PERGAMON Computers and Mathematics with Applications 37 (1999) 89–97

An International Journal
computers & mathematics with applications

GP_G-Stability of Runge-Kutta Methods for Generalized Delay Differential Systems

BIAO YANG*
Department of Mathematics, Shanghai Normal University
Shanghai 200234, P.R. China

L. QIU AND T. MITSUI
Graduate School of Human Informatics, Nagoya University
Nagoya 464-8601, Japan

(Received June 1998; accepted July 1998)

Abstract—The GP_G-stability of Runge-Kutta methods for the numerical solutions of the systems of delay differential equations is considered. The stability behaviour of implicit Runge-Kutta methods (IRK) is analyzed for the solution of the system of linear test equations with multiple delay terms. After an establishment of a sufficient condition for asymptotic stability of the solutions of the system, a criterion of numerical stability of IRK with the Lagrange interpolation process is given for any stepsize of the method.

Keywords—Generalized delay differential system, Runge-Kutta method, GP_G-stability.

1. INTRODUCTION

In 1994, in 't Hout [1] considered the stability of θ-methods for systems of differential equations with a single delay. For the initial value problem

$$\mathbf{y}'(t) = L\mathbf{y}(t) + M\mathbf{y}(t-\tau), \quad t \geq 0, \qquad \mathbf{y}(t) = \Phi(t), \quad t \leq 0, \tag{1.1}$$

he presented a necessary and sufficient condition for the asymptotic stability. Here τ stands for a positive constant delay, L and $M \in \mathbf{C}^{d\times d}$, $\mathbf{y}(t) = (y_1(t), y_2(t), \dots, y_d(t))^\top$ denotes the d-dimensional unknown vector, and $\phi(t)$ is a given vector-valued initial function. Furthermore he proved that the θ-methods applied to the asymptotically stable system is numerically asymptotically stable if and only if $1/2 \leq \theta \leq 1$. In the same year, Koto [2] gave another sufficient condition of the asymptotic stability of IRK for above systems. In 1995, Kuang [3] considered the stability of linear multistep methods (LMM) for the system of multiple delay

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \sum_{j=1}^{m} B_j\mathbf{y}_j(t-\tau_j), \quad t \geq 0, \qquad \mathbf{y}(t) = \Phi(t), \quad t \leq 0, \tag{1.2}$$

*Supported by Science and Technology Foundation of Shanghai Higher Education.

0898-1221/99/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(99)00089-9

where τ_j ($j = 1, \dots, m$) denote positive constant delays, and proved that LMM is GP_m-stable if and only if it is A-stable. Here the GP_m-stability means an extended version of the GP-stability for the case of multiple delay. In 1997, in 't Hout [4] also investigated the numerical stability of Runge-Kutta methods with a certain interpolation process for the systems (1.1), and obtained that an A-stable Runge-Kutta method preserves the asymptotic stability properties of analytic solutions of the systems (1.1). In particular, when L and M reduce to scalar numbers, stability of numerical methods has been widely studied (see [5–9] and references therein).

On the other hand, in 1991 Lu [10] treated a simple but very interesting system of delay differential equations with double delay given by

$$\begin{aligned} u'(t) &= a_1 u(t) + b_1 v(t - \tau_1), \\ v'(t) &= a_2 v(t) + b_2 u(t - \tau_2), \end{aligned} \tag{1.3}$$

or, in a vector form, by

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(t_\tau), \tag{1.4}$$

where matrices A and B are given by

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix},$$

and the vectors mean

$$\mathbf{y}(t) = (u(t), v(t))^\top, \qquad \mathbf{y}(t_\tau) = (u(t - \tau_1), v(t - \tau_2))^\top.$$

He showed that the solution of (1.4) satisfies the condition $\lim_{t\to\infty} y(t) = 0$ if

$$\Re(a_i) < 0, \quad i = 1, 2 \qquad \text{and} \qquad |b_1 b_2| < \Re(a_1)\Re(a_2), \tag{1.5}$$

and the θ-methods are asymptotically stable if and only if $1/2 \le \theta \le 1$.

