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Journal of Functional Analysis 257 (2009) 428–463

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Multi-variable subordination distributions for free additive convolution

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Received 5 November 2008; accepted 29 December 2008

Available online 26 January 2009

Communicated by D. Voiculescu

Abstract

Let k be a positive integer and let $\mathcal{D}_c(k)$ denote the space of joint distributions for k-tuples of selfadjoint elements in C^*-probability space. The paper studies the concept of "subordination distribution of $\mu \boxplus \nu$ with respect to ν" for $\mu, \nu \in \mathcal{D}_c(k)$, where $\boxplus$ is the operation of free additive convolution on $\mathcal{D}_c(k)$. The main tools used in this study are combinatorial properties of R-transforms for joint distributions and a related operator model, with operators acting on the full Fock space.

Multi-variable subordination turns out to have nice relations to a process of evolution towards $\boxplus$-infinite divisibility on $\mathcal{D}_c(k)$ that was recently found by Belinschi and Nica (arXiv: 0711.3787). Most notably, one gets better insight into a connection which this process was known to have with free Brownian motion.

Keywords: Free additive convolution; R-transform; Subordination distribution; Non-crossing partition; Operator model on full Fock space

1. Introduction and statements of results

The free additive convolution $\boxplus$ is a binary operation on the space of probability distributions on $\mathbb{R}$, reflecting the addition operation for free random variables in a noncommutative probability space. A significant fact in its theory (see [9,15,16]) is that the Cauchy transform of the distribution $\mu \boxplus \nu$ is subordinated to the Cauchy transforms of μ and of ν, as analytic functions

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[1] Research supported by a Discovery Grant from NSERC, Canada.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.12.022

on the upper half-plane $\mathbb{C}^+$. Thus (choosing for instance to discuss subordination with respect to ν) one has an analytic subordination function $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$G_{\mu \boxplus \nu}(z) = G_\nu\big(\omega(z)\big), \quad \forall z \in \mathbb{C}^+,$$

where $G_{\mu \boxplus \nu}$ and G_ν are the Cauchy transforms of $\mu \boxplus \nu$ and of ν, respectively. Moreover, the subordination function ω can be identified as the reciprocal Cauchy transform of a uniquely determined probability distribution σ on $\mathbb{R}$. Following [11], this σ will be denoted as "$\mu \boxminus \nu$." The name used in [11] for $\sigma = \mu \boxminus \nu$ is "s-free additive convolution of μ and ν," in relation to a suitably tailored concept of "s-freeness" that is also introduced in [11]. Since s-freeness is only marginally addressed in the present paper, $\mu \boxminus \nu$ will just be called here the *subordination distribution* of $\mu \boxplus \nu$ with respect to ν.

The goal of the present paper is to introduce and study the analogue for $\mu \boxminus \nu$ in a multi-variable framework where μ, ν become joint distributions of k-tuples of selfadjoint elements in a C^*-probability space. The particular case $k = 1$ corresponds of course to the framework of probability distributions on $\mathbb{R}$ as discussed above, with μ, ν compactly supported. The main tool used in the paper is the R-transform for joint distributions. In particular, the k-variable version of $\mu \boxminus \nu$ is introduced in Definition 1.1 below via an extension of the formula which describes $R_{\mu \boxminus \nu}$ in terms of R_μ and R_ν in the case $k = 1$. (The 1-variable motivation behind Definition 1.1 is presented in Section 2.1.)

It is convenient to write the definition for the k-variable version of $\mu \boxminus \nu$ by allowing μ and ν to be any linear functionals on $\mathbb{C}\langle X_1, \dots, X_k\rangle$ (the algebra of polynomials in non-commuting indeterminates $X_1, \dots, X_k$) such that $\mu(1) = \nu(1) = 1$. The set of all such "purely algebraic" distributions will be denoted by $\mathcal{D}_{\text{alg}}(k)$. The main interest of the paper is in the smaller set of "noncommutative C^*-distributions with compact support"

$$\mathcal{D}_{\text{c}}(k) := \left\{ \mu \in \mathcal{D}_{\text{alg}}(k) \;\middle|\; \begin{array}{l} \mu \text{ can appear as joint distribution for a } k\text{-tuple} \\ \text{of selfadjoint elements in a } C^*\text{-probability space} \end{array} \right\}.$$

But in order to define $\boxminus$ on $\mathcal{D}_{\text{c}}(k)$ it comes in handy to first define it as a binary operation on $\mathcal{D}_{\text{alg}}(k)$, and then prove that $\mu \boxminus \nu \in \mathcal{D}_{\text{c}}(k)$ whenever $\mu, \nu \in \mathcal{D}_{\text{c}}(k)$.

In the next definition and throughout the paper, k is a positive integer denoting "the number of variables" that one is working with.

Definition 1.1. Let μ, ν be distributions in $\mathcal{D}_{\text{alg}}(k)$. The *subordination distribution* of $\mu \boxplus \nu$ with respect to ν is the distribution $\mu \boxminus \nu \in \mathcal{D}_{\text{alg}}(k)$ uniquely determined by the requirement that its R-transform is

$$R_{\mu \boxminus \nu} = R_\mu\big(z_1(1 + M_\nu), \dots, z_k(1 + M_\nu)\big) \cdot (1 + M_\nu)^{-1}. \tag{1.1}$$

In (1.1) M_ν is the moment series of ν and $(1 + M_\nu)^{-1}$ is the inverse of $1 + M_\nu$ with respect to multiplication, in the algebra $\mathbb{C}\langle\langle z_1, \dots, z_k\rangle\rangle$ of power series in the non-commuting indeterminates $z_1, \dots, z_k$. (A more detailed review of the notations used here can be found in Section 2.3.)

Remark 1.2. 1° From Eq. (1.1) it is clear that the R-transform of $\mu \boxminus \nu$ depends linearly on the one of μ. Since the R-transform linearizes $\boxplus$, this amounts to a form of "$\boxplus$-linearity" in the way how $\mu \boxminus \nu$ depends on μ. More precisely one has

$$(\mu_1 \boxplus \mu_2) \boxminus \nu = (\mu_1 \boxminus \nu) \boxplus (\mu_2 \boxminus \nu), \quad \forall \mu_1, \mu_2, \nu \in \mathcal{D}_{\mathrm{alg}}(k), \tag{1.2}$$

or, when looking at $\boxplus$-convolution powers,

$$\left(\mu^{\boxplus t}\right) \boxminus \nu = (\mu \boxminus \nu)^{\boxplus t}, \quad \forall \mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k),\ \forall t > 0. \tag{1.3}$$

2° The series $R_\mu(z_1(1 + M_\nu), \ldots, z_k(1 + M_\nu))$ appearing in (1.1) bears a resemblance to a well-known "functional equation for the R-transform" (see [13, Lecture 16]), which says that

$$R_\mu\big(z_1(1 + M_\mu), \ldots, z_k(1 + M_\mu)\big) = M_\mu, \quad \forall \mu \in \mathcal{D}_{\mathrm{alg}}(k).$$

One can actually invoke this functional equation in the particular case of Definition 1.1 when $\nu = \mu$, to obtain that

$$R_{\mu \boxminus \mu} = M_\mu \cdot (1 + M_\mu)^{-1}, \quad \mu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{1.4}$$

The series $M_\mu \cdot (1 + M_\mu)^{-1}$ is called the *η-series* of μ, and plays an important role in the study of connections between free and Boolean probability. In particular, the relation between R-transforms and η-series yields a special bijection $\mathbb{B} : \mathcal{D}_{\mathrm{alg}}(k) \to \mathcal{D}_{\mathrm{alg}}(k)$, defined as follows: for every $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$, $\mathbb{B}(\mu)$ is the unique distribution in $\mathcal{D}_{\mathrm{alg}}(k)$ which has

$$R_{\mathbb{B}(\mu)} = \eta_\mu. \tag{1.5}$$

$\mathbb{B}$ is called the *Boolean Bercovici–Pata bijection* (first put into evidence in the 1-variable case in [7], then extended to multi-variable framework in [5]). This bijection has the important property that it carries $\mathcal{D}_{\mathrm{c}}(k)$ into itself and that $\mathbb{B}(\mathcal{D}_{\mathrm{c}}(k))$ is precisely the set of distributions in $\mathcal{D}_{\mathrm{c}}(k)$ which are infinitely divisible with respect to $\boxplus$ (cf. Theorem 1 in [5]).

By comparing (1.4) to (1.5), one draws the conclusion that

$$\mu \boxminus \mu = \mathbb{B}(\mu), \quad \forall \mu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{1.6}$$

Eq. (1.6) can be generalized to a nice formula describing $\mu_1 \boxminus \mu_2$ in the case when both μ_1 and μ_2 are $\boxplus$-convolution powers of the same μ; see Proposition 5.3 below.

3° One can rewrite Eq. (1.1) as

$$R_{\mu \boxminus \nu} \cdot (1 + M_\nu) = R_\mu\big(z_1(1 + M_\nu), \ldots, z_k(1 + M_\nu)\big), \tag{1.7}$$

and then one can equate coefficients in the series on the two sides of (1.7), in order to obtain an explicit combinatorial formula for the coefficients of $R_{\mu \boxminus \nu}$. This in turn can be used to obtain an explicit formula for the moments of $\mu \boxminus \nu$, which is stated next. In Theorem 1.3, $NC(n)$ is the set of non-crossing partitions of $\{1, \ldots, n\}$ (cf. review of $NC(n)$ terminology in Section 2.2). The notation "$(i_1, \ldots, i_n) \mid V$" stands for "$(i_{v(1)}, \ldots, i_{v(m)})$," where $V = \{v(1), \ldots, v(m)\}$ is a non-empty subset of $\{1, \ldots, n\}$ (listed with $v(1) < \cdots < v(m)$) and $i_1, \ldots, i_n$ are some indices in $\{1, \ldots, k\}$.

Theorem 1.3. *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ let us denote the coefficients of $z_{i_1} \cdots z_{i_n}$ in the series R_μ and R_ν by $\alpha_{(i_1,\ldots,i_n)}$ and $\beta_{(i_1,\ldots,i_n)}$, respectively. Then for every $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ one has*

$$(\mu \boxminus \nu)(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} \Biggl(\prod_{\substack{V \text{ outer} \\ \text{block of } \pi}} \alpha_{(i_1,\ldots,i_n)|V} \Biggr) \cdot \Biggl(\prod_{\substack{W \text{ inner} \\ \text{block of } \pi}} \alpha_{(i_1,\ldots,i_n)|W} + \beta_{(i_1,\ldots,i_n)|W} \Biggr). \tag{1.8}$$

Based on the moment formula from Theorem 1.3 one can find an "operator model on the full Fock space" for $\boxminus$. This is a recipe which starts from the data stored in the R-transforms R_μ and R_ν, and uses creation and annihilation operators on the full Fock space over $\mathbb{C}^{2k}$ in order to produce a k-tuple of operators with distribution $\mu \boxminus \nu$. The precise description of how this works appears in Theorem 4.4. Once the full Fock space model is in place it is easy to see that one can in fact upgrade it to a more general operator model for $\boxminus$, not making specific reference to the full Fock space, and described as follows.

Theorem 1.4. *Let $\mathcal{H}$ be a Hilbert space, let Ω be a unit-vector in $\mathcal{H}$, and let φ be the vector-state defined by Ω on $B(\mathcal{H})$. Suppose that $A_1, \ldots, A_k, B_1, \ldots, B_k \in B(\mathcal{H})$ are such that $\{A_1, \ldots, A_k\}$ is free from $\{B_1, \ldots, B_k\}$ in $(B(\mathcal{H}), \varphi)$, and let μ, ν denote the joint distributions of the k-tuples $A_1, \ldots, A_k$ and respectively $B_1, \ldots, B_k$. Let moreover $P \in B(\mathcal{H})$ denote the orthogonal projection onto the 1-dimensional subspace $\mathbb{C}\Omega$ of $\mathcal{H}$, and consider the operators*

$$C_i := A_i + (1 - P)B_i(1 - P) \in B(\mathcal{H}), \quad 1 \leqslant i \leqslant k. \tag{1.9}$$

Then the joint distribution of $C_1, \ldots, C_k$ with respect to φ is equal to $\mu \boxminus \nu$.

Now, any given pair of distributions $\mu, \nu \in \mathcal{D}_{\mathrm{c}}(k)$ can be made to appear in the setting of Theorem 1.4, in such a way that the operators $A_1, \ldots, A_k, B_1, \ldots, B_k$ involved in the theorem are all selfadjoint. (This is done via a standard free product construction – cf. Remark 4.11 below.) Since in this case the operators $C_1, \ldots, C_k$ from Eq. (1.9) are selfadjoint as well, one thus obtains the following corollary, giving the desired fact that $\boxminus$ can be defined as a binary operation on $\mathcal{D}_{\mathrm{c}}(k)$.

Corollary 1.5. *If μ, ν are in $\mathcal{D}_{\mathrm{c}}(k)$ then $\mu \boxminus \nu$ belongs to $\mathcal{D}_{\mathrm{c}}(k)$ as well.*

Remark 1.6. In the 1-variable framework, the study of $\boxminus$ was started in [11]. That paper gives an operator model for $\mu \boxminus \nu$ obtained via an "s-free product" construction for Hilbert spaces, and where $\mu \boxminus \nu$ appears as the distribution of the sum of two "s-free operators" with distributions μ and ν, respectively. By using Theorem 1.4, it is easy to find a k-variable analogue for this fact – that is, one can make $\mu \boxminus \nu$ appear as the distribution of the sum of two s-free k-tuples on an s-free product Hilbert space. The way how this is done is outlined in Remark 4.12 below.

The next part of the introduction (from Remark 1.7 to Proposition 1.10) explains how $\boxminus$ relates to the work in [6] concerning evolution towards $\boxplus$-infinite divisibility and its connection to the free Brownian motion.

Remark 1.7. Here is a brief summary of relevant results from [6]. One considers a family of bijective transformations $\{\mathbb{B}_t \mid t \geqslant 0\}$ of $\mathcal{D}_{\mathrm{alg}}(k)$ defined by

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus 1/(1+t)}, \quad \forall t \geqslant 0,\ \forall \mu \in \mathcal{D}_{\mathrm{alg}}(k),$$

where the $\boxplus$-powers and $\uplus$-powers are taken in connection to free and respectively Boolean convolution. The transformations $\mathbb{B}_t$ form a semigroup ($\mathbb{B}_{s+t} = \mathbb{B}_s \circ \mathbb{B}_t$, $\forall s, t \geqslant 0$), each of them carries $\mathcal{D}_{\mathrm{c}}(k)$ into itself, and at $t = 1$ one has $\mathbb{B}_1 = \mathbb{B}$, the Boolean Bercovici–Pata bijection that was also encountered in Remark 1.2.2. Thus for a fixed $\mu \in \mathcal{D}_{\mathrm{c}}(k)$ the process $\{\mathbb{B}_t(\mu) \mid t \geqslant 0\}$ can be viewed as some kind of "evolution of μ towards $\boxplus$-infinite divisibility" (since $\mathbb{B}_t(\mu)$ is infinitely divisible for all $t \geqslant 1$).

On the other hand let us recall that the free Brownian motion started at a distribution $\nu \in \mathcal{D}_{\mathrm{c}}(k)$ is the process $\{\nu \boxplus \gamma^{\boxplus t} \mid t \geqslant 0\}$, where $\gamma \in \mathcal{D}_{\mathrm{c}}(k)$ is the joint distribution of a standard semicircular system (a free family of k centered semicircular elements of variance 1). The paper [6] puts into evidence a certain transformation $\Phi : \mathcal{D}_{\mathrm{alg}}(k) \to \mathcal{D}_{\mathrm{alg}}(k)$ which carries $\mathcal{D}_{\mathrm{c}}(k)$ into itself and has the property that

$$\Phi\left(\nu \boxplus \gamma^{\boxplus t}\right) = \mathbb{B}_t\left(\Phi(\nu)\right), \quad \forall \nu \in \mathcal{D}_{\mathrm{alg}}(k),\ \forall t \geqslant 0. \tag{1.10}$$

In other words, (1.10) says that a relation of the form "$\Phi(\nu) = \mu$" is not affected when ν evolves under the free Brownian motion while μ evolves under the action of the semigroup $\{\mathbb{B}_t \mid t \geqslant 0\}$. The transformation Φ from [6] turns out to be related to subordination distributions, as follows.

