Inequalities for Proper Contractions and Strictly Dissipative Operators

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ABSTRACT

Some inequalities for proper contractions and strictly dissipative operators on a Hilbert space are proved. These inequalities for operators are related to the classical Harnack's inequalities for analytic functions with a positive real part.

1. INEQUALITIES FOR PROPER CONTRACTIONS

Throughout this paper, $\mathscr{H}$ denotes a complex Hilbert space. By an operator we mean a bounded linear transformation of $\mathscr{H}$ into itself. An operator A with $\|A\| < 1$ is called a *proper contraction*. For an operator A, we denote its adjoint by A^*, and use the notation

$$\operatorname{Re} A = \frac{A + A^*}{2}, \qquad \operatorname{Im} A = \frac{A - A^*}{2i}.$$

For two Hermitian operators A, B on $\mathscr{H}$, we write $A \geqslant B$ or $B \leqslant A$ to indicate that the inner product $\langle (A - B)x, x \rangle$ is $\geqslant 0$ for all $x \in \mathscr{H}$. If, in addition to $A \geqslant B$, $A - B$ is invertible (i.e., has a bounded inverse), then we write the strict inequality $A > B$ or $B < A$. The identity operator on $\mathscr{H}$ is denoted by I.

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PROPOSITION 1. *For any proper contraction A on $\mathscr{H}$, the following inequalities hold*:

$$\frac{A^*A - \|A\|I}{1 - \|A\|} \leqslant \operatorname{Re} A \leqslant \frac{A^*A + \|A\|I}{1 + \|A\|}, \tag{1}$$

$$\frac{A^*A - \|A\|I}{1 - \|A\|} \leqslant \operatorname{Im} A \leqslant \frac{A^*A + \|A\|I}{1 + \|A\|}, \tag{2}$$

$$\frac{1 - \|A\|}{1 + \|A\|}(I - A^*)(I - A) \leqslant I - A^*A \leqslant \frac{1 + \|A\|}{1 - \|A\|}(I - A^*)(I - A), \tag{3}$$

$$\pm \operatorname{Re} A \leqslant \frac{\|A\|}{1 - \|A\|^2}(A^* - iI)(A + iI), \tag{4}$$

$$\pm \operatorname{Im} A \leqslant \frac{\|A\|}{1 - \|A\|^2}(I - A^*)(I - A). \tag{5}$$

Proof. Let $x \in \mathscr{H}$. Since $\|A\| < 1$, we have

$$\|Ax\|^2 - (1 + \|A\|)\|Ax\| \cdot \|x\| + \|A\| \cdot \|x\|^2$$
$$= (\|A\| \cdot \|x\| - \|Ax\|)(\|x\| - \|Ax\|) \geqslant 0,$$

or

$$(1 + \|A\|)\|Ax\| \cdot \|x\| \leqslant \|Ax\|^2 + \|A\| \cdot \|x\|^2.$$

Therefore

$$(1 + \|A\|)\operatorname{Re}\langle Ax, x\rangle \leqslant (1 + \|A\|)\|Ax\| \cdot \|x\| \leqslant \|Ax\|^2 + \|A\| \cdot \|x\|^2,$$

or

$$(1 + \|A\|)\operatorname{Re}\langle Ax, x\rangle \leqslant \langle A^*Ax, x\rangle + \|A\|\langle x, x\rangle.$$

As $\operatorname{Re}\langle Ax, x\rangle = \langle(\operatorname{Re} A)x, x\rangle$, the last inequality may be written as

$$\langle(\operatorname{Re} A)x, x\rangle \leqslant \left\langle \frac{A^*A + \|A\|I}{1+\|A\|}x, x\right\rangle,$$

which holds for all $x \in \mathscr{H}$. This proves the right-hand inequality in (1).

Similarly, we have for all $x \in \mathscr{H}$

$$\|A\|\cdot\|x\|^2 - (1-\|A\|)\|Ax\|\cdot\|x\| - \|Ax\|^2$$
$$= (\|A\|\cdot\|x\| - \|Ax\|)(\|x\| + \|Ax\|) \geqslant 0,$$

or

$$(1-\|A\|)\|Ax\|\cdot\|x\| \leqslant \|A\|\cdot\|x\|^2 - \|Ax\|^2.$$

Therefore

$$-(1-\|A\|)\operatorname{Re}\langle Ax, x\rangle \leqslant (1-\|A\|)\|Ax\|\cdot\|x\| \leqslant \|A\|\cdot\|x\|^2 - \|Ax\|^2$$

or

$$(1-\|A\|)\operatorname{Re}\langle Ax, x\rangle \geqslant \|Ax\|^2 - \|A\|\cdot\|x\|^2,$$

which can be written

$$\langle(\operatorname{Re} A)x, x\rangle \geqslant \left\langle \frac{A^*A - \|A\|I}{1-\|A\|}x, x\right\rangle.$$

This proves the left-hand inequality in (1).