In the present paper, we are concerned with the numerical solution and its stability of a more generalized initial value problem of Lu's given by

$$\mathbf{y}'(t) = L\mathbf{y}(t) + M\mathbf{y}(t_\tau), \quad t \ge 0, \qquad \mathbf{y}(t) = \Phi(t), \quad t \le 0. \tag{1.6}$$

Here L and M again denote constant complex matrices, $\mathbf{y}(t) = (y_1(t), y_2(t), \dots, y_d(t))^\top$ means the unknowns, and its multiple delay value is given as $\mathbf{y}(t_\tau) = (y_1(t - \tau_1), y_2(t - \tau_2), \dots, y_d(t - \tau_d))^\top$ with positive constants τ_j ($j = 1, 2, \dots, d$). We will devote ourselves to the application of IRK methods to (1.6). First, we will give a sufficient condition of asymptotic stability of solutions of the system (1.6). Second, under the condition, we will prove that an A-stable Runge-Kutta method with the Lagrange interpolation process is GP_G-stable when applied to the system. The GP_G-stability is a stability concept corresponding to GP-stability of scalar DDE.

2. IMPLICIT RUNGE-KUTTA METHOD

Before dealing with the numerical stability of IRK for systems of DDEs, we consider the following initial value problem of ordinary differential equations (ODEs).

$$y'(t) = f(t, y(t)), \quad t > 0, \qquad y(0) = y_0, \tag{2.1}$$

where f is a given function and $y(t)$ is the unknown for $t > 0$.

When an s-stage IRK with the stepsize h is applied to the problem (2.1), it reads

$$\begin{aligned} K_i &= hf\left(t_n + c_i h, y_n + \sum_{j=1}^{s} a_{ij} K_j\right), & i &= 1, 2, \dots, s \\ y_{n+1} &= y_n + \sum_{i=1}^{s} b_i K_i, & n &= 0, 1, 2, \dots. \end{aligned} \tag{2.2}$$

Here y_n stands for an approximation of the solution $y(t_n)$ at the step-point $t_n = nh$, and the formula parameters a_{ij}, b_i, and c_i are also expressed in the Butcher tableau given by

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^\top \end{array}$$

where $A, \mathbf{b}$ and $\mathbf{c}$ are s-dimensional square matrix and vectors, respectively. Furthermore, by introducing the s-dimensional vector $\mathbf{e} = (1, 1, \ldots, 1)^\top$, we assume

$$\mathbf{e}^\top \mathbf{b} = 1 \quad \text{and} \quad A\mathbf{e} = \mathbf{c}.$$

As is well known, the linear stability analysis is carried out through the application of IRK (2.2) to the following linear test equation

$$y'(t) = \lambda y(t), \qquad \Re\lambda < 0, \quad y(0) = y_0. \tag{2.3}$$

It implies the numerical recurrence formula (e.g., [11])

$$y_{n+1} = r(z) y_n, \qquad n = 0, 1, \ldots, \quad (z = \lambda h) \tag{2.4}$$

and the rational function $r(z)$, called the stability factor of IRK, is given by

$$\begin{aligned} r(z) &= 1 + z\mathbf{b}^\top (I - zA)^{-1}\mathbf{e} \\ &= \frac{\det\left[I - z\left(A - \mathbf{e}\mathbf{b}^\top\right)\right]}{\det[I - zA]}, \qquad \text{if } \det[I - zA] \neq 0. \end{aligned} \tag{2.5}$$

Thus we arrive at the following definition (e.g., [11]).

