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Universal enveloping algebras and universal derivations of Poisson algebras

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ABSTRACT

Let k be an arbitrary field of characteristic 0. It is shown that for any $n \geqslant 1$ the universal enveloping algebras of the Poisson symplectic algebra $P_n(k)$ and the Weyl algebra $A_n(k)$ are isomorphic and the canonical isomorphism between them easily leads to the Moyal product. A basis of the universal enveloping algebra P^e of a free Poisson algebra $P = k\{x_1, \ldots, x_n\}$ is constructed and it is proved that the left dependence of a finite number of elements of P^e over P^e is algorithmically recognizable. We describe the Poisson dependence of any two elements of a free Poisson algebra in characteristic 0 in the language of universal derivatives. The Fox derivatives on free Poisson algebras are defined and it is proved that an analogue of the Jacobian Conjecture for two generated free Poisson algebras is equivalent to the two-dimensional classical Jacobian Conjecture. A new proof of the tameness of automorphisms of two generated free Poisson algebras is also given.

1. Introduction

The main property of the universal (multiplicative) enveloping algebra $U_{\mathfrak{M}}(A)$ of an algebra A in a variety of algebras $\mathfrak{M}$ is that the notion of an A-bimodule in $\mathfrak{M}$ is equivalent to the notion of a left module over the associative algebra $U_{\mathfrak{M}}(A)$ (see, for example [12]). The universal enveloping algebra of a Poisson algebra was studied in [18]. A linear basis of the universal enveloping algebra is constructed in [19] for Poisson polynomial algebras, i.e. for Poisson brackets on the polynomial algebras

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$k[x_1, \ldots, x_n]$. In general case the validity of an analogue of the Poincare–Birkhoff–Witt theorem is still open.

We study the universal enveloping algebras of Poisson symplectic algebras and free Poisson algebras. For any Poisson algebra P denote by P^e its universal enveloping algebra. Note that the universal enveloping algebra A^e of an associative algebra A in the variety of associative algebras is $A \otimes_k A^{op}$ where A^{op} is an anti-isomorphic copy of A (see, for example [31,32]). It is proved that the universal enveloping algebras of the Poisson symplectic algebra P_n and the Weyl algebra A_n are isomorphic. The canonical isomorphism between P_n^e and A_n^e naturally leads to the Moyal product (Lemma 6).

It is well known that the universal enveloping algebras of free Lie algebras are free associative algebras. P. Cohn [8] proved that every left ideal of a free associative algebra is a free left module. In fact, it follows from this theorem [32] that subalgebras of free Lie algebras are free [24,33] and automorphisms of finitely generated free Lie algebras are tame [7].

Let $P = k\{x_1, \ldots, x_n\}$ be the free Poisson algebra in the variables $x_1, \ldots, x_n$. We construct a linear basis of the universal enveloping algebra P^e. Unfortunately left ideals of P^e are not free left P^e-modules, i.e., an analogue of Cohn's theorem [8] for free associative algebras is not true in this case. We prove a weaker result which says that the left dependence of a finite set of elements of P^e over P^e is algorithmically recognizable. It is well known [8] that any two elements of a free associative algebra either generate a free associative algebra in two variables or commute. An analogue of this for Poisson algebras in characteristic 0 was recently proved in [14]. We describe the Poisson dependence of any two elements of a free Poisson algebra over a field of characteristic 0 in the language of universal derivatives. It allows us to prove that an analogue of the Jacobian Conjecture for two generated free Poisson algebras is equivalent to the two-dimensional classical Jacobian Conjecture. A new proof of the tameness of automorphisms of two generated free Poisson algebras is also given.

This paper is organized as follows. In Section 2 we give the definition of Poisson modules and universal enveloping algebras of Poisson algebras by generators and defining relations. We define also the universal derivations of Poisson algebras. Section 3 is devoted to studying the universal enveloping algebras of P_n and A_n. In Section 4 we construct a linear basis of the universal enveloping algebra P^e of the free Poisson algebra $P = k\{x_1, \ldots, x_n\}$. We introduce a degree function on the universal enveloping algebra P^e and describe its associated graded algebra $gr\, P^e$. In Section 5 we consider the left dependence of elements over P^e and study in details the left dependence of two elements of a special type over $gr\, P^e$. Section 6 is devoted to studying the universal derivations and two generated subalgebras of free Poisson algebras. In Section 7 we give some comments and formulate some open problems.

All vector spaces are considered over an arbitrary fixed field k of characteristic 0. In the statements of algorithmic character we assume that k is constructive.

2. Universal enveloping algebras

Recall that a vector space P over k endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called *a Poisson algebra* if P is a commutative associative algebra under $x \cdot y$, P is a Lie algebra under $\{x, y\}$, and P satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

Let P be a Poisson algebra over k. A vector space V over k is called a *Poisson module* over P (or *Poisson P-module*) if there are two bilinear maps

$$P \times V \longrightarrow V \quad \big((x, v) \mapsto x \cdot v\big)$$

and

$$P \times V \longrightarrow V \quad \big((x, v) \mapsto \{x, v\}\big),$$

such that the relations

$$(x \cdot y) \cdot v = x \cdot (y \cdot v),$$
$$\bigl\{\{x, y\}, v\bigr\} = \bigl\{x, \{y, v\}\bigr\} - \bigl\{y, \{x, v\}\bigr\},$$
$$\{x \cdot y, v\} = y \cdot \{x, v\} + x \cdot \{y, v\},$$
$$\{x, y\} \cdot v = \{x, y \cdot v\} - y \cdot \{x, v\}$$

hold for all $x, y \in P$ and $v \in V$.

The first relation means that every Poisson P-module is a usual (left) module over the associative and commutative algebra P. Sometimes we consider P as an associative and commutative algebra with $x \cdot y$ and a module over this algebra will be called a module over the commutative algebra P.

Let V be a Poisson module over a Poisson algebra P. For every $x \in P$ we denote by M_x the operator of multiplication by x acting on V, i.e., $M_x(v) = x \cdot v$ for any $v \in V$. For every $x \in P$ we define also the Hamiltonian operator H_x on V by $H_x(v) = \{x, v\}$. Then the Poisson module relations can be written as

$$M_{xy} = M_x M_y,$$
$$H_{\{x,y\}} = H_x H_y - H_y H_x,$$
$$H_{xy} = M_y H_x + M_x H_y,$$
$$M_{\{x,y\}} = H_x M_y - M_y H_x,$$

respectively.

We consider only Poisson algebras with an identity element 1 and unitary modules over Poisson algebras, i.e. $M_1 = id$.

Now we give the definition of the universal (multiplicative) enveloping algebra P^e of a Poisson algebra P. Let $m_P = \{m_a \mid a \in P\}$ and $h_P = \{h_a \mid a \in P\}$ be two copies of the vector space P endowed with two linear isomorphisms $m : P \longrightarrow m_P$ $(a \mapsto m_a)$ and $h : P \longrightarrow h_P$ $(a \mapsto h_a)$. Then P^e is an associative algebra over k, with an identity 1, generated by two linear spaces m_P and h_P and defined by the relations

$$m_{xy} = m_x m_y,$$
$$h_{\{x,y\}} = h_x h_y - h_y h_x,$$
$$h_{xy} = m_y h_x + m_x h_y,$$
$$m_{\{x,y\}} = h_x m_y - m_y h_x = [h_x, m_y],$$
$$m_1 = 1$$

for all $x, y \in P$. The last relation comes from our agreement to consider only unitary P-modules. Note that $h_1 = 0$.

