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Analysis of a system of fractional differential equations

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Abstract

We prove existence and uniqueness theorems for the initial value problem for the system of fractional differential equations $D^\alpha[\bar{x}(t) - \bar{x}(0)] = A\bar{x}(t)$, $\bar{x}(0) = \bar{x}_0$, where D^α denotes standard Riemann–Liouville fractional derivative, $0 < \alpha < 1$, $\bar{x}(t) = [x_1(t), \ldots, x_n(t)]^t$ and A is a square matrix. The unique solution to this initial value problem turns out to be $E_\alpha(t^\alpha A)\bar{x}_0$, where E_α denotes the Mittag–Leffler function generalized for matrix arguments. Further we analyze the system $D^\alpha[\bar{x}(t) - \bar{x}(0)] = \bar{f}(t, \bar{x})$, $\bar{x}(0) = \bar{x}_0$, $0 < \alpha < 1$, and investigate dependence of the solutions on the initial conditions.

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1. Introduction

The existence and uniqueness of solutions of initial value problems for fractional order differential equations have been studied in the literature [2,3,7,9]. In this paper we present analysis of the system of fractional differential equations

$$D^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = A\bar{x}(t), \quad \bar{x}(0) = \bar{x}_0, \ 0 < \alpha < 1,$$

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where D^α denotes Riemann–Liouville derivative operator and A denotes a square matrix having real entries. The unique solution to this initial value problem turns out to be $E_\alpha(t^\alpha A)\bar{x}_0$, where E_α denotes the Mittag–Leffler function generalized for matrices. Further we discuss the initial value problem for nonautonomous nonlinear system

$$D^\alpha\big[\bar{x}(t) - \bar{x}_0\big] = \bar{f}(t, \bar{x}), \quad \bar{x}(0) = \bar{x}_0,\ 0 < \alpha < 1,$$

where $\bar{f} : W(\subset \mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n$. It is shown that for $\bar{f}$ bounded, continuous and Lipschitz in the second variable, there exists unique solution (locally). The dependence of solutions on initial conditions has also been discussed.

2. Preliminaries and notations

Riemann–Liouville derivative and integral are defined below [7,9].

Definition 2.1. Let f be a continuous function defined on $[a, b]$, and $n - 1 \leqslant \alpha < n$, $n \in \mathbb{N}$. Then the expression

$$D^\alpha_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt, \quad x > a, \tag{1}$$

is called left-sided fractional derivatives of order α.

Definition 2.2. Let f be a continuous function defined on $[a, b]$, and $\alpha > 0$. Then the expression

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{-\alpha+1}}\, dt, \quad x > a, \tag{2}$$

is called as left-sided fractional integral of order α.

Without loss of generality we will work with $D^\alpha_{a+} f(x)$, $I^\alpha_{a+} f(x)$ and unless mentioned otherwise, we denote D^α_{a+} by $D^\alpha_a f(x)$ and $I^\alpha_{a+} f(x)$ by $I^\alpha_a f(x)$, respectively. Also $D^\alpha f(x)$ and $I^\alpha f(x)$ refer to $D^\alpha_{0+} f(x)$ and $I^\alpha_{0+} f(x)$.

Theorem 2.1 [5]. *Let $T \in L(\mathbb{R}^n)$, have real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$. Then there exists a basis of $\mathbb{R}^n$ in which the matrix representation of T assumes Jordan form, i.e., the matrix of T is made of diagonal blocks of the form* $\operatorname{diag}[C_1, C_2, \ldots, C_r]$, *where each C_i consists of diagonal blocks of the form*

$$\begin{bmatrix} \lambda_i & 0 & \ldots & 0 & 0 \\ 1 & \lambda_i & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \lambda_i \end{bmatrix}.$$

Theorem 2.2 [5]. *Let $T \in L(\mathbb{R}^n)$ have nonreal eigenvalues $\mu_j = a_j + ib_j$, $j = 1, \dots, r$, with multiplicity. Then there exists a basis of $\mathbb{R}^n$, where T has matrix form* $\operatorname{diag}[\hat{C}_1, \hat{C}_2, \dots, \hat{C}_r]$, *where $\hat{C}_i$ consists of the diagonal blocks of the type*

$$\begin{bmatrix} D & 0 & \dots & 0 & 0 \\ I_2 & D & \dots & 0 & 0 \\ 0 & I_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_2 & D \end{bmatrix}, \quad D = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}, \ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 2.3 [5, Theorem 2, p. 129]. *Let $T \in L(\mathbb{R}^n)$. Then $\mathbb{R}^n$ has a basis giving T a matrix representation composed of diagonal blocks of type C_i and/or matrices $\hat{C}_i$, where C_i and $\hat{C}_i$ are as defined in the preceding theorems.*