DEFINITION 2.1. *Let $R(q)$ be a function of q.*

a. *If any real negative q implies $|R(q)| < 1$, then $R(q)$ is said to be A_0-acceptable.*
b. *If any complex q of negative real part implies $|R(q)| < 1$, then $R(q)$ is said to be A-acceptable.*
c. *If an A-acceptable $R(q)$ further satisfies the condition $\lim_{\Re q \to -\infty} |R(q)| = 0$, then $R(q)$ is said to be L-acceptable.*

By virtue of this definition, we call an IRK A_0-stable, A-stable, and L-stable when it provides the stability factor being A_0-acceptable, A-acceptable, and L-acceptable, respectively.

3. A SUFFICIENT CONDITION OF ASYMPTOTIC STABILITY

For the systems (1.6), we will briefly give a sufficient condition of the asymptotic stability. Here we mean the asymptotic stability by

$$\text{every solution of (1.6) satisfies } \lim_{t\to\infty} y(t) = 0.$$

The following is readily seen (e.g., [10]).

LEMMA 3.1. *The system is asymptotically stable if and only if all the roots of the characteristic equation*

$$\det\left[\zeta I - L - Me^{-\zeta\mathrm{T}}\right] = 0 \tag{3.1}$$

have negative real parts. Here $e^{-\zeta\mathrm{T}}$ stands for the matrix $\mathrm{diag}\{e^{-\zeta\tau_1}, e^{-\zeta\tau_2}, \ldots, e^{-\zeta\tau_d}\}$.

Below we give a sufficient condition of the asymptotic stability which will provide a basis of numerical stability analysis. Hereafter, we adopt the inner product of d-dimensional vectors and the vector norm induced from it. The matrix norm is the subordinate one of the vector norm. Moreover the Hermitian conjugate of the matrix L will be denoted by L^*.

THEOREM 3.1. *If the matrices L and M in (1.6) satisfy the condition*

$$\|M\| < -\frac{1}{2}\lambda_{\max}(L + L^*), \tag{3.2}$$

then the system is asymptotically stable. Here $\lambda_{\max}$ denotes the maximum eigenvalue of a matrix.

The quantity $-(1/2)\lambda_{\max}(L+L^*)$, the maximum eigenvalue of the "real part" of the matrix L, has a crucial role in the present paper. Thus we will denote it by $\eta(L)$. The proof is similar as in [12]. However, here we will give it for the reference.

PROOF. By virtue of Lemma 3.1, it suffices to show the negativeness of the real part of characteristic roots. Equation (3.1) implies the existence of a unit vector $\mathbf{x} \in \mathbf{C}^d$ satisfying

$$\left(\zeta I - L - Me^{-\zeta\mathrm{T}}\right)\mathbf{x} = \mathbf{0}.$$

Let H_1 and H_2 be the real and imaginary parts, respectively, of the matrix L. That is, $L = H_1 + \mathrm{i}H_2$, $H_1 = (1/2)(L + L^*)$ and $H_2 = (1/2\mathrm{i})(L - L^*)$. Put $\zeta = a + ib$. The inner product with $\mathbf{x}$ of the above equation yields

$$\zeta - \langle H_1\mathbf{x}, \mathbf{x}\rangle - \mathrm{i}\langle H_2\mathbf{x}, \mathbf{x}\rangle = \left\langle Me^{-a\mathrm{T}}e^{-ib\mathrm{T}}\mathbf{x}, \mathbf{x}\right\rangle,$$

which further implies

$$\begin{aligned}(a - \langle H_1\mathbf{x}, \mathbf{x}\rangle)^2 + (b - \langle H_2\mathbf{x}, \mathbf{x}\rangle)^2 &= \left|\left\langle Me^{a\mathrm{T}}e^{-ib\mathrm{T}}\mathbf{x}, \mathbf{x}\right\rangle\right|^2 \\ &\le \|M\|^2 \cdot \|e^{-a\mathrm{T}}\|^2 \cdot \|e^{-ib\mathrm{T}}\|^2 = \|M\|^2 \cdot \|e^{-a\mathrm{T}}\|^2.\end{aligned}$$