Theorem 1.8. *For every distribution $\nu \in \mathcal{D}_{\mathrm{alg}}(k)$ one has that*

$$\gamma \boxminus \nu = \mathbb{B}\left(\Phi(\nu)\right), \tag{1.11}$$

where γ is as above (*the joint distribution of a standard semicircular system*) *and $\mathbb{B}$ is the Boolean Bercovici–Pata bijection.*

Remark 1.9. 1° Eq. (1.11) thus offers an alternative description for Φ:

$$\Phi(\nu) = \mathbb{B}^{-1}(\gamma \boxminus \nu), \quad \nu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{1.12}$$

It is worth noting that the two main properties of Φ obtained in [6] (formula (1.10) and the fact that Φ maps $\mathcal{D}_{\mathrm{c}}(k)$ into itself) are very easy to derive by starting from this description and by invoking the suitable properties of subordination distributions; see Proposition 5.7.

2° It is also worth noting that one has a simple explicit formula for how $\mu \boxminus \nu$ itself evolves under the action of the $\mathbb{B}_t$. This formula pops up when one compares the explicit descriptions for the free and the Boolean cumulants of $\mu \boxminus \nu$ (see Remark 3.8.1 and Proposition 5.1 below), and is described as follows.

Proposition 1.10. *Let μ, ν be two distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. Then for every $t \geqslant 0$ one has*:

$$\mathbb{B}_t(\mu \boxminus \nu) = \mu \boxminus \left(\mu^{\boxplus t} \boxplus \nu\right). \tag{1.13}$$

The final part of the introduction discusses two other interesting algebraic properties of $\boxminus$, obtained by extrapolating functional equations which are known to be satisfied by subordination functions in the 1-variable framework. One of these two properties extends a remarkable formula for the sum of the subordination functions of $\mu \boxplus \nu$ with respect to μ and to ν (see e.g. Theorem 4.1 in [4]). This formula can be equivalently written in terms of the η-series of $\mu \boxminus \nu$ and $\nu \boxminus \mu$, and in this form it goes through to the multi-variable framework, as follows.

Proposition 1.11. *One has that*

$$\eta_{\mu \boxminus \nu} + \eta_{\nu \boxminus \mu} = \eta_{\mu \boxplus \nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{1.14}$$

Another property of $\boxminus$ comes from the functional equation satisfied by the subordination function of a convolution power $\nu^{\boxplus p}$ with respect to ν, where ν is a probability measure on $\mathbb{R}$ and $p \in [1, \infty)$ (see Theorem 2.5 in [3]). This too can be translated into a formula involving η-series, which goes through to multi-variable framework. More precisely, the subordination distribution of $\nu^{\boxplus p}$ with respect to ν can be considered for any $\nu \in \mathcal{D}_{\mathrm{alg}}(k)$ and $p \in [1, \infty)$ (see Definition 6.3 below), and the following statement holds.

Proposition 1.12. *For every $\nu \in \mathcal{D}_{\mathrm{alg}}(k)$ and $p \geqslant 1$, the subordination distribution of $\nu^{\boxplus p}$ with respect to ν is equal to $(\mathbb{B}(\nu))^{\boxplus (p-1)}$.*

In particular, for distributions in $\mathcal{D}_{\mathrm{c}}(k)$ one gets the following corollary.

Corollary 1.13. *Let ν be a distribution in $\mathcal{D}_{\mathrm{c}}(k)$. Then for every $p \geqslant 1$ the subordination distribution of $\nu^{\boxplus p}$ with respect to ν is still in $\mathcal{D}_{\mathrm{c}}(k)$, and is $\boxplus$-infinitely divisible.*

One can also put into evidence other natural situations when subordination distributions in $\mathcal{D}_{\mathrm{c}}(k)$ are sure to be $\boxplus$-infinitely divisible. In particular, an immediate consequence of Remark 1.2.1 (combined with Corollary 1.5) is that $\mu \boxminus \nu$ is $\boxplus$-infinitely divisible whenever $\mu, \nu \in \mathcal{D}_{\mathrm{c}}(k)$ and μ is itself $\boxplus$-infinitely divisible; see Corollary 4.13 below.

Remark 1.14. After circulating the first version of this paper, I was made aware of the connection between the results obtained here and the paper [2] of Anshelevich, where a two-variable extension of the transformation Φ from Remark 1.7 is being studied. More precisely, [2] introduces a map

$$\mathcal{D}_{\mathrm{alg}}(k) \times \mathcal{D}_{\mathrm{alg}}(k) \ni (\rho, \psi) \mapsto \Phi[\rho, \psi] \in \mathcal{D}_{\mathrm{alg}}(k)$$

with the property that $\Phi[\rho, \psi] \in \mathcal{D}_{\mathrm{c}}(k)$ for every $\rho, \psi \in \mathcal{D}_{\mathrm{c}}(k)$ such that ρ is $\boxplus$-infinitely divisible, and with the property that

$$\Phi[\gamma, \psi] = \Phi(\psi), \quad \forall \psi \in \mathcal{D}_{\mathrm{alg}}(k) \tag{1.15}$$

(where $\gamma \in \mathcal{D}_{\mathrm{c}}(k)$ is the same as in Remark 1.7). The formula which gives the translation between the results of the present paper and those of [2] is

$$\Phi[\rho, \psi] = \mathbb{B}^{-1}(\rho \boxminus \psi), \quad \forall \rho, \psi \in \mathcal{D}_{\mathrm{alg}}(k). \tag{1.16}$$

Eq. (1.16) can be used to explain why the argument ρ in $\Phi[\rho, \psi]$ is naturally chosen to be $\boxplus$-infinitely divisible: as observed right before the present remark, one has in this situation that $\rho \boxminus \psi$ is $\boxplus$-infinitely divisible, hence that $\mathbb{B}^{-1}(\rho \boxminus \psi) \in \mathcal{D}_c(k)$ for every $\psi \in \mathcal{D}_c(k)$. (Another explanation for why ρ is naturally taken to be infinitely divisible is presented in Remark 10 of [2].)

By using the formula (1.16), the description of Φ given in the above Remark 1.9.1 is reduced to (1.15), and it is also easily seen that Proposition 1.10 of the present paper is equivalent to Theorem 11(b) from [2].

The scope of [2] is different from (albeit overlapping with) the one of the present paper, and the methods of proof are different, invoking e.g. results about conditionally positive definite functionals, or about a multi-variable version of monotonic convolution – see specifics in Section 4.2 of [2].

Remark 1.15 *(Organization of the paper).* Besides the introduction, the paper has five other sections. Section 2 contains a review of some background and notations. Section 3 derives explicit combinatorial formulas for the free and Boolean cumulants of $\mu \boxminus \nu$, and uses them in order to obtain the moment formula announced in Theorem 1.3. Section 4 is devoted to operator models, and to the proof of Theorem 1.4. Section 5 discusses in more detail the relations to the transformations $\mathbb{B}_t$ that were stated in Theorem 1.8 and Proposition 1.10. Finally, Section 6 discusses in more detail the statements made in Propositions 1.11, 1.12, and in Corollary 1.13.

2. Background and notations

2.1. Motivation from 1-variable framework

Remark 2.1. Recall that for a probability distribution μ on $\mathbb{R}$, the *Cauchy transform* of μ is the analytic function G_μ defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

The *reciprocal Cauchy transform* F_μ is defined by

$$F_\mu(z) = 1/G_\mu(z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.2}$$

It is easily checked that G_μ maps the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ to the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$; as a consequence of this, F_μ can be viewed as an analytic self-map of $\mathbb{C}^+$. The measure μ is uniquely determined by G_μ (hence by F_μ as well); and more precisely, μ can be retrieved from G_μ by a procedure called "Stieltjes inversion formula" (see e.g. [1]).

Let $\mathfrak{F}$ denote the set of all analytic self-maps of $\mathbb{C}^+$ that can arise as F_μ for some probability measure μ on $\mathbb{R}$. One has a very nice intrinsic description of $\mathfrak{F}$, that

$$\mathfrak{F} = \left\{ F : \mathbb{C}^+ \to \mathbb{C}^+ \;\middle|\; F \text{ is analytic and } \lim_{t \to \infty} \frac{F(it)}{it} = 1 \right\}. \tag{2.3}$$

(For a nice review of this and other properties of $\mathfrak{F}$ one can consult Section 2 of [12] or Section 5 of [8].)

As mentioned in Section 1, the starting point of this paper is that for any two probability measures μ, ν on $\mathbb{R}$, there exists a *subordination function* $\omega \in \mathfrak{F}$ such that

$$G_{\mu \boxplus \nu}(z) = G_\nu\big(\omega(z)\big), \quad z \in \mathbb{C}^+. \tag{2.4}$$

With μ, ν, ω as in (2.4), it is natural to consider the unique probability measure σ on $\mathbb{R}$ such that $F_\sigma = \omega$. This σ was studied in [11], where it is called the *s-free convolution* of μ and ν, and is denoted by $\mu \boxminus \nu$. The name "s-free convolution" appears in [11] in connection to a suitably tailored concept of "s-freeness" that is also introduced in [11]. Since s-freeness is only marginally addressed in the present paper, we will refer to $\mu \boxminus \nu$ by just calling it the *subordination distribution of* $\mu \boxplus \nu$ *with respect to* ν. We will only look at $\mu \boxminus \nu$ in the special case when μ and ν are compactly supported; in this case $\mu \boxminus \nu$ is compactly supported as well (as one sees by examining the operator model obtained for $\mu \boxminus \nu$ in [11]).

Remark 2.2. If μ is a compactly supported probability measure on $\mathbb{R}$, then in particular μ has moments of all orders:

$$m_n := \int_{-\infty}^{\infty} t^n \, d\mu(t), \quad n \in \mathbb{N},$$

and one can form the *moment series* of μ,

$$M_\mu(z) = \sum_{n=1}^{\infty} m_n z^n. \tag{2.5}$$

In (2.5), M_μ can be viewed as an analytic function on a neighbourhood of 0, but in the present paper it is preferable to treat it as a formal power series in z. It is immediate that M_μ is connected to the Cauchy transform G_μ by the formula

$$1 + M_\mu(1/z) = z \cdot G_\mu(z), \tag{2.6}$$

where in (2.6) it is convenient to also treat G_μ as a power series (obtained by writing $1/(t-z) = \sum_{n=1}^{\infty} t^{n-1}/z^n$ and then integrating term by term on the right-hand side of (2.1), for $z \in \mathbb{C}^+$ with $|z|$ large enough).

In the study of free additive convolution a fundamental object is Voiculescu's R-transform, which has the linearizing property that $R_{\mu \boxplus \nu} = R_\mu + R_\nu$. For a compactly supported probability measure μ on $\mathbb{R}$, the R-transform R_μ can be viewed as a power series, defined in terms of M_μ as the unique solution of the equation

$$R_\mu\big(z\big(1 + M_\mu(z)\big)\big) = M_\mu(z) \tag{2.7}$$

(equation in $\mathbb{C}[[z]]$, where M_μ is given as data and R_μ is the unknown). For the next proposition it is more convenient to write the definition of R_μ by emphasizing its relation to the Cauchy transform G_μ. On these lines one first defines the so-called *K-transform* of μ, which is simply the inverse under composition

$$K_\mu := G_\mu^{\langle -1 \rangle}. \tag{2.8}$$

K_μ is a Laurent series of the form $K_\mu(z) = \frac{1}{z} + \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \cdots$, and one has[2]

$$R_\mu(z) = z\left(K_\mu(z) - \frac{1}{z}\right). \tag{2.9}$$

In Proposition 2.3, Eq. (2.9) will be used in the equivalent form giving K_μ in terms of R_μ,

$$K_\mu(z) = \frac{1 + R_\mu(z)}{z}. \tag{2.10}$$

Proposition 2.3. *Let μ, ν be compactly supported probability measures on $\mathbb{R}$, and let the probability measure $\mu \boxminus \nu$ be defined as in Remark* 2.1. *Then*

$$R_{\mu \boxminus \nu}(z) = \frac{R_\mu(z(1 + M_\nu(z)))}{1 + M_\nu(z)}. \tag{2.11}$$

Proof. Let us denote for brevity $\mu \boxminus \nu =: \sigma$. From how $\mu \boxminus \nu$ is defined we have that

$$G_{\mu \boxplus \nu} = G_\nu \circ F_\sigma. \tag{2.12}$$

By taking inverses under composition on both sides of (2.12) one finds that $K_{\mu \boxplus \nu} = F_\sigma^{\langle -1 \rangle} \circ K_\nu$, hence that $F_\sigma \circ K_{\mu \boxplus \nu} = K_\nu$; this in turn implies that $G_\sigma \circ K_{\mu \boxplus \nu} = 1/K_\nu$, and that $K_{\mu \boxplus \nu} = K_\sigma \circ (1/K_\nu)$. So one gets the formula:

$$K_{\mu \boxplus \nu}(w) = K_\sigma\big(1/K_\nu(w)\big) \tag{2.13}$$

(equality of Laurent series in an indeterminate w).

In (2.13) let us next replace the K-transforms of $\mu \boxplus \nu$ and of σ in terms of the corresponding R-transforms, by using Eq. (2.10). On the left-hand side we obtain

$$K_{\mu \boxplus \nu}(w) = \frac{1 + R_{\mu \boxplus \nu}(w)}{w} = \frac{1 + R_\mu(w) + R_\nu(w)}{w} = \frac{R_\mu(w)}{w} + K_\nu(w),$$

while on the right-hand side we obtain

$$K_\sigma\big(1/K_\nu(w)\big) = \frac{1 + R_\sigma(1/K_\nu(w))}{1/K_\nu(w)} = K_\nu(w) + K_\nu(w) \cdot R_\sigma\big(1/K_\nu(w)\big).$$

After making these replacements and after subtracting $K_\nu(w)$ out of both sides of (2.13) one arrives to

$$\frac{R_\mu(w)}{w} = K_\nu(w) \cdot R_\sigma\big(1/K_\nu(w)\big). \tag{2.14}$$

[2] The original definition of the R-transform, made in [14], simply has $\mathcal{R}_\mu(z) = K_\mu(z) - 1/z$. The present paper uses the shifted version $R_\mu(z) = z\mathcal{R}_\mu(z)$, which is more convenient for extension to a multi-variable framework.

Finally, in (2.14) let us make the substitution $z = 1/K_\nu(w)$, with inverse $w = G_\nu(1/z) = z(1 + M_\nu(z))$; this substitution converts (2.14) into

$$\frac{R_\mu(z(1+M_\nu(z)))}{z(1+M_\nu(z))} = \frac{1}{z} \cdot R_\sigma(z),$$

and (2.11) follows. □

2.2. Non-crossing partitions

Notation 2.4 *(NC(n) terminology).* Let n be a positive integer.

1° Let $\pi = \{V_1, \ldots, V_p\}$ be a partition of $\{1, \ldots, n\}$ – i.e. $V_1, \ldots, V_p$ are pairwise disjoint non-empty sets (called the *blocks* of π), and $V_1 \cup \cdots \cup V_p = \{1, \ldots, n\}$. We say that π is *non-crossing* if for every $1 \leqslant i < j < i' < j' \leqslant n$ such that i is in the same block with i' and j is in the same block with j', it necessarily follows that all of i, i', j, j' are in the same block of π. The set of all non-crossing partitions of $\{1, \ldots, n\}$ will be denoted by $NC(n)$.

2° Let π be a partition in $NC(n)$. Since π is, after all, a set of subsets of $\{1, \ldots, n\}$, it will be convenient to write "$V \in \pi$" as a shorthand for "V is a block of π." In the same vein, various calculations throughout the paper will use functions "$c : \pi \to \{1, 2\}$." Such a function is thus a recipe for assigning a number $c(V) \in \{1, 2\}$ to every block V of π, and will be referred to as a *colouring* of π.

3° For $\pi \in NC(n)$, the number of blocks of π will be denoted by $|\pi|$.

4° Let π be a partition in $NC(n)$, and let V be a block of π. If there exists a block W of π such that $\min(W) < \min(V)$ and $\max(W) > \max(V)$, then one says that V is an *inner* block of π. In the opposite case one says that V is an *outer* block of π.

5° Every partition $\pi \in NC(n)$ has a special colouring $\mathrm{o}_\pi : \pi \to \{1, 2\}$ which will be called the *inner/outer colouring* of π, and is defined by

$$\mathrm{o}_\pi(V) = \begin{cases} 1 & \text{if } V \text{ is outer,} \\ 2 & \text{if } V \text{ is inner,} \end{cases} \qquad V \in \pi. \tag{2.15}$$

Remark 2.5. $NC(n)$ is partially ordered by *reverse refinement*: for $\pi, \rho \in NC(n)$ one writes "$\pi \leq \rho$" to mean that every block of ρ is a union of blocks of π. The minimal and maximal element of $(NC(n), \leq)$ are denoted by 0_n (the partition of $\{1, \ldots, n\}$ into n singleton blocks) and respectively 1_n (the partition of $\{1, \ldots, n\}$ into only one block).