By the definition of the universal (multiplicative) enveloping algebras [12] the notion of a bimodule over an algebra is equivalent to the notion of a left module over its universal enveloping algebra. Let V be an arbitrary Poisson P-module. Then V becomes a left P^e-module under the action

$$m_x v = x \cdot v, \qquad h_x v = \{x, v\},$$

for all $x \in P$ and $v \in V$. Conversely, if V is a left P^e-module then the same formulas turn V to a Poisson P-module.

Corollary 1. *The category of unitary Poisson modules over a Poisson algebra P and the category of* (*left*) *unitary modules over the universal enveloping algebra P^e are equivalent.*

The first example of a Poisson P-module is $V = P$ under the actions $x \cdot v$ and $\{x, v\}$. Since $1 \in P$ and $m_x 1 = x$ it follows that the mapping

$$m : P \longrightarrow P^e \quad (x \mapsto m_x)$$

is an injection. Therefore we identify m_x with x. After this identification the essential part of the defining relations of P_e are

$$h_{\{x,y\}} = h_x h_y - h_y h_x, \tag{1}$$

$$h_{xy} = y h_x + x h_y, \tag{2}$$

$$\{x, y\} = h_x y - y h_x = [h_x, y], \tag{3}$$

for all $x, y \in P$. From (3) it follows that

$$[h_x, y] = [x, h_y]. \tag{4}$$

Let Ω_P be the left ideal of P^e generated by all h_x where $x \in P$. Consider the mapping

$$\Delta : P \longrightarrow \Omega_P \quad (x \mapsto h_x).$$

It follows from (1) and (2) that Δ is a derivation of the Poisson algebra P with coefficients on the Poisson P-module Ω_P.

Lemma 1. *Δ is the universal derivation of the Poisson algebra P and Ω_P is the universal differential module of P.*

Proof. Recall that by the definition it means (see, for example [31,32]) that for any Poisson P-module V and for any derivation $d : P \to V$ there exists a unique homomorphism $\tau : \Omega_P \to V$ of Poisson P-modules such that $d = \tau\Delta$.

We follow the construction of the universal differential modules by generators and defining relations (see, for example [5]). Let $\overline{P}$ be a copy of the vector space P. Consider the left P^e-module $V = P^e \otimes \overline{P}$. Denote by $\phi : P \to V$ the mapping defined by $\phi(x) = 1 \otimes \bar{x}$ for all $x \in P$. Let W be a submodule of V generated by all $h_x \otimes \bar{y} - h_y \otimes \bar{x} - 1 \otimes \overline{\{x, y\}}$ and $y \otimes \bar{x} + x \otimes \bar{y} - 1 \otimes \overline{xy}$, where $x, y \in P$. Put $M = V/W$ and denote by $\psi : V \to M$ the natural homomorphism. It is well known that [5,31,32] the mapping $D = \psi\phi : P \to M$ is the universal derivation and M is the universal differential module of P.

Since V is a P^e-module freely generated by the vector space $\overline{P}$ there exists a P^e-module homomorphism $\eta : V \to \Omega_P$ such that $\eta(\bar{x}) = h_x$ for all $x \in P$. It follows from (1) and (2) that $W \subseteq \mathit{Ker}\,\eta$. Consequently, η induces the homomorphism $\theta : M \to \Omega_P$. Note that from (1)–(3) it follows that Ω_P is a P^e-module generated by symbols h_x and defined by relations (1) and (2). Consequently, θ is an isomorphism. $\square$

3. Universal enveloping algebras of P_n and A_n

For an integer $n \geqslant 1$ the Poisson symplectic algebra P_n is the usual polynomial algebra $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = 0, \qquad \{y_i, y_j\} = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leqslant i, j \leqslant n$.

By A_n we denote the Weyl algebra of index n, i.e., A_n is an associative algebra given by generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and defined relations

$$[X_i, Y_j] = \delta_{ij}, \qquad [X_i, X_j] = 0, \qquad [Y_i, Y_j] = 0,$$

where $1 \leqslant i, j \leqslant n$.

Let $k[z_1, z_2, \ldots, z_n]$ be the polynomial algebra in the variables $z_1, z_2, \ldots, z_n$. Denote by $\mathcal{D}(k[z_1, z_2, \ldots, z_n])$ the algebra of polynomial differential operators on $k[z_1, z_2, \ldots, z_n]$. Note that A_n is isomorphic to $\mathcal{D}(k[z_1, z_2, \ldots, z_n])$:

$$Y_i \mapsto z_i, \qquad X_i \mapsto \partial_i = \frac{\partial}{\partial z_i}, \quad 1 \geqslant i \geqslant n.$$

The statement of the next lemma was also mentioned in [18].

Lemma 2. $P_n^e \simeq \mathcal{D}(k[x_1, \ldots, x_n, y_1, \ldots, y_n]) \simeq A_{2n}$.

Proof. It follows directly from (1)–(4) that P_n^e is generated by $x_i, y_i, h_{x_i}, h_{y_i}$ and satisfies the relations

$$[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 0, \qquad [h_{x_i}, h_{x_j}] = [h_{x_i}, h_{y_j}] = [h_{y_i}, h_{y_j}] = 0,$$
$$[h_{x_i}, y_i] = \{x_i, y_i\} = 1, \qquad [h_{y_i}, x_i] = \{y_i, x_i\} = -1,$$

for all i, j.

The Weyl algebra A_{2n} is generated by $X_1, \ldots, X_{2n}, Y_1, \ldots, Y_{2n}$ and defined by the relations

$$[X_i, Y_j] = \delta_{ij}, \qquad [X_i, X_j] = 0, \qquad [Y_i, Y_j] = 0,$$

where $1 \leqslant i, j \leqslant 2n$. Consequently, there exists a surjective homomorphism $\phi : A_{2n} \longrightarrow P_n^e$ such that

$$X_i \mapsto x_i, \qquad X_{i+n} \mapsto y_i, \qquad Y_i \mapsto h_{y_i}, \qquad Y_{i+n} \mapsto -h_{x_i}$$

for all $1 \leqslant i \leqslant n$. Since A_{2n} is simple it follows that ϕ is an isomorphism. $\square$

Now let A be an arbitrary associative algebra over k with an identity. Then the universal enveloping algebra A^e of A is $A \otimes_k A^{op}$, where A^{op} is an algebra anti-isomorphic to A. Algebra A^{op} is considered together with an anti-isomorphism $' : A \longrightarrow A^{op}$. In fact, each operator of the left multiplication l_x is represented by $x \otimes 1$ and each operator of the right multiplication r_x is represented by $1 \otimes x'$ in $A \otimes_k A^{op}$. Following the main property of the universal enveloping algebras (see [12]) we get that the category of associative unitary A-bimodules and the category of left unitary A^e-modules are equivalent.

Note that A_n^{op} is isomorphic to A_n again. An isomorphism, can be chosen, for example, as $\varphi : A_n \longrightarrow A_n^{op}$ with $\varphi(X_i) = X_i'$, $\varphi(Y_i) = -Y_i'$ for all i. Then, $A_n^e = A_n \otimes_k A_n^{op} \simeq A_n \otimes_k A_n \simeq A_{2n}$ and consequently, $P_n^e \simeq A_n^e$. We proved

Theorem 1. *For every integer $n \geqslant 1$ the universal enveloping algebras of the Poisson symplectic algebra P_n and the Weyl algebra A_n are isomorphic to the Weyl algebra A_{2n}, i.e.*

$$P_n^e \simeq A_n^e \simeq A_{2n}.$$

Corollary 2. *The category of unitary Poisson modules over the Poisson symplectic algebra P_n and the category of unitary bimodules over the Weyl algebra A_n are equivalent.*

The notion of bimodules in the case of Poisson algebras corresponds to the notion of Poisson modules, as follows from the commutativity and anti-commutativity of operations.