Suppose that $a \ge 0$. The above inequality gives

$$|a - \langle H_1\mathbf{x}, \mathbf{x}\rangle| \le \|M\|.$$

On the other hand, the condition (3.2) implies $\langle H_1\mathbf{x}, \mathbf{x}\rangle < 0$, which, together with the above inequality, gives

$$a - \langle H_1\mathbf{x}, \mathbf{x}\rangle \le \|M\|.$$

Thus we have

$$-\eta(L) \le -\langle H_1\mathbf{x}, \mathbf{x}\rangle \le \|M\|.$$

This contradicts our assumption, and the proof is completed. ∎

4. CHARACTERISTIC EQUATION OF IRK FOR DDES

Now we express the numerical process when an IRK is applied to (1.6). The numerical solution is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \sum_{i=1}^{s} b_i K_{n,i}, \tag{4.1}$$

while the stage values $K_{n,i}$ are computed by

$$K_{n,i} = h\left[L\left(\mathbf{y}_n + \sum_{j=1}^{s} a_{ij}K_{n,j}\right) + M\left(\mathbf{y}_{n-m(\tau)} + \sum_{j=1}^{s} a_{ij}K_{n-m(\tau),j}\right)\right], \qquad (i = 1, 2, \dots, s). \tag{4.2}$$

Here for each i, $\delta_i \in [0,1)$ denotes the real number given by $\delta_i = m_i - \tau_i/h$ for a certain integer m_i, and the approximated delayed values $\mathbf{y}_{n-m(\tau)}$ and $K_{n-m(\tau),j}$ are calculated through the Lagrange interpolation process. That is,

$$\begin{aligned} \mathbf{y}_{n-m(\tau)} &= \left(\sum_{p_1=-q}^{r} L_{p_1}(\delta_1) y_{n-m_1+p_1}^{(1)}, \dots, \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) y_{n-m_d+p_d}^{(d)} \right)^{\top}, \\ K_{n-m(\tau),j} &= \left(\sum_{p_1=-q}^{r} L_{p_1}(\delta_1) K_{n-m_1+p_1,j}^{(1)}, \dots, \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) K_{n-m_d+p_d,j}^{(d)} \right)^{\top} \end{aligned}$$

for $j = 1, 2, \dots, s$ and the Lagrange interpolation multiplier is

$$L_p(\delta) = \prod_{k=-q, k\neq p}^{r} \frac{\delta - k}{p - k}.$$

Here the superscript (l) indicates the l^{th} component of the vector and the initial function $\Phi(t)$ will be referred whenever the argument t is negative. Note that the integers m_i are supposed to be greater than or equal to $r+1$. It is possible when the stepsize h becomes sufficiently small.

Next we will convert equations (4.1) and (4.2) into a more compact form for the stability analysis. By introducing (sd)-dimensional vectors as

$$\begin{aligned} \mathbf{K}_n &= \left(K_{n,1}^{\top}, K_{n,2}^{\top}, \dots, K_{n,s}^{\top} \right)^{\top}, \\ \mathbf{K}_{n-m(\tau)}^{\#} &= \left(K_{n-m(\tau),1}^{\top}, \dots, K_{n-m(\tau),s}^{\top} \right)^{\top}, \end{aligned}$$

and two matrices $\mathcal{L} = Lh$ and $\mathcal{M} = Mh$, we can express (4.1) and (4.2) in a matrix form

$$\begin{aligned} &\begin{pmatrix} I_{sd} - A \otimes \mathcal{L} & 0 \\ -\mathbf{b}^{\top} \otimes I_d & I_d \end{pmatrix} \begin{pmatrix} \mathbf{K}_n \\ \mathbf{y}_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{e} \otimes \mathcal{L} \\ 0 & I_d \end{pmatrix} \begin{pmatrix} \mathbf{K}_{n-1} \\ \mathbf{y}_n \end{pmatrix} \\ &+ \begin{pmatrix} A \otimes \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{K}_{n-m(\tau)}^{\#} \\ \mathbf{y}_{n+1-m(\tau)} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{e} \otimes \mathcal{M} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{K}_{n-1-m(\tau)}^{\#} \\ \mathbf{y}_{n-m(\tau)} \end{pmatrix}. \end{aligned} \tag{4.3}$$