Since $\operatorname{Im} A = \operatorname{Re}(-iA)$, (2) is obtained by replacing A in (1) by $-iA$.

The left-hand inequality in (1) may be written as

$$(1-\|A\|)(I - A - A^* + A^*A) \leqslant (1+\|A\|)(I - A^*A). \tag{6}$$

The right-hand inequality in (1) may be written as

$$(1-\|A\|)(I - A^*A) \leqslant (1+\|A\|)(I - A - A^* + A^*A). \tag{7}$$

Then (3) follows from (6) and (7).

As $\operatorname{Re} A = \operatorname{Im}(iA)$, (4) will follow from (5) on replacing A in (5) by iA. It remains to prove (5).

By Cauchy-Schwarz inequality, we can write for $x \in \mathscr{H}$

$$\begin{aligned}
&2\|A\|\operatorname{Re}\langle Ax, x\rangle \pm (1-\|A\|^2)\operatorname{Im}\langle Ax, x\rangle \\
&\quad \leqslant \left\{4\|A\|^2 + (1-\|A\|^2)^2\right\}^{1/2}\left\{(\operatorname{Re}\langle Ax, x\rangle)^2 + (\operatorname{Im}\langle Ax, x\rangle)^2\right\}^{1/2} \\
&\quad = (1+\|A\|^2)|\langle Ax, x\rangle| \leqslant (1+\|A\|^2)\|Ax\|\cdot\|x\|.
\end{aligned}$$

On the other hand, since $\|A\| < 1$,

$$\begin{aligned}
&(1+\|A\|^2)\|Ax\|\cdot\|x\| - \|A\|(\|Ax\|^2 + \|x\|^2) \\
&\quad = (\|Ax\| - \|A\|\cdot\|x\|)(\|x\| - \|A\|\cdot\|Ax\|) \leqslant 0,
\end{aligned}$$

or

$$(1+\|A\|^2)\|Ax\|\cdot\|x\| \leqslant \|A\|(\|Ax\|^2 + \|x\|^2).$$

It follows that

$$2\|A\|\operatorname{Re}\langle Ax, x\rangle \pm (1-\|A\|^2)\operatorname{Im}\langle Ax, x\rangle \leqslant \|A\|(\langle A^*Ax, x\rangle + \langle x, x\rangle).$$

Hence

$$\begin{aligned}
\pm(1-\|A\|^2)\langle(\operatorname{Im} A)x, x\rangle &\leqslant \|A\|\left\{\langle A^*Ax, x\rangle + \langle x, x\rangle - 2\langle(\operatorname{Re} A)x, x\rangle\right\} \\
&= \|A\|\langle(I-A^*)(I-A)x, x\rangle
\end{aligned}$$

for all $x \in \mathscr{H}$, and therefore (5). This completes the proof. ■

2. INEQUALITIES FOR STRICTLY DISSIPATIVE OPERATORS

An operator B on $\mathscr{H}$ is said to be *strictly dissipative* if $\operatorname{Im} B > 0$ (i.e., if $\operatorname{Im} B$ is an invertible positive operator).

PROPOSITION 2. *Let B be a strictly dissipative operator on $\mathscr{H}$, and let*

$$A=(B-iI)(B+iI)^{-1}. \tag{8}$$

Then $\|A\|<1$ and the following inequalities hold:

$$\pm(I-B^*B)\leqslant\frac{2\|A\|}{1-\|A\|^2}(I-B^*)(I-B), \tag{9}$$

$$\pm\operatorname{Re}B\leqslant\frac{2\|A\|}{1-\|A\|^2}I, \tag{10}$$

$$\frac{1-\|A\|}{1+\|A\|}I\leqslant\operatorname{Im}B\leqslant\frac{1+\|A\|}{1-\|A\|}I, \tag{11}$$

$$2\frac{1-\|A\|}{1+\|A\|}\operatorname{Im}B\leqslant(I+B^*)(I+B)\leqslant2\frac{1+\|A\|}{1-\|A\|}\operatorname{Im}B. \tag{12}$$