3. Analysis of a system of fractional differential equations

In the present paper using methods of linear algebra [4,6] we study the following system of fractional differential equations:

$$D^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = A\bar{x}(t), \quad \bar{x}(0) = \bar{x}_0, \ t \in [0, \chi^*], \ \chi^* > 0, \ 0 < \alpha < 1, \tag{3}$$

where $\bar{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^t$, A is $n \times n$ real matrix and

$$D^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = \big(D^\alpha\big[x_1(t) - x_1(0)\big], D^\alpha\big[x_2(t) - x_2(0)\big], \dots, D^\alpha\big[x_n(t) - x_n(0)\big]\big)^t.$$

We now proceed to solve the initial value problem (3) in various cases. In Theorem 3.1, we discuss the case when matrix A has real and distinct eigenvalues. Theorems 3.2 and 3.3 deal with the cases of complex eigenvalues and repeated eigenvalues, respectively. Using these theorems, the most general case has been treated where matrix A can have any type of eigenvalues (cf. Theorem 3.4)

Theorem 3.1. *Let $A \in L(\mathbb{R}^n)$ have distinct real eigenvalues. Then given $\bar{x}_0 \in \mathbb{R}^n$, $\exists \chi > 0$, such that the system* (3) *has unique solution defined on* $[0, \chi]$.

Proof. Suppose $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n\}$ are the (distinct) eigenvectors corresponding to the distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$; so that $A\bar{g}_j = \lambda_j \bar{g}_j$, $j = 1, 2, \dots, n$. If all the eigenvalues are real and distinct then $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n\}$ forms a basis of $\mathbb{R}^n$. Let T be the operator on $\mathbb{R}^n$ having A as the matrix representation in the standard basis. Let B be the matrix representation of T in $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n\}$. Then $B = \operatorname{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$ and $B = QAQ^{-1}$, where $Q^{-1} = P^t$, $P = [p_{ij}]$ and $\bar{g}_i = \sum_j p_{ij}\bar{e}_j$, where $\bar{e}_j$, $j = 1, \dots, n$, denotes the standard basis. Define $\bar{y} = Q\bar{x}$,

$$D^\alpha\big[\bar{y}(t) - \bar{y}(0)\big] = QD^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = QA\bar{x}(t) = QAQ^{-1}\bar{y}(t) = B\bar{y}(t),$$

where $\bar{y}(0) = Q\bar{x}(0) = \bar{y}_0$. Since B is diagonal,

$$D^\alpha\big[y_i(t) - y_i(0)\big] = \lambda_i y_i(t), \quad y_i(0) = (\bar{y}_0)_i, \ i = 1, 2, \dots, n.$$

Here $f_i(t, y_i): \Omega_i \to \mathbb{R}$, where $\Omega_i = [0, \chi^*] \times [(\bar{y}_0)_i - l_i, (\bar{y}_0)_i + l_i]$ for $l_i > 0$. Note $f_i(t, y_i) = \lambda_i y_i$ which is Lipschitz in the second variable. Hence in view of Theorems 2.1 and 2.2 in [3], there exists unique solution $y_i : [0, \chi_i] \to \mathbb{R}$ solving

$$D^\alpha\big[y_i(t) - y_i(0)\big] = \lambda_i y_i(t), \quad y_i(0) = \big[Q\bar{x}(0)\big]_i = (\bar{y}_0)_i,$$

where

$$\chi_i = \min\left\{\chi^*, \left(\frac{l_i \Gamma(\alpha+1)}{\|f_i\|_\infty}\right)^{1/\alpha}\right\}, \quad i = 1, \dots, n.$$

Let $\chi = \min\{\chi_1, \chi_2, \dots, \chi_n\}$. Then $\bar{x}(t) = [Q^{-1}\bar{y}(t)]$ uniquely solves Eq. (3), where $t \in [0, \chi]$. $\square$