Assume a solution of the above recurrence equation of the type

$$\begin{pmatrix} \mathbf{K}_n \\ \mathbf{y}_{n+1} \end{pmatrix} = z^n \mathbf{x},$$

where $\mathbf{x}$ is a complex $(s+1)$ d-dimensional vector

$$\mathbf{x} = \left(\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(d)}, \xi^{(d+1)}, \dots, \xi^{((s+1)d)} \right)^{\top}$$

and z is a complex variable. A calculation leads to

$$\begin{aligned} \begin{pmatrix} \mathbf{K}_{n-m(\tau)}^{\#} \\ \mathbf{y}_{n+1-m(\tau)} \end{pmatrix} = \Bigg(& \sum_{p_1=-q}^{r} L_{p_1}(\delta_1) z^{n-m_1+p_1} \xi^{(1)}, \dots, \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) z^{n-m_d+p_d} \xi^{(d)}, \dots, \\ & \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) z^{n-m_d+p_d} \xi^{((s+1)d)} \Bigg)^{\top}. \end{aligned}$$

Hence, by introducing a d-dimensional diagonal matrix T_m as

$$T_m = \operatorname{diag}\left(\sum_{p_1=-q}^{r} L_{p_1}(\delta_1) z^{-m_1+p_1}, \sum_{p_2=-q}^{r} L_{p_2}(\delta_2) z^{-m_2+p_2}, \dots, \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) z^{-m_d+p_d} \right)$$

and furthermore an $(s+1)$ d-dimensional diagonal matrix $\mathcal{T}$ as

$$\mathcal{T} = \operatorname{diag}(T_m, T_m, \dots, T_m),$$

we obtain

$$\begin{pmatrix} \mathbf{K}^{\#}_{n-m(\tau)} \\ \mathbf{y}_{n+1-m(\tau)} \end{pmatrix} = z^n \mathcal{T} \mathbf{x}.$$

Substitution of the above results into (4.3) yields

$$\begin{aligned} &\left\{ z^n \begin{pmatrix} I_{sd} - A \otimes \mathcal{L} & 0 \\ -\mathbf{b}^\top \otimes I_d & I_d \end{pmatrix} - z^{n-1} \begin{pmatrix} 0 & \mathbf{e} \otimes \mathcal{L} \\ 0 & I_d \end{pmatrix} \right. \\ &\left. -z^n \begin{pmatrix} A \otimes \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{T} - z^{n-1} \begin{pmatrix} 0 & \mathbf{e} \otimes \mathcal{M} \\ 0 & 0 \end{pmatrix} \mathcal{T} \right\} \mathbf{x} = \mathbf{0}. \end{aligned} \tag{4.4}$$

The condition which assures the existence of a nonzero $\mathbf{x}$ in (4.4) implies the characteristic equation of IRK applied to (1.6) with the Lagrange interpolation as follows.

$$\det \begin{pmatrix} (I_{sd} - A \otimes \mathcal{L})z - (A \otimes \mathcal{M})\mathcal{T}z & -(\mathbf{e} \otimes \mathcal{L}) - (\mathbf{e} \otimes \mathcal{M})\mathcal{T} \\ (-\mathbf{b}^\top \otimes I_d)z & I_d z - I_d \end{pmatrix} = 0. \tag{4.5}$$

To reduce the expression (4.5) simpler, we define the following four matrices.