Let $\rho = \{W_1, \ldots, W_q\}$ be a fixed partition in $NC(n)$. It is easy to see that one has a natural poset isomorphism

$$\bigl\{\pi \in NC(n) \bigm| \pi \leq \rho\bigr\} \ni \pi \mapsto (\pi_1, \ldots, \pi_q) \in NC\bigl(|W_1|\bigr) \times \cdots \times NC\bigl(|W_q|\bigr) \tag{2.16}$$

where for every $1 \leqslant j \leqslant q$ the partition $\pi_j \in NC(|W_j|)$ is obtained by restricting π to W_j and by re-denoting the elements of W_j, in increasing order, so that they become $1, 2, \ldots, |W_j|$. This is a particular case of a more general factorization property satisfied by the intervals of the poset $(NC(n), \leq)$ – see [13, Lecture 9].

Remark 2.6. This paper also makes use of another partial order relation on $NC(n)$, which was introduced in [5] and is denoted by "$\ll$." For $\pi, \rho \in NC(n)$ one writes "$\pi \ll \rho$" to mean that $\pi \leq \rho$ and that, in addition, the following condition is fulfilled:

$$\begin{cases} \text{For every block } W \text{ of } \rho \text{ there exists a block} \\ V \text{ of } \pi \text{ such that } \min(W), \max(W) \in V. \end{cases} \tag{2.17}$$

It is immediately verified that "$\ll$" is indeed a partial order relation on $NC(n)$. It is much coarser than the reversed refinement order. For instance, the inequality $\pi \ll 1_n$ is not holding for all $\pi \in NC(n)$, but it rather amounts to the condition that the numbers 1 and n belong to the same block of π (or equivalently, that π has a unique outer block). At the other end of $NC(n)$, the inequality $\pi \gg 0_n$ can only take place when $\pi = 0_n$. The remaining part of Section 2.2 reviews a couple of other properties of $\ll$ that will be used later on in the paper.

Definition 2.7. Let π, ρ be partitions in $NC(n)$ such that $\pi \ll \rho$. A block V of π is said to be ρ-*special* when there exists a block W of ρ such that $\min(V) = \min(W)$ and $\max(V) = \max(W)$.

Proposition 2.8. *Let $\pi \in NC(n)$ be such that $\pi \ll 1_n$, and consider the set of partitions*

$$\{\rho \in NC(n) \mid \pi \ll \rho \ll 1_n\}. \tag{2.18}$$

Then $\rho \mapsto \{V \in \pi \mid V \text{ is } \rho\text{-special}\}$ is a one-to-one map from the set (2.18) *to the set of subsets of π.*[3] *The image of this map is equal to $\{\mathfrak{V} \subseteq \pi \mid \mathfrak{V} \ni V_0\}$, where V_0 denotes the unique outer block of π.*

For the proof of Proposition 2.8, the reader is referred to Proposition 2.13 and Remark 2.14 of [5].

Remark 2.9 *(Interval partitions).* A partition π of $\{1, \ldots, n\}$ is said to be an *interval partition* if every block V of π is of the form $V = [i, j] \cap \mathbb{Z}$ for some $1 \leqslant i \leqslant j \leqslant n$. The set of all interval partitions of $\{1, \ldots, n\}$ will be denoted by $\mathrm{Int}(n)$. It is clear that $\mathrm{Int}(n) \subseteq NC(n)$, and it is easily verified that every interval partition is a maximal element of the poset $(NC(n), \ll)$. It is moreover easy to see (left as exercise to the reader) that for every $\pi \in NC(n)$ there exists a unique $\rho \in \mathrm{Int}(n)$ such that $\pi \ll \rho$; the blocks of this special interval partition ρ are in some sense the "convex hulls" of the outer blocks of π.

2.3. Power series in k non-commuting indeterminates

Notation 2.10. We will denote by $\mathbb{C}\langle\langle z_1, \ldots, z_k\rangle\rangle$ the set of power series with complex coefficients in the non-commuting indeterminates $z_1, \ldots, z_k$, and we will use the notation $\mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ for the set of series in $\mathbb{C}\langle\langle z_1, \ldots, z_k\rangle\rangle$ which have vanishing constant term. The general form of a series $f \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ is thus

$$f(z_1, \ldots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{k} \alpha_{(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n} \tag{2.19}$$

where the coefficients $\alpha_{(i_1, \ldots, i_n)}$ are from $\mathbb{C}$.

[3] According to the conventions made in Notation 2.4.2, "subset of π" stands here for "set of blocks of π."

Definition 2.11 *(Coefficients for series in* $\mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$*).*

1° For $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ we will denote by

$$\mathrm{Cf}_{(i_1,\ldots,i_n)} : \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle \to \mathbb{C} \tag{2.20}$$

the linear functional which extracts the coefficient of $z_{i_1} \cdots z_{i_n}$ in a series $f \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$. Thus for f written as in Eq. (2.19) we have $\mathrm{Cf}_{(i_1,\ldots,i_n)}(f) = \alpha_{(i_1,\ldots,i_n)}$.

2° Suppose we are given a positive integer n, some indices $i_1, \ldots, i_n \in \{1, \ldots, k\}$, and a partition $\pi \in NC(n)$. We define a (generally non-linear) functional

$$\mathrm{Cf}_{(i_1,\ldots,i_n);\pi} : \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle \to \mathbb{C}, \tag{2.21}$$

as follows. For every block $V = \{v(1), \ldots, v(m)\}$ of π, with $1 \leqslant v(1) < \cdots < v(m) \leqslant n$, let us use the notation

$$(i_1, \ldots, i_n) \mid V := (i_{v(1)}, \ldots, i_{v(m)}) \in \{1, \ldots, k\}^m.$$

Then we define

$$\mathrm{Cf}_{(i_1,\ldots,i_n);\pi}(f) := \prod_{V\in\pi} \mathrm{Cf}_{(i_1,\ldots,i_n)|V}(f), \quad \forall f \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle. \tag{2.22}$$

(For example if we had $n = 5$ and $\pi = \{\{1, 4, 5\}, \{2, 3\}\}$, and if $i_1, \ldots, i_5$ would be some fixed indices in $\{1, \ldots, k\}$, then the above formula would become

$$\mathrm{Cf}_{(i_1,i_2,i_3,i_4,i_5);\pi}(f) = \mathrm{Cf}_{(i_1,i_4,i_5)}(f) \cdot \mathrm{Cf}_{(i_2,i_3)}(f),$$

$f \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$.) The quantities $\mathrm{Cf}_{(i_1,\ldots,i_n);\pi}(f)$ will be referred to as *generalized coefficients* of the series f.

3° Suppose that the positive integer n, the indices $i_1, \ldots, i_n \in \{1, \ldots, k\}$ and the partition $\pi \in NC(n)$ are as above, and that in addition we are also given a colouring $c : \pi \to \{1, 2\}$. Then for any two series $f_1, f_2 \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ we define their *mixed generalized coefficient* corresponding to $(i_1, \ldots, i_n)$, π and c via the formula

$$\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(f_1, f_2) := \prod_{V\in\pi} \mathrm{Cf}_{(i_1,\ldots,i_n)|V}(f_{c(V)}). \tag{2.23}$$

(For example if we had $n = 5$, $\pi = \{\{1, 4, 5\}, \{2, 3\}\}$ and $c : \pi \to \{1, 2\}$ defined by $c(\{1, 4, 5\}) = 1$, $c(\{2, 3\}) = 2$, then (2.23) would become

$$\mathrm{Cf}_{(i_1,i_2,i_3,i_4,i_5);\pi}(f) = \mathrm{Cf}_{(i_1,i_4,i_5)}(f_1) \cdot \mathrm{Cf}_{(i_2,i_3)}(f_2),$$

for $f_1, f_2 \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ and $1 \leqslant i_1, \ldots, i_5 \leqslant k$.)

Remark 2.12. It is clear that for every $n \geqslant 1$, $1 \leqslant i_1, \ldots, i_n \leqslant k$, $\pi \in NC(n)$ and $f \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ one has

$$\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(f, f) = \mathrm{Cf}_{(i_1,\ldots,i_n);\pi}(f),$$

for no matter what colouring c of π. Let us also record here the obvious expansion formula

$$\mathrm{Cf}_{(i_1,\dots,i_n);\pi}(f_1+f_2)=\sum_{c:\pi\to\{1,2\}}\mathrm{Cf}_{(i_1,\dots,i_n);\pi;c}(f_1,f_2), \tag{2.24}$$

holding for every $n\geqslant 1$, $1\leqslant i_1,\dots,i_n\leqslant k$, $\pi\in NC(n)$, and $f_1, f_2\in\mathbb{C}_0\langle\langle z_1,\dots,z_k\rangle\rangle$.

Definition 2.13 *(Review of the series M_μ, R_μ, η_μ).* Let μ be a distribution in $\mathcal{D}_{\mathrm{alg}}(k)$.

1° We will denote by M_μ the series in $\mathbb{C}_0\langle\langle z_1,\dots,z_k\rangle\rangle$ defined by

$$M_\mu(z_1,\dots,z_k):=\sum_{n=1}^{\infty}\sum_{i_1,\dots,i_n=1}^{k}\mu(X_{i_1}\cdots X_{i_n})z_{i_1}\cdots z_{i_n}. \tag{2.25}$$

M_μ is called the *moment series* of μ, and its coefficients (the numbers $\mu(X_{i_1}\cdots X_{i_n})$, with $n\geqslant 1$ and $1\leqslant i_1,\dots,i_n\leqslant k$) are called the *moments* of μ.

2° The *η-series* of μ is

$$\eta_\mu:=M_\mu(1+M_\mu)^{-1}\in\mathbb{C}_0\langle\langle z_1,\dots,z_k\rangle\rangle, \tag{2.26}$$

where $(1+M_\mu)^{-1}$ is the inverse of $1+M_\mu$ under multiplication in $\mathbb{C}\langle\langle z_1,\dots,z_k\rangle\rangle$. The coefficients of η_μ are called the *Boolean cumulants* of μ.

3° There exists a unique series $R_\mu\in\mathbb{C}_0\langle\langle z_1,\dots,z_k\rangle\rangle$ which satisfies the functional equation

$$R_\mu\big(z_1(1+M_\mu),\dots,z_k(1+M_\mu)\big)=M_\mu. \tag{2.27}$$

Indeed, it is easily seen that Eq. (2.27) amounts to a recursion which determines uniquely the coefficients of R_μ in terms of those of M_μ. The series R_μ is called the *R-transform* of μ, and its coefficients are called the *free cumulants* of μ. (See the discussion in [13, Lecture 16], and specifically Theorem 16.15 and Corollary 16.16 of that lecture.)

Remark 2.14. It is very useful that one has explicit summation formulas which express the moments of a distribution $\mu\in\mathcal{D}_{\mathrm{alg}}(k)$ either in terms of its free cumulants or in terms of its Boolean cumulants. These are sometimes referred to as *moment–cumulant* formulas. They say that for every $n\geqslant 1$ and $1\leqslant i_1,\dots,i_n\leqslant k$ one has

$$\mu(X_{i_1}\cdots X_{i_n})=\sum_{\pi\in NC(n)}\mathrm{Cf}_{(i_1,\dots,i_n);\pi}(R_\mu) \tag{2.28}$$

and respectively

$$\mu(X_{i_1}\cdots X_{i_n})=\sum_{\pi\in \mathrm{Int}(n)}\mathrm{Cf}_{(i_1,\dots,i_n);\pi}(\eta_\mu) \tag{2.29}$$

(where (2.28), (2.29) use the notations for generalized coefficients from Definition 2.11.2, and $\mathrm{Int}(n)$ is the set of interval-partitions from Remark 2.9). Moreover, a similar summation formula can be used in order to express the Boolean cumulants of μ in terms of its free cumulants; it says that for every $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ one has

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(\eta_\mu) = \sum_{\substack{\pi \in NC(n),\\ \pi \ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi}(R_\mu). \tag{2.30}$$

(For a more detailed discussion of the relation between R_μ and η_μ see Section 3 of [5], where Eq. (2.30) appears in Proposition 3.9.)

3. The approach to $\boxminus$ via R-transforms

The goal of this section is to derive explicit combinatorial formulas for the free and Boolean cumulants of $\mu \boxminus \nu$, and then use them in order to obtain the moment formula announced in Theorem 1.3.

Remark 3.1. Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. Consider the subordination distribution $\mu \boxminus \nu$, and recall that its R-transform satisfies the equation

$$R_{\mu \boxminus \nu} \cdot (1 + M_\nu) = R_\mu\big(z_1(1 + M_\nu), \ldots, z_k(1 + M_\nu)\big). \tag{3.1}$$

If we denote for convenience

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_\mu) =: \alpha_{(i_1,\ldots,i_n)}, \quad \forall n \geqslant 1,\ 1 \leqslant i_1, \ldots, i_n \leqslant k,$$

then the series on the right-hand side of (3.1) is written more precisely as

$$\sum_{m=1}^{\infty} \sum_{j_1,\ldots,j_m=1}^{k} \alpha_{(j_1,\ldots,j_m)} z_{j_1}(1 + M_\nu) \cdots z_{j_m}(1 + M_\nu). \tag{3.2}$$

Let us fix an $n \geqslant 1$ and some indices $1 \leqslant i_1, \ldots, i_n \leqslant k$, and let us look at the coefficient of $z_{i_1} \cdots z_{i_n}$ in the infinite sum from (3.2). Clearly, a term $\alpha_{(j_1,\ldots,j_m)} z_{j_1}\ (1 + M_\nu) \cdots z_{j_m}(1 + M_\nu)$ contributes to this coefficient if and only if $m \leqslant n$ and there exist $1 = s(1) < s(2) < \cdots < s(m) \leqslant n$ such that

$$j_1 = i_{s(1)}, \qquad j_2 = i_{s(2)}, \ldots, j_m = i_{s(m)}. \tag{3.3}$$

In the case when (3.3) holds let us denote $\{s(1), s(2), \ldots, s(m)\} =: S$, and let us refer to the intervals of integers

$$\big(s(1), s(2)\big) \cap \mathbb{Z}, \quad \ldots, \quad \big(s(m-1), s(m)\big) \cap \mathbb{Z}, \quad \big(s(m), n\big] \cap \mathbb{Z}$$

by calling them the *gaps of S*; with this notation the contribution of $\alpha_{(j_1,\ldots,j_m)} z_{j_1}(1 + M_\nu) \cdots z_{j_m}(1 + M_\nu)$ to the coefficient of $z_{i_1} \cdots z_{i_n}$ in (3.2) is written as

$$\alpha_{(i_1,\ldots,i_n)|S} \cdot \prod_{\substack{G=\{p,\ldots,q\}\\ \text{gap of } S}} \nu(X_{i_p} \cdots X_{i_q})$$

(we make the convention that if G is an empty gap of S then the corresponding product $\nu(X_{i_p} \cdots X_{i_q})$ is taken to be equal to 1). Since the set S appearing above can be any subset of $\{1, \dots, n\}$ which contains 1, we come to the conclusion that

$$\begin{aligned}
&\mathrm{Cf}_{(i_1,\dots,i_n)}\big(R_\mu\big(z_1(1+M_\nu),\dots,z_k(1+M_\nu)\big)\big) \\
&\quad = \sum_{\substack{S \subseteq \{1,\dots,n\} \\ \text{such that } S \ni 1}} \Bigg(\alpha_{(i_1,\dots,i_n)|S} \cdot \prod_{\substack{G=\{p,\dots,q\} \\ \text{gap of } S}} \nu(X_{i_p} \cdots X_{i_q})\Bigg). \qquad (3.4)
\end{aligned}$$

By equating coefficients in the series on the two sides of (3.1) and by employing (3.4) one obtains explicit formulas for the coefficients of $R_{\mu \boxplus \nu}$, as shown in the next lemma and proposition.

Lemma 3.2. *Consider the same notations as in Remark* 3.1. *For every $n \geqslant 1$ and $1 \leqslant i_1, \dots, i_n \leqslant k$ one has that*

$$\mathrm{Cf}_{(i_1,\dots,i_n)}(R_{\mu \boxplus \nu}) = \sum_{\substack{S \subseteq \{1,\dots,n\} \\ \text{such that } S \ni 1,n}} \Bigg(\alpha_{(i_1,\dots,i_n)|S} \cdot \prod_{\substack{G=\{p,\dots,q\} \\ \text{gap of } S}} \nu(X_{i_p} \cdots X_{i_q})\Bigg). \qquad (3.5)$$

Proof. We will prove the required formula (3.5) by induction on n.