Therefore P_n^e is isomorphic to A_n^e. We use this isomorphism to get the Moyal product. In fact, there are many isomorphisms between A_n^e and P_n^e. We choose "the most canonical" one between them in the next lemma.

Lemma 3. *There exists a unique isomorphism $\theta : A_n^e \longrightarrow P_n^e$ such that*

$$\theta(X_i \otimes 1) = x_i + 1/2h_{x_i}, \qquad \theta\big(1 \otimes X_i'\big) = x_i - 1/2h_{x_i},$$
$$\theta(Y_i \otimes 1) = y_i + 1/2h_{y_i}, \qquad \theta\big(1 \otimes Y_i'\big) = y_i - 1/2h_{y_i},$$

for all $1 \leqslant i \leqslant n$.

Proof. The existence of θ follows from (1)–(3). For example,

$$\begin{aligned}
\big[\theta(X_i \otimes 1), \theta(Y_i \otimes 1)\big] &= [x_i + 1/2h_{x_i}, y_i + 1/2h_{y_i}] \\
&= 1/2[x_i, h_{y_i}] + 1/2[h_{x_i}, y_i] + 1/4[h_{x_i}, h_{y_i}] = \{x_i, y_i\} = 1, \\
\big[\theta\big(1 \otimes X_i'\big), \theta\big(1 \otimes Y_i'\big)\big] &= [x_i - 1/2h_{x_i}, y_i - 1/2h_{y_i}] \\
&= -1/2[x_i, h_{y_i}] - 1/2[h_{x_i}, y_i] + 1/4[h_{x_i}, h_{y_i}] = -\{x_i, y_i\} = -1, \\
\big[\theta(X_i \otimes 1), \theta(X_j \otimes 1)\big] &= [x_i + 1/2h_{x_i}, x_j + 1/2h_{x_j}] \\
&= 1/2[x_i, h_{x_j}] + 1/2[h_{x_i}, x_j] + 1/4[h_{x_i}, h_{x_j}] = 0,
\end{aligned}$$

if $i \neq j$. Obviously, θ is surjective. And it is also injective since A_n^e is simple. $\square$

Denote by L the linear space $kx_1 + \cdots + kx_n + ky_1 + \cdots + ky_n$. We define a linear bijection

$$w : P_n \to A_n$$

by

$$w(l_1 l_2 \ldots l_p) = 1/p! \sum_{\pi \in S_p} l_{\pi(1)} l_{\pi(2)} \ldots l_{\pi(p)}$$

for all $l_1, l_2, \ldots, l_p \in kx_1 + \cdots + kx_n + ky_1 + \cdots + ky_n$. This mapping is called the symmetrization [10].

Let us introduce some notations. Let $\mathbb{Z}_+$ be the set of all nonnegative integers. For every $\alpha = (i_1, i_2, \ldots, i_t) \in \mathbb{Z}_+^t$ we put $|\alpha| = i_1 + i_2 + \cdots + i_t$ and $\alpha! = i_1!i_2!\ldots i_t!$. If $b = (b_1, b_2, \ldots, b_t)$, where $b_1, b_2, \ldots, b_t$ is a subset of commuting elements of an algebra B, then we put

$$b^\alpha = b_1^{i_1} b_2^{i_2} \ldots b_t^{i_t}.$$

We set

$$\partial_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad \partial_y = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right),$$
$$\partial = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right),$$

$$h_x = (h_{x_1}, h_{x_2}, \dots, h_{x_n}), \qquad h_y = (h_{y_1}, h_{y_2}, \dots, h_{y_n}),$$
$$h = (h_{x_1}, \dots, h_{x_n}, h_{y_1}, \dots, h_{y_n}),$$

where $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ are usual partial derivatives of P_n.

Lemma 4. *If $f \in P_n$ then*

$$\theta\big(w(f) \otimes 1\big) = \sum_{\gamma \in \mathbb{Z}_+^{2n}} \frac{1}{\gamma! 2^{|\gamma|}} \partial^\gamma(f) h^\gamma = \sum_{\gamma \in \mathbb{Z}_+^{2n}} \frac{1}{\gamma! 2^{|\gamma|}} \partial^\gamma(f) h^\gamma.$$

Proof. By the linearity, it is sufficient to prove the statement of the lemma only for elements of the form $f = l^p$, where $l \in kx_1 + \dots + kx_n + ky_1 + \dots + ky_n$. In this case $w(f) = w(l)^p$. Hence

$$\theta\big(w(f) \otimes 1\big) = \theta\big(w(l)^p \otimes 1\big) = \big(\theta(w(l) \otimes 1)\big)^p = (l + 1/2h_l)^p.$$

Note that l and $h(l)$ commute. Consequently,

$$\theta\big(w(f) \otimes 1\big) = \sum_{i=0}^{p} \binom{p}{i} \frac{1}{2^i} l^{p-i} h_l^i = \sum_{i=0}^{p} \frac{1}{i! 2^i} \frac{\partial^i f}{\partial l^i} h_l^i.$$

Let $l = a_1x_1 + \dots + a_nx_n + b_1y_1 + \dots + b_ny_n$ and put $c = (a_1, \dots, a_n, b_1, \dots, b_n)$. The elements $h_{x_1}, \dots, h_{x_n}, h_{y_1}, \dots, h_{x_n}$ commute. Using multinomial coefficients we can write

$$h_l^i = \sum_{\lambda \in \mathbb{Z}_+^{2n},\, |\lambda| = i} \frac{i!}{\lambda!} c^\lambda h^\lambda.$$

Note that $\partial^\lambda(f) = c^\lambda \frac{\partial^i f}{\partial l^i}$ if $|\lambda| = i$. Consequently,

$$\theta\big(w(f) \otimes 1\big) = \sum_{\gamma \in \mathbb{Z}_+^{2n}} \frac{1}{\gamma! 2^{|\gamma|}} \partial^\gamma(f) h^\gamma.$$

The second equality of the lemma can be proved similarly. □

If $\alpha = (i_1, \dots, i_n, j_1, \dots, j_n) \in \mathbb{Z}_+^{2n}$ then we put $\alpha_1 = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ and $\alpha_2 = (j_1, \dots, j_n) \in \mathbb{Z}_+^n$. We often write $\alpha = (\alpha_1, \alpha_2)$ and put $\alpha^* = (\alpha_2, \alpha_1)$.

Lemma 5. *If $f \in P_n$ then*

$$\frac{1}{\gamma!} h^\gamma f = \sum_{\alpha + \beta = \gamma} \frac{(-1)^{|\alpha_2|}}{\alpha! \beta!} \partial^{\alpha^*}(f) h^\beta$$

for all $\gamma \in \mathbb{Z}_+^{2n}$.