$$\begin{aligned} \mathcal{Q}_1(z) &= z(I_{sd} - A \otimes (\mathcal{L} + \mathcal{M}T_m)), \\ \mathcal{Q}_2(z) &= -\mathbf{e} \otimes (\mathcal{L} + \mathcal{M}T_m), \\ \mathcal{Q}_3(z) &= -(\mathbf{b}^\top \otimes I_d)z, \\ \mathcal{Q}_4(z) &= (z-1)I_d. \end{aligned}$$

Thus we have another expression of the characteristic equation as

$$\det \begin{pmatrix} \mathcal{Q}_1(z) & \mathcal{Q}_2(z) \\ \mathcal{Q}_3(z) & \mathcal{Q}_4(z) \end{pmatrix} = 0. \tag{4.6}$$

The following lemma can be readily seen.

LEMMA 4.1. *If* $\det \mathcal{Q}_1(z)$ *does not vanish, the equation (4.6) is equivalent to*

$$\det \left[\mathcal{Q}_4(z) - \mathcal{Q}_3(z)\mathcal{Q}_1(z)^{-1} \cdot \mathcal{Q}_2(z) \right] = 0. \tag{4.7}$$

Note that due to the definition of $\mathcal{L}$ and $\mathcal{M}$ we can assert the nonsingularity of $\mathcal{Q}_1(z)$ for sufficiently small h. We can, henceforth, regard (4.7) as the final form of the characteristic equation.

5. GP_G-STABILITY OF IRK METHODS

Since delay differential equations include the delay argument, numerical stability strongly depends on the relationship of the stepsize with the magnitude of the delay. Therefore, we will give two definitions of numerical stability for DDEs.

DEFINITION 5.1. *A numerical method for DDEs (1.6) is said to be* P_G*-stable if, for an asymptotically stable system, the numerical solutions* $\{\mathbf{y}_n\}$ *satisfies*

$$\lim_{n \to \infty} \mathbf{y}_n = \mathbf{0}$$

for every stepsize h *of integral fraction of all* τ_i.

However, we are treating the several delay case, which does not always imply existence of the stepsize h of integral fraction of *all* τ_i. We need, therefore, to introduce the following.

DEFINITION 5.2. *A numerical method for DDEs (1.6) is called GP_G-stable if, for an asymptotically stable system, the condition $\mathbf{y}_n \to \mathbf{0}$ holds as $n \to \infty$ for every $h > 0$.*

Since in Section 3, we have obtained a sufficient condition of the asymptotic stability of linear DDEs (1.6), we refer to the condition (3.2) as our basis of numerical stability.

To attain a criterion of GP_G-stability of IRK for DDEs, an estimation of the magnitude is significant on the effect induced from the Lagrange interpolation process. Thus we focus on the polynomial

$$\gamma(z,\delta) = \sum_{p=-q}^{r} L_p(\delta) z^{p+q}.$$

Fortunately, Strang [13] and Iserles and Strang [14] established the following result.

LEMMA 5.1. *Whenever $|z| = 1$ and $0 \le \delta < 1$, the estimation*

$$|\gamma(z,\delta)| \le 1$$

holds if and only if the relationship $q \le r \le q+2$ is valid. Moreover, if $q+r > 0$ and $q \le r \le q+2$, $|z| = 1$ and $0 < \delta < 1$ holds, then $|\gamma(z,\delta)| = 1$ if and only if $z = 1$.

First, investigate the magnitude of

$$R(z;\delta) = \sum_{p=-q}^{r} L_p(\delta) \cdot z^{p-m} = \gamma(z,\delta) z^{-(m+q)}$$

when the integer m is greater than r. In the case of z on the unit circle, Lemma (5.1) implies $|R(z,\delta)| \le 1$ for every $\delta \in [0,1)$. When $|z| = \infty$, we have $|R(z,\delta)| = 0$ since the stepsize h can be so small that $m \ge s+1$ holds. Thus the maximum modulus principle for an analytic function yields

$$|R(z,\delta)| \le 1, \qquad \text{for } |z| \ge 1 \text{ and } \delta \in [0,1). \tag{5.1}$$

Taking advantage of the above estimation, we have the following lemma.

