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An alternative proof of the Barker, Berman, Plemmons (BBP) result on diagonal stability and extensions

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Abstract

We revisit the theorem of Barker, Berman and Plemmons on the existence of a diagonal quadratic Lyapunov function for a stable linear time-invariant (LTI) dynamical system [G.P. Barker, A. Berman, R.J. Plemmons, Positive diagonal solutions to the Lyapunov equations, Linear and Multilinear Algebra 5(3) (1978) 249–256]. We use recently derived results to provide an alternative proof of this result and to derive extensions.

Keywords: Diagonal stability; Common quadratic Lyapunov functions; Copositive diagonal Lyapunov functions

1. Introduction

The stability theory for linear time-invariant (LTI) differential equations of the form

$$\dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n} \tag{1}$$

is well-established and several equivalent conditions for the asymptotic stability of such systems have been derived. In particular, it is well known that the asymptotic stability of (1) is equivalent to the spectrum of the matrix A being contained within the open left half of the complex plane. Such matrices are referred to as *stable* or *Hurwitz* matrices. This in turn is equivalent to the existence of a positive definite solution $P = P^{\mathrm{T}} > 0$ of the *Lyapunov matrix inequality*

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$$A^{\mathrm{T}}P + PA < 0. \tag{2}$$

For any solution $P = P^{\mathrm{T}} > 0$ of (2), the function $V(x) = x^{\mathrm{T}}Px$ on $\mathbb{R}^n$ is a *quadratic Lyapunov function* (QLF) for the system (1), which is said to be *quadratically stable*.

A question that has attracted a great deal of attention in the past concerns what additional conditions on A are required so that there exists a diagonal matrix D satisfying (2) [10,11,9,1,7]. If such a matrix exists, the system is said to be *diagonally stable*, and (by an abuse of the language) the associated Lyapunov function $V(x) = x^{\mathrm{T}}Dx$ is called a diagonal Lyapunov function. Previous work on diagonal stability has followed several lines of inquiry, the main thrust of which is documented in [10,11,9,1,7]. Perhaps the best known complete solution to this problem was given by Barker, Berman and Plemmons in 1978 [1]; here, it was shown that $A \in \mathbb{R}^{n\times n}$ is diagonally stable if and only if AX has at least one negative diagonal entry for all non-zero positive semi-definite $X \in \mathbb{R}^{n\times n}$. Following the publication of this paper, a number of authors attempted to derive algebraic conditions that could be used to verify its main result in practice. Noteworthy efforts in this direction have appeared in the work of Kraaijvanger [9], Wanat [15] and others. The primary contribution of the present paper is to describe a simple proof of the original result in [1], as well as opening the way to a number of extensions of this basic result. In particular, the approach given here allows several problems related to the diagonal stability problem to be treated in a similar way. Some initial results obtained using this perspective are presented in later sections.

2. Notation and preliminary results

We shall use the following notation. The vector $x \in \mathbb{R}^n$ is said to be positive if all its entries are positive. This is denoted $x \succ 0$. If the entries of x are zero or positive then x is said to be non-negative. This is denoted $x \succeq 0$. The Hadamard (entry-wise) product of x and y is denoted $x \circ y$. Note that if $x \circ y \succeq 0$ then x and y are in the same closed orthant. Also, for $i = 1, \ldots, n$, e_i denotes the column vector in $\mathbb{R}^n$, whose ith entry is 1 with all other entries zero. Throughout the paper, for symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, we shall use the notation $\langle A, B\rangle = \mathrm{Trace}(AB)$ to denote the usual inner product on the space of symmetric matrices. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is positive semi-definite if $x^{\mathrm{T}}Px \geqslant 0$ for all $x \in \mathbb{R}^n$, and is positive definite if $x^{\mathrm{T}}Px > 0$ for all $x \neq 0$ in $\mathbb{R}^n$. We shall use the notations $P \geqslant 0$, $P > 0$ to denote positive semi-definiteness and positive definiteness respectively.