Proof. From (8), we have

$$I-A^*A=(B^*-iI)^{-1}\{(B^*-iI)(B+iI)-(B^*+iI)(B-iI)\}(B+iI)^{-1}$$

or

$$I-A^*A=4(B^*-iI)^{-1}(\operatorname{Im}B)(B+iI)^{-1}. \tag{13}$$

Since $\operatorname{Im}B>0$, (13) implies that $I-A^*A>0$, so $\|A\|<1$.
Again by (8),

$$\operatorname{Re}A=\tfrac{1}{2}(B^*-iI)^{-1}\{(B^*-iI)(B-iI)+(B^*+iI)(B+iI)\}(B+iI)^{-1},$$

or

$$\operatorname{Re}A=(B^*-iI)^{-1}(B^*B-I)(B+iI)^{-1}. \tag{14}$$

As

$$\begin{aligned}A+iI&=[(B-iI)+i(B+iI)](B+iI)^{-1}\\&=(1+i)(B-I)(B+iI)^{-1},\end{aligned}$$

we have

$$(A^* - iI)(A + iI) = 2(B^* - iI)^{-1}(B^* - I)(B - I)(B + iI)^{-1}. \quad (15)$$

As $\|A\| < 1$, the inequality (4) holds. In view of (14) and (15), we can rewrite (4) as

$$\pm(B^* - iI)^{-1}(I - B^*B)(B + iI)^{-1}$$
$$\leqslant \frac{2\|A\|}{1 - \|A\|^2}(B^* - iI)^{-1}(I - B^*)(I - B)(B + iI)^{-1},$$

and therefore (9).

Next, from (8) we derive

$$\operatorname{Im} A = \frac{1}{2i}(B^* - iI)^{-1}\{(B^* - iI)(B - iI) - (B^* + iI)(B + iI)\}(B + iI)^{-1},$$

or

$$\operatorname{Im} A = -2(B^* - iI)^{-1}(\operatorname{Re} B)(B + iI)^{-1}. \quad (16)$$

As

$$I - A = [(B + iI) - (B - iI)](B + iI)^{-1} = 2i(B + iI)^{-1},$$

we have

$$(I - A^*)(I - A) = 4(B^* - iI)^{-1}(B + iI)^{-1}. \quad (17)$$

Then, from (5), (16), and (17), we obtain

$$\pm 2(B^* - iI)^{-1}(\operatorname{Re} B)(B + iI)^{-1} \leqslant \frac{4\|A\|}{1 - \|A\|^2}(B^* - iI)^{-1}(B + iI)^{-1}$$

and therefore (10).

From (3), (13), and (17), we have

$$4\frac{1-\|A\|}{1+\|A\|}(B^*-iI)^{-1}(B+iI)^{-1}\leqslant 4(B^*-iI)^{-1}(\operatorname{Im} B)(B+iI)^{-1}$$

$$\leqslant 4\frac{1+\|A\|}{1-\|A\|}(B^*-iI)^{-1}(B+iI)^{-1},$$

which is equivalent to (11).

Again from (8), we derive easily

$$\frac{A^*A+\|A\|I}{1+\|A\|}=(B^*-iI)^{-1}\left\{I+B^*B-2\frac{1-\|A\|}{1+\|A\|}\operatorname{Im} B\right\}(B+iI)^{-1} \quad (18)$$

and

$$\frac{A^*A-\|A\|I}{1-\|A\|}=(B^*-iI)^{-1}\left\{I+B^*B-2\frac{1+\|A\|}{1-\|A\|}\operatorname{Im} B\right\}(B+iI)^{-1}. \quad (19)$$

Then from (2), (16), (18) and (19), we obtain

$$I+B^*B-2\frac{1+\|A\|}{1-\|A\|}\operatorname{Im} B\leqslant -2\operatorname{Re} B\leqslant I+B^*B-2\frac{1-\|A\|}{1+\|A\|}\operatorname{Im} B,$$

which proves (12). ∎

3. OPERATOR-VALUED ANALYTIC FUNCTIONS

We consider now operator-valued analytic functions [8, pp. 92–94] of a complex variable. The following result is a generalization of the classical Harnack's inequalities.