LEMMA 5.2. *Assume that the condition (3.2) holds and every integer m_i is greater than r. Then we have that all the eigenvalues of the matrix $L + MT_m$ have negative real part for $|z| \ge 1$. That is, $\Re(\lambda_i(L + MT_m)) < 0$ for $i = 1,2,\dots,d$ and $|z| \ge 1$.*

PROOF. As $|z| \ge 1$ and $m_i \ge s+1$ ($i = 1,2,\dots,d$), the estimation (5.1) implies

$$\|T_m\| = \left\| \operatorname{diag}\left(\sum_{p_1=-q}^{r} L_{p_1}(\delta_1) z^{-m_1+p_1}, \dots, \sum_{p_d=-q}^{r} L_{p_d}(\delta_d) z^{-m_d+p_d} \right) \right\| \le 1.$$

Let $\lambda = a+ib$ be an eigenvalue of $L+MT_m$. There exists a vector $\mathbf{x} \in \mathbf{C}^d$ of unit length satisfying $(L + MT_m)\mathbf{x} = \lambda\mathbf{x}$. Taking the inner product with x, we obtain

$$\lambda - \langle L\mathbf{x},\mathbf{x}\rangle - \langle MT_m\mathbf{x},\mathbf{x}\rangle = 0,$$

which, with the Schwartz' inequality, yields

$$a = \langle H_1\mathbf{x},\mathbf{x}\rangle + r \cdot \cos\phi \le \eta(L) + \|M\|.$$

Here we put $\langle MT_m\mathbf{x},\mathbf{x}\rangle = re^{i\phi}$. This completes the proof. ∎

Now we give our main theorem.

THEOREM 5.1. *Assume that the condition (3.2) holds and $q \le r \le q+2$. An IRK applied to the system (1.6) with the Lagrange interpolation process is GP_G-stable if and only if it is A-stable.*

PROOF. Assume that IRK is A-stable. It suffices to show that all the roots of its characteristic equation (4.6) have the modulus less than unity. Assume there exists a root $\tilde{z}$ of (4.6) with the modulus greater than or equal to unity. Lemma 5.2 implies also $\Re\lambda_i(\mathcal{L} + \mathcal{M}T_m) < 0$ for positive h. Thus we have for $|\tilde{z}| \ge 1$

$$\begin{aligned}\det\left[Q_1(\tilde{z})\right] &= \tilde{z}^{sd}\det\left[I_{sd} - A\otimes\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right]\\ &= \tilde{z}^{sd}\prod_{j=1}^{d}\det\left[I_s - A\cdot\lambda_j\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right] \ne 0.\end{aligned}$$

Here $\tilde{T}_m$ means the matrix T_m substituted with $\tilde{z}$ in the variable z.

Now we can apply Lemma 4.1, which derives the equation of $\tilde{z}$ as

$$\begin{aligned}&\det\left[I_d(\tilde{z}-1) - \left(\mathbf{b}^\top\otimes I_d\right)\tilde{z}\left\{\tilde{z}\left[I_{sd} - A\otimes\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right]\right\}^{-1}\left(\mathbf{e}\otimes\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right)\right]\\ &\quad= \det\left[(\tilde{z}-1)I_d - \left(\mathbf{b}^\top\otimes I_d\right)\left[I_{sd} - A\otimes\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right]^{-1}\left(\mathbf{e}\otimes\left(\mathcal{L}+\mathcal{M}\tilde{T}_m\right)\right)\right] = 0\end{aligned} \tag{5.2}$$

for $|\tilde{z}| \ge 1$. In the determinat of (5.2) we can notice the block of matrices of the form $\mathcal{L}+\mathcal{M}\tilde{T}_m$. Actually, denoting it by $Q(\tilde{z})$, we can establish

$$\det[\tilde{z}I_d - r(Q(\tilde{z}))] = 0 \tag{5.3}$$

in place of (5.2), where

$$r(Q(\tilde{z})) = I_d + \left(\mathbf{b}^\top\otimes I_d\right)\left(I_{sd} - A\otimes Q(\tilde{z})\right)^{-1}(\mathbf{e}\otimes Q(\tilde{z})).$$