Theorem 3.2. (i) *Consider the system of equations*

$$D^\alpha\big[\bar{x}(t) - \bar{x}_0\big] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{4}$$

where $a, b \in \mathbb{R}$, $\bar{x}(0) = \bar{x}_0$, $t \in [0, \chi^*]$, $\chi^* > 0$, $0 < \alpha < 1$. *Define* $z(t) = x_1(t) + i x_2(t)$. *Then the equation*

$$D^\alpha\big[z(t) - z_0\big] = \mu z, \quad z(0) = z_0 = x_1(0) + i x_2(0), \ \mu = a + ib, \tag{5}$$

is equivalent to Eq. (4). *It can be shown that the complex equation* (5) *has unique solution. Hence the theorem.*

(ii) *Consider the system*

$$D^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = A\bar{x}(t), \quad \bar{x}(0) = \bar{x}_0, \ 0 < \alpha < 1.$$

$A \in L(\mathbb{R}^2)$ *and* A *has eigenvalues* $a \pm ib$, $a, b \in \mathbb{R}$. *Then there exists a matrix* Q *such that*

$$A = Q \begin{bmatrix} a & -b \\ b & a \end{bmatrix} Q^{-1}.$$

Define $\bar{y}(t) = Q^{-1}\bar{x}(t)$,

$$D^\alpha\big[\bar{y}(t) - \bar{y}(0)\big] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} y(t), \quad \bar{y}(0) = \bar{y}_0. \tag{6}$$

Equation (6) *has unique solution in view of case* (i). *Hence the result.*

Theorem 3.3. *Consider the system*

$$D^\alpha\big[\bar{x}(t) - \bar{x}(0)\big] = A\bar{x}(t), \quad \bar{x}(0) = \bar{x}_0, \ 0 < \alpha < 1, \ t \in [0, \chi^*], \ \chi^* > 0, \tag{7}$$

where A *is the elementary Jordan matrix*

$$\begin{bmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda \end{bmatrix}.$$

Then

$$D^\alpha\big[x_1(t)-x_1(0)\big]=\lambda x_1(t),$$
$$D^\alpha\big[x_2(t)-x_2(0)\big]=x_1(t)+\lambda x_2(t),$$
$$\vdots$$
$$D^\alpha\big[x_n(t)-x_n(0)\big]=x_{n-1}(t)+\lambda x_n(t).$$

Consider $D^\alpha[x_1(t)-x_1(0)]=\lambda x_1(t)$, $x_1(0)=(\bar{x}_0)_1$. *Here* $f_1(t,x_1)=\lambda x_1$ *is defined on* $\Omega_1=[0,\chi^*]\times[x_1(0)-l_1,x_1(0)+l_1]$ *for* $l_1>0$. f_1 *is continuous and Lipschitz in the second variable. Hence it has unique solution* $x_1(t)$, $t\in[0,\chi_1]$, *where*

$$\chi_1=\min\left\{\chi^*,\left(\frac{l_1\Gamma(\alpha+1)}{\|f_1\|_\infty}\right)^{1/\alpha}\right\}.$$

Consider $D^\alpha[x_2(t)-x_2(0)]=x_1(t)+\lambda x_2(t)$, *where now* $x_1(t)$ *is known function. Here* $f_2(t,x_2)=x_1(t)+\lambda x_2$ *is defined on* $\Omega_2=[0,\chi^*]\times[x_2(0)-l_2,x_2(0)+l_2]$ *for* $l_2>0$. f_2 *is continuous and Lipschitz in the second variable. Hence it has unique solution* $x_2(t)$, $t\in[0,\chi_2]$, *where*

$$\chi_2=\min\left\{\chi^*,\left(\frac{l_2\Gamma(\alpha+1)}{\|f_2\|_\infty}\right)^{1/\alpha}\right\}.$$

Now $x_1(t)$ *and* $x_2(t)$ *are known functions which will be substituted in the equation*

$$D^\alpha\big[x_3(t)-x_3(0)\big]=x_2(t)+\lambda x_3(t),$$

and so on. Thus the system of equations given in Eq. (7) *has a unique solution on* $[0,\chi]$, *where* $\chi=\min\{\chi_1,\chi_2,\ldots,\chi_n\}$.