PROPOSITION 3. *Let F be an operator-valued analytic function on the open unit disk $\Delta=\{z\in\mathbf{C}:|z|<1\}$ such that for each $z\in\Delta$, $F(z)$ is an* *operator on $\mathscr{H}$ with* $\operatorname{Re} F(z) > 0$ *and* $F(0) = I$. *Then*

$$\frac{1-|z|}{1+|z|} I \leqslant \operatorname{Re} F(z) \leqslant \frac{1+|z|}{1-|z|} I, \tag{20}$$

$$-\frac{2|z|}{1-|z|^2} I \leqslant \operatorname{Im} F(z) \leqslant \frac{2|z|}{1-|z|^2} I \tag{21}$$

hold for all $z \in \Delta$.

Proof. For each $x \in \mathscr{H}$ with $\|x\| = 1$, let f_x be defined on Δ by

$$f_x(z) = \langle F(z)x, x \rangle.$$

Then f_x is a complex-valued analytic function on Δ, $f_x(0) = \|x\|^2 = 1$. Since $\operatorname{Re} F(z) > 0$ and

$$\operatorname{Re}\langle F(z)x, x \rangle = \langle [\operatorname{Re} F(z)]x, x \rangle,$$

we have

$$\operatorname{Re} f_x(z) > 0 \qquad \text{for} \quad z \in \Delta.$$

By the classical Harnack's inequalities, we have

$$\frac{1-|z|}{1+|z|} \leqslant \operatorname{Re} f_x(z) \leqslant \frac{1+|z|}{1-|z|}$$

and

$$-\frac{2|z|}{1-|z|^2} \leqslant \operatorname{Im} f_x(z) \leqslant \frac{2|z|}{1-|z|^2}$$

for $z \in \Delta$. In other words, we have

$$\frac{1-|z|}{1+|z|} \langle x, x \rangle \leqslant \langle [\operatorname{Re} F(z)]x, x \rangle \leqslant \frac{1+|z|}{1-|z|} \langle x, x \rangle$$

and

$$-\frac{2|z|}{1-|z|^2}\langle x,x\rangle \leqslant \langle [\operatorname{Im} F(z)]x,x\rangle \leqslant \frac{2|z|}{1-|z|^2}\langle x,x\rangle$$

for all $x \in \mathscr{H}$ with $\|x\| = 1$. This proves (20) and (21). ∎

From the proof of Proposition 1, it is clear that each of (1) and (2) is just a reformulation of (3), and that (4) is a reformulation of (5). Now we are going to give another proof of the inequalities (3) and (5) by applying Proposition 3.

Second proof of (3) *and* (5). Let A be an operator on $\mathscr{H}$ with $\|A\| < 1$. Define

$$F(z) = (I + zA)(I - zA)^{-1} = I + 2\sum_{n=1}^{\infty} A^n z^n$$

for $z \in \Delta$. Then F is an operator-valued analytic function on Δ, $F(0) = I$. Furthermore, from

$$\begin{aligned}\operatorname{Re} F(z) &= \tfrac{1}{2}(I + zA)(I - zA)^{-1} + \tfrac{1}{2}(I - \bar{z}A^*)^{-1}(I + \bar{z}A^*)\\ &= \tfrac{1}{2}(I - \bar{z}A^*)^{-1}\{(I - \bar{z}A^*)(I + zA)\\ &\qquad + (I + \bar{z}A^*)(I - zA)\}(I - zA)^{-1}\\ &= (I - \bar{z}A^*)^{-1}(I - |z|^2A^*A)(I - zA)^{-1} \qquad (22)\end{aligned}$$

and $I - |z|^2A^*A > 0$ for $z \in \Delta$ (since $\|A\| < 1$), we infer that $\operatorname{Re} F(z) > 0$ for $z \in \Delta$.