Given m LTI systems, $\dot{x} = A_i x$, $i \in \{1, \ldots, m\}$, if a positive definite matrix $P = P^{\mathrm{T}} > 0$ exists such that

$$A_i^{\mathrm{T}}P + PA_i = -Q_i < 0, \quad i \in \{1, \ldots, m\}, \tag{3}$$

then $V(x) = x^{\mathrm{T}}Px$ defines a *common quadratic Lyapunov function* (CQLF), for the m LTI systems, and P is said to be a *common Lyapunov solution* for A_i, $i \in \{1, \ldots, m\}$. If one or more of the matrices Q_i, $i \in \{1, \ldots, m\}$ are positive semi-definite, then $V(x)$ is said to be a *weak CQLF* and P is called a *weak common Lyapunov solution*.

The basic idea that we exploit in this paper is based upon the following recently observed fact [14]. Let $B_i = -e_i e_i^{\mathrm{T}}$, $i = 1, \ldots, n$ be the diagonal matrices in $\mathbb{R}^{n\times n}$, whose ith diagonal element is -1, with all other entries zero. A matrix $A \in \mathbb{R}^{n\times n}$ is diagonally stable, if and only if $A, B_1, \ldots, B_n$ admit a weak common Lyapunov solution. While this result is easily deduced, it is nevertheless important for a number of reasons. Firstly, it establishes a direct link between diagonal stability and the concept of strict positive realness, which plays a central role in control theory; this line of research has been pursued in [14]. A scalar-valued rational function $H(s)$ of a complex variable s is said to be positive real (PR) if and only if $H(s)$ is real for real s and H maps the open right half plane into the closed right half plane. If there is some $\epsilon > 0$ such that $H(s-\epsilon)$ is PR, then $H(s)$ is said to be strictly positive real (SPR). Also, a matrix-valued function $H(s)$ is said to be PR if $x^*H(s)x$ is PR for every complex vector x, and is said to be SPR if there is some $\epsilon > 0$ such that $x^*H(s-\epsilon)x$ is PR for every complex vector x.

3. Main result

In Theorem 3.1, we present an elementary proof of the main result of [1]. First of all, we state the following lemma, which is a relatively straightforward extension of results presented in [6] for sets of Hurwitz matrices. The argument we present here is a simple adaptation of that presented in [8] for the case of two Hurwitz matrices.

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and let $M_i \in \mathbb{R}^{n\times n}$, for $i = 1,\dots,k$. Then there exists a positive definite P satisfying*

$$A^{\mathrm{T}}P + PA < 0, \quad M_i^{\mathrm{T}}P + PM_i \leqslant 0 \quad i = 1,\dots,k \tag{4}$$

if and only if there do not exist positive semi-definite matrices $X, Y_1,\dots,Y_k$, with $X \neq 0$ such that

$$AX + XA^{\mathrm{T}} + \sum_{i=1}^{k}(M_iY_i + Y_iM_i^{\mathrm{T}}) = 0. \tag{5}$$

Proof. Recall that $\langle A, B\rangle = \mathrm{Trace}(AB)$ denotes the usual inner product on the space of symmetric matrices in $\mathbb{R}^{n\times n}$. Consider the following four statements.

(i) There exists a positive definite solution to the set of inequalities (4).
(ii) There exists a symmetric matrix H such that $\langle H, AX + XA^{\mathrm{T}}\rangle < 0$ for all non-zero $X = X^{\mathrm{T}} \geqslant 0$ and $\langle H, M_iY_i + Y_iM_i^{\mathrm{T}}\rangle \leqslant 0$ for all non-zero $Y_i = Y_i^{\mathrm{T}} \geqslant 0$, and all $i = 1,\dots,k$.
(iii) There exist no positive semi-definite matrices $X, Y_1,\dots,Y_k$ with $X \neq 0$ satisfying (5).
(iv) The two pointed, convex cones

$$\mathscr{C}_A = \{AX + XA^{\mathrm{T}} : X = X^{\mathrm{T}} \geqslant 0\} \tag{6}$$

and

$$\mathscr{C}_B = -\left\{\sum_{i=1}^{k} M_iY_i + Y_iM_i^{\mathrm{T}} : Y_i = Y_i^{\mathrm{T}} \geqslant 0 \text{ for } i = 1,\dots,k\right\} \tag{7}$$

intersect only at the origin.

We wish to prove the equivalence of (i) and (iii). First of all, we shall show that (i) and (ii) are equivalent.