Theorem 3.4. *Consider the initial value problem*

$$D^\alpha\big[\bar{x}(t)-\bar{x}(0)\big]=A\bar{x}(t),\quad \bar{x}(0)=\bar{x}_0, \tag{8}$$

where $t\in[0,\chi^*]$, $\chi^*>0$, $0<\alpha<1$ *and* $A\in L(\mathbb{R}^n)$. *Then* $\exists\chi>0$ *and a unique solution to Eq.* (8) *defined on* $[0,\chi]$.

Proof. In view of Theorem 2.3, there exists a basis of $\mathbb{R}^n$, in which the differential equation becomes

$$D^\alpha\big[\bar{y}(t)-\bar{y}(0)\big]=B\bar{y}(t),\quad \bar{y}(0)=\bar{y}_0,$$

where B is composed of diagonal blocks of the type C_i and $\hat{C}_j$, as defined in Theorems 2.1 and 2.2. In this basis the system decouples into simpler subsystems. Then in view of Theorems 3.1–3.3, $\exists\chi>0$ and a unique solution to the initial value problem under consideration defined on $[0,\chi]$. The solution to initial value problem (8) can be obtained by simple formula $\bar{x}(t)=[Q^{-1}]\bar{y}(t)$, where Q is defined in Theorem 3.1. □

3.1. Illustrative examples

It is proved [5,7] that the initial value problem

$$D^\alpha\bigl[y - y(0)\bigr] - \lambda y = g(x), \quad 0 < \alpha < 1,\ y(0) = c_0,$$

has the following unique solution:

$$y(x) = \int_0^x t^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha) g(x-t)\, dt + c_0 E_{\alpha,1}(\lambda x^\alpha), \tag{9}$$

where $E_{\alpha,\beta}$ denotes two parameter Mittag–Leffler function [7].

Example 1. Consider the following system, where $0 < \alpha < 1$:

$$D^\alpha\bigl[x_1 - x_1(0)\bigr] = -x_1 - 3x_2,$$
$$D^\alpha\bigl[x_2 - x_2(0)\bigr] = 2x_2, \quad x_1(0) = 1,\ x_2(0) = 0.$$

Here

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix},$$

having the eigenvalues -1, and 2. Choose the eigenvectors $\bar{g}_1 = (1,0)^t$, $\bar{g}_2 = (-1,1)^t$. Then

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = Q^{-1} \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} Q,$$

where

$$Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Define $\bar{y} = Q\bar{x}$. Then the system of equations in $\bar{y}$ is decoupled, namely

$$D^\alpha\bigl(y_1 - y_1(0)\bigr) = -y_1,\ y_1(0) = 1,$$
$$D^\alpha\bigl(y_2 - y_2(0)\bigr) = 2y_2,\ y_2(0) = 0.$$

In view of Eq. (9), $y_1(t) = E_{\alpha,1}(-t^\alpha)$, $y_2(t) = 0$. Hence $x_1(t) = E_{\alpha,1}(-t^\alpha)$, $x_2(t) = 0$.

Example 2. Consider the following system of equations, where $0 < \alpha < 1$:

$$D^\alpha\bigl(x_1(t) - x_1(0)\bigr) = \lambda x_1(t), \quad x_1(0) = c_1,$$
$$D^\alpha\bigl(x_2(t) - x_2(0)\bigr) = x_1(t) + \lambda x_2(t), \quad x_2(0) = c_2,$$
$$D^\alpha\bigl(x_3(t) - x_3(0)\bigr) = x_2(t) + \lambda x_3(t), \quad x_3(0) = c_3.$$

In view of Eq. (9), $x_1(t) = E_{\alpha,1}(\lambda t^\alpha)c_1$, $x_2(t) = E_{\alpha,1}(\lambda t^\alpha)c_2 + \int_0^t x_1(t-\tau)\tau^{\alpha-1} \times E_{\alpha,\alpha}(\lambda\tau^\alpha)\, d\tau$, $x_3(t) = E_{\alpha,1}(\lambda t^\alpha)c_3 + \int_0^t x_2(t-\tau)\tau^{\alpha-1} E_{\alpha,\alpha}(\lambda\tau^\alpha)\, d\tau$.

