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New Grüss' type inequalities for functions of bounded variation and applications

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ABSTRACT

Some new sharp Grüss' type inequalities for functions of bounded variation and applications for selfadjoint operators in Hilbert spaces are given.

1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt.$$

In 1935, Grüss [1] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \qquad -\infty < n \le g \le N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

For other similar results, see [2–9] and the references therein.

The importance of Grüss' inequality in various approximation results has been described in [2,5,7]. Motivated by those applications, in this paper we provide other Grüss' type inequalities and use them to obtain new inequalities involving functions of selfadjoint operators on Hilbert spaces.

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2. New bounds

Theorem 1. *Let $f : [a, b] \to \mathbb{C}$ be of bounded variation on $[a, b]$ and $g : [a, b] \to \mathbb{C}$ a Lebesgue integrable function on $[a, b]$. Then*

$$|C(f, g)| \le \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \tag{2.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. We start with the following equality of interest that follows from Sonin's identity [10, p. 246],

$$C(f, g) = \frac{1}{b-a} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2} \right] \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt. \tag{2.2}$$

Taking the modulus in (2.2) and utilizing the triangle inequality, we get

$$\begin{aligned} |C(f, g)| &\le \frac{1}{b-a} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\le \frac{1}{2(b-a)} \int_a^b [|f(t) - f(a)| + |f(b) - f(t)|] \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\le \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt, \end{aligned} \tag{2.3}$$

since, f being of bounded variation, we have that $|f(t) - f(a)| + |f(b) - f(t)| \le \bigvee_a^b(f)$ for all $t \in [a, b]$.

Consider now the functions $f, g : [a, b] \to \mathbb{R}$ with $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ and $g(t) = t - \frac{a+b}{2}$. Observe that f is of bounded variation and $\bigvee_a^b(f) = 2$. The function g is integrable on $[a, b]$ and $\int_a^b g(s)ds = 0$. Also $\int_a^b |g(t)| dt = \frac{1}{4}(b-a)^2$ and $\int_a^b f(t)g(t)dt = \frac{1}{4}(b-a)^2$. Inserting these functions in (2.1) produces the same quantity $\frac{1}{4}(b-a)$ in both sides. □

We denote the variance of the function $f : [a, b] \to \mathbb{C}$ which is square integrable on $[a, b]$ by $D(f)$ and defined as

$$D(f) = \left[C\left(f, \bar{f}\right)\right]^{1/2} = \left[\frac{1}{b-a} \int_a^b |f(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b f(t) dt \right|^2 \right]^{1/2}, \tag{2.4}$$

where $\bar{f}$ denotes the complex conjugate function of f.

Corollary 1. *If the function $f : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then*

$$D(f) \le \frac{1}{2} \bigvee_a^b (f). \tag{2.5}$$

The constant $\frac{1}{2}$ is best possible in (2.5).

Proof. If we apply Theorem 1 for $g = \bar{f}$ we get

$$D^2(f) \le \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt. \tag{2.6}$$

By the Cauchy–Bunyakovsky–Schwarz integral inequality we have

$$\frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \le D(f). \tag{2.7}$$

On making use of (2.6) and (2.7) we deduce the desired inequality (2.5).

Now, if we choose $f : [a, b] \to \mathbb{R}$ with $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, then we obtain in both sides of (2.5) the same quantity 1, which proves the sharpness of the constant $\frac{1}{2}$. □

Now we can state the following result when both functions are of bounded variation:

Corollary 2. *If $f, g : [a, b] \to \mathbb{C}$ are of bounded variation on $[a, b]$, then*

$$|C(f, g)| \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g) . \tag{2.8}$$

The constant $\frac{1}{4}$ is best possible in (2.8).

Proof. On making use of Theorem 1 and Corollary 1 we have successively

$$\begin{aligned} |C(f, g)| &\leq \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b (f) D(g) \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g). \end{aligned}$$

The case of equality is obtained in (2.8) for $f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right), t \in [a, b]$. □

Remark 1. We can consider the following quantity associated with a complex valued square integrable function on $[a, b]$, $f : [a, b] \to \mathbb{C}$,

$$E(f) := |C(f, f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2 \right|^{1/2} .$$

Utilizing the above results we can state for functions of bounded variation that

$$\begin{aligned} E^2(f) &\leq \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b (f) D(f) \leq \frac{1}{4} \left[\bigvee_a^b (f) \right]^2 . \end{aligned} \tag{2.9}$$

If we consider $G(f) := |C(f, |f|)|^{1/2}$, where $f : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then we also have

$$\begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)|\, ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b (f) D(|f|) \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (|f|) \leq \frac{1}{4} \left[\bigvee_a^b (f) \right]^2 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b (|f|) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b (|f|) D(f) \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (|f|) \leq \frac{1}{4} \left[\bigvee_a^b (f) \right]^2 . \end{aligned} \tag{2.11}$$

The following representation is of interest in itself. The result was first obtained in [6] (see also [7]).