To see this note that, as A is Hurwitz, there exists a positive definite P satisfying (4) if and only if there exists a symmetric H satisfying (4). Also

$$A^{\mathrm{T}}H + HA < 0, \quad M_i^{\mathrm{T}}H + HM_i \leqslant 0 \quad 1 \leqslant i \leqslant k$$

if and only if for any non-zero positive semi-definite matrices $X, Y_1,\dots,Y_k$

$$\langle A^{\mathrm{T}}H + HA, X\rangle = \langle H, AX + XA^{\mathrm{T}}\rangle < 0,$$

and

$$\langle M_i^{\mathrm{T}} H + HM_i, Y_i\rangle = \langle H, M_i Y_i + Y_i M_i^{\mathrm{T}}\rangle \leqslant 0, \quad 1 \leqslant i \leqslant k.$$

The equivalence of statements (iii) and (iv) is immediate. It is also immediate that (ii) implies (iii), therefore the proof of the lemma will be completed by showing that (iv) implies (ii), since this will demonstrate that (i) and (iii) are equivalent. To this end, define the truncated cone $\widetilde{\mathscr{C}}_A$

$$\widetilde{\mathscr{C}}_A = \{AX + XA^{\mathrm{T}} : X = X^{\mathrm{T}} \geqslant 0,\ \mathrm{Trace}(X) = 1\}. \qquad \square \tag{8}$$

Theorem 2.39 from [13] establishes that two disjoint convex sets C_1, C_2 are strongly separated if both sets are closed and one of them is bounded. Note that $\widetilde{\mathscr{C}}_A$ is convex, closed and bounded, and disjoint from the convex closed set $\mathscr{C}_B$. Hence $\widetilde{\mathscr{C}}_A$ and $\mathscr{C}_B$ are strongly separated, meaning that there is a symmetric matrix H and $\alpha_1 < \alpha_2$ so that $\widetilde{\mathscr{C}}_A$ is contained in $\{M = M^{\mathrm{T}} : \langle M, H\rangle \leqslant \alpha_1\}$ and $\mathscr{C}_B$ is contained in $\{M = M^{\mathrm{T}} : \langle M, H\rangle \geqslant \alpha_2\}$. Since $\mathscr{C}_B$ is a cone, if $\langle M, H\rangle < 0$ for some $M \in \mathscr{C}_B$, then $\langle kM, H\rangle < \alpha_2$ for sufficiently large k which is a contradiction. It follows that $\langle M, H\rangle \geqslant 0$ for all $M \in \mathscr{C}_B$ which is the second statement in (ii). Also, as $0 \in \mathscr{C}_B$, it follows that $\alpha_2 \leqslant 0$ and hence that $\alpha_1 < 0$. Thus $\langle M, H\rangle < 0$ for all $M \in \widetilde{\mathscr{C}}_A$. Furthermore if $M \in \mathscr{C}_A$ then there is $k > 0$ such that $kM \in \widetilde{\mathscr{C}}_A$, and therefore also $\langle M, H\rangle < 0$ for all $M \in \mathscr{C}_A$. The statement (ii) now follows and hence the lemma.

We are now ready to present the main result of this section.

Theorem 3.1. *The Hurwitz matrix $A \in \mathbb{R}^{n\times n}$ is diagonally stable if and only if AX has a negative diagonal entry for every non-zero $X = X^{\mathrm{T}} \geqslant 0$, $X \in \mathbb{R}^{n\times n}$.*

Proof. (a) *Necessity*: Suppose that A is diagonally stable, with the diagonal Lyapunov solution D. It follows immediately that, for any non-zero $X = X^{\mathrm{T}} \geqslant 0$, we have $\mathrm{Trace}(DAX) < 0$. Therefore AX must have a negative diagonal entry for all non-zero positive semi-definite X.