For $n = 1$, (3.5) states that $\mathrm{Cf}_{(i_1)}(R_{\mu \boxplus \nu}) = \alpha_{i_1}$, $\forall 1 \leqslant i_1 \leqslant k$; this is indeed true, as one sees by equating the coefficients of z_{i_1} on the two sides of (3.1).

Induction step. We fix an integer $n \geqslant 2$, we assume that (3.5) holds for $1, 2, \dots, n-1$ and we prove that it also holds for n. So let $i_1, \dots, i_n$ be some indices in $\{1, \dots, k\}$. The coefficient of $z_{i_1} \cdots z_{i_n}$ in $R_{\mu \boxplus \nu} \cdot (1 + M_\nu)$ is equal to:

$$\mathrm{Cf}_{(i_1,\dots,i_n)}(R_{\mu \boxplus \nu}) + \sum_{m=1}^{n-1} \mathrm{Cf}_{(i_1,\dots,i_m)}(R_{\mu \boxplus \nu}) \cdot \nu(X_{i_{m+1}} \cdots X_{i_n}). \qquad (3.6)$$

For every $1 \leqslant m \leqslant n-1$ the induction hypothesis gives us that

$$\begin{aligned}
&\mathrm{Cf}_{(i_1,\dots,i_m)}(R_{\mu \boxplus \nu}) \cdot \nu(X_{i_{m+1}} \cdots X_{i_n}) \\
&\quad = \sum_{\substack{S \subseteq \{1,\dots,m\} \\ \text{such that } S \ni 1,m}} \alpha_{(i_1,\dots,i_m)|S} \cdot \Bigg(\prod_{\substack{G=\{p,\dots,q\} \\ \text{gap of } S}} \nu(X_{i_p} \cdots X_{i_q})\Bigg) \cdot \nu(X_{i_{m+1}} \cdots X_{i_n}).
\end{aligned}$$

In the latter expression the separate factor $\nu(X_{i_{m+1}} \cdots X_{i_n})$ can be incorporated into the product over the gaps of S, via the simple trick of treating S as a subset of $\{1, \dots, n\}$ rather than a subset of $\{1, \dots, m\}$. (Indeed, in this way S gets the additional gap $\{m+1, \dots, n\}$, with corresponding factor $\nu(X_{i_{m+1}} \cdots X_{i_n})$.) When this is done and when the resulting formula for $\mathrm{Cf}_{(i_1,\dots,i_m)}(R_{\mu \boxplus \nu}) \cdot \nu(X_{i_{m+1}} \cdots X_{i_n})$ is substituted in (3.6), we find that

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}\big(R_{\mu\boxplus\nu}\cdot(1+M_\nu)\big)$$
$$=\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\mu\boxplus\nu})+\sum_{\substack{S\subseteq\{1,\ldots,n\}\\ \text{such that } 1\in S \text{ and } n\notin S}}\alpha_{(i_1,\ldots,i_n)|S}\cdot\Bigg(\prod_{\substack{G=\{p,\ldots,q\}\\ \text{gap of } S}}\nu(X_{i_p}\cdots X_{i_q})\Bigg). \tag{3.7}$$

Finally, we equate the right-hand sides of Eqs. (3.7) and (3.4), and the required formula for $\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\mu\boxplus\nu})$ follows. □

Proposition 3.3. *Let μ,ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $n\geqslant 1$ and $1\leqslant i_1,\ldots,i_n\leqslant k$ one has*

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\mu\boxplus\nu})=\sum_{\substack{\pi\in NC(n),\\ \pi\ll 1_n}}\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu,R_\nu), \tag{3.8}$$

where the inner/outer colouring o_π is as in Notation 2.4.5*, and the generalized coefficient $\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu,R_\nu)$ is as in Definition* 2.11.3.

Proof. We will use the various notations introduced in Remark 3.1 and Lemma 3.2 above.

Let us pick a subset $S\subseteq\{1,\ldots,n\}$ such that $S\ni 1,n$, and let us prove that

$$\alpha_{(i_1,\ldots,i_n)|S}\cdot\Bigg(\prod_{\substack{G=\{p,\ldots,q\}\\ \text{gap of } S}}\nu(X_{i_p}\cdots X_{i_q})\Bigg)=\sum_{\substack{\pi\in NC(n)\\ \text{such that } S\in\pi}}\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu,R_\nu). \tag{3.9}$$

In order to verify (3.9), let us write explicitly $S=\{s(1),s(2),\ldots,s(m)\}$ with $1=s(1)<s(2)<\cdots<s(m)=n$; then the gaps of S are listed as $G_1,\ldots,G_{m-1}$, with

$$G_j=\{p_j,\ldots,q_j\}=\big(s(j),s(j+1)\big)\cap\mathbb{Z}\quad\text{for } 1\leqslant j\leqslant m-1,$$

and the left-hand side of (3.9) becomes

$$\alpha_{(i_1,\ldots,i_n)|S}\cdot\prod_{j=1}^{m-1}\nu(X_{i_{p_j}}\cdots X_{i_{q_j}}) \tag{3.10}$$

(with the same convention as used above, that "$\nu(X_{i_{p_j}}\cdots X_{i_{q_j}})$" is to be read as 1 in the case when $G_j=\emptyset$). Now in (3.10) let us use the free moment–cumulant formula (2.28) to express the moments $\nu(X_{i_{p_j}}\cdots X_{i_{q_j}})$ in terms of the coefficients of R_ν; we get

$$\alpha_{(i_1,\ldots,i_n)|S}\cdot\prod_{j=1}^{m-1}\Bigg(\sum_{\pi_j\in NC(|G_j|)}\mathrm{Cf}_{(i_{p_j},\ldots,i_{q_j});\pi_j}(R_\nu)\Bigg)$$
$$=\sum_{\substack{\pi_1\in NC(|G_1|),\ldots,\\ \pi_{m-1}\in NC(|G_{m-1}|)}}\Bigg(\mathrm{Cf}_{(i_1,\ldots,i_n)|S}(R_\mu)\cdot\prod_{j=1}^{m-1}\mathrm{Cf}_{(i_1,\ldots,i_n)|G_j);\pi_j}(R_\nu)\Bigg). \tag{3.11}$$

But a family of non-crossing partitions $\pi_1 \in NC(|G_1|), \dots, \pi_{m-1} \in NC(|G_{m-1}|)$ is naturally assembled, together with S, into one non-crossing partition $\pi \in NC(n)$; and all partitions $\pi \in NC(n)$ such that $S \in \pi$ are obtained in this way, without repetitions. Moreover, when $\pi_1, \dots, \pi_{m-1}$ and S are assembled together into π, it is clear that the big product from (3.11) becomes just $\mathrm{Cf}_{(i_1,\dots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\nu)$. Hence the substitution $(\pi_1, \dots, \pi_{m-1}) \leftrightarrow \pi$ leads to the right-hand side of (3.9), and this completes the proof that (3.9) holds.

Finally, we sum over S on both sides of (3.9), with S running in the collection of all subsets of $\{1, \dots, n\}$ which contain 1 and n. The sum on the left-hand side gives $\mathrm{Cf}_{(i_1,\dots,i_n)}(R_{\mu\boxplus\nu})$ by Lemma 3.2, while the sum on the right-hand side takes us precisely to the right-hand side of (3.8), as we wanted. □

It will come in handy to also have an extended version of the formula found in Proposition 3.3, which covers the generalized coefficients "$\mathrm{Cf}_{(i_1,\dots,i_n);\rho}$" of the R-transform of $\mu\boxplus\nu$. This is presented in Lemma 3.6, and uses the following extension for the concept of inner/outer colouring of a non-crossing partition.

Notation 3.4. Let n be a positive integer and let π, ρ be partitions in $NC(n)$ such that $\pi \ll \rho$. We denote by $\mathrm{o}_{\pi,\rho}$ the colouring of π defined by

$$\mathrm{o}_{\pi;\rho}(V) = \begin{cases} 1, & \text{if } V \text{ is } \rho\text{-special}, \\ 2, & \text{if } V \text{ is not } \rho\text{-special}, \end{cases} \qquad V \in \pi, \tag{3.12}$$

where the concept of "being ρ-special" for a block of π is as in Definition 2.7.

Remark 3.5. Let π be a partition in $NC(n)$ and let ρ be the unique interval-partition with the property that $\rho \gg \pi$. Then the colouring $\mathrm{o}_{\pi,\rho}$ defined above is just the usual inner/outer colouring o_π – indeed, in this case a block V of π is ρ-special if and only if it is outer.

Lemma 3.6. *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $n \geqslant 1$, $\rho \in NC(n)$ and $1 \leqslant i_1, \dots, i_n \leqslant k$ one has*

$$\mathrm{Cf}_{(i_1,\dots,i_n);\rho}(R_{\mu\boxplus\nu}) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll \rho}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\nu). \tag{3.13}$$

Proof. Let us write explicitly $\rho = \{W_1, \dots, W_q\}$. Then

$$\begin{aligned}
&\mathrm{Cf}_{(i_1,\dots,i_n);\rho}(R_{\mu\boxplus\nu}) \\
&\quad = \prod_{j=1}^{q} \mathrm{Cf}_{(i_1,\dots,i_n)|W_j}(R_{\mu\boxplus\nu}) \\
&\quad = \prod_{j=1}^{q} \Biggl(\sum_{\substack{\pi_j \in NC(|W_j|), \\ \pi_j \ll 1_{|W_j|}}} \mathrm{Cf}_{((i_1,\dots,i_n)|W_j);\pi_j;\mathrm{o}_{\pi_j}}(R_\mu, R_\nu) \Biggr) \\
&\quad = \sum_{\substack{\pi_1 \in NC(|W_1|), \pi_1 \ll 1_{|W_1|}, \dots, \\ \pi_q \in NC(|W_q|), \pi_q \ll 1_{|W_q|}}} \Biggl(\prod_{j=1}^{q} \mathrm{Cf}_{((i_1,\dots,i_n)|W_j);\pi_j;\mathrm{o}_{\pi_j}}(R_\mu, R_\nu) \Biggr).
\end{aligned} \tag{3.14}$$

Now let us consider the bijection (2.16) from Remark 2.5. It is immediate that if $\pi \leftrightarrow (\pi_1, \ldots, \pi_q)$ via this bijection, then

$$\prod_{j=1}^{q} \mathrm{Cf}_{((i_1,\ldots,i_n)|W_j);\pi_j;\mathrm{o}_{\pi_j}}(R_\mu, R_\nu) = \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\nu).$$

Thus when in (3.14) we perform the change of variable given by the bijection from (2.16), we arrive precisely to the right-hand side of (3.13), as required. $\square$

On our way towards the formula for moments stated in Theorem 1.3 we next put into evidence an explicit formula for the Boolean cumulants of $\mu \boxplus \nu$.

Proposition 3.7. *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ one has*

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(\eta_{\mu\boxplus\nu}) = \sum_{\substack{\pi \in NC(n),\, \pi \ll 1_n \\ \text{with outer block } V_o}} \sum_{\substack{c:\pi \to \{1,2\} \\ \text{such that } c(V_o)=1}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(R_\mu, R_\nu). \tag{3.15}$$

Moreover, for every $\pi \in NC(n)$, $\pi \ll 1_n$ with outer block V_o, one has:

$$\sum_{\substack{c:\pi \to \{1,2\} \\ \text{such that } c(V_o)=1}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(R_\mu, R_\nu) = \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu). \tag{3.16}$$

Hence Eq. (3.15) *can also be written in the form*

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(\eta_{\mu\boxplus\nu}) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu). \tag{3.17}$$

Proof. It is immediate that the left-hand side of (3.16) is merely the expansion as a sum for the product which defines $\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu)$. Hence the only non-trivial point in this proof is to verify that (3.15) holds.

By using how $\mathrm{Cf}_{(i_1,\ldots,i_n)}(\eta_{\mu\boxplus\nu})$ is written in terms of the coefficients of $R_{\mu\boxplus\nu}$ (cf. Eq. (2.30) in Remark 2.14), then by invoking Lemma 3.6 and by performing an obvious change in the order of summation we get that

$$\begin{aligned}
\mathrm{Cf}_{(i_1,\ldots,i_n)}(\eta_{\mu\boxplus\nu}) &= \sum_{\substack{\rho\in NC(n), \\ \rho\ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\rho}(R_{\mu\boxplus\nu}) \\
&= \sum_{\substack{\rho\in NC(n), \\ \rho\ll 1_n}} \left(\sum_{\substack{\pi \in NC(n), \\ \pi \ll \rho}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\nu) \right) \\
&= \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \left(\sum_{\substack{\rho\in NC(n) \\ \text{such that } \pi \ll \rho \ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\nu) \right).
\end{aligned}$$

In order to conclude the proof we are left to show that for every partition $\pi \in NC(n)$ with $\pi \ll 1_n$ and with outer block denoted V_0 one has

$$\sum_{\substack{\rho \in NC(n) \\ \text{such that } \pi \ll \rho \ll 1_n}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\nu) = \sum_{\substack{c:\pi \to \{1,2\} \\ \text{such that } c(V_o)=1}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;c}(R_\mu, R_\nu). \tag{3.18}$$

And indeed, recall from Proposition 2.8 that we have a bijection

$$\begin{aligned} \bigl\{\rho \in NC(n) \bigm| \pi \ll \rho \ll 1_n\bigr\} &\to \{\mathfrak{V} \subseteq \pi \mid \mathfrak{V} \ni V_0\} \\ \rho &\mapsto \{V \in \pi \mid V \text{ is } \rho\text{-special}\}. \end{aligned}$$

When comparing this bijection against the formula which defined $\mathrm{o}_{\pi,\rho}$ in Notation 3.4, it is immediate that the map $\rho \mapsto \mathrm{o}_{\pi,\rho}$ is itself a bijection from $\{\rho \in NC(n) \mid \pi \ll \rho \ll 1_n\}$ onto the set of colourings $\{c : \pi \to \{1,2\} \mid c(V_0) = 1\}$, and (3.18) immediately follows. □

Remark 3.8. 1° When considered together, Eqs. (3.17) and (3.8) give that

$$\eta_{\mu \boxminus \nu} = R_{\mu \boxminus (\mu \boxplus \nu)}; \tag{3.19}$$

the latter formula is in turn telling us that

$$\mathbb{B}(\mu \boxminus \nu) = \mu \boxminus (\mu \boxplus \nu), \tag{3.20}$$

where $\mathbb{B}$ is the Boolean Bercovici–Pata bijection on $\mathcal{D}_{\mathrm{alg}}(k)$. Eq. (3.20) is a special case of Proposition 1.10; but actually the general case of Proposition 1.10 easily follows from here, as explained in the proof of Proposition 5.1 below.

2° In the same way as the statement of Proposition 3.3 was extended to the one of Lemma 3.6, the formula found in Proposition 3.7 can be extended to

$$\mathrm{Cf}_{(i_1,\dots,i_n);\rho}(\eta_{\mu \boxminus \nu}) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll \rho}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\mu + R_\nu), \tag{3.21}$$

holding for every $n \geqslant 1$, $\rho \in NC(n)$, and $1 \leqslant i_1, \dots, i_n \leqslant k$. Eq. (3.21) can be obtained from (3.17) by an argument similar to the one used in the proof of Lemma 3.6; but in fact we do not need to repeat that argument, we can simply infer (3.21) by using Lemma 3.6 itself, in conjunction to Eq. (3.19) from the first part of the present remark.

It is now easy to obtain the moment formula stated in Theorem 1.3.