Proof. Put $H_a(f) = [h_a, f]$ for all $a, f \in P$ and put also $H = (h_{x_1}, \dots, h_{x_n}, h_{y_1}, \dots, h_{y_n})$. Using the relations

$$h_{x_i} f = [h_{x_i}, f] + f h_{x_i}, \qquad h_{y_i} f = [h_{y_i}, f] + f h_{y_i},$$

and multinomial coefficients, we get

$$\frac{1}{\gamma!}h^{\gamma} f = \sum_{\alpha+\beta=\gamma} \frac{1}{\alpha!\beta!} H^{\alpha}(f) h^{\beta}.$$

Note that

$$H_{x_i}(f) = [h_{x_i}, f] = \frac{\partial}{\partial y_i}(f), \qquad H_{y_i}(f) = [h_{y_i}, f] = -\frac{\partial}{\partial x_i}(f)$$

for all i. Consequently, $H^{\alpha}(f) = (-1)^{|\alpha_2|}\partial^{\alpha^*}(f)$. □

Consider the mapping

$$\rho_w : P_n \longrightarrow P_n^e, \quad f \mapsto w(f) \otimes 1, \ f \in P_n.$$

Lemma 6. *The product*

$$f *_w g = \rho_w^{-1}\big(\rho_w(f)\rho_w(g)\big)$$

is the Moyal product on P_n, i.e.

$$f *_w g = \sum_{\alpha \in \mathbb{Z}_+^{2n}} \frac{(-1)^{|\alpha_2|}}{\alpha! 2^{|\alpha}} \partial^{\alpha}(f) \partial^{\alpha^*}(g).$$

Proof. A direct calculation gives

$$\begin{aligned}
\theta\big(w(f) \otimes 1\big)\theta\big(w(g) \otimes 1\big) &= \sum_{\gamma,\delta \in \mathbb{Z}_+^{2n}} \frac{1}{\gamma!\delta! 2^{|\gamma+\delta|}} \partial^{\gamma}(f) h^{\gamma} \partial^{\delta}(g) h^{\delta} \\
&= \sum_{\gamma,\delta \in \mathbb{Z}_+^{2n}} \sum_{\alpha+\beta=\gamma} \frac{(-1)^{|\alpha_2|}}{\alpha!\beta!\delta! 2^{|\gamma+\delta|}} \partial^{\gamma}(f) \partial^{\alpha^*+\delta}(g) h^{\beta+\delta} \\
&= \sum_{\mu} \left(\sum_{\beta+\delta=\mu} \sum_{\alpha} \frac{(-1)^{|\alpha_2|}}{\alpha!\beta!\delta! 2^{|\alpha+\beta+\delta|}} \partial^{\alpha+\beta}(f) \partial^{\alpha^*+\delta}(g) \right) h^{\mu} \\
&= \sum_{\mu} \frac{1}{\mu! 2^{|\mu|}} \partial^{\mu} \left(\sum_{\alpha} \frac{(-1)^{|\alpha_2|}}{\alpha! 2^{|\alpha}} \partial^{\alpha}(f) \partial^{\alpha^*}(g) \right) h^{\mu} \\
&= \sum_{\mu} \frac{1}{\mu! 2^{|\mu|}} \partial^{\mu}(f *_w g) h^{\mu}.
\end{aligned}$$

Consequently,

$$\rho_w^{-1}\big(\rho_w(f)\rho_w(g)\big) = \rho_w^{-1}\big(\theta\big(w(f) \otimes 1\big)\theta\big(w(g) \otimes 1\big)\big) = f *_w g$$

for all $f, g \in P_n$. □

4. Enveloping algebras of free Poisson algebras

Let g be a Lie algebra with a linear basis $e_1, e_2, \ldots, e_k, \ldots$. The Poisson symmetric algebra $PS(g)$ of g is the usual polynomial algebra $k[e_1, e_2, \ldots, e_k, \ldots]$ endowed with the Poisson bracket defined by

$$\{e_i, e_j\} = [e_i, e_j]$$

for all i, j, where $[x, y]$ is the multiplication in the Lie algebra g.

Denote by $P = k\{x_1, x_2, \ldots, x_n\}$ the free Poisson algebra over k in the variables $x_1, x_2, \ldots, x_n$. From now on let $g = Lie\langle x_1, x_2, \ldots, x_n\rangle$ be the free Lie algebra with free (Lie) generators $x_1, x_2, \ldots, x_n$. It is well known (see, for example [22]) that the Poisson symmetric algebra $PS(g)$ is the free Poisson algebra $P = k\{x_1, x_2, \ldots, x_n\}$ in the variables $x_1, x_2, \ldots, x_n$.

By deg we denote the standard degree function of the homogeneous algebra P, i.e. $\deg(x_i) = 1$, where $1 \leqslant i \leqslant n$. Note that

$$\deg\{f, g\} = \deg f + \deg g$$

if f and g are homogeneous and $\{f, g\} \neq 0$. By $\deg_{x_i}$ we denote the degree function on P with respect to x_i. If f is homogeneous with respect to each $\deg_{x_i}$, where $1 \leqslant i \leqslant n$, then f is called multihomogeneous.

Let us choose a multihomogeneous linear basis

$$x_1, x_2, \ldots, x_n, \quad [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n], \quad \big[[x_1, x_2], x_3\big], \ldots$$

of a free Lie algebra g and denote the elements of this basis by

$$e_1, e_2, \ldots, e_m, \ldots. \tag{5}$$

The algebra $P = k\{x_1, x_2, \ldots, x_n\}$ coincides with the polynomial algebra on the elements (5). Consequently, the set of all words of the form

$$u = e^{\alpha} = e_1^{i_1} e_2^{i_2} \ldots e_m^{i_m}, \tag{6}$$

where $0 \leqslant i_k$, $1 \leqslant k \leqslant m$, and $m \geqslant 0$, forms a linear basis of P. The basis (6) is multihomogeneous since so is (5).

Let $k\{x_1, x_2, \ldots, x_n, y\}$ be the free Poisson algebra in the variables $x_1, x_2, \ldots, x_n, y$. Denote by W the set of all homogeneous of degree one with respect to y elements of $k\{x_1, x_2, \ldots, x_n, y\}$.

Theorem 2. *Let $P = k\{x_1, x_2, \ldots, x_n\}$ be the free Poisson algebra over a field k in the variables $x_1, x_2, \ldots, x_n$ and let P^e be its universal enveloping algebra. Then the following statements are true*:

(i) *The subalgebra A (with identity) of P^e generated by $h_{x_1}, h_{x_2}, \ldots, h_{x_n}$ is the free associative algebra in the variables $h_{x_1}, h_{x_2}, \ldots, h_{x_n}$*;

(ii) *The left commutative P-module P^e is isomorphic to the left commutative P-module $P \otimes_k A$.*

Proof. Recall that $P\{x_1, x_2, \ldots, x_n, y\}$ is the Poisson symmetric algebra of the free Lie algebra $Lie\langle x_1, x_2, \ldots, x_n, y\rangle$. The elements of the form

$$\big\{x_{i_1}, \big\{x_{i_2}, \ldots, \{x_{i_k}, y\} \ldots\big\}\big\} = h_{x_{i_1}} h_{x_{i_2}} \ldots h_{x_{i_k}}(y)$$

are linearly independent in the free Lie algebra $Lie\langle x_1, x_2, \ldots, x_n, y\rangle$. Consequently, the elements of the form

$$h_{x_{i_1}} h_{x_{i_2}} \ldots h_{x_{i_k}} \tag{7}$$

are linearly independent in P^e.

Using (1)–(4), it can be easily shown that every element of P^e can be written as a linear combination of elements pw, where $p \in P$ and w is an element of the form (7).