(b) *Sufficiency*: The main idea of the proof of sufficiency is to re-write the condition that A is diagonally stable as the condition that the matrices $A, B_1, \ldots, B_{n-1}$ have a common quadratic Lyapunov function (CQLF), where $B_i = -e_i e_i^{\mathrm{T}}$ for $1 \leqslant i \leqslant n-1$, as was observed and proved in [14]. Now, suppose that A is not diagonally stable. It follows from Lemma 3.1 that one can find real symmetric positive semi-definite matrices $X, Y_1, \ldots, Y_{n-1}$, with $X \neq 0$, so that

$$AX + XA^{\mathrm{T}} + \sum_{i=1}^{n-1}(B_i Y_i + Y_i B_i) = 0. \tag{9}$$

It follows immediately from (9) that the (n, n) entry of $AX + XA^{\mathrm{T}}$ must be zero. Note that $B_i Y_i + Y_i B_i = 0$ cannot be true for all i as this would imply that $AX + XA^{\mathrm{T}} = 0$, which contradicts the assumptions that A is Hurwitz and $X \neq 0$. Note also that when $B_i Y_i + Y_i B_i$ is non-zero, the diagonal entry of this matrix is strictly negative. Hence, if there is no diagonal Lyapunov solution for A, then all the diagonal entries of AX are non-negative. Therefore if AX has at least one negative entry for every positive semi-definite X, A must be diagonally stable. This completes the proof. $\square$

4. Extensions of BBP result

The main result of the previous section can be readily adapted in a number of ways using very similar arguments to those given above. As an example of this, we present a result in this section which provides necessary and sufficient conditions for so-called *copositive Lyapunov function* existence. In the following section, we describe another application of the methods here to the class of positive dynamical systems.

4.1. Copositive diagonal quadratic Lyapunov functions

We next consider a problem motivated by the stability of linear systems whose trajectories are confined to the positive orthant of $\mathbb{R}^n$; so-called *positive systems*. An LTI system $\Sigma_A : \dot{x} = Ax$ is positive if A is a *Metzler* matrix, meaning that the off-diagonal entries of A are all non-negative [4]. For systems of this class, the existence of *copositive* Lyapunov functions is of interest. For such functions, we only require that the usual Lyapunov conditions are satisfied for state-values in the non-negative orthant. We shall give below a necessary and sufficient condition for the existence of a diagonal copositive Lyapunov solution for a Hurwitz matrix in $\mathbb{R}^{n\times n}$. Formally, we are interested in the existence of a diagonal matrix $D > 0$ such that $x^{\mathrm{T}}(A^{\mathrm{T}}D + DA)x < 0$ for all non-zero $x \succeq 0$.

Before stating the following result, we recall the definition of a *completely positive* matrix [3].

Definition 4.1. A matrix $X \in \mathbb{R}^{n\times n}$ is said to be completely positive if there exists some positive integer p and a non-negative matrix Y in $\mathbb{R}^{n\times p}$ such that $X = YY^{\mathrm{T}}$.

Note that any completely positive matrix is both non-negative and positive semi-definite (such matrices are said to be *doubly non-negative*), and that the set of all completely positive matrices in $\mathbb{R}^{n\times n}$ is a convex cone. Note also that if $X = YY^{\mathrm{T}}$ with $Y = (y_1, \ldots, y_p)$ where $y_i \in \mathbb{R}^n$ for $i = 1, \ldots, p$, then $X = \sum_{i=1}^{p} y_i y_i^{\mathrm{T}}$. It follows immediately from Caratheodory's Theorem [12] that any completely positive matrix can be written as the sum of at most $N = \frac{n(n+1)}{2} + 1$ matrices of the form $xx^{\mathrm{T}},\ x \succeq 0$.

Theorem 4.1. *Let A in $\mathbb{R}^{n\times n}$ be Hurwitz. There exists a positive definite, diagonal matrix P satisfying*

$$x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x < 0 \quad \text{for all non-zero } x \succeq 0 \text{ in } \mathbb{R}^n, \tag{10}$$

if and only if AX has a negative diagonal entry for every non-zero completely positive matrix X in $\mathbb{R}^{n\times n}$.

Proof. Let $N = \frac{n(n+1)}{2} + 1$. It is straightforward to adapt the proof of Lemma 3.1 to show that there is some diagonal P satisfying (10) if and only if there do not exist positive semi-definite matrices $Y_1, \ldots, Y_n$, $X = \sum_{k=1}^{N} x_k x_k^{\mathrm{T}}$, where $x_k \succeq 0$ for $k = 1, \ldots, N$ and $X \neq 0$, satisfying (5). Combining this observation with the remarks made after Definition 4.1, the present result follows in the same manner as Theorem 3.1.