By Proposition 3, the inequalities (20) and (21) hold for $z \in \Delta$. Using (22) and

$$\begin{aligned}\operatorname{Im} F(z) &= \frac{1}{2i}(I - \bar{z}A^*)^{-1}\{(I - \bar{z}A^*)(I + zA)\\ &\qquad - (I + \bar{z}A^*)(I - zA)\}(I - zA)^{-1}\\ &= 2(I - \bar{z}A^*)^{-1}(\operatorname{Im} zA)(I - zA)^{-1}, \qquad (23)\end{aligned}$$

we derive from (20) and (21)

$$\frac{1-|z|}{1+|z|}I \leqslant (I-\bar{z}A^*)^{-1}(I-|z|^2A^*A)(I-zA)^{-1} \leqslant \frac{1+|z|}{1-|z|}I \tag{24}$$

and

$$-\frac{|z|}{1-|z|^2}I \leqslant (I-\bar{z}A^*)^{-1}(\operatorname{Im} zA)(I-zA)^{-1} \leqslant \frac{|z|}{1-|z|^2}I, \tag{25}$$

which hold for all $z \in \Delta$.

Now, if $\|A\| < |z| < 1$, then (24) and (25) remain true when A is replaced by $z^{-1}A$. Therefore

$$\frac{1-|z|}{1+|z|}I \leqslant (I-A^*)^{-1}(I-A^*A)(I-A)^{-1} \leqslant \frac{1+|z|}{1-|z|}I \tag{26}$$

and

$$-\frac{|z|}{1-|z|^2}I \leqslant (I-A^*)^{-1}(\operatorname{Im} A)(I-A)^{-1} \leqslant \frac{|z|}{1-|z|^2}I \tag{27}$$

hold for $\|A\| < |z| < 1$. Then (3) and (5) are obtained by letting $|z| \downarrow \|A\|$ in (26) and (27). ■

This second proof of the inequalities (3) and (5) is more enlightening than the first one. It is interesting to connect Propositions 1 and 2 with the classical Harnack's inequalities in complex analysis. In the next section, we shall see an alternative connection with Harnack's inequalities via functional calculus.

4. HARNACK'S INEQUALITIES IN FUNCTIONAL CALCULUS

For a complex-valued function f analytic on the open unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and for an operator A on $\mathscr{H}$ with $\|A\| < 1$, we denote by $f(A)$ the operator on $\mathscr{H}$ defined, as in functional calculus, by the usual Riesz-

Dunford contour integral [2, p. 568]

$$f(A)=\frac{1}{2\pi i}\int_{\Gamma} f(z)(zI-A)^{-1}\,dz, \tag{28}$$

where Γ is a positively oriented simple closed rectifiable contour lying in Δ and encircling the spectrum of A.

The following is an analogue of the classical Harnack's inequalities in functional calculus:

PROPOSITION 4. *Let f be a complex-valued function analytic on the open unit disk Δ such that $f(0)=1$ and $\operatorname{Re} f(z)>0$ for all $z\in\Delta$. Then for any operator A on $\mathcal{H}$ with $\|A\|<1$, we have*

$$\frac{1-\|A\|}{1+\|A\|}I \leqslant \operatorname{Re} f(A) \leqslant \frac{1+\|A\|}{1-\|A\|}I, \tag{29}$$

$$-\frac{2\|A\|}{1-\|A\|^2}I \leqslant \operatorname{Im} f(A) \leqslant \frac{2\|A\|}{1-\|A\|^2}I. \tag{30}$$

Proof. See [5]. ■

Let us consider the function $f(z)=(1+z)/(1-z)$, which verifies the hypothesis of Proposition 4. For an operator A with $\|A\|<1$, we have $f(A)=(I+A)(I-A)^{-1}$ and therefore

$$\operatorname{Re} f(A)=(I-A^*)^{-1}(I-A^*A)(I-A)^{-1},$$

$$\operatorname{Im} f(A)=2(I-A^*)^{-1}(\operatorname{Im} A)(I-A)^{-1}.$$

Then we see that the inequalities (3) and (5) are precisely the special case $f(z)=(1+z)(1-z)^{-1}$ of (29) and (30) respectively.

Consider now an operator B with $\operatorname{Im} B>0$, and let A be given by (8). Then $\|A\|<1$ and

$$B=i(I+A)(I-A)^{-1}.$$

So for $f(z)=(1+z)(1-z)^{-1}$, we have $f(A)=-iB$. Then

$$\operatorname{Im} B=\operatorname{Re}(-iB)=\operatorname{Re} f(A),$$

$$\operatorname{Re} B=\operatorname{Re}[if(A)]=-\operatorname{Im} f(A).$$

These relations show that (11) and (10) are also special cases of (29) and (30) respectively.

To conclude this paper, we mention that, besides Harnack's inequalities, several related classical results in complex analysis can be extended to operators on a Hilbert space (see [1, 3, 4, 6, 7]).

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