Proposition 3.9. *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $n \geqslant 1$ and $1 \leqslant i_1, \dots, i_n \leqslant k$ one has*

$$(\mu \boxminus \nu)(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu). \tag{3.22}$$

Proof. By using the Boolean moment–cumulant formula (Eq. (2.29) in Remark 2.14), then by invoking Remark 3.8.2 and by performing an obvious change in the order of summation we get that

$$(\mu \boxplus \nu)(X_{i_1} \cdots X_{i_n}) = \sum_{\rho \in \mathrm{Int}(n)} \mathrm{Cf}_{(i_1,\ldots,i_n);\rho}(\eta_{\mu \boxplus \nu})$$

$$= \sum_{\rho \in \mathrm{Int}(n)} \Bigl(\sum_{\substack{\pi \in NC(n),\\ \pi \ll \rho}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\mu + R_\nu) \Bigr)$$

$$= \sum_{\pi \in NC(n)} \Bigl(\sum_{\substack{\rho \in \mathrm{Int}(n),\\ \rho \gg \pi}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_{\pi,\rho}}(R_\mu, R_\mu + R_\nu) \Bigr). \tag{3.23}$$

But for every $\pi \in NC(n)$ there exists a unique partition $\rho \in \mathrm{Int}(n)$ such that $\rho \gg \pi$, and for this ρ we have $\mathrm{o}_{\pi,\rho} = \mathrm{o}_\pi$ (as observed in Remark 3.5). Thus the sum over ρ in (3.23) consists of just one term, $\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu)$, and (3.22) follows. □

Remark 3.10. 1° A summation of the same type as in Eq. (3.22), which uses coefficients from two series and distinguishes between the inner and outer blocks of $\pi \in NC(n)$, has previously appeared in the theory of c-free convolution – see e.g. the third displayed equation in [10, p. 366]. This connection is not pursued in the present paper, but c-free convolution is heavily used in [2] (which relates to the present paper in the way explained in Remark 1.14).

2° In the proof of Theorem 4.4 we will also need the equivalent form of Eq. (3.22) where, for every $\pi \in NC(n)$, the product defining $\mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, R_\mu + R_\nu)$ is expanded into a sum. It is immediate (left as exercise to the reader) to check that the formula for the moments of $\mu \boxplus \nu$ will then look as follows:

$$(\mu \boxplus \nu)(X_{i_1} \cdots X_{i_n}) = \sum_{(\pi,c)} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(R_\mu, R_\nu), \tag{3.24}$$

where the index set for the sum on the right-hand side of (3.24) is

$$\left\{ (\pi, c) \;\middle|\; \begin{array}{l} \pi \in NC(n),\ c : \pi \to \{1, 2\},\ \text{such that} \\ c(V) = 1 \text{ for every outer block } V \text{ of } \pi \end{array} \right\}.$$

Remark 3.11. Let μ and $(\mu_N)_{N \geqslant 1}$ be in $\mathcal{D}_{\mathrm{alg}}(k)$. If

$$\lim_{N \to \infty} \mu_N(X_{i_1} \cdots X_{i_n}) = \mu(X_{i_1} \cdots X_{i_n}), \quad \forall n \geqslant 1,\ \forall 1 \leqslant i_1, \ldots, i_n \leqslant k, \tag{3.25}$$

then one says that the sequence $(\mu_N)_{N \geqslant 1}$ *converges in moments* to μ (denoted simply as $\mu_N \to \mu$). Due to the moment–cumulant formulas from Remark 2.14, this is equivalent to convergence in coefficients for the R-transforms R_{μ_N} to R_μ, or for the η-series η_{μ_N} to η_μ.

Now, from the fact that one has polynomial expressions giving the moments of $\mu \boxplus \nu$ in terms of the free cumulants of μ and of ν it is immediate that the operation $\boxplus$ is well-behaved under taking limits in moments in $\mathcal{D}_{\mathrm{alg}}(k)$. That is, if $\mu, \nu, (\mu_N)_{N=1}^\infty$ and $(\nu_N)_{N=1}^\infty$ are distributions in $\mathcal{D}_{\mathrm{alg}}(k)$ such that $\mu_N \to \mu$ and $\nu_N \to \nu$, then it follows that $\mu_N \boxplus \nu_N \to \mu \boxplus \nu$. The same conclusion could have been of course derived directly from Proposition 3.3, or from Proposition 3.7.

4. The approach to $\boxplus$ via operator models

This section puts into evidence a full Fock space model for $\mu \boxplus \nu$, then uses this model in order to obtain Theorem 1.4 stated in the introduction of the paper.

The full Fock space model is given in Theorem 4.4, and is just a variation of the "standard" full Fock space model for the R-transform (as presented for instance in [13, Lecture 21]). In order to avoid tedious notations involving formal operators on the full Fock space, we will only consider this model in the special case when the R-transforms R_μ and R_ν are polynomials. A generalization of Theorem 4.4 could be obtained from this special case by doing approximations in distribution (a very similar procedure to how Theorem 21.4 is extended to Theorem 21.7 in [13, Lecture 21]). However, for the situation at hand it is actually more convenient to incorporate the necessary approximations in distribution directly into the proof of Theorem 4.10 below, where the full Fock space model is upgraded to the more general framework of Theorem 1.4.

Notation 4.1. Let $\mathcal{F}$ be the full Fock space over $\mathbb{C}^{2k}$,

$$\mathcal{F} := \mathbb{C} \oplus \mathbb{C}^{2k} \oplus \left(\mathbb{C}^{2k}\right)^{\otimes 2} \oplus \cdots \oplus \left(\mathbb{C}^{2k}\right)^{\otimes n} \oplus \cdots.$$

The vector $1 \oplus 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \cdots$ is called the *vacuum-vector* of $\mathcal{F}$ and is denoted by Ω. We will let $P_\Omega \in B(\mathcal{F})$ denote the orthogonal projection onto the 1-dimensional space $\mathbb{C}\Omega \subseteq \mathcal{F}$. The vector-state $T \mapsto \langle T\Omega, \Omega\rangle$ defined by Ω on $B(\mathcal{F})$ will be referred to as *vacuum-state*.

We fix an orthonormal basis for $\mathbb{C}^{2k}$, which we denote as $e'_1, \dots, e'_k, e''_1, \dots, e''_k$. This leads to a natural choice of orthonormal basis for $\mathcal{F}$,

$$\{\Omega\} \cup \left\{\xi_1 \otimes \cdots \otimes \xi_n \;\middle|\; n \geqslant 1,\ \xi_1, \dots, \xi_n \in \left\{e'_1, \dots, e'_k, e''_1, \dots, e''_k\right\}\right\}. \tag{4.1}$$

For every $1 \leqslant i \leqslant k$ the left creation operators with e'_i and e''_i will be denoted by L'_i and L''_i, respectively. So $L'_i \in B(\mathcal{F})$ is the isometry which acts on the orthonormal basis (4.1) by

$$L'_i(\Omega) = e'_i, \qquad L'_i(\xi_1 \otimes \cdots \otimes \xi_n) = e'_i \otimes \xi_1 \otimes \cdots \otimes \xi_n,$$

and similar formulas hold for L''_i. Moreover, we will denote by $\mathfrak{M}'$ and $\mathfrak{M}''$ the sets of operators in $B(\mathcal{F})$ defined by

$$\begin{cases} \mathfrak{M}' := \{L'_{i_1} \cdots L'_{i_n} \mid n \geqslant 1,\ 1 \leqslant i_1, \dots, i_n \leqslant k\}, \\ \mathfrak{M}'' := \{L''_{i_1} \cdots L''_{i_n} \mid n \geqslant 1,\ 1 \leqslant i_1, \dots, i_n \leqslant k\}. \end{cases} \tag{4.2}$$

The full Fock space model from Theorem 4.4 will use some special monomials "$S_1^* M_1 \cdots S_n^* M_n$" formed with the isometries $L'_1, \dots, L'_k, L''_1, \dots, L''_k$ and their adjoints, which are described in the next lemma.

Lemma 4.2. *Given a positive integer n and some fixed indices $i_1, \dots, i_n \in \{1, \dots, k\}$.*

1° *Let π be a partition in $NC(n)$ and let $c : \pi \to \{1, 2\}$ be a colouring. For every $m \in \{1, \dots, n\}$ let $V = \{v(1), v(2), \dots, v(p)\}$ (with $v(1) < v(2) < \cdots < v(p)$) denote the block of π which contains m, and define*

$$S_m := \begin{cases} L'_{i_m} & \textit{if } c(V) = 1, \\ L''_{i_m} & \textit{if } c(V) = 2, \end{cases} \tag{4.3}$$

$$M_m = \begin{cases} L'_{i_{v(p)}} \cdots L'_{i_{v(2)}} L'_{i_{v(1)}} & \textit{if } m = \max(V) (= v(p)) \textit{ and } c(V) = 1, \\ L''_{i_{v(p)}} \cdots L''_{i_{v(2)}} L''_{i_{v(1)}} & \textit{if } m = \max(V) \textit{ and } c(V) = 2, \\ 1_{B(\mathcal{F})} & \textit{if } m \neq \max(V). \end{cases} \tag{4.4}$$

Then $S_1^* M_1 \cdots S_n^* M_n \Omega = \Omega$.

2° *Suppose that* $S_1, \ldots, S_n, M_1, \ldots, M_n \in B(\mathcal{F})$ *are such that*

(i) $S_m \in \{L'_{i_m}, L''_{i_m}\}$, $1 \leqslant m \leqslant n$;

(ii) $M_m \in \{1_{B(\mathcal{F})}\} \cup \mathfrak{M}' \cup \mathfrak{M}''$, $1 \leqslant m \leqslant n$ *(with* $\mathfrak{M}', \mathfrak{M}''$ *as in* (4.2)); *and*

(iii) $S_1^* M_1 \cdots S_n^* M_n \Omega = \Omega$.

Then there exist a partition $\pi \in NC(n)$ *and a colouring* $c : \pi \to \{1, 2\}$ *such that* $S_1, \ldots, S_n$, $M_1, \ldots, M_n$ *are obtained from* π *and* c *via the recipe described in part* 1° *of the lemma.*

Remark 4.3. 1° Here is a concrete example of how the recipe from Lemma 4.2 works. Say for instance that $n = 5$. Let $i_1, \ldots, i_5$ be some indices in $\{1, \ldots, k\}$, and consider the monomial

$$(L'_{i_1})^* (L''_{i_2})^* (L''_{i_3})^* L''_{i_3} L''_{i_2} (L'_{i_4})^* (L'_{i_5})^* L'_{i_5} L'_{i_4} L'_{i_1}. \tag{4.5}$$

Note that the product in (4.5) reduces upon simplifications to $1_{B(\mathcal{F})}$, so in particular it fixes Ω. Lemma 4.2 views this product as being $S_1^* M_1 \cdots S_5^* M_5$, where

$$\begin{cases} S_1 = L'_{i_1}, \quad S_2 = L''_{i_2}, \quad S_3 = L''_{i_3}, \quad S_4 = L'_{i_4}, \quad S_5 = L'_{i_5}, \quad \text{and} \\ M_1 = M_2 = M_4 = 1_{B(\mathcal{F})}, \quad M_3 = L''_{i_3} L''_{i_2}, \quad M_5 = L'_{i_5} L'_{i_4} L'_{i_1}. \end{cases}$$

Moreover, these $S_1, \ldots, S_5, M_1, \ldots, M_5$ correspond in Lemma 4.2 to the partition $\pi = \{V_1, V_2\} \in NC(5)$ with $V_1 = \{1, 4, 5\}$, $V_2 = \{2, 3\}$, and to the colouring $c : \pi \to \{1, 2\}$ defined by $c(V_1) = 1$, $c(V_2) = 2$.

2° The proof of Lemma 4.2 is very similar to the corresponding argument concerning the standard full Fock space model for the R-transform, as presented e.g. in [13, Lecture 21]. Because of this, I will only explain (in the remaining part of this remark) how one makes the connection to the arguments from [13], and will leave the details as exercise to the reader.

Besides $\mathfrak{M}'$ and $\mathfrak{M}''$ from (4.2), let us also use the notation

$$\mathfrak{M} := \{1_{B(\mathcal{F})}\} \cup \{S_1 \cdots S_\ell \mid \ell \geqslant 1, \ S_1, \ldots, S_\ell \in \{L'_1, \ldots, L'_k, L''_1, \ldots, L''_k\}\}. \tag{4.6}$$

Suppose that the following data is given: a positive integer n, some indices $i_1, \ldots, i_n \in \{1, \ldots, k\}$, and a function $b : \{1, \ldots, n\} \to \{1, 2\}$. Let the isometries $S_1 \in \{L'_{i_1}, L''_{i_1}\}, \ldots, S_n \in \{L'_{i_n}, L''_{i_n}\}$ be picked via the rule that

$$S_m = \begin{cases} L'_{i_m} & \text{if } b(m) = 1, \\ L''_{i_m} & \text{if } b(m) = 2, \end{cases} \quad 1 \leqslant m \leqslant n, \tag{4.7}$$

and consider the following problem: describe all possible ways of choosing $(M_1, \dots, M_n) \in \mathfrak{M}^n$ such that $S_1^* M_1 \cdots S_n^* M_n \Omega = \Omega$.[4] The solution to this problem is that the n-tuples $(M_1, \dots, M_n)$ with the required property are canonically parametrized by $NC(n)$. For the description of how to construct the n-tuple $(M_1, \dots, M_n) \in \mathfrak{M}^n$ canonically associated to a partition $\pi \in NC(n)$, and for the explanation why this construction works, see the discussion on pp. 342, 343 and Exercises 21.20–21.22 on pp. 356, 357 of [13]. The statement of Lemma 4.2 is merely an adjustment of this procedure (for how to construct $(M_1, \dots, M_n)$ by starting from π), where one has to take into account the following additional detail: $M_1, \dots, M_n$ are now only allowed to run in the smaller set $\{1_{B(\mathcal{F})}\} \cup \mathfrak{M}' \cup \mathfrak{M}''$ (instead of all of $\mathfrak{M}$). This imposes a compatibility condition between π and the function $b : \{1, \dots, n\} \to \{1, 2\}$ that was used in (4.7) – specifically, that b must be constant along the blocks of π (and hence must correspond to a colouring c of π).

Theorem 4.4. *Let μ, ν be distributions in $\mathcal{D}_{\text{alg}}(k)$ such that the R-transforms R_μ and R_ν are polynomials*:

$$\begin{cases} R_\mu(z_1, \dots, z_k) = \displaystyle\sum_{n=1}^{N} \sum_{i_1, \dots, i_n = 1}^{k} \alpha_{(i_1, \dots, i_n)} z_{i_1} \cdots z_{i_n}, \\ R_\nu(z_1, \dots, z_k) = \displaystyle\sum_{n=1}^{N} \sum_{i_1, \dots, i_n = 1}^{k} \beta_{(i_1, \dots, i_n)} z_{i_1} \cdots z_{i_n} \end{cases} \tag{4.8}$$

(*where N is a common upper bound for the degrees of R_μ and R_ν*). *In the framework of Notation* 4.1, *consider the operator $T \in B(\mathcal{F})$ defined by*

$$T = 1_{B(\mathcal{F})} + \sum_{n=1}^{N} \sum_{i_1, \dots, i_n = 1}^{k} \alpha_{(i_1, \dots, i_n)} L'_{i_n} \cdots L'_{i_1} + \sum_{n=1}^{N} \sum_{i_1, \dots, i_n = 1}^{k} \beta_{(i_1, \dots, i_n)} L''_{i_n} \cdots L''_{i_1}, \tag{4.9}$$

and make the notations

$$A_i := (L'_i)^* T, \qquad B_i := (L''_i)^* T, \quad 1 \leqslant i \leqslant k, \tag{4.10}$$

followed by

$$C_i := A_i + (1 - P_\Omega) B_i (1 - P_\Omega), \quad 1 \leqslant i \leqslant k. \tag{4.11}$$

Then the joint distribution of $C_1, \dots, C_k$ with respect to the vacuum-state on $B(\mathcal{F})$ is equal to $\mu \boxplus \nu$.

Remark 4.5. By comparing the framework of Theorem 4.4 with the "standard" full Fock space model for the R-transform (as presented for instance in Theorem 21.4 of [13]), one sees that the operators $A_1, \dots, A_k, B_1, \dots, B_k$ defined by Eq. (4.10) give the standard full Fock space model for the free product $\mu \star \nu \in \mathcal{D}_{\text{alg}}(2k)$. In particular one has that $\{A_1, \dots, A_k\}$ is free from

[4] It is easy to see that this condition is in fact equivalent to the requirement that the product $S_1^* M_1 \cdots S_n^* M_n$ simplifies to $1_{B(\mathcal{F})}$ after repeated use of the relations $(L'_i)^* L'_i = (L''_i)^* L''_i = 1_{B(\mathcal{F})}$, $1 \leqslant i \leqslant k$.

$\{B_1, \ldots, B_k\}$ with respect to the vacuum-state on $B(\mathcal{F})$, and the joint distributions of the k-tuples $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are equal to μ and to ν, respectively.

Another way of phrasing this same remark is that the full Fock space model for $\mu \boxplus \nu$ is obtained by merely performing an extra step (specifically, by considering the operators $C_1, \ldots, C_k$ defined by Eq. (4.11)) in the standard full Fock space model for $\mu \star \nu$.