Let B_1 be the linear basis (5) of the free Lie algebra $g = Lie\langle x_1, x_2, \ldots, x_n\rangle$. Denote by B_2 the set of all elements of the form wy, where w is an element of the form (7). Note that the set of elements $B_1 \cup B_2$ is linearly independent. We can choose a set of elements B_3 of degree $\geqslant 2$ in y such that $B_1 \cup B_2 \cup B_3$ is a linear basis of $Lie\langle x_1, x_2, \ldots, x_n, y\rangle$. Then $P\{x_1, x_2, \ldots, x_n, y\}$ is a polynomial algebra in the set of variables $B_1 \cup B_2 \cup B_3$. Consequently, W is a free left module over the polynomial algebra $k[B_1]$ and B_2 is a set of free generators. Note that $P = k[B_1]$. □

Corollary 3. *Every nonzero element u of the universal enveloping algebra P^e can be uniquely written in the form*

$$u = \sum_{i=1}^{k} p_i w_i, \tag{8}$$

where $0 \neq p_i \in P$ for all i and $w_1, w_2, \ldots, w_k$ are different elements of the form (7).

Put $h_{x_i} < h_{x_j}$ if $i < j$. Let u, v be two elements of the form (7). Then put $u < v$ if $\deg u < \deg v$ or $\deg u = \deg v$ and u precedes v in the lexicographical order. We can assume $w_1 < w_2 < \cdots < w_k$ in (8). Then w_k is called the *leading monomial* of u and p_k is called the *leading coefficient* of u. We will write $w_k = ldm(u)$ and $p_k = ldc(u)$. The *leading term* of u is defined by $ldt(u) = ldc(u)\, ldm(u)$.

Lemma 7. *If u and v are arbitrary nonzero elements of P^e then $ldc(uv) = ldc(u)\, ldc(v)$ and $ldm(uv) = ldm(u)\, ldm(v)$.*

Proof. Note that if u and v are two elements of the form (8) then to put the product uv into the form (8) again we need to use only the relations (3). This means that h_{x_i} and $y \in P$ commute modulo terms of smaller degrees in the variables $h_{x_1}, h_{x_2}, \ldots, h_{x_n}$. Consequently, we can put uv into the form (8) with the leading monomial $ldm(u)\, ldm(v)$ and the leading coefficient $ldc(u)\, ldc(v)$. □

Now we introduce a degree function $hdeg$ (or h-degree function) on P^e. Let u be an element of P^e written in the form (8). Then we put $hdeg\, u = \max_{i=1}^{k} \deg w_i$ and $hdeg\, 0 = -\infty$. We say that u is homogeneous with respect to $hdeg$ if $\deg w_1 = \deg w_2 = \cdots = \deg w_k$. It follows directly from Lemma 6 and (3) that

$$hdeg\, uv = hdeg\, u + hdeg\, v$$

for every u and v from P^e, i.e., $hdeg$ is a degree function on P^e. Denote by $\bar{u}$ the highest homogeneous part of u with respect to $hdeg$.

Denote by U_i the subset of all elements u of P^e with $hdeg\, u \leqslant i$. Then,

$$P = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_k \subset \cdots,$$

is a filtration of P^e, i.e., $U_i U_j \subseteq U_{i+j}$ for all $i, j \geqslant 0$. Put also

$$gr\, P^e = gr\, U_0 \oplus gr\, U_1 \oplus gr\, U_2 \oplus \cdots \oplus gr\, U_k \oplus \cdots,$$

where $gr\, U_0 = P$ and $gr\, U_i = U_i / U_{i-1}$ for all $i \geqslant 1$. Denote by $\varphi_i : U_i \to gr\, U_i$ the natural projection for every $i \geqslant 1$ and put $\varphi_0 = id : P \to P$. We define also

$$\varphi = \{\varphi_i\}_{i \geqslant 0} : P^e \to gr\, P^e \tag{9}$$

by $\varphi(u) = \varphi_i(u)$ if $u \in U_i \setminus U_{i-1}$ for every $i \geqslant 1$ and $\varphi(u) = u$ if $u \in P$.

The multiplication of P^e induces a multiplication on $gr\,P^e$ and the graded vector space $gr\,P^e$ becomes an algebra.

Consider $B = P \otimes_k A$ as a tensor product of associative algebras. Then B is a free associative algebra over P in the variables $h_{x_1}, h_{x_2}, \dots, h_{x_n}$.

Theorem 3. *The graded algebra $gr\,P^e$ is isomorphic to $B = P \otimes_k A$.*

Proof. By (3), P is in the center of the algebra $gr\,P^e$ and $gr\,P^e$ is generated by $\varphi(h_{x_1}), \varphi(h_{x_2}), \dots, \varphi(h_{x_n})$ as an algebra over P. Note that $B = P \otimes_k A$ is a free associative algebra over P. Then there is a P-algebra homomorphism $\psi : B \to gr\,P^e$ such that $\psi(h_{x_i}) = \varphi(h_{x_i})$ for all i.

Let T_s be the space of homogeneous with respect to *hdeg* elements of P^e of degree $s \geqslant 1$ and let B_s be the space of homogeneous of degree s elements of B. There is an obvious isomorphism between the spaces T_s and A_s established by the P-module homomorphism in Theorem 2. Note that $U_s = U_{s-1} + T_s$, $U_s/U_{s-1} \simeq T_s \simeq B_s$, and $\psi_{|B_s} : B_s \to gr\,U_s$ is an isomorphism of P-modules. Consequently, $Ker(\psi) = 0$ and ψ is an isomorphism of algebras. $\square$

5. The left dependence

We use the notations of Section 4.

Lemma 8. *Let $u \in P^e$ and $hdeg\, u = m$. Then there exists $v \in P^e$ such that $\lambda^{m+1} u = v\lambda$.*

Proof. By (3), $\lambda u = u\lambda + u_1$, where u_1 has degree less than m. Consequently, $\lambda^{m+1} u = w\lambda$ by induction on m. $\square$

Let u and v be two elements of the form (7). We write $u \ll v$ if u is a left divisor of v, i.e., $v = tu$ for some t of the form (7).

Definition 1. Let u and v be nonzero elements of P^e such that $ldm(u) = ldm(v)$. Let $r = gcd(ldc(u), ldc(v))$ be the greatest common divisor of $ldc(u)$ and $ldc(v)$. Then put

$$(u, v)_c = \big(ldc(v)/r\big)u - \big(ldc(u)/r\big)v.$$

Note that $(u, v)_c = 0$ or $ldm((u, v)_c) < ldm(u) = ldm(v)$.

Lemma 9. *Let $s_1, s_2, \dots, s_k$ be a finite set of nonzero elements of P^e. If $ldm(s_i)$ and $ldm(s_j)$ are not comparable with respect to $\ll$ for every $i \neq j$ then the elements $s_1, s_2, \dots, s_k$ are left independent over P^e.*

Proof. Suppose that

$$\sum_{r=1}^{k} u_r s_r = 0. \tag{10}$$

By Lemma 7, $ldm(u_r s_r) = ldm(u_r)\, ldm(s_r)$ for every r. Suppose that Eq. (10) is not trivial, i.e., at least one of the coefficients u_r is nonzero. Then, comparing the leading monomials of the summands, we conclude that $ldm(u_i)\, ldm(s_i) = ldm(u_j)\, ldm(s_j) \neq 0$ for some $i \neq j$. It is possible if and only if $ldm(s_i) \ll ldm(s_j)$ or $ldm(s_i) \gg ldm(s_j)$. This contradicts the condition of the lemma. $\square$

Lemma 10. *Let $s_1, s_2, \dots, s_k$ be a finite set of nonzero elements of P^e. Suppose that $ldm(s_i) \gg ldm(s_j)$ and $ldm(s_i) = t \cdot ldm(s_j)$. Put $s_i' = (s_i, s_j)_c$. Then the elements $s_1, s_2, \dots, s_k$ are left dependent over P^e if and only if the elements $s_1, s_2, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_k$ are left dependent over P^e.*

Proof. Consider Eq. (10). By Lemma 8, there exists a number m such that $\lambda^m u_i = v_i\lambda$ for every $\lambda \in P$ and i. We choose

$$\lambda = ldc(s_j)/\mu_{ij},$$

where $\mu_{ij} = gcd(ldc(s_i), ldc(s_j))$.

