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A note on the colour class of a self-complementary graph

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Abstract

We prove that any self-complementary graph G contains no proper overfull subgraph H such that $\Delta(H)=\Delta(G)$ and $|V(H)|<|V(G)|$. Moreover, a self-complementary graph G is overfull if, and only if, it is regular. This is a support for the following conjecture: *a self-complementary graph G is Class 2 if, and only if, G is regular.*

All graphs we consider are simple. $V(G)$ and $E(G)$ denote the vertex and edge set of a graph G, $n=|V(G)|$ is the order and $e=|E(G)|$ is the size of G. The degree of a vertex v in G is denoted by $d(v,G)$. For a subset X of V we denote by G_X the subgraph of G induced by the set X.

A graph $G=(V,E)$ is said to be *self-complementary* if it is isomorphic with its complement (cf. [6]) $\bar{G}=(V,\bar{E})$. Any permutation $\sigma: V\to V$ such that $xy\in E$ if, and only if, $\sigma(x)\sigma(y)\notin E$ is called *self-complementing permutation of G*.

Let $\chi'(G)$ be the least number of colours sufficient for colouring properly the edges of a graph G. By a very well-known theorem of Vizing [7] there are two possibilities: either $\chi'(G)=\Delta(G)$ (and G is called *Class 1*) or $\chi'(G)=\Delta(G)+1$ (and G is called *Class 2*), where $\Delta(G)$ is the maximum vertex degree in G. Holyer [3] proved that the problem of determining the class of a graph is NP-complete.

A graph G of order n is said to be *overfull* if $|E|>\lfloor n/2\rfloor\Delta(G)$. Since in any colour class we may have at the most $\lfloor n/2\rfloor$ edges, it is clear that the overfull graphs are Class 2. The following conjecture has been stated in [8].

Conjecture 1 (Wojda and Zwonek [8]). A self-complementary graph G is Class 2 if, and only if, G is regular.

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If a self-complementary graph G has a complementing permutation that is a cycle, then we say that G is *s.c.-cyclic* (see [6]). The following result has been proved in [8].

Theorem 1 (Wojda and Zwonek [8]). *If G is an s.c.-cyclic self-complementary graph then G is Class* 1.

Niessen noticed [5] that, since $\Delta(G) \geqslant (n-1)/2$ for every self-complementary graph G of order n, Conjecture 1 is related to the following conjecture made by Chetwynd and Hilton.

Conjecture 2 (Chetwynd and Hilton [1] and Hilton [2]). Let G be a graph of order n with the maximum vertex degree $\Delta(G) > n/3$. Then G is Class 2 if, and only if, G contains an overfull subgraph H with the maximum vertex degree $\Delta(H) = \Delta(G)$.

The following proposition has been given in [8] without a proof, we prove it here for completeness.

Proposition 2 (Wojda and Zwonek [8]). *A self-complementary graph G is overfull if, and only if, G is regular.*

Proof. If a self-complementary graph G of order n is regular then $\Delta(G) = \delta(G) = (n-1)/2$ and $e(G) = n(n-1)/4 > (n-1)^2/4 = \lfloor n/2 \rfloor (n-1)/2$, hence G is overfull.

Let us suppose now that G is overfull. Then $e(G) = n(n-1)/4 > \lfloor n/2 \rfloor \Delta(G) = ((n-1)/2)\Delta(G)$ and therefore, $\Delta(G) < n/2$. Since G is self-complementary, we have $\Delta(G) + \delta(G) = n-1$ and, by consequence, $\Delta(G) = \delta(G) = (n-1)/2$. □

In this paper we prove.

Theorem 3. *A self-complementary graph G does not contain any overfull subgraph H with $\Delta(H) = \Delta(G)$ and $|V(H)| < |V(G)|$.*

Since for any non trivial self-complementary graph G of order n we have $n \geqslant 4$, $\Delta(G) + \delta(G) = n-1$ and $e(G) = n(n-1)/4$, Theorem 3 is a consequence of the following lemma.

Lemma 4. *Let $G = (V, E)$ be a graph of order $n \geqslant 4$, such that $e(G) \leqslant n(n-1)/4$ and $\delta(G) \geqslant n - \Delta(G) - 1$. Then G does not contain any overfull subgraph H such that $\Delta(H) = \Delta(G)$ and $|V(H)| < n$.*

Proof. Suppose that H is an overfull subgraph of a self-complementary graph G of order n such that $\Delta(H) = \Delta(G)$ and $|V(H)| < n$. It is clear that the subgraph F of G induced in G by the vertex set of H is also an overfull subgraph of G and $\Delta(F) = \Delta(G)$, so without loss of generality we may assume that H is an induced subgraph of G, say $H=G_{V-W}$, with $W\subset V$ and $|W|=l>0$.

Since H is overfull with $\Delta=\Delta(H)=\Delta(G)$ we have $e(H)>\Delta\lfloor(n-l)/2\rfloor$. It is a very well known fact that every overfull graph has odd order, thus $\lfloor(n-l)/2\rfloor=(n-l-1)/2$ and

$$e(H)>\Delta\frac{n-l-1}{2}. \tag{1}$$

For a vertex w from W and a subset X of the vertex set V, let us denote by $e(w,X)$ the number of edges which have w for one end vertex, while the second end vertex is in the subset X. Thus the number p of edges of G which are not the edges of H satisfies

$$\begin{aligned} p &= \sum_{w\in W} e(w,V-W)+\frac{1}{2}\sum_{w\in W} e(w,W) \\ &= \sum_{w\in W}(e(w,V-W)+e(w,W))-\frac{1}{2}\sum_{w\in W} e(w,W) \\ &= \sum_{w\in W} d(w,G)-\frac{1}{2}\sum_{w\in W} d(w,G_W)\geqslant l(n-\Delta-1)-\frac{1}{2}l(l-1) \\ &= \frac{l}{2}(2n-2\Delta-l-1) \end{aligned}$$

and

$$e(H)\leqslant\frac{n(n-1)}{4}-p\leqslant\frac{n(n-1)}{4}-\frac{1}{2}(2n-2\Delta-l-1). \tag{2}$$

Since $\Delta(H)=\Delta$ we have $n-l\geqslant\Delta+1$, thus

$$l\leqslant n-\Delta-1. \tag{3}$$

We conclude from (1) and (2) that

$$\Delta\frac{n-l-1}{2}<e(H)\leqslant\frac{n(n-1)}{4}-\frac{l}{2}(2n-2\Delta-l-1),$$

hence

$$f(l)=2l^2+l(6\Delta-4n+2)+(n-1)(n-2\Delta)>0.$$

Since $1\leqslant l\leqslant n-\Delta-1$, $f(l)>0$ implies $\max\{f(l)\colon 1\leqslant l\leqslant n-\Delta-1\}=\max\{f(1),f(n-\Delta-1)\}>0$. Therefore, $f(1)=(n-4)(n-2\Delta-1)>0$ or $f(n-\Delta-1)=-(n-2\Delta)(n-2\Delta-1)>0$. In each case one gets very easily a contradiction, and the lemma follows. □

For further reading

The following reference is also of interest to the reader: [4].

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