3.2. Nonautonomous case

Theorem 3.5 (Existence). *Let $f_i : W \to \mathbb{R}$ be continuous, $i = 1, 2, \ldots, n$, where*

$$W = [0, \chi^*] \times \prod_{j=1}^{n} \bigl[x_j(0) - l_j, x_j(0) + l_j\bigr], \quad \chi^* > 0,\ l_j > 0,\ \forall j,$$

and $\bar{f} = (f_1, f_2, \ldots, f_n)$. Then the nonautonomous initial value problem

$$D^\alpha\bigl[\bar{x}(t) - \bar{x}(0)\bigr] = \bar{f}(t, \bar{x}), \quad \bar{x}(0) = \bar{x}_0,\ 0 < \alpha < 1, \tag{10}$$

has a solution $\bar{x}(t) : [0, \chi] \to \mathbb{R}^n$, where

$$\chi = \min\left\{\chi^*, \left(\frac{l\Gamma(\alpha+1)}{\|\bar{f}\|_\infty}\right)^{1/\alpha}\right\}, \quad l = \min\{l_1, l_2, \ldots, l_n\}.$$

Proof. Denote $A(\bar{x}) = (A_1(\bar{x}), A_2(\bar{x}), \ldots, A_n(\bar{x}))$, where

$$A_i\bigl[\bar{x}(t)\bigr] = x_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i\bigl(s, \bar{x}(s)\bigr)\, ds. \tag{11}$$

Define $U = \{\bar{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)),\ x_i \in C[0, \chi]\colon\ |x_i(t) - x_i(0)| < l\}$. Clearly U is a nonempty, convex, closed subset of $C([0, \chi]^n)$. Since f_i's are continuous on the compact set W, they are uniformly continuous on W. Thus, given an arbitrary $\epsilon > 0$, we can find $\delta > 0$ such that

$$\bigl|f_i(t, \bar{x}) - f_i(t, \bar{z})\bigr| < \frac{\epsilon}{\chi^\alpha}\Gamma(\alpha+1) \tag{12}$$

whenever $\|\bar{x} - \bar{z}\|_\infty < \delta$.

Now let $\bar{x}, \bar{z} \in U$ such that $\|\bar{x} - \bar{z}\|_\infty < \delta$. Then in view of Eq. (12),

$$\begin{aligned}
\bigl\|A\bar{x}(t) - A\bar{z}(t)\bigr\|_\infty &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-s)^{\alpha-1}\bigl[\bar{f}\bigl(s, \bar{x}(s)\bigr) - \bar{f}\bigl(s, \bar{z}(s)\bigr)\bigr]\, ds \right\|_\infty \\
&= \sup_{1 \leqslant i \leqslant n} \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1}\bigl[f_i\bigl(s, \bar{x}(s)\bigr) - f_i\bigl(s, \bar{z}(s)\bigr)\bigr]\, ds \right| \\
&\leqslant \sup_{1 \leqslant i \leqslant n} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\bigl|f_i\bigl(s, \bar{x}(s)\bigr) - f_i\bigl(s, \bar{z}(s)\bigr)\bigr|\, ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\chi (\chi-s)^{\alpha-1} \sup_{\substack{1<i<n \\ 0<t<s}} \bigl|f_i\bigl(t, \bar{x}(t)\bigr) - f_i\bigl(t, \bar{z}(t)\bigr)\bigr|\, ds \\
&\leqslant \frac{\epsilon\Gamma(\alpha+1)}{\chi^\alpha\Gamma(\alpha)} \int_0^\chi (\chi-s)^{\alpha-1}\, ds = \frac{\epsilon\chi^\alpha}{\chi^\alpha} = \epsilon,
\end{aligned}$$

proving the continuity of the operator A. Moreover, for $\bar{x} \in U$ and $t \in [0, \chi]$, we find

$$\begin{aligned}
\left\| A\bar{x}(t) - \bar{x}(0) \right\|_\infty &= \frac{1}{\Gamma(\alpha)} \sup_{1 \leqslant i \leqslant n} \left| \int_0^t (t-s)^{\alpha-1} f_i\big(s, \bar{x}(s)\big)\, ds \right| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\chi (\chi - s)^{\alpha-1} \sup_{\substack{1<i<n \\ 0<t<s}} \left| f_i\big(t, \bar{x}(t)\big) \right| ds \\
&\leqslant \frac{\|\bar{f}\|_\infty}{\Gamma(\alpha+1)} \int_0^\chi (\chi - s)^{\alpha-1}\, ds \\
&\leqslant \frac{\|\bar{f}\|_\infty \chi^\alpha}{\Gamma(\alpha+1)} \leqslant \frac{\|\bar{f}\|_\infty}{\Gamma(\alpha+1)} \frac{l\Gamma(\alpha+1)}{\|\bar{f}\|_\infty} = l.
\end{aligned}$$