Note that

$$\lambda s_i = s_i' + \bigl(ldc(s_i)/\mu_{ij}\bigr)s_j.$$

Then (10) is equivalent to

$$\sum_{r=1}^{k} v_r\lambda s_r = \sum_{r\neq i,j} v_r\lambda s_r + v_i s_i' + \bigl(v_j\lambda + v_i\bigl(ldc(s_i)/\mu_{ij}\bigr)\bigr)s_j = 0.$$

Obviously, this relation is trivial if and only if (10) is trivial. □

Theorem 4. *In the universal enveloping algebra P^e of the free Poisson algebra $P = k\{x_1, x_2, \ldots, x_n\}$ the left dependence of a finite system of elements is algorithmically recognizable.*

Proof. Let $s_1, s_2, \ldots, s_k$ be a finite set of nonzero elements of P^e. If $ldm(s_i)$ and $ldm(s_j)$ are not comparable with respect to $\ll$ for every $i \neq j$ then the elements $s_1, s_2, \ldots, s_k$ are left independent over P^e by Lemma 9.

If $ldm(s_i) \gg ldm(s_j)$ for some $i \neq j$ then we can change s_i by s_i', according to Lemma 10. Note that $ldm(s_i) > ldm(s_i')$. If $s_i' = 0$ then the elements $s_1, s_2, \ldots, s_k$ are left dependent. If $s_i' \neq 0$ then we will apply the same discussions to the new system of elements $s_1, s_2, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_k$. This process stabilizes after a finite number of steps since the set of all leading monomials is well ordered. □

Recall that $B = P \otimes_k A$ is the free associative algebra over P in the variables $y_1 = h_{x_1},\ y_2 = h_{x_2}, \ldots, y_n = h_{x_n}$. Considering B as an algebra over P, we define a degree function Deg on B by $Deg\, y_i = 1$, where $1 \leqslant i \leqslant n$.

Denote by L the Lie subalgebra of the Lie algebra $A^{(-)}$ generated by $y_1, y_2, \ldots, y_n$. Then L is a free Lie algebra and $y_1, y_2, \ldots, y_n$ are free generators of L. We need the next purely Lie-theoretical statement.

Lemma 11. *Let f be a homogeneous nonlinear element of L such that $f = f_1y_1 + f_2y_2 + \cdots + f_ny_n$ and $f_n \neq 0$ in A. Then there exists $i \leqslant n-1$ such that $ldm(f_i) > ldm(f_n)$.*

Proof. Recall that a nonempty associative word u in the alphabet $y_1, y_2, \ldots, y_n$ is called a *Lindon–Shirshov* word (see, for example [6]) if for every nonempty words v and w the equality $u = vw$ implies $vw > wv$. It is well known that if $f \in L$ then $ldm(f)$ is a Lindon–Shirshov word [6].

Suppose that for every nonzero f_i, where $i \leqslant n-1$, $ldm(f_i) \leqslant ldm(f_n)$, then $ldm(f) = ldm(f_n)y_n = u$ and u is a Lindon–Shirshov word. Put $v = ldm(f_n)$, then $vy_n > y_nv$. Since y_n is the greatest symbol of the alphabet it follows that $v = y_nw$ and $wy_n > y_nw$. Continuing the same discussions we can get that $u = y_n^s$ for some $s \geqslant 2$. Note that y_n^s is not a Lindon–Shirshov word. □

Lemma 12. *Let f and g be nonzero homogeneous with respect to Deg elements of $P \otimes_k L \subseteq P \otimes_k A = B$. If f and g are left dependent over B then $Deg\, f = Deg\, g$.*

Proof. Suppose that $Deg\, f \geqslant Deg\, g$. The elements f and g are left dependent in a free associative algebra over P. Then there exist a nonzero $\lambda \in P$ and a homogeneous of degree $Deg\, f - Deg\, g$ element $T \in B$ such that $\lambda f = Tg$. Changing f by λf we may assume that

$$f = Tg. \tag{11}$$

Every nonzero homogeneous element $b \in B$ can be represented as

$$b = \beta_1 \otimes a_1 + \beta_2 \otimes a_2 + \cdots + \beta_s \otimes a_s$$

with the least possible s, $ldm(a_1) < ldm(a_2) < \cdots < ldm(a_s)$, and $ldc(a_1) = ldc(a_2) = \cdots = ldc(a_s) = 1$. We call this representation of b a *short representation*. From the minimality of s it follows that $\beta_1, \beta_2, \ldots, \beta_s$ are linearly independent elements of P and $a_1, a_2, \ldots, a_s$ are linearly independent elements of A. If $b \in P \otimes_k L$ then we can easily find a short representation of b with an additional condition $a_i \in L$ for all i. Let

$$f = \alpha_1 \otimes l_1 + \alpha_2 \otimes l_2 + \cdots + \alpha_s \otimes l_s,$$

$$g = \beta_1 \otimes m_1 + \beta_2 \otimes m_2 + \cdots + \beta_t \otimes m_t,$$

$$T = \gamma_1 \otimes W_1 + \gamma_2 \otimes W_2 + \cdots + \gamma_r \otimes W_r$$

be short representations of f, g, T such that $l_i, m_j \in L$ for all i, j. Then,

$$f = \sum_{i,j} \gamma_i \beta_j W_i m_j.$$

From this representation of f we can get every other short representation of f by linear transformations over k. Consequently, l_i belongs to the left ideal of A generated by $m_1, m_2, \ldots, m_t$ for all i. It follows from (11) that $\alpha_s = \gamma_r \beta_t$ and $ldm(l_s) = ldm(W_r)\, ldm(m_t)$. Consequently,

$$l_s = g_1 m_1 + g_2 m_2 + \cdots + g_t m_t$$

and $ldm(g_1), ldm(g_2), \ldots, ldm(g_{t-1}) \leqslant ldm(g_t) = W_r$.

It follows from [28,29] that l_s belongs to the Lie subalgebra generated by $m_1, m_2, \ldots, m_t$. We fix $h \in Lie\langle z_1, z_2, \ldots, z_t\rangle$ such that $l_s = h(m_1, m_2, \ldots, m_t)$. We may assume that h is homogeneous since $m_1, m_2, \ldots, m_t$ are homogeneous and have the same degrees. In this case their linear independence implies [32] their freeness in the Lie algebra L. Hence the left A-submodule generated by these elements is free with the same free generators [32]. If $h = h_1 z_1 + h_2 z_2 + \cdots + h_t z_t$ then $h_i(m_1, m_2, \ldots, m_t) = g_i$ for all i. Put $z_1 < z_2 < \cdots < z_t$. Note that

$$ldm\big(h_i(m_1, m_2, \ldots, m_t)\big) = ldm(h_i)\big(ldm(m_1), ldm(m_2), \ldots, ldm(m_t)\big)$$

since $ldm(m_1) < ldm(m_2) < \cdots < ldm(m_t)$ and have one and the same degrees. The same idea gives $ldm(h_i) < ldm(h_j)$ if and only if $ldm(g_i) < ldm(g_j)$. Then,

$$ldm(h_1), ldm(h_2), \ldots, ldm(h_{t-1}) \leqslant ldm(h_t).$$

If h is not linear then this contradicts Lemma 11, therefore h is linear. It is possible if and only if $Deg\, T = 0$, hence $Deg\, f = Deg\, g$. □

6. Universal derivations

As before, $P = k\{x_1, x_2, \ldots, x_n\}$ is the free Poisson algebra in the variables $x_1, x_2, \ldots, x_n$. Denote by Ω_P the left ideal of P^e generated by $h_{x_1}, h_{x_2}, \ldots, h_{x_n}$. By Theorem 2,

$$\Omega_P = P^e h_{x_1} \oplus P^e h_{x_2} \oplus \cdots \oplus P^e h_{x_n},$$

i.e., Ω_P is a free left P^e-module. Note that

$$P^e = P \oplus \Omega_P.$$

Consider

$$H : P \to \Omega_P$$

such that $H(p) = h_p$ for all $p \in P$. By Lemma 1, H is the universal derivation of P and Ω_P its universal differential module.

