi_1), \ldots, (P_r, \chi_r)$ of $G_1, \ldots, G_r$, respectively such that one endpoint of each path is labeled by $\{d_1, \ldots, d_k\}$. Now, consider the graph $G$ formed by taking the enhanced union of $\mathcal{K}$ with $\mathcal{S}_1, \ldots, \mathcal{S}_r$ such that the identified nodes are $d_1, \ldots, d_{j-1}$ and $d_j$ is adjacent to all first nodes of the spirals.

We can form a width $k$ tree-decomposition $(T, \chi)$ of $G$ as follows: Let $P_i = w_{i,0}, \ldots, w_{i,j_i}$ such that $\chi_i(w_{i,0}) = \{d_1, \ldots, d_k\}$. Then, let $T$ be the union of the $P_i$ formed by identifying all $w_{i,0}$. As well, we define $\chi$ to be the union of the $\chi_i$'s. It is not difficult to verify that $(T, \chi)$ is a tree-decomposition with the appropriate properties. The fact that $G$ is $j$-connected can be proved in a manner similar to that used in Lemma 5.1.

**Lemma 5.3.** **SUBGRAPH ISOMORPHISM** on partial $k$-trees is NP-complete when $G$ is $g$-connected, $H$ is $k$-connected, and $g$ is less than $k$. 
Proof. Let \((I_1, \ldots, I_{3m}, B)\) be an instance of 3-Partition. Let \(G\) be the enhanced star-union of a \(k\)-star \(\mathcal{X}\) and \(g\)-spirals \(\mathcal{Y}_1, \ldots, \mathcal{Y}_m\) of length \(I_1, \ldots, I_{3m}\) respectively with \(d_1, \ldots, d_{g-1}\) the identified nodes and \(d_g\) the clique node adjacent to all first nodes in all \(\mathcal{Y}_i\). By Lemma 5.2, \(G\) is \(g\)-connected and has tree-width \(k\). Let \(H\) be the enhanced star-union of a \(k\)-star \(\mathcal{X}'\) and \(k\)-spirals \(\mathcal{Y}_1, \ldots, \mathcal{Y}_m\) all of length \(B\) with \(d'_1, \ldots, d'_{k-1}\) the identified nodes and \(d'_k\) the clique node adjacent to all first exterior nodes in all \(\mathcal{Y}_i\). By Lemma 5.2, \(H\) is \(k\)-connected and has tree-width \(k\). We claim that \((I_1, \ldots, I_{3m}, B)\) has a solution if and only if \(G\) is a subgraph of \(H\); the argument is very similar to that in Lemma 4.4 except that there is an additional edge from the first node of each spiral to the clique.

If there is a solution to \((I_1, \ldots, I_{3m}; B)\), then there is a partition \(C_1, \ldots, C_m\) of \(I_1, \ldots, I_{3m}\) each with three elements such that the sum of the elements in any \(C_i\) is \(B\). Notice that \(C_i\) corresponds to a set of spirals of total length \(B\). Then the mapping from \(G\) to \(H\) is specified as follows: \(f(d_i) = d'_i\), pendant nodes in \(G\) map to pendant nodes in \(H\) in an arbitrary manner, and the exterior nodes of \(G\) in spirals associated with \(C_i\) appear as a contiguous set of nodes in some spiral of \(H\).

For the converse, suppose \(G\) is isomorphic to a subgraph of \(H\) and let \(f : V(G) \rightarrow V(H)\) be the subgraph isomorphism. Clearly \(f\) is a bijection since \(|V(G)| = |V(H)|\); we wish to conclude that exterior nodes of \(G\) map to exterior nodes of \(H\).

Notice that identified nodes in \(G\) have degree \(k-1+3+mB\), the node \(d_g\) has degree \(k-1+3+3m\), all other clique nodes have degree \(k+2\), and all exterior nodes have degree at most \(k+1\). In \(H\), identified nodes have degree \(k-1+3+mB\), the node \(d'_k\) has degree \(k-1+3+m\), and all exterior nodes have degree at most \(k+1\). Therefore, simply because of degree constraints, for \(1 \leq i \leq g\), \(f(d_i) \in \{d'_1, \ldots, d'_{k-1}\}\); without loss of generality assume that \(f(d_i) = d'_i\). Then it follows, again from degree constraints, that for \(g < i \leq k\), clique nodes \(d_i\) must map to clique nodes \(d'_j\) for \(j > g\). From this we conclude that clique nodes in \(G\) map to clique nodes in \(H\). As well, pendant nodes of \(G\) must map to pendant nodes of \(H\); if pendant node \(p\) of \(G\) maps to an exterior node \(w\) of \(H\), then the pre-image of an exterior neighbour \(v\) of \(w\) does not exist in \(G\). Therefore, exterior nodes of \(G\) map to exterior nodes of \(H\).

Now, the image under \(f\) of the exterior nodes of a particular spiral in \(G\) must form a consecutive set of exterior nodes in \(H\). Consequently, each of the spirals \(\mathcal{Y}_i\) must be contained entirely in \(\mathcal{Y}_j\) for some \(j\). Since \(f\) is a bijection, this forms a partition of \(\{I_1, \ldots, I_{3m}\}\) into sets of size \(B\) each specified by the \(\mathcal{Y}_j\), as required. \(\square\)

Theorem 3.1 is a consequence of Lemmas 4.6 and 5.3. Theorem 3.2 follows from the fact that the constructions above entail at most \(k\) nodes having unbounded degree, the remainder of the nodes having degree at most \(k+2\).

6. Conclusions and directions for further research

In this paper we have shown that the subgraph isomorphism problem for partial \(k\)-trees is NP-complete when either the source graph is not \(k\)-connected or there are at
least \( k \) vertices of unbounded degree. It is straightforward to verify that all graphs used in our reductions are chordal. Therefore, we have actually shown something stronger, namely that subgraph isomorphism is NP-complete even for bounded clique size chordal graphs.

One open question is the complexity of the problem on graphs of bounded pathwidth when only \( H \) is \( k \)-connected. A second open problem is that of determining the minimum number of unbounded degree nodes which makes these problems NP-complete. In particular, is it sufficient to have only one unbounded degree node to ensure NP-completeness?

Since partial \( k \)-paths are a subclass of partial \( k \)-trees and since topological embedding and minor containment are generalizations of subgraph isomorphism, our results immediately imply the NP-completeness of these problems for these classes of graphs. It would be interesting to know whether or not there are other problems for which the connectivity and degree of \( k \)-paths or \( k \)-trees yields such a fine distinction in complexity.

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