Thus, we have shown that $A\bar{x} \in U$, i.e., A maps the set U to itself. Then we look at the set of functions $A(U) := \{A\bar{x}(t)\colon\ \bar{x}(t) \in U\}$. For $\bar{z}(t) \in A(U)$ we find that, for all $t \in [0, \chi]$,

$$\begin{aligned}
\left\| \bar{z}(t) \right\|_\infty &= \left\| A\bar{x}(t) \right\|_\infty + \frac{1}{\Gamma(\alpha)} \sup_{1 \leqslant i \leqslant n} \left| \int_0^t (t-s)^{\alpha-1} f_i\big(s, \bar{x}(s)\big)\, ds \right| \\
&\leqslant \left\| \bar{x}(0) \right\|_\infty + \frac{1}{\Gamma(\alpha)} \int_0^\chi (\chi - s)^{\alpha-1} \sup_{\substack{1<i<n \\ 0<t<s}} \left| f_i\big(t, \bar{x}(t)\big)\, ds \right| \\
&\leqslant \left\| \bar{x}(0) \right\|_\infty + \frac{\|\bar{f}\|_\infty}{\Gamma(\alpha)} \int_0^\chi (\chi - s)^{\alpha-1}\, ds = \left\| \bar{x}(0) \right\|_\infty + \frac{\|\bar{f}\|_\infty}{\Gamma(\alpha+1)} \chi^\alpha,
\end{aligned}$$

which implies $A(U)$ is bounded in pointwise sense. Moreover, for $0 \leqslant t_1 \leqslant t_2 \leqslant \chi$,

$$\begin{aligned}
&\left\| A\bar{x}(t_1) - A\bar{x}(t_2) \right\|_\infty \\
&\quad = \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} \bar{f}\big(s, \bar{x}(s)\big)\, ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} \bar{f}\big(s, \bar{x}(s)\big)\, ds \right\|_\infty \\
&\quad = \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] \bar{f}\big(s, \bar{x}(s)\big)\, ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \bar{f}\big(s, \bar{x}(s)\big)\, ds \right\|_\infty \\
&\quad \leqslant \frac{\|\bar{f}\|_\infty}{\Gamma(\alpha)} \left[\int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \right]
\end{aligned}$$

$$\leqslant \frac{\|\bar f\|_\infty}{\Gamma(\alpha+1)}\left[2(t_2-t_1)^\alpha - t_1^\alpha - t_2^\alpha\right] \leqslant \frac{\|\bar f\|_\infty}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.$$

Thus, if $|t_2-t_1|<\delta$, then

$$\left\|A\bar x(t_1)-A\bar x(t_2)\right\|_\infty \leqslant 2\frac{\|f\|_\infty}{\Gamma(\alpha+1)}\delta^\alpha.$$

Since the expression on the right-hand side is independent of $\bar x$, we see that the set $A(U)$ is equicontinuous. Then, the Arzela–Ascoli theorem [1] implies that every sequence of functions in $A(U)$ is relatively compact. Then by Schauder fixed point theorem [8], A has a fixed point $\bar x:[0,\chi]\to\mathbb{R}$ which is a solution of Eq. (10). □

Theorem 3.6 (Uniqueness). *Let $f_i: W\to\mathbb{R}$ be bounded, where*

$$W=[0,\chi^*]\times\prod_{j=1}^{n}\left[x_j(0)-l_j, x_j(0)+l_j\right], \quad \chi^*>0,\ l_j>0.$$

If $\bar f=(f_1,f_2,\dots,f_n)$ satisfies Lipschitz condition with respect to the second variable, i.e.,

$$\left\|\bar f(t,\bar x)-\bar f(t,\bar z)\right\|_\infty \leqslant L\|\bar x-\bar z\|_\infty,$$

then the initial value problem (10) *has unique solution $\bar x(t):[0,\chi]\to\mathbb{R}$, where χ is as defined in Theorem* 3.4.