Recall that a set of elements $f_1, f_2, \ldots, f_k$ of the free Poisson algebra P is called *Poisson free* or *Poisson independent* if the Poisson subalgebra of P generated by these elements is the free Poisson algebra with free generators $f_1, f_2, \ldots, f_k$. Otherwise these elements are called *Poisson dependent*.

Lemma 13. *Let $f_1, f_2, \ldots, f_k$ be arbitrary elements of the free Poisson algebra P over a field k of characteristic* 0. *If the elements $f_1, f_2, \ldots, f_k$ are Poisson dependent then the elements $H(f_1), H(f_2), \ldots, H(f_k)$ are left dependent over P^e.*

Proof. Let $F = F(z_1, z_2, \ldots, z_k)$ be a nonzero element of $T = k\{z_1, z_2, \ldots, z_k\}$ with the minimal degree such that $F(f_1, f_2, \ldots, f_k) = 0$. Suppose that

$$H(F) = u_1 H(z_1) + u_2 H(z_2) + \cdots + u_k H(z_k)$$

in Ω_T. We may assume that $u_1 = u_1(z_1, z_2, \ldots, z_k) \neq 0$. Note that $\deg u_1 < \deg F$. Consequently,

$$0 = H\big(F(f_1, f_2, \ldots, f_k)\big) = u'_1 H(f_1) + u'_2 H(f_2) + \cdots + u'_k H(f_k),$$

where $u'_i = u_i(f_1, f_2, \ldots, f_k)$ for all i. If $u'_1 \neq 0$ then the last equation gives a nontrivial dependence of $H(f_1), H(f_2), \ldots, H(f_k)$.

Suppose that $u'_1 = 0$. Note that $u_1 = t + w$, where $t \in T$ and $w \in \Omega_T$, since $U(T) = T \oplus \Omega_T$. Obviously, $t(f_1, f_2, \ldots, f_k) \in P$ and it easily follows from (1)–(2) that $w(f_1, f_2, \ldots, f_k) \in \Omega_P$. Then, $t(f_1, f_2, \ldots, f_k) = 0$ and $w(f_1, f_2, \ldots, f_k) = 0$ since $0 = u'_1 = t(f_1, f_2, \ldots, f_k) + w(f_1, f_2, \ldots, f_k) \in P \oplus \Omega_P$. If $t \neq 0$ then this contradicts the minimality of $\deg F$ since $\deg t \leqslant \deg u_1 < \deg F$. If $w \neq 0$ then, continuing the same discussions, we get a nontrivial dependence of $H(f_1), H(f_2), \ldots, H(f_k)$ over P^e. $\square$

Theorem 5. *Let f and g be arbitrary elements of the free Poisson algebra $P = k\{x_1, x_2, \ldots, x_n\}$ in the variables $x_1, x_2, \ldots, x_n$ over a field k of characteristic zero. Then the following conditions are equivalent*:

(i) *f and g are Poisson dependent*;
(ii) *$H(f)$ and $H(g)$ are left dependent over P^e*;
(iii) *f and g are polynomially dependent, i.e., they are algebraically dependent in the polynomial algebra P*;
(iv) *there exists $a \in P$ such that $f, g \in k[a]$*;
(v) *$\{f, g\} = 0$ in P.*

Proof. By Lemma 12, (i) implies (ii). The conditions (iii), (iv), and (v) are equivalent [16,35]. Obviously, (iii) implies (i).

To prove the theorem it is sufficient to show that (ii) implies (iii). Note that $f=0$ if and only if $H(f)=0$. Suppose that $(H(f),H(g))\neq 0$ and

$$uH(f)+vH(g)=0,\quad (u,v)\neq 0.$$

Then obviously

$$\overline{\bar{u}\overline{H(f)}+\bar{v}\overline{H(g)}}=0,\quad (\bar{u},\bar{v})\neq 0,$$

or equivalently,

$$\varphi(u)\varphi\big(H(f)\big)+\varphi(v)\varphi\big(H(g)\big)=0,\quad \big(\varphi(u),\varphi(v)\big)\neq 0$$

in the algebra $B=P\otimes_k A$, where $\varphi: P^e\to gr\,P^e$ is the gradation mapping (9). So, $\varphi(H(f))$ and $\varphi(H(g))$ are left dependent over B.

Let $l=l(x_1,x_2,\ldots,x_n)$ be an arbitrary element of the free Lie algebra $g=Lie\langle x_1,x_2,\ldots,x_n\rangle$. Then, $H_l=h_l=l(h_{x_1},h_{x_2},\ldots,h_{x_n})$ by (1). Hence $H_l=l(y_1,y_2,\ldots,y_n)\in L$. For every $i\geqslant 1$ denote by ∂_{e_i} the usual partial derivation of the polynomial algebra P in the variables (5). It can be easily checked that

$$H(a)=\sum_{i\geqslant 1}\partial_{e_i}(a)H(e_i)\in PL$$

for every $a\in P$. Consequently, $\varphi(H(f)),\varphi(H(g))\in P\otimes_k L$.

So, the homogeneous nonzero elements $\varphi(H(f))$ and $\varphi(H(g))$ of $P\otimes_k L$ are left dependent over B. Then $Deg\,\varphi(H(f))=Deg\,\varphi(H(g))$ by Lemma 11. Recall that B is a free associative algebra over P. Hence there exist nonzero elements $\lambda,\mu\in P$ such that $\lambda\varphi(H(f))=\mu\varphi(H(g))$ or equivalently, $\lambda\overline{H(f)}=\mu\overline{H(g)}$ and $hdeg(\lambda H(f)-\mu H(g))<hdeg\,H(f)=hdeg\,H(f)$.

Using Lemma 7, it is easy to show that $H(f)$ and $\lambda H(f)-\mu H(g)$ are left dependent over P^e again. If $\lambda H(f)-\mu H(g)\neq 0$ then $\varphi(H(f))$ and $\varphi(\lambda H(f)-\mu H(g))$ are homogeneous nonzero elements of $P\otimes_k L$ left dependent over B. This contradicts the statement of Lemma 11 since $Deg\,\varphi(\lambda H(f)-\mu H(g))<Deg\,\varphi(H(f))$. Consequently, $\lambda H(f)-\mu H(g)=0$. Then,

$$\sum_{i\geqslant 1}\big(\lambda\partial_{e_i}(f)-\mu\partial_{e_i}(g)\big)H(e_i)=0$$

and hence

$$\lambda\partial_{e_i}(f)-\mu\partial_{e_i}(g)=0$$

for all $i\geqslant 1$. It is well known (see, for example [23]) that in this case f and g are algebraically dependent in the polynomial algebra P. $\square$

The equivalence of the conditions (i) and (iii) of Theorem 5 was proved recently in [14]. The condition (ii) plays a central role in the given proof as well as in further proofs.