By the spectral mapping theorem, we have

$$\lambda(r(Q(\tilde{z}))) = r(\lambda(Q(\tilde{z})),$$

which, together with equation (5.3), the inequality $\Re(\lambda(Q(\tilde{z})) < 0$ and the A-stability of IRK, implies

$$|\tilde{z}| = |r(\lambda(Q(\tilde{z}))| < 1.$$

This contradicts the assumption $|\tilde{z}| \ge 1$.

Converse is obviously seen, and the proof is completed. ∎

6. CONCLUDING REMARKS

The condition (3.2) can be easily applied in actual applications. As a matter of fact, numerical evaluations of the spectral radius as well as of the maximum eigenvalue are available. Thus our condition may contribute to the stability judgement, and the numerical solutions based on IRK will be of practical value. However, the condition is only a sufficient one, and we cannot yet discriminate all the stable cases. It is a future problem.

REFERENCES

1. K.J. in 't Hout, The stability of θ-methods for systems of delay differential equations, *Ann. Numer. Math.* **1**, 323–334, (1994).
2. T. Koto, A stability property of A-stable natural Runge-Kutta methods for systems of delay differential equations, *BIT* **34**, 262–267, (1994).

3. J. Kuang and H. Tian, The numerical treatment of linear systems with many delays, Report in AMS meeting #904, Kent State Univ., Nov. 3–4, 1995.
4. K.J. in 't Hout, Stability analysis of Runge-Kutta methods for systems of delay differential equations, *IMA J. Numer. Anal.* **17**, 17–27, (1997).
5. A.N. Al-Mutib, Stability properties of numerical methods for solving delay differential equations, *J. Comput. Appl. Math.* **10** (1), 71–79, (1981).
6. V.K. Barwell, Special stability problems for functional differential equations, *BIT* **15**, 130–135, (1975).
7. K.J. in 't Hout, A new interpolation procedure for adapting Runge-Kutta methods for delay differential equations, *BIT* **32**, 634–649, (1992).
8. Z. Jackiewicz, Asymptotic stability analysis of θ-methods for functional differential equations, *Numer. Math.* **43**, 389–396, (1984).
9. M. Zennaro, P-stability properties of Runge-Kutta methods for delay differential equations, *Numer. Math.* **49**, 305–318, (1986).
10. L. Lu, Numerical stability of θ-methods for systems of differential equation with several delay terms, *J. Comput. Appl. Math.* **34**, 291–304, (1991).
11. J.D. Lambert, *Numerical Methods for Ordinary Differential Systems*, John-Wiley, Chichester, (1991).
12. J. Kuang and H. Tian, The asymptotic behaviour of theoretical and numerical solutions for nonlinear differential systems with several delay terms, *Journal of Shanghai Teachers Univ., Natural Sciences* **24**, 1–7, (1994).
13. G. Strang, Trigonometric polynomials and difference methods of maximum accuracy, *J. Math. Phys.* **41**, 147–154, (1962).
14. A. Iserles and G. Strang, The optimal accuracy of difference schemes, *Trans. Amer. Math. Soc.* **277**, 799–803, (1983).
15. K.J. in 't Hout and M.N. Spijker, Stability analysis of numerical methods for delay differential equations, *Numer. Math.* **59**, 807–814, (1991).
16. J. Kuang and H. Tian, The asymptotical stability of theoretical and numerical solution for delay differential equations with many delays, *Journal of Shanghai Teachers Univ.* A special issue of Math., (1994).