Proof. Let $l=\min\{l_1,l_2,\dots,l_n\}$ and

$$U=\left\{\bar x(t)=\left(x_1(t),x_2(t),\dots,x_n(t)\right),\ x_i(t)\in C[0,\chi],\ \forall i:\ \left|x_i(t)-x_i(0)\right|<l\right\}.$$

We use the operator $A(\bar x)=(A_1(\bar x),A_2(\bar x),\dots,A_n(\bar x))$, where A_i is defined in Eq. (11) and recall that it maps the nonempty, convex, and closed set U to itself. Further A is a continuous operator. We prove that, for every $n\in\mathbb{N}\cup\{0\}$ and for every $\bar x,\bar z\in U$,

$$\|A^n\bar x-A^n\bar z\|_\infty \leqslant \frac{(L\chi^\alpha)^n}{\Gamma(n\alpha+1)}\|\bar x-\bar z\|_\infty. \tag{13}$$

In the following steps, we use the Lipschitz condition on f and the induction hypothesis and find

$$\begin{aligned}
&\|A^n\bar x-A^n\bar z\|_\infty\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-s)^{\alpha-1}\sup_{\substack{0\leqslant t\leqslant s\\1\leqslant i\leqslant n}}\left|f_i\left(t,A^{n-1}\bar x(t)\right)-f_i\left(t,A^{n-1}\bar z(t)\right)\right|ds\\
&\quad\leqslant \frac{L}{\Gamma(\alpha)}\int_0^\chi(\chi-s)^{\alpha-1}\sup_{0\leqslant t\leqslant s}\left\|A^{n-1}\bar x(t)-A^{n-1}\bar z(t)\right\|_\infty ds\\
&\quad\leqslant \frac{L^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))}\int_0^\chi(\chi-s)^{\alpha-1}s^{\alpha(n-1)}\sup_{\substack{0\leqslant t\leqslant s\\1\leqslant i\leqslant n}}\left|x_i(t)-z_i(t)\right|ds
\end{aligned}$$

$$\leqslant \frac{L^n}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \sup_{\substack{0\leqslant t\leqslant \chi \\ 1\leqslant i\leqslant n}} \left|x_i(t)-z_i(t)\right| \int_0^{\chi} (\chi-s)^{\alpha-1} s^{\alpha(n-1)}\, ds$$

$$= \frac{L^n \|\bar{x}-\bar{z}\|_\infty}{\Gamma(\alpha)\Gamma(1+\alpha(n-1))} \frac{\Gamma(\alpha)\Gamma(1+\alpha(n-1))}{\Gamma(1+\alpha n)} \chi^{n\alpha} \leqslant \frac{(L\chi^\alpha)^n}{\Gamma(n\alpha+1)} \|\bar{x}-\bar{z}\|_\infty.$$

It is however, known that [7,9]

$$\sum_{n=0}^{\infty} \frac{(L\chi^\alpha)^n}{\Gamma(n\alpha+1)} =: E_\alpha(L\chi^\alpha)$$

is the Mittag–Leffler function of order α, evaluated at $L\chi^\alpha$. Therefore, in view of the Banach fixed point theorem [3] A has unique fixed point which is the solution of the differential equation (10). □

3.3. Mittag–Leffler function for matrices

We consider Mittag–Leffler function for matrices, namely

$$E_\alpha(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+1)}, \tag{14}$$

where A is $n \times n$ matrix. It is easy to show that this series converges absolutely for all square matrices in the uniform norm, where uniform norm of $n \times n$ matrix A is defined to be

$$\|A\| = \max\bigl\{\left|A(x)\right|/|x| \leqslant 1\bigr\}.$$

Theorem 3.7. *The unique solution to the initial value problem*

$$D^\alpha\bigl[\bar{x}(t)-\bar{x}_0\bigr] = A\bar{x}(t), \quad \bar{x}(0)=\bar{x}_0,\ 0<\alpha<1,\ t\in[0,\chi],\ \chi>0,$$

where A is $n \times n$ matrix, is $E_\alpha(t^\alpha A)\bar{x}_0$.