For every $p\in P=k\{x_1,x_2,\ldots,x_n\}$ the Fox derivatives $\frac{\partial p}{\partial x_i}$ (see [31,32]) are uniquely defined by

$$H(p)=\frac{\partial p}{\partial x_i}h_{x_1}+\frac{\partial p}{\partial x_i}h_{x_2}+\cdots+\frac{\partial p}{\partial x_i}h_{x_n},\quad \frac{\partial p}{\partial x_i}\in P^e,$$

for all $1 \leqslant i \leqslant n$. For every endomorphism ψ of the free Poisson algebra P we define the Jacobian matrix $J(\psi) = [u_{ij}]$ with $u_{ij} = \frac{\partial \psi(x_i)}{\partial x_j}$ for all $1 \leqslant i, j \leqslant n$. It is easy to show [31] that $J(\psi)$ is invertible over P^e if ψ is an automorphism. The reverse statement is an analogue of the classical Jacobian Conjecture [11] for free Poisson algebras.

Theorem 6. *Let ψ be an endomorphism of the free Poisson algebra $P = k\{x, y\}$ in the variables x, y over a field k of characteristic 0. If $J(\psi)$ is invertible over P^e then $\psi(x), \psi(y) \in k[x, y]$.*

Proof. Put $\psi(x) = f$ and $\psi(y) = g$. Note that $J(\psi)$ is invertible over P^e if and only if Ω_P is the free P^e module with basis $H(f)$ and $H(g)$. It is sufficient to prove that $hdeg(H(f)) = hdeg(H(g)) = 1$.

Suppose that $hdeg\, H(f) + hdeg\, H(g) \geqslant 3$. Note that $\Omega_P = P^e H(f) + P^e H(g) = P^e h_x + P^e h_y$. Consequently, $P^e H(f) + P^e H(g)$ contains two elements h_x and h_y of h-degree 1. Hence there exists $(u, v) \neq 0$ such that

$$\overline{u\overline{H(f)} + v\overline{H(g)}} = 0.$$

As in the proof of Theorem 5, $hdeg\, H(f) = hdeg\, H(g)$ and there exist $0 \neq \lambda,\ \mu \in P$ such that $hdeg(\lambda H(f) - \mu H(g)) < hdeg\, H(f)$. Put $T = \lambda H(f) - \mu H(g)$. Note that $hdeg\, H(f) + hdeg\, T \geqslant 3$. By Lemma 8, it is not difficult to find a nonzero $\eta \in P$ such that $\eta h_x, \eta h_y \in P^e H(f) + P^e T$. Again, as in the proof of Theorem 5, we get $hdeg\, H(f) = hdeg\, T$. This is a contradiction. $\square$

Using Jung's theorem [13] and Theorem 6 we get

Corollary 4. *Automorphisms of free Poisson algebras in two variables over a field of characteristic zero are tame.*

The first proof of this result was given in [15] and the other two proofs recently appeared also in [17,14].

Corollary 5. *The two-dimensional Jacobian Conjecture for free Poisson algebras is equivalent to the two-dimensional Jacobian Conjecture for polynomial algebras in characteristic zero.*

An analogue of the Jacobian Conjecture is true for free Lie algebras [20,25,29,36], for free associative algebras [9,21], and for free nonassociative algebras [31,34].

7. Comments and problems

In Section 3 we proved that the universal enveloping algebras of the Poisson symplectic algebra P_n and the Weyl algebra A_n are isomorphic. It is not difficult to show that this result is true also for fields of positive characteristic. A. Belov-Kanel and M. Kontsevich [3] formulated the next problem.

Problem 1. The automorphism group of the Weyl algebra of index n is isomorphic to the group of polynomial symplectomorphisms of a $2n$-dimensional affine space, i.e.,

$$Aut\, A_n \simeq Aut\, P_n.$$

This problem was posed in [3] for fields of characteristic zero but it makes sense in positive characteristic also [1,4,26,27].

Every automorphism φ of P_n can be uniquely extended to an automorphism φ^* of the universal enveloping algebra P_n^e by $x \mapsto \varphi(x), h_x \mapsto h_{\varphi(x)}$ for all $x \in P_n$. Similarly, every automorphism ψ of A_n can be uniquely extended to an automorphism ψ° of the universal enveloping algebra A_n^e by $x \otimes 1 \mapsto \psi(x) \otimes 1,\ 1 \otimes x' \mapsto 1 \otimes \psi(x)'$ for all $x \in A_n$. By means of $\varphi \mapsto \varphi^*$ and $\psi \mapsto \psi^\circ$ we can identify the groups of automorphisms $Aut\, P_n$ and $Aut\, A_n$ with the corresponding subgroups of $Aut\, P_n^e$ and $Aut\, A_n^e$, respectively. By means of the canonical isomorphism θ from Section 3 we can identify P_n^e and A_n^e and consider $Aut\, P_n$ and $Aut\, A_n$ as subgroups of $Aut\, A_{2n}$.

Problem 2. Is it true that $Aut\, P_n$ and $Aut\, A_n$ are conjugate?

First of all it is interesting to know the answer to the question: Are $Aut\, P_n$ and $Aut\, A_n$ conjugate in $Aut\, A_{2n}$? It seems the structure of an isomorphism between $Aut\, P_n$ and $Aut\, A_n$ is very complicated if it exists.

We say that the *subalgebra membership problem* is decidable for an algebra A if there is an effective procedure that defines for any element $a \in A$ and for any finitely generated subalgebra B of A whether a belong to B or not. Some relations between the distortion of subalgebras and the subalgebra membership problem were established in [2] for polynomial, free associative, and free Lie algebras. It can be easily derived from [24,33] that the subalgebra membership problem is decidable for free Lie algebras. Moreover, finitely generated subalgebras of free Lie algebras are residually finite [28]. The subalgebra membership problem is still open in the case of free Poisson algebras.

Problem 3. Is the subalgebra membership problem decidable for free Poisson algebras?

The subalgebra membership problem for free associative algebras is undecidable [30]. In fact, if $A = k\langle x_1, \ldots, x_n\rangle$ is a free associative algebra, then the structure of the left ideals of the universal enveloping algebra $A^e = A \otimes_k A^{op}$ is very difficult [30]. The left ideal membership problem is algorithmically undecidable and the left dependence of a finite set of elements of A^e is algorithmically unrecognizable [30]. The next problem is closely related to Problem 3.

Problem 4. Is the left ideal membership problem decidable over the universal enveloping algebras of free Poisson algebras?

By Theorem 4, the left dependence of a finite set of elements over the universal enveloping algebras of free Poisson algebras is algorithmically recognizable. This is a positive result in the direction of solving Problems 3 and 4. These problems are also related to the next problem.

Problem 5. Is the freeness of a finite set of elements of free Poisson algebras algorithmically recognizable?

Recall that the freeness of a finite set of elements is algorithmically recognizable in the case of free Lie algebras (see, for example [29]) and unrecognizable in the case of free associative algebras [30]. In Lemma 13 we proved that the Poisson dependence implies the left dependence of universal derivatives.

Problem 6. Let $f_1, f_2, \ldots, f_k$ be arbitrary elements of the free Poisson algebra P over a field k of characteristic 0. Is it true that the left dependence of $H(f_1), H(f_2), \ldots, H(f_k)$ over P^e implies the Poisson dependence of $f_1, f_2, \ldots, f_k$.

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