Note that the diagonal entries of $A(x_1 x_1^{\mathrm{T}} + \cdots + x_N x_N^{\mathrm{T}})$ are simply $Ax_1 \circ x_1 + \cdots + Ax_N \circ x_N$. By exploiting the linearity of the Hadamard product, and by noting that any vector y in the positive orthant is given by $y = De$ for some diagonal $D \geqslant 0$, where e is the vector of all ones, the previous theorem actually says that A has a copositive diagonal Lyapunov solution, if and only if, e and $\left(\sum_{k=1}^{N} D_k A D_k\right) e$ are never in the same orthant, for any diagonal matrices, $D_1, \ldots, D_N$, not all zero, with non-negative diagonal entries. □

5. Example – Positive systems

We shall now apply the result of Theorem 4.1 to linear systems whose trajectories are confined to the positive orthant; that is, we wish to consider the stability of $\dot{x} = Ax$ where A is a Metzler, Hurwitz matrix (i.e. $-A$ is an M-matrix) [5]. This problem has been well studied. Here, we shall use Theorem 4.1 to demonstrate the known result that a diagonal copositive Lyapunov function always exists for such a system.

Formally, let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz and let $N = \frac{n(n+1)}{2} + 1$. We shall now show that there exists a $D > 0$ such that $x^{\mathrm{T}}(A^{\mathrm{T}}D + DA)x < 0$ for all $x \succeq 0$, $x \neq 0$. Theorem 4.1 implies that such a D exists if and only if e, the vector of all ones, and $\left(\sum_{k=1}^{N} D_k A D_k\right) e$ are never in the same orthant, for all diagonal matrices $D_1, \ldots, D_N$ with non-negative diagonal entries.

First of all, recall the following basic fact concerning Metzler matrices.

If $M \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz, then for every non-zero $x \in \mathbb{R}^n$, there is some $i \in \{1, \ldots, n\}$ such that $x_i(Mx)_i < 0$ (M reverses the sign of some entry of x) [2].

If we write $d_i^{(k)}$ for the ith diagonal entry of the matrix D_k, then the (i, j) entry of $\sum_{k=1}^{N} D_k A D_k$, is

$$a_{ij}\left(\sum_{k=1}^{N} d_i^{(k)} d_j^{(k)}\right),$$

for $1 \leqslant i, j \leqslant n$. To simplify notation, define

$$\gamma_{ij} = \sum_{k=1}^{N} d_i^{(k)} d_j^{(k)} \quad \text{for } i \neq j,$$

$$\gamma_i = \left(\sum_{k=1}^{N} (d_i^{(k)})2\right)^{1/2}, \tag{11}$$

for $1 \leqslant i, j \leqslant n$. It follows from the Cauchy–Schwartz inequality that $\gamma_{ij} \leqslant \gamma_i\gamma_j$ for all $i \neq j$. Thus, if we define $D = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, then

$$\left(\sum_{k=1}^{N} D_k A D_k\right) \preceq DAD. \tag{12}$$

But, as A is Metzler and Hurwitz and $D \neq 0$, it follows that some entry of $(DAD)e$ must be negative. It follows from (12) that some entry of $\left(\sum_k D_k A D_k\right) e$ must also be negative and hence A has a copositive diagonal Lyapunov function by Theorem 4.1.

6. Concluding remarks

In the present paper we have presented a novel approach to the problem of diagonal matrix stability by recasting it as an existence question for common quadratic Lyapunov functions (CQLFs). This approach has led directly to a new proof of the classical result of Barker, Berman and Plemmons (BBP) on the existence of diagonal solutions to the Lyapunov inequality, as well as to some novel extensions of the BBP result. While progress has been made recently on questions pertaining to diagonal stability, there are still numerous fundamental issues unresolved. In particular, easily verifiable algebraic conditions for diagonal stability are only known for very low dimensions and the precise nature of the relationship between diagonal stability and other strong concepts of matrix stability such as D-stability is still unclear. It is hoped that the approach described here, combined with the considerable literature on the CQLF existence problem, may lead to further progress in the area of diagonal stability in the future.

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