Lemma 1. *If $v : [a, b] \to \mathbb{C}$ and $h : [a, b] \to \mathbb{C}$ are such that one is continuous and the other of bounded variation on $[a, b]$, then we have the identity*

$$\frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t)\, dh(t)}{b-a} - \int_a^b v(t) dh(t) = \int_a^b h(t) dv(t) - \frac{v(b) - v(a)}{b-a} \int_a^b h(t) dt. \tag{2.12}$$

The proof can be easily done integrating by parts in the left hand side and performing the required calculations.

We can provide now the following corollaries of Theorem 1:

Corollary 3. *If $v : I \to \mathbb{C}$ is differentiable on the interior of the interval I denoted by $\mathring{I}$ and $[a, b] \subset \mathring{I}$, v' is of bounded variation on $[a, b]$ and $h : [a, b] \to \mathbb{C}$ is integrable on $[a, b]$, then we have the inequality*

$$\left|\frac{v(b)\int_a^b (t-a)dh\,(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)dh(t)\right| \le \frac{1}{2}\bigvee_a^b (v') \int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h\,(s)\,ds\right| dt. \quad (2.13)$$

Corollary 4. *If $v : [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$ and $h : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then we have the inequality*

$$\left|\frac{v(b)\int_a^b (t-a)dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)dh(t)\right| \le \frac{1}{2}\bigvee_a^b (h) \int_a^b \left|v'(t) - \frac{v(b)-v(a)}{b-a}\right| dt. \quad (2.14)$$

The constant $\frac{1}{2}$ is best possible in (2.14).

3. Applications for functions of selfadjoint operators

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,.\rangle)$ with the spectrum $Sp\,(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann–Stieltjes integral* [11, p. 256]:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f\,(\lambda)\,d\,(\langle E_\lambda x, y\rangle)\,, \quad (3.1)$$

for any $x, y \in H$. The function $g_{x,y}\,(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and $g_{x,y}\,(m-0) = 0$ and $g_{x,y}(M) = \langle x, y\rangle$ for any $x, y \in H$. It is also well known that $g_x\,(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be its* spectral family. *If $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous on $[m, M]$, then we have the inequality*

$$\left|\left\langle\left[\frac{f(m)\,(M1_H - A) + f(M)\,(A - m1_H)}{M-m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \le \frac{1}{2}\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt$$
$$\le \frac{1}{2}\,\|x\|\,\|y\|\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt \quad (3.2)$$

for any $x, y \in H$.

Proof. If we apply the inequality (2.14) for $v(t) = f(t)$ and $h(t) = \langle E_t x, y\rangle$ where $t \in \mathbb{R}$ and $x, y \in H$ we get

$$\left|\frac{v(M)\int_{m-0}^M (t-m)\,d\,\langle E_t x, y\rangle + f(m)\int_{m-0}^M (M-t)\,d\,\langle E_t x, y\rangle}{M-m} - \int_{m-0}^M f(t)d\,\langle E_t x, y\rangle\right|$$
$$\le \frac{1}{2}\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt. \quad (3.3)$$

Since by the spectral representation (3.1) we have

$$\int_{m-0}^M (t-m)\,d\,\langle E_t x, y\rangle = \langle (A - 1_H m)\,x, y\rangle\,, \qquad \int_{m-0}^M (M-t)\,d\,\langle E_t x, y\rangle = \langle (M1_H - A)\,x, y\rangle$$

and

$$\int_{m-0}^M f(t)d\,\langle E_t x, y\rangle = \langle f(A)x, y\rangle\,,$$

then we get from (3.3) the first part of (3.2).

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \ge 0$ for any $x \in H$, then $|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\,\langle Py, y\rangle$ for any $x, y \in H$.

Further, if $d : m-s = t_0 < t_1 < \cdots < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m-s, M]$, with $s > 0$, then we have by the Schwarz inequality for nonnegative operators that

$$\begin{aligned}\bigvee_{m-s}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right) &= \sup_d \left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, y\right\rangle\right|\right\} \\ &\le \sup_d \left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle^{1/2}\right]\right\} := I.\end{aligned}$$

By the Cauchy–Bunyakovsky–Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}I &\le \sup_d \left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle\right]^{1/2}\right\} \\ &\le \sup_d \left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle\right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle\right]^{1/2} \\ &= \left[\bigvee_{m-s}^{M}\left(\langle E_{(\cdot)}x, x\rangle\right)\right]^{1/2}\left[\bigvee_{m-s}^{M}\left(\langle E_{(\cdot)}y, y\rangle\right)\right]^{1/2} \\ &= \left(\|x\|^2 - \langle E_{m-s}x, x\rangle\right)^{1/2}\left(\|y\|^2 - \langle E_{m-s}y, y\rangle\right)^{1/2}\end{aligned}$$

for any $x, y \in H$ and $s > 0$. Taking $s \to 0+$ we get last part of (3.2). □

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