Proof of Theorem 4.4. For the whole proof we fix a positive integer n and some indices $1 \leqslant i_1, \ldots, i_n \leqslant k$, for which we will show that

$$\langle C_{i_1} \cdots C_{i_n} \Omega, \Omega \rangle = (\mu \boxplus \nu)(X_{i_1} \cdots X_{i_n}). \tag{4.12}$$

From (4.9)–(4.11) it follows that every C_i $(1 \leqslant i \leqslant k)$ can be written as a sum of products of the form

$$Q \cdot S^* \cdot (\gamma M) \cdot Q, \tag{4.13}$$

where $Q \in \{1_{B(\mathcal{F})}, 1_{B(\mathcal{F})} - P_\Omega\}$, $S \in \{L'_i, L''_i\}$, and γM is a term from the sum defining T (where $\gamma \in \mathbb{C}$ and $M \in \{1_{B(\mathcal{F})}\} \cup \mathfrak{M}' \cup \mathfrak{M}''$). Of course, there are some restrictions on what combinations of Q, S and γM can go together in (4.13): if $Q = 1_{B(\mathcal{F})}$ then $S = L'_i$ and γM is either $1_{B(\mathcal{F})}$ or of the form $\alpha_{(j_1,\ldots,j_m)} L'_{j_m} \cdots L'_{j_1}$, while $Q = 1_{B(\mathcal{F})} - P_\Omega$ goes with $S = L''_i$ and with γM being either $1_{B(\mathcal{F})}$ or of the form $\beta_{(j_1,\ldots,j_m)} L''_{j_m} \cdots L''_{j_1}$. A precise count thus gives that every C_i splits into a sum of $2 \cdot (1 + k + \cdots + k^N)$ terms of the form (4.13). When one writes each of $C_{i_1}, \ldots, C_{i_n}$ as a sum in this way and expands the product, the inner product on the left-hand side of (4.12) is thus broken into a sum of $(2 \cdot (1 + k + \cdots + k^N))^n$ terms of the form

$$\big\langle \big(Q_1 S_1{}^*(\gamma_1 M_1) Q_1\big) \cdots \big(Q_n S_n{}^*(\gamma_n M_n) Q_n\big)\Omega, \Omega\big\rangle. \tag{4.14}$$

Now let us fix one of the possible choices of operators $Q_i, S_i, \gamma_i M_i$ $(1 \leqslant i \leqslant n)$ in (4.14), and let us look at the $4n$ vectors

$$\xi_1 = Q_n \Omega, \quad \xi_2 = M_n Q_n \Omega, \quad \ldots, \quad \xi_{4n} = Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega \tag{4.15}$$

obtained by successively applying the operators $Q_n, M_n, S_n{}^*, Q_n, \ldots, Q_1, M_1, S_1{}^*, Q_1$ to Ω. It is clear that each of these $4n$ vectors either is 0 or belongs to the orthonormal basis (4.1) for $\mathcal{F}$; and consequently, the inner product (4.14) is equal to

$$\begin{cases} \gamma_1 \cdots \gamma_n & \text{if } Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega = \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

Let us moreover observe that if $Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega = \Omega$, then we also have $S_1^* M_1 \cdots S_n^* M_n \Omega = \Omega$. This is because when one successively applies $Q_n, M_n, \ldots, S_1{}^*, Q_1$ to Ω, the projections $Q_1, \ldots, Q_n$ used on the way either leave invariant the vector presented to them, or send it to 0 (but cannot actually do the latter, as $Q_1 S_1^* \cdots M_n Q_n \Omega = \Omega \neq 0$).

By invoking Lemma 4.2 we thus see that if an inner product as in (4.14) is to be different from 0, then there have to exist a partition $\pi \in NC(n)$ and a colouring $c : \pi \to \{1, 2\}$ such that $S_1, M_1, \ldots, S_n, M_n$ are defined in terms of π and c in the way described in Lemma 4.2. It is immediate that in this case the numbers $\gamma_1, \ldots, \gamma_n$ from (4.16) are identified as $\alpha_{(j_1,\ldots,j_m)}$'s and

$\beta_{(j_1,\ldots,j_m)}$'s (coefficients of the R-transforms of μ and of ν) in such a way that their product becomes

$$\gamma_1 \cdots \gamma_n = \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(R_\mu, R_\nu). \tag{4.17}$$

Conversely, let π be a partition in $NC(n)$, let c be a colouring of π, and consider the operators $S_1, M_1, \ldots, S_n, M_n$ defined in terms of π and c in the way described in Lemma 4.2. Observe that there exists a unique way of choosing projections $Q_1, \ldots, Q_n \in \{1_{B(\mathcal{F})}, 1_{B(\mathcal{F})} - P_\Omega\}$ so that the S_j, M_j, Q_j for $1 \leqslant j \leqslant n$ give together an inner product as in (4.14). To be precise, for every $1 \leqslant j \leqslant n$ the projection Q_j is chosen as follows: consider the block V of π which contains the number j, and put

$$Q_j = \begin{cases} 1_{B(\mathcal{F})} & \text{if } c(V) = 1, \\ 1_{B(\mathcal{F})} - P_\Omega & \text{if } c(V) = 2. \end{cases} \tag{4.18}$$

Note that whereas Lemma 4.2 ensures that $S_1^* M_1 \cdots S_n^* M_n \Omega = \Omega$, it may still happen that (with Q_js defined by (4.18)) the vector $Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega$ is equal to 0. It is easy (though perhaps notationally tedious) to check that

$$Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega = \begin{cases} \Omega & \text{if } c(V) = 1 \text{ for every outer block of } \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

The verification of (4.19) is left as exercise to the reader. Informally speaking, what makes (4.19) hold is that in a sequence of $4n$ vectors obtained as in (4.15) one reaches Ω precisely at the positions where the outer blocks of π begin and end – hence these are the positions where Q_j has a chance to make a difference, and cause the vector $Q_1 S_1^* M_1 Q_1 \cdots Q_n S_n^* M_n Q_n \Omega$ to vanish.

Summarizing the above discussion, one sees that

$$\langle C_{i_1} \cdots C_{i_n} \Omega, \Omega \rangle = \sum_{(\pi,c)} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;c}(R_\mu, R_\nu), \tag{4.20}$$

where the index set for the sum on the right-hand side of (4.20) is

$$\left\{ (\pi, c) \;\middle|\; \begin{array}{l} \pi \in NC(n),\ c : \pi \to \{1,2\},\ \text{such that} \\ c(V) = 1 \text{ for every outer block } V \text{ of } \pi \end{array} \right\}.$$

But the sum on the right-hand side of (4.20) is precisely the expression observed for $(\mu \boxplus \nu)(X_{i_1} \cdots X_{i_n})$ in Remark 3.10.2, and this concludes the proof. $\square$

Let us now go towards the proof of Theorem 1.4. It will be convenient to adopt a slightly different point of view on the vacuum projection, which does not make explicit use of vectors in a Hilbert space, and is described as follows.

Definition 4.6. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. A *vacuum-projection* for φ is an element $P \in \mathcal{A}$ such that $P = P^2 \neq 0$ and such that

$$PAP = \varphi(A)P, \quad \forall A \in \mathcal{A}. \tag{4.21}$$

Remark 4.7. 1° The main example of vacuum-projection is of course provided by the situation when $\mathcal{A} = B(\mathcal{H})$, the functional φ is the vector-state associated to a unit vector $\xi_0 \in \mathcal{H}$, and P is the orthogonal projection onto the 1-dimensional subspace $\mathbb{C}\xi_0$ of $\mathcal{H}$.

2° Let $(\mathcal{A}, \varphi)$ and P be as in Definition 4.6. Observe that $\varphi(P) = 1$ (as seen by making $A = P$ in Eq. (4.21)). Let us also observe that

$$\varphi(PB) = \varphi(B) = \varphi(BP), \quad \forall B \in \mathcal{A}. \tag{4.22}$$

In order to verify the first of these two equalities we set $A = (1_{\mathcal{A}} - P)B$ and find that

$$\varphi(A)P = PAP = P(1_{\mathcal{A}} - P)BP = 0,$$

which implies that $\varphi(A) = 0$ and hence that $\varphi(B) = \varphi(PB)$. The verification of the second equality in (4.22) is analogous.

Lemma 4.8. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $P \in \mathcal{A}$ be a vacuum-projection for φ. Then*

$$\varphi(T_1 P T_2 P \cdots P T_n) = \prod_{i=1}^{n} \varphi(T_i), \quad \forall n \geqslant 2 \text{ and } T_1, \ldots, T_n \in \mathcal{A}. \tag{4.23}$$

Proof. By induction on n. For $n = 2$ we write

$$\begin{aligned}
\varphi(T_1 P T_2) &= \varphi(T_1 P T_2 P) \quad \big(\text{by (4.22)}\big) \\
&= \varphi\big(T_1 \cdot \varphi(T_2) P\big) \quad \big(\text{by (4.21)}\big) \\
&= \varphi(T_2)\varphi(T_1 P) \\
&= \varphi(T_2)\varphi(T_1) \quad \big(\text{by (4.22)}\big).
\end{aligned}$$

The induction step "$n \Rightarrow n+1$" is immediately obtained by writing $T_1 P T_2 P \cdots P T_n P T_{n+1}$ as $T_1 P T_2'$ with $T_2' := T_2 P \cdots P T_n P T_{n+1}$ and by repeating the above calculation, followed by the induction hypothesis. □

Lemma 4.9. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $T_1, \ldots, T_\ell, P$ be in $\mathcal{A}$, where P is a vacuum-projection for φ. Suppose moreover that for every $N \geqslant 1$ we are given a noncommutative probability space $(\mathcal{A}_N, \varphi_N)$ and elements $T_1^{(N)}, \ldots, T_\ell^{(N)}, P^{(N)} \in \mathcal{A}_N$, such that $P^{(N)}$ is a vacuum-projection for φ_N. If the ℓ-tuples $T_1^{(N)}, \ldots, T_\ell^{(N)}$ converge in moments for $N \to \infty$ to $T_1, \ldots, T_\ell$, then the $(\ell+1)$-tuples $T_1^{(N)}, \ldots, T_\ell^{(N)}, P^{(N)}$ converge in moments for $N \to \infty$ to the $(\ell+1)$-tuple $T_1, \ldots, T_\ell, P$.*

Proof. It clearly suffices to verify that, for any $n \geqslant 2$ and any choice of noncommutative polynomials $f_1, \ldots, f_n \in \mathbb{C}\langle X_1, \ldots, X_\ell\rangle$, the sequence

$$\varphi_N\big(f_1\big(T_1^{(N)}, \ldots, T_\ell^{(N)}\big) P^{(N)} f_2\big(T_1^{(N)}, \ldots, T_\ell^{(N)}\big) P^{(N)} \cdots P^{(N)} f_n\big(T_1^{(N)}, \ldots, T_\ell^{(N)}\big)\big), \quad N \geqslant 1,$$

converges for $N \to \infty$ to $\varphi(f_1(T_1, \dots, T_\ell)Pf_2(T_1, \dots, T_\ell)P \cdots Pf_n(T_1, \dots, T_\ell))$. But in view of Lemma 4.8 the latter convergence amounts to

$$\lim_{N\to\infty} \prod_{i=1}^{n} \varphi_N\big(f_i\big(T_1^{(N)}, \dots, T_\ell^{(N)}\big)\big) = \prod_{i=1}^{n} \varphi\big(f_i(T_1, \dots, T_\ell)\big),$$

which is an immediate consequence of the given hypothesis. □

Theorem 4.10. *Let two distributions $\mu, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$ be given. Suppose that $(\mathcal{A}, \varphi)$ is a noncommutative probability space and that $A_1, \dots, A_k, B_1, \dots, B_k \in \mathcal{A}$ are such that $\{A_1, \dots, A_k\}$ is free from $\{B_1, \dots, B_k\}$, such that the joint distribution of $A_1, \dots, A_k$ is equal to μ, and such that the joint distribution of $B_1, \dots, B_k$ is equal to ν. Suppose in addition that $P \in \mathcal{A}$ is a vacuum-projection for φ, and consider the elements*

$$C_i := A_i + (1_{\mathcal{A}} - P)B_i(1_{\mathcal{A}} - P), \quad 1 \leqslant i \leqslant k. \tag{4.24}$$

Then the joint distribution of $C_1, \dots, C_k$ with respect to φ is equal to $\mu \boxplus \nu$.

Proof. For $n \geqslant 1$ and $1 \leqslant i_1, \dots, i_n \leqslant k$ we will denote the coefficients of $z_{i_1} \cdots z_{i_n}$ in the series R_μ and R_ν by $\alpha_{(i_1,\dots,i_n)}$ and $\beta_{(i_1,\dots,i_n)}$, respectively.

Let N be a positive integer. Consider the distributions $\mu_N, \nu_N \in \mathcal{D}_{\mathrm{alg}}(k)$ which are uniquely determined by the requirement that their R-transforms are

$$\begin{cases} R_{\mu}(z_1, \dots, z_k) = \displaystyle\sum_{n=1}^{N} \sum_{i_1,\dots,i_n=1}^{k} \alpha_{(i_1,\dots,i_n)} z_{i_1} \cdots z_{i_n}, \\ R_{\nu}(z_1, \dots, z_k) = \displaystyle\sum_{n=1}^{N} \sum_{i_1,\dots,i_n=1}^{k} \beta_{(i_1,\dots,i_n)} z_{i_1} \cdots z_{i_n}. \end{cases} \tag{4.25}$$

Let us consider the standard full Fock space model, exactly as described in Theorem 21.4 of [13], for the free product $\mu_N * \nu_N \in \mathcal{D}_{\mathrm{alg}}(2k)$. This gives us a noncommutative probability space $(\mathcal{A}_N, \varphi_N)$ and elements $A_1^{(N)}, \dots, A_k^{(N)}, B_1^{(N)}, \dots, B_k^{(N)} \in \mathcal{A}_N$ such that $\{A_1^{(N)}, \dots, A_k^{(N)}\}$ is free from $\{B_1^{(N)}, \dots, B_k^{(N)}\}$, such that the joint distribution of $A_1^{(N)}, \dots, A_k^{(N)}$ is equal to μ_N, and such that the joint distribution of $B_1^{(N)}, \dots, B_k^{(N)}$ is equal to ν_N. Since the full Fock space model is constructed by using a true vacuum-state on a Hilbert space, we also get at the same time a vacuum-projection $P^{(N)} \in \mathcal{A}_N$.