Proof.

$$D^\alpha\bigl[\bigl[(E_\alpha(t^\alpha A)\bigr]\bar{x}_0-\bar{x}_0\bigr] = D^\alpha\left[\frac{t^\alpha A}{\Gamma(\alpha+1)} + \frac{(t^\alpha A)^2}{\Gamma(2\alpha+1)} + \cdots\right]\bar{x}_0.$$

But the series

$$\left[\frac{t^\alpha A}{\Gamma(\alpha+1)} + \frac{(t^\alpha A)^2}{\Gamma(2\alpha+1)} + \cdots\right]$$

is uniformly convergent on $[0,\chi]$ as

$$\left\|\frac{(t^\alpha A)^k}{\Gamma(k\alpha+1)}\right\| \leqslant \frac{\|\chi^\alpha A\|^k}{\Gamma(k\alpha+1)}, \quad \forall k,$$

and the series $\sum_{k=1}^{\infty} \frac{\|\chi^\alpha A\|^k}{\Gamma(k\alpha+1)}$ is convergent. Hence

$$D^\alpha\left[\sum_{n=1}^{\infty}\frac{(t^\alpha A^n)}{\Gamma(\alpha n+1)}\right]=\sum_{n=1}^{\infty}\frac{A^n D^\alpha[t^{\alpha n}]}{\Gamma(\alpha n+1)}=A\left[I+\frac{At^\alpha}{\Gamma(\alpha+1)}+\cdots\right].$$

Therefore

$$D^\alpha\big[\big[E_\alpha(t^\alpha A)\big]\bar{x}_0-\bar{x}_0\big]=AE_\alpha(t^\alpha A)\bar{x}_0. \qquad \square$$

Remark. For $\alpha = 1$, we recover the standard result for a system of ordinary differential equations [5] viz. the unique solution for the system of equations $D[\bar{x}(t)] = [x_1'(t), \ldots, x_n'(t)] = A\bar{x}(t)$, $\bar{x}(0) = \bar{x}_0$ is $e^{tA}\bar{x}_0$.

4. Dependence of solution on initial condition

Theorem 4.1. *Let the functions $f_i : W \to \mathbb{R}$, $i = 1, 2, \ldots, n$, where*

$$W=[0,\chi^*]\times\prod_{j=1}^{n}\big[x_j(0)-l_j,x_j(0)+l_j\big], \quad \chi^*>0,\ l_j>0,$$

be bounded. Let $\bar{f} = (f_1, f_2, \ldots, f_n)$ be Lipschitz in the second variable with Lipschitz constant L. Let $\bar{y}(t)$ and $\bar{z}(t)$ be the solutions of the initial value problems

$$D^\alpha\big[\bar{x}(t)-\bar{x}(0)\big]=\bar{f}(t,\bar{x}), \quad \bar{x}(0)=\bar{x}_0,$$
$$D^\alpha\big[\bar{z}(t)-\bar{z}(0)\big]=\bar{f}(t,\bar{z}), \quad \bar{z}(0)=\bar{z}_0,$$

respectively, where $0 < \alpha < 1$. Then

$$\big\|\bar{x}(t)-\bar{z}(t)\big\|_\infty \leqslant \|\bar{x}_0-\bar{z}_0\|_\infty E_\alpha(Lt^\alpha), \tag{15}$$

where E_α is the Mittag–Leffler function.

Proof. Consider the following iterated sequence defined for $m = 1, 2, \ldots$:

$$(A^0\bar{x}_0)(t)=\bar{x}_0, \quad (A^m\bar{x}_0)(t)=\bar{x}_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bar{f}\big(s,(A^{m-1}\bar{x}_0)(s)\big)\,ds,$$

and

$$(A^0\bar{z}_0)(t)=\bar{z}_0, \quad (A^m\bar{z}_0)(t)=\bar{z}_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bar{f}\big(s,(A^{m-1}\bar{z}_0)(s)\big)\,ds.$$

It can be shown that

$$\big\|(A^m\bar{x}_0)(t)-(A^m\bar{z}_0)(t)\big\|_\infty \leqslant \|\bar{x}_0-\bar{z}_0\|_\infty\sum_{k=1}^{m}\frac{(Lt^\alpha)^k}{\Gamma(k\alpha+1)}, \quad \forall m.$$

Hence

$$\lim_{m\to\infty}\big\|(A^m\bar{x}_0)(t)-(A^m\bar{z}_0)(t)\big\|_\infty \leqslant \|\bar{x}_0-\bar{z}_0\|_\infty E_\alpha(Lt^\alpha).$$

Therefore $\|\bar{y}(t)-\bar{z}(t)\|_\infty \leqslant \|\bar{x}_0-\bar{z}_0\|_\infty E_\alpha(Lt^\alpha)$. $\square$

Remark. In the case when $\alpha = 1$, Eq. (15) becomes the well-known Gronwall inequality [5].

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