We now make $N \to \infty$. From how μ_N and ν_N were constructed it is immediate that we have limits in moments $\mu_N \to \mu$ and $\nu_N \to \nu$. This implies that we also have the limit in moments $\mu_N * \nu_N \to \mu * \nu$, or in terms of operators that the $(2k)$-tuples $A_1^{(N)}, \dots, A_k^{(N)}, B_1^{(N)}, \dots, B_k^{(N)}$ converge in moments for $N \to \infty$ to the $(2k)$-tuple $A_1, \dots, A_k, B_1, \dots, B_k$. By invoking Lemma 4.9 we upgrade this to the fact that the $(2k+1)$-tuples $A_1^{(N)}, \dots, A_k^{(N)}, B_1^{(N)}, \dots, B_k^{(N)}, P^{(N)}$ converge in moments for $N \to \infty$ to the $(2k+1)$-tuple $A_1, \dots, A_k, B_1, \dots, B_k, P$. The latter convergence implies in turn that the k-tuple $C_1, \ldots, C_k$ defined in (4.24) is the limit in moments for the k-tuples $C_1^{(N)}, \ldots, C_k^{(N)}$, where for $1 \leqslant i \leqslant k$ and $N \geqslant 1$ we put

$$C_i^{(N)} := A_i^{(N)} + \big(1_{\mathcal{A}_N} - P^{(N)}\big) B_i^{(N)} \big(1_{\mathcal{A}_N} - P^{(N)}\big) \in \mathcal{A}_N. \tag{4.26}$$

But for every $N \geqslant 1$, the operators $C_1^{(N)}, \ldots, C_k^{(N)}$ provide (as observed at the end of Remark 4.5) the full Fock space model for the subordination distribution $\mu_N \boxminus \nu_N$. Hence the conclusion of the preceding paragraph can be read as follows: the joint distribution of $C_1, \ldots, C_k$ is the $N \to \infty$ limit of the distributions $\mu_N \boxminus \nu_N$. Since it was noticed in Remark 3.11 that $(\mu_N \boxminus \nu_N)_{N=1}^{\infty}$ converges in moments to $\mu \boxminus \nu$, the conclusion of the theorem follows. □

Remark 4.11. Suppose now that $\mu, \nu \in \mathcal{D}_c(k)$, i.e. they can appear as joint distributions for k-tuples of selfadjoint elements in some C^*-probability spaces. By considering the GNS representations of these C^*-probability spaces, one finds Hilbert spaces $\mathcal{H}, \mathcal{K}$, unit vectors $\xi_o \in \mathcal{H}$, $\zeta_o \in \mathcal{K}$, and k-tuples of selfadjoint operators $A_1, \ldots, A_k \in B(\mathcal{H})$, $B_1, \ldots, B_k \in B(\mathcal{K})$ such that μ is the joint distribution of $A_1, \ldots, A_k$ with respect to the vector-state defined by ξ_o on $B(\mathcal{H})$, while ν is the joint distribution of $B_1, \ldots, B_k$ with respect to the vector-state defined by ζ_o on $B(\mathcal{K})$. Let us denote

$$\mathcal{H}^o := \mathcal{H} \ominus \mathbb{C}\xi_o, \qquad \mathcal{K}^o := \mathcal{K} \ominus \mathbb{C}\zeta_o,$$

and let us consider the "free product" Hilbert space

$$\begin{aligned} \mathcal{M} := \mathbb{C}\Omega \oplus \big(\mathcal{H}^o \oplus \mathcal{K}^o\big) \oplus \big(\big(\mathcal{H}^o \otimes \mathcal{K}^o\big) \oplus \big(\mathcal{K}^o \otimes \mathcal{H}^o\big)\big) \\ \oplus \big(\big(\mathcal{H}^o \otimes \mathcal{K}^o \otimes \mathcal{H}^o\big) \oplus \big(\mathcal{K}^o \otimes \mathcal{H}^o \otimes \mathcal{K}^o\big)\big) \oplus \cdots \end{aligned} \tag{4.27}$$

(direct sum of all possible alternating tensor products of copies of $\mathcal{H}^o$ and $\mathcal{K}^o$). Then $A_1, \ldots, A_k, B_1, \ldots, B_k$ extend naturally to selfadjoint operators $\widetilde{A}_1, \ldots, \widetilde{A}_k$, $\widetilde{B}_1, \ldots, \widetilde{B}_k \in B(\mathcal{M})$ such that $\{\widetilde{A}_1, \ldots, \widetilde{A}_k\}$ is free from $\{\widetilde{B}_1, \ldots, \widetilde{B}_k\}$ with respect to the vacuum-state defined by Ω on $B(\mathcal{M})$ and such that (with respect to the same state) the joint distributions of $\widetilde{A}_1, \ldots, \widetilde{A}_k$ and of $\widetilde{B}_1, \ldots, \widetilde{B}_k$ are equal to μ and ν, respectively (see e.g. [17, Section 1.5]).

Theorem 4.10 clearly applies in the situation described in the preceding paragraph, and tells us that if $P_\Omega \in B(\mathcal{M})$ is the orthogonal projection onto $\mathbb{C}\Omega$ and if we put

$$\widetilde{C}_i = \widetilde{A}_i + (1 - P_\Omega)\widetilde{B}_i(1 - P_\Omega), \quad 1 \leqslant i \leqslant k, \tag{4.28}$$

then the joint distribution of $\widetilde{C}_1, \ldots, \widetilde{C}_k$ is equal to $\mu \boxminus \nu$. Since the $\widetilde{C}_i$ are selfadjoint, this provides us with a proof that (as stated in Corollary 1.5) the subordination distribution $\mu \boxminus \nu$ does indeed belong to $\mathcal{D}_c(k)$.

Remark 4.12. In the framework and notations of the preceding remark, consider the subspace $\mathcal{L}$ of $\mathcal{M}$ defined by:

$$\mathcal{L} := \mathbb{C}\Omega \oplus \mathcal{H}^o \oplus \big(\mathcal{K}^o \otimes \mathcal{H}^o\big) \oplus \big(\mathcal{H}^o \otimes \mathcal{K}^o \otimes \mathcal{H}^o\big) \oplus \cdots \tag{4.29}$$

(direct sum of all alternating tensor products of copies of $\mathcal{H}^o$ and $\mathcal{K}^o$ which end in $\mathcal{H}^o$). In the terminology of [11], this is the *s-free product space* of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, considered with respect to the special unit vectors $\xi_o \in \mathcal{H}$ and $\zeta_o \in \mathcal{K}$.

Observe that $\mathcal{L}$ is invariant for the operators $\widetilde{C}_1, \ldots, \widetilde{C}_k$ from (4.28); this happens because $\mathcal{L}$ is in fact invariant both for $\widetilde{A}_i$ and for $(1 - P_\Omega)\widetilde{B}_i(1 - P_\Omega)$, $1 \leqslant i \leqslant k$. It follows that the restrictions of $\widetilde{C}_1, \ldots, \widetilde{C}_k$ to $\mathcal{L}$ also provide us with an operator model for $\mu \boxminus \nu$, with respect to the vector-state defined by Ω on $B(\mathcal{L})$. By analyzing this operator model a bit further, one can moreover relate to the concept of "s-freeness" from [11], in the way outlined in the next paragraph.

For every $1 \leqslant i \leqslant k$ let $\widehat{A}_i$ and $\widehat{B}_i$ denote the restrictions to $\mathcal{L}$ of the operators $\widetilde{A}_i$ and respectively $(1 - P_\Omega)\widetilde{B}_i(1 - P_\Omega)$. Let us consider the subalgebras $\mathcal{A}, \mathcal{B}$ of $B(\mathcal{L})$ which are generated by $\{1_{B(\mathcal{L})}, \widehat{A}_1, \ldots, \widehat{A}_k\}$ and respectively by $\{1_{B(\mathcal{L})} - P_\Omega, \widehat{B}_1, \ldots, \widehat{B}_k\}$. (Note that $\mathcal{B}$ is not a unital subalgebra of $B(\mathcal{L})$, but it has its own unit $1_{\mathcal{B}} = 1_{B(\mathcal{L})} - P_\Omega$, where P_Ω is viewed here as a 1-dimensional projection in $B(\mathcal{L})$.) Finally, let us select (and fix) an arbitrary unit vector $\theta_o \in \mathcal{H}^o \subseteq \mathcal{L}$, and let φ and ψ be the vector-states defined on $B(\mathcal{L})$ by Ω and by θ_o, respectively. It is not hard to verify that the algebras $\mathcal{A}$ and $\mathcal{B}$ are s-free in $(B(\mathcal{L}), \varphi, \psi)$, in the sense of Definition 7.1 from [11]. It is moreover immediate that the joint distribution of $\widehat{A}_1, \ldots, \widehat{A}_k$ in $(\mathcal{A}, \varphi \mid \mathcal{A})$ is equal to μ, while the joint distribution of $\widehat{B}_1, \ldots, \widehat{B}_k$ in $(\mathcal{B}, \psi \mid \mathcal{B})$ is equal to ν. Thus $\mu \boxminus \nu$ has been realized as the joint distribution of $\widehat{A}_1 + \widehat{B}_1, \ldots, \widehat{A}_k + \widehat{B}_k$, where the k-tuples $\widehat{A}_1, \ldots, \widehat{A}_k$ and $\widehat{B}_1, \ldots, \widehat{B}_k$ are s-free and have distributions μ and ν, respectively.

The verification of the s-freeness of $\mathcal{A}$ and $\mathcal{B}$ in the preceding paragraph is left as an exercise. A reader who is interested in s-freeness may also find it as an amusing (not hard) exercise to start from this latter description of $\mu \boxminus \nu$ and see, conversely, how the statement of Theorem 1.4 can be obtained from there.

We conclude this section by observing that (as a supplement to the fact that $\mu \boxminus \nu \in \mathcal{D}_c(k)$ whenever $\mu, \nu \in \mathcal{D}_c(k)$), there exist natural situations when $\mu \boxminus \nu$ is sure to be infinitely divisible.

Corollary 4.13. *Let μ, ν be two distributions in $\mathcal{D}_c(k)$.*

1° *If μ is $\boxplus$-infinitely divisible, then so is $\mu \boxminus \nu$.*
2° *Suppose that "μ is a $\boxplus$-summand of ν in $\mathcal{D}_c(k)$," in the sense that there exists $\nu' \in \mathcal{D}_c(k)$ such that $\nu = \mu \boxplus \nu'$. Then $\mu \boxminus \nu$ is infinitely divisible.*

Proof. 1° The hypothesis that μ is $\boxplus$-infinitely divisible is equivalent to the fact that, for every $t > 0$, the convolution power $\mu^{\boxplus t}$ (which can always be defined in $\mathcal{D}_{\mathrm{alg}}(k)$) still belongs to $\mathcal{D}_c(k)$. But then, by invoking Remark 1.2.1 and Corollary 1.5 one finds that

$$(\mu \boxminus \nu)^{\boxplus t} = \left(\mu^{\boxplus t} \boxminus \nu\right) \in \mathcal{D}_c(k), \quad \forall t > 0,$$

which means that $\mu \boxminus \nu$ is infinitely divisible as well.

2° One has $\mu \boxminus \nu = \mu \boxminus (\mu \boxplus \nu') = \mathbb{B}(\mu \boxminus \nu')$ (where at the second equality sign we used Remark 3.8.1). Since $\mu \boxminus \nu' \in \mathcal{D}_c(k)$ (by Corollary 1.5), and since $\mathbb{B}$ carries $\mathcal{D}_c(k)$ onto the set of $\boxplus$-infinitely divisible distributions in $\mathcal{D}_c(k)$, the conclusion follows. $\square$

5. Relations with the transformations $\mathbb{B}_t$

Proposition 5.1. *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$. For every $t > 0$ one has that*

$$\mathbb{B}_t(\mu \boxminus \nu) = \mu \boxminus \left(\mu^{\boxplus t} \boxplus \nu\right). \tag{5.1}$$

Proof. We first prove by induction that

$$\mathbb{B}_m(\mu \boxminus \nu) = \mu \boxminus \left(\mu^{\boxplus m} \boxplus \nu\right), \quad \forall m \in \mathbb{N}. \tag{5.2}$$

The base case $m = 1$ of the induction is provided by formula (3.20) in Remark 3.8.1. The induction step "$m \Rightarrow m + 1$" also follows immediately by using the same formula:

$$\begin{aligned}
\mathbb{B}_{m+1}(\mu \boxminus \nu) &= \mathbb{B}\big(\mathbb{B}_m(\mu \boxminus \nu)\big) \quad (\text{since } \mathbb{B}_{m+1} = \mathbb{B} \circ \mathbb{B}_m) \\
&= \mathbb{B}\big(\mu \boxminus \left(\mu^{\boxplus m} \boxplus \nu\right)\big) \quad (\text{by the induction hypothesis}) \\
&= \mu \boxminus \big(\mu \boxplus \left(\mu^{\boxplus m} \boxplus \nu\right)\big) \quad \big(\text{by Eq. (3.20)}\big) \\
&= \mu \boxminus \left(\mu^{\boxplus (m+1)} \boxplus \nu\right).
\end{aligned}$$

Now we move to proving that (5.1) holds for arbitrary $t > 0$. It suffices to fix $n \in \mathbb{N}$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ and to verify that

$$\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\mathbb{B}_t(\mu \boxminus \nu)}) = \mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\mu \boxminus (\mu^{\boxplus t} \boxplus \nu)}), \quad \forall t > 0. \tag{5.3}$$

For both sides of (5.3) one has explicit writings as sums indexed by non-crossing partitions. Indeed, Remark 4.4 from [6] tells us that the left-hand side of (5.3) is equal to

$$\sum_{\substack{\rho \in NC(n), \\ \rho \ll 1_n}} t^{|\rho|-1} \, \mathrm{Cf}_{(i_1,\ldots,i_n);\rho}(R_{\mu \boxminus \nu}), \tag{5.4}$$

while the right-hand side of (5.3) can be written (by Proposition 3.3 and by taking into account the additivity of the R-transform) in the form

$$\sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\mu, tR_\mu + R_\nu). \tag{5.5}$$

Rather than pursuing a detailed combinatorial analysis of the sums in (5.4) and (5.5) we can simply exploit the obvious fact that (for our fixed n and $i_1, \ldots, i_n$) both these sums are polynomial functions of t. Two polynomial functions that agree (as shown by (5.2)) for all $m \in \mathbb{N}$ must in fact agree for all $t > 0$, and (5.3) follows. $\square$

Remark 5.2. As an application of Proposition 5.1, we will next see how the formula "$\mu \boxminus \mu = \mathbb{B}(\mu)$" from Remark 1.2.2 extends to a formula for $(\mu^{\boxplus s}) \boxminus (\mu^{\boxplus t})$, where $s, t \geqslant 0$. In order to cover the cases when $s = 0$ or $t = 0$, we will denote by $\delta \in \mathcal{D}_{\mathrm{alg}}(k)$ the "noncommutative Dirac distribution at 0" which has all moments equal to 0. Then, clearly, $R_\delta = \eta_\delta = 0 \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$; as a consequence one has $\delta^{\boxplus t} = \delta^{\uplus t} = \delta$, hence $\mathbb{B}_t(\delta) = \delta$ for every $t > 0$.

Moreover, it is clear that δ is the neutral element for both the operations $\boxplus$ and $\uplus$ on $\mathcal{D}_{\mathrm{alg}}(k)$, which justifies the convention that

$$\mu^{\boxplus 0} = \mu^{\uplus 0} = \delta, \quad \forall \mu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{5.6}$$

Concerning subordination distributions it is easy to check, directly from Definition 1.1, that

$$\mu \boxminus \delta = \mu \quad \text{and} \quad \delta \boxminus \mu = \delta, \quad \forall \mu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{5.7}$$

Proposition 5.3. *Let μ be a distribution in $\mathcal{D}_{\mathrm{alg}}(k)$. Then for every $s, t \geqslant 0$ one has*

$$\left(\mu^{\boxplus s}\right) \boxminus \left(\mu^{\boxplus t}\right) = \left(\mathbb{B}_t(\mu)\right)^{\boxplus s}. \tag{5.8}$$

Proof. First observe that

$$\begin{aligned} \mu \boxminus \left(\mu^{\boxplus t}\right) &= \mu \boxminus \left(\left(\mu^{\boxplus t}\right) \boxplus \delta\right) \quad (\delta \text{ neutral element for } \boxplus) \\ &= \mathbb{B}_t(\mu \boxminus \delta) \quad (\text{by Proposition 5.1}) \\ &= \mathbb{B}_t(\mu) \quad \big(\text{by (5.7)}\big). \end{aligned}$$

Then recall from Remark 1.2.1 that $(\mu^{\boxplus s}) \boxminus (\mu^{\boxplus t}) = (\mu \boxminus (\mu^{\boxplus t}))^{\boxplus s}$, and (5.8) follows. □

Remark 5.4. The remaining part of this section discusses the relation to free Brownian motion stated in Theorem 1.8. Same as in Remark 1.7, we denote by $\gamma \in \mathcal{D}_{\mathrm{c}}(k)$ the joint distribution of a free family of k centered semicircular elements of variance 1. A fundamental property of γ is that its R-transform is

$$R_\gamma(z_1, \ldots, z_k) = z_1^2 + \cdots + z_k^2 \tag{5.9}$$

(see e.g. [13, Example 11.21.2 on p. 187]). More generally, for every $t > 0$ let γ_t denote the distribution of a free family of k centered semicircular elements of variance t. It is immediate that

$$R_{\gamma_t}(z_1, \ldots, z_k) = R_\gamma(\sqrt{t} z_1, \ldots, \sqrt{t} z_k) = t\left(z_1^2 + \cdots + z_k^2\right);$$

hence $R_{\gamma_t} = t R_\gamma$, which shows that $\gamma_t = \gamma^{\boxplus t}$ for every $t > 0$.

Proposition 5.5. *Let ν be a distribution in $\mathcal{D}_{\mathrm{alg}}(k)$. One has that*

$$R_{\gamma \boxminus \nu}(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \big(1 + M_\nu(z_1, \ldots, z_k)\big) z_i. \tag{5.10}$$

Proof. For $n \geqslant 3$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ one has

$$
\begin{aligned}
&\mathrm{Cf}_{(i_1,\ldots,i_n)}(R_{\gamma \boxplus \nu}) \\
&\quad = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \mathrm{Cf}_{(i_1,\ldots,i_n);\pi;\mathrm{o}_\pi}(R_\gamma, R_\nu) \quad \text{(by Proposition 3.3)} \\
&\quad = \sum_{\substack{\pi \in NC(n) \\ \text{such that } \{1,n\} \in \pi}} \delta_{i_1,i_n} \cdot \prod_{\substack{W \in \pi \\ W \neq \{1,n\}}} \mathrm{Cf}_{(i_1,\ldots,i_n)|W}(R_\nu) \quad \text{(because of the special form of } R_\gamma\text{).}
\end{aligned}
$$

But the set of partitions $\pi \in NC(n)$ which have $\{1, n\}$ as a block is in natural bijection with $NC(n-2)$; when we follow through with this bijection, the above sequence of equalities is continued with

$$
\begin{aligned}
&= \delta_{i_1,i_n} \cdot \sum_{\rho \in NC(n-2)} \prod_{W \in \rho} \mathrm{Cf}_{(i_2,\ldots,i_{n-1})|W}(R_\nu) \\
&= \delta_{i_1,i_n} \cdot \nu(X_{i_2} \cdots X_{i_{n-1}}) \quad \big(\text{by the moment–cumulant formula (2.28)}\big) \\
&= \mathrm{Cf}_{(i_1,\ldots,i_n)}\left(\sum_{i=1}^{k} z_i \big(1 + M_\nu(z_1, \ldots, z_k)\big) z_i \right).
\end{aligned}
$$

The above calculation shows that the series on the two sides of Eq. (5.10) have identical coefficients of length $\geqslant 3$. It is immediately verified that the coefficients of length 1 and 2 also coincide (each of the two series has vanishing linear part and quadratic part equal to $\sum_{i=1}^{k} z_i^2$), and this completes the proof. □

Corollary 5.6. *The transformation $\Phi : \mathcal{D}_{\mathrm{alg}}(k) \to \mathcal{D}_{\mathrm{alg}}(k)$ from* [6] *satisfies*

$$
\gamma \boxplus \nu = \mathbb{B}\big(\Phi(\nu)\big), \quad \forall \nu \in \mathcal{D}_{\mathrm{alg}}(k). \tag{5.11}
$$

Proof. In [6] the distribution $\Phi(\nu)$ is defined via the prescription that its η-series is

$$
\eta_{\Phi(\nu)}(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \big(1 + M_\nu(z_1, \ldots, z_k)\big) z_i. \tag{5.12}
$$

Comparing this to Proposition 5.5 we see that $\eta_{\Phi(\nu)}$ coincides with the R-transform of $\gamma \boxplus \nu$, and Eq. (5.11) follows. □

It is worth noting that the two main facts proved about Φ in [6] can be easily obtained from the prespective of subordination distributions, as explained in the next proposition. (The two statements of this proposition originally appeared as Theorem 6.2 and respectively as Corollary 7.10 in [6].)

Proposition 5.7.

1° *For every* $\nu \in \mathcal{D}_{\mathrm{alg}}(k)$ *and* $t > 0$ *one has that*

$$\Phi(\nu \boxplus \gamma_t) = \mathbb{B}_t\big(\Phi(\nu)\big). \tag{5.13}$$

2° *The transformation* Φ *maps the subset* $\mathcal{D}_{\mathrm{c}}(k)$ *of* $\mathcal{D}_{\mathrm{alg}}(k)$ *into itself.*

Proof. 1° Since the Boolean Bercovici–Pata bijection is one-to-one on $\mathcal{D}_{\mathrm{alg}}(k)$, it will suffice to prove that

$$\mathbb{B}\big(\Phi(\nu \boxplus \gamma_t)\big) = \mathbb{B}\big(\mathbb{B}_t\big(\Phi(\nu)\big)\big).$$

And indeed, starting from the right-hand side of the above equation we can go as follows:

$$\begin{aligned}
\mathbb{B}\big(\mathbb{B}_t\big(\Phi(\nu)\big)\big) &= \mathbb{B}_t\big(\mathbb{B}\big(\Phi(\nu)\big)\big) \quad (\text{because } \mathbb{B} \circ \mathbb{B}_t = \mathbb{B}_{t+1} = \mathbb{B}_t \circ \mathbb{B}) \\
&= \mathbb{B}_t(\gamma \boxminus \nu) \quad (\text{by Corollary 5.6}) \\
&= \gamma \boxminus \big(\gamma^{\boxplus t} \boxplus \nu\big) \quad (\text{by Proposition 5.1}) \\
&= \gamma \boxminus (\nu \boxplus \gamma_t) \quad \big(\text{because } \gamma^{\boxplus t} = \gamma_t\big) \\
&= \mathbb{B}\big(\Phi(\nu \boxplus \gamma_t)\big) \quad (\text{by Corollary 5.6}).
\end{aligned}$$

2° Since $\mathbb{B}$ is one-to-one, it will suffice to show that for $\nu \in \mathcal{D}_{\mathrm{c}}(k)$ one has $\mathbb{B}(\Phi(\nu)) \in \mathbb{B}(\mathcal{D}_{\mathrm{c}}(k))$. The latter set is precisely the set of distributions in $\mathcal{D}_{\mathrm{c}}(k)$ which are $\boxplus$-infinitely divisible (cf. Theorem 1 in [5]). In view of (5.11), what we have thus to prove is the implication "$\nu \in \mathcal{D}_{\mathrm{c}}(k) \Rightarrow \gamma \boxminus \nu$ is infinitely divisible." But γ is itself infinitely divisible (since $\gamma^{\boxplus t} = \gamma_t \in \mathcal{D}_{\mathrm{c}}(k)$, $\forall t > 0$), so the required implication follows from Corollary 4.13.1. □

6. Properties originating from functional equations

Remark 6.1. In this remark we briefly return to the 1-variable framework and notations from Section 2.1, and review the two functional equations that are to be extended to multi-variable framework. Recall in particular that for a probability measure μ on $\mathbb{R}$, $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ denotes the reciprocal Cauchy transform of μ. In the case when μ is compactly supported $F_\mu(z)$ can be viewed as a Laurent series in z, related to the η-series of μ by the formula

$$F_\mu(z) = z\left(1 - \eta_\mu\left(\frac{1}{z}\right)\right). \tag{6.1}$$

In order to verify (6.1), one writes $F_\mu = 1/G_\mu$, $\eta_\mu = M_\mu/(1 + M_\mu)$, and uses the relation between M_μ and G_μ that was recorded in Eq. (2.6) in Section 2.1.

1° Let μ, ν be two probability measures on $\mathbb{R}$, and let ω_1, ω_2 be the subordination functions of $\mu \boxplus \nu$ with respect to μ and to ν, respectively. A remarkable equation satisfied by these functions (see e.g. Theorem 4.1 in [4]) is that

$$\omega_1(z) + \omega_2(z) = z + F_{\mu \boxplus \nu}(z), \quad z \in \mathbb{C}^+. \tag{6.2}$$

But $\omega_1 = F_{\nu \boxminus \mu}$ and $\omega_2 = F_{\mu \boxminus \nu}$, hence (6.2) amounts to

$$F_{\mu \boxminus \nu}(z) + F_{\nu \boxminus \mu}(z) = z + F_{\mu \boxplus \nu}(z), \quad z \in \mathbb{C}^+. \tag{6.3}$$

Let us moreover replace the reciprocal Cauchy transforms in (6.3) by η-series, by using Eq. (6.1). Then (6.3) becomes

$$\eta_{\mu \boxminus \nu} + \eta_{\nu \boxminus \mu} = \eta_{\mu \boxplus \nu},$$

and in this form it goes through to the multi-variable framework of $\mathcal{D}_{\text{alg}}(k)$, as shown in Proposition 6.2 below.

2° Let ν be a probability measure on $\mathbb{R}$. Then for every $p \geqslant 1$ one can consider the probability measure $\nu^{\boxplus p}$, and in Theorem 2.5 of [3] it was shown that one has

$$G_{\nu^{\boxplus p}}(z) = G_\nu\left(\frac{1}{p}z + \left(1 - \frac{1}{p}\right)F_{\nu^{\boxplus p}}(z)\right), \quad z \in \mathbb{C}^+. \tag{6.4}$$

In other words, Eq. (6.4) says that the Cauchy transform of $\nu^{\boxplus p}$ is subordinated to the one of ν, with subordination function ω defined by

$$\omega(z) = \frac{1}{p}z + \left(1 - \frac{1}{p}\right)F_{\nu^{\boxplus p}}(z), \quad z \in \mathbb{C}^+. \tag{6.5}$$

It is immediate that ω from (6.5) belongs to the set $\mathfrak{F}$ of reciprocal Cauchy transforms from Eq. (2.3) of Section 2.1, hence there exists a unique probability measure σ on $\mathbb{R}$ such that $F_\sigma = \omega$. It is natural to call this σ the "subordination distribution of $\nu^{\boxplus p}$ with respect to ν." (If $p \geqslant 2$ then σ is just $\nu^{\boxplus(p-1)} \boxminus \nu$, but for $1 \leqslant p < 2$ this point of view does not always work, as the probability measure $\nu^{\boxplus(p-1)}$ might not be defined.) So then Eq. (6.5) becomes

$$F_\sigma(z) = \frac{1}{p}z + \left(1 - \frac{1}{p}\right)F_{\nu^{\boxplus p}}(z), \quad z \in \mathbb{C}^+,$$

and upon writing the reciprocal Cauchy transforms in terms of η-series this takes us to

$$\eta_\sigma = \frac{p-1}{p} \cdot \eta_{\nu^{\boxplus p}}. \tag{6.6}$$

This latter formula is the one that will be extended to the framework of $\mathcal{D}_{\text{c}}(k)$ – see Corollary 6.4 and Remark 6.5 below.

Proposition 6.2. *For every $\mu, \nu \in \mathcal{D}_{\text{alg}}(k)$ one has that*

$$\eta_{\mu \boxplus \nu} = \eta_{\mu \boxminus \nu} + \eta_{\nu \boxminus \mu}. \tag{6.7}$$

Proof. We fix $n \in \mathbb{N}$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ and compare the coefficients of $z_{i_1} \cdots z_{i_n}$ for the series on the two sides of Eq. (6.7). By using the relation between R and η and the linearizing property of R, and then by invoking Eq. (2.24) in Remark 2.12 we find that

$$\mathrm{Cf}_{(i_1,\dots,i_n)}(\eta_{\mu\boxplus\nu}) = \sum_{\substack{\pi\in NC(n),\\ \pi\ll 1_n}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi}(R_\mu + R_\nu)$$
$$= \sum_{\substack{\pi\in NC(n),\\ \pi\ll 1_n}} \sum_{c:\pi\to\{1,2\}} \mathrm{Cf}_{(i_1,\dots,i_n);\pi;c}(R_\mu, R_\nu).$$

In the latter double sum, the colourings c of π can be subdivided according to whether $c(V_0)=1$ or $c(V_0)=2$, where V_0 is the unique outer block of π. This leads to an equality of the form

$$\mathrm{Cf}_{(i_1,\dots,i_n)}(\eta_{\mu\boxplus\nu}) = \Sigma_1 + \Sigma_2,$$

where Σ_1 is exactly as on the right-hand side of Eq. (3.15) from Proposition 3.7, and Σ_2 is the counterpart of Σ_1 with the roles of μ and ν being reversed. We are only left to invoke Proposition 3.7 to conclude that

$$\Sigma_1 + \Sigma_2 = \mathrm{Cf}_{(i_1,\dots,i_n)}(\eta_{\mu\boxminus\nu}) + \mathrm{Cf}_{(i_1,\dots,i_n)}(\eta_{\nu\boxminus\mu}) = \mathrm{Cf}_{(i_1,\dots,i_n)}(\eta_{\mu\boxminus\nu} + \eta_{\nu\boxminus\mu}),$$

and (6.7) follows. □

When discussing the multi-variable analogue for Eq. (6.6) it is convenient to note that there is no problem to generally talk about the "subordination distribution of λ with respect to ν" for any $\lambda, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$.

Definition 6.3. Let two distributions $\lambda, \nu \in \mathcal{D}_{\mathrm{alg}}(k)$ be given. Consider the distribution $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$ which is uniquely determined by the requirement that

$$R_\mu = R_\lambda - R_\nu \tag{6.8}$$

(or equivalently, via the requirement that $\mu \boxplus \nu = \lambda$). Then the *subordination distribution of λ with respect to ν* is, by definition, equal to $\mu \boxminus \nu$.

Corollary 6.4.

1° *For every $\nu \in \mathcal{D}_{\mathrm{alg}}(k)$ and every $p \geqslant 1$, the subordination distribution of $\nu^{\boxplus p}$ with respect to ν is equal to $(\mathbb{B}(\nu))^{\boxplus(p-1)}$.*

2° *Let ν be a distribution in $\mathcal{D}_{\mathrm{c}}(k)$. Then, for every $p \geqslant 1$, the subordination distribution of $\nu^{\boxplus p}$ with respect to ν belongs to $\mathcal{D}_{\mathrm{c}}(k)$ as well, and is moreover $\boxplus$-infinitely divisible.*

Proof. 1° According to Definition 6.3, the distribution in question is $\nu^{\boxplus(p-1)} \boxminus \nu$. Thus we only need to invoke the particular case of Proposition 5.3 where $s = p-1$ and $t = 1$.

2° This follows from part 1° of the corollary and the fact that $\mathbb{B}(\nu)$ is $\boxplus$-infinitely divisible (which implies that any convolution power $(\mathbb{B}(\nu))^{\boxplus t}$, $t \geqslant 0$, lives in $\mathcal{D}_{\mathrm{c}}(k)$ and is itself infinitely divisible). □

Remark 6.5. It is an easy exercise (left to the reader) to verify the identity

$$\left(\mathbb{B}(\nu)\right)^{\boxplus(p-1)} = \left(\nu^{\boxplus p}\right)^{\uplus(p-1)/p}, \quad \forall \nu \in \mathcal{D}_{\mathrm{alg}}(k),\ \forall p \in [1,\infty). \tag{6.9}$$

So if we denote the subordination distribution of $\nu^{\boxplus p}$ with respect to ν by σ, then by invoking Corollary 6.4 and by taking the η-series of the distribution on the right-hand side of (6.9) we obtain that $\eta_\sigma = ((p-1)/p) \cdot \eta_{\nu^{\boxplus p}}$. Thus Corollary 6.4 gives indeed a multi-variable generalization of Eq. (6.6) from Remark 6.1.2.

References

[1] N.I. Akhiezer, The Classical Moment Problem and Some Related Problems in Analysis, Hafner, New York, 1965.
[2] M. Anshelevich, Free evolution on algebras with two states, preprint, 2008, available at http://arXiv.org, under arXiv: 0803.4280.
[3] S.T. Belinschi, H. Bercovici, Atoms and regularity for measures in a partially defined free convolution semigroup, Math. Z. 248 (2004) 665–674.
[4] S.T. Belinschi, H. Bercovici, A new approach to subordination results in free probability, J. Anal. Math. 101 (2007) 357–366.
[5] S.T. Belinschi, A. Nica, η-series and a Boolean Bercovici–Pata bijection for bounded k-tuples, Adv. Math. 217 (2008) 1–41.
[6] S.T. Belinschi, A. Nica, Free Brownian motion and evolution towards $\boxplus$-infinite divisibility for k-tuples, Internat. J. Math., in press, available at http://arXiv.org under arXiv: 0711.3787.
[7] H. Bercovici, V. Pata, Stable laws and domains of attraction in free probability theory, with an appendix by P. Biane, The density of free stable distributions, Ann. of Math. 149 (1999) 1023–1060.
[8] H. Bercovici, D. Voiculescu, Free convolution of measures with unbounded support, Indiana Univ. Math. J. 42 (1993) 733–773.
[9] P. Biane, Processes with free increments, Math. Z. 227 (1998) 143–174.
[10] M. Bozejko, M. Leinert, R. Speicher, Convolution and limit theorems for conditionally free random variables, Pacific J. Math. 175 (1996) 357–388.
[11] R. Lenczewski, Decompositions of the free additive convolution, J. Funct. Anal. 246 (2007) 330–365.
[12] H. Maassen, Addition of freely independent random variables, J. Funct. Anal. 106 (1992) 409–438.
[13] A. Nica, R. Speicher, Lectures on the Combinatorics of Free Probability, London Math. Soc. Lecture Note Ser., vol. 335, Cambridge Univ. Press, 2006.
[14] D. Voiculescu, Addition of certain non-commuting random variables, J. Funct. Anal. 66 (1986) 323–346.
[15] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory I, Comm. Math. Phys. 155 (1993) 71–92.
[16] D. Voiculescu, The coalgebra of the free difference quotient and free probability, Int. Math. Res. Not. 2 (2000) 79–106.
[17] D.V. Voiculescu, K.J. Dykema, A. Nica, Free Random Variables, CRM Monogr. Ser., vol. 1, Amer. Math. Soc., 1992.