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Note

A Note on the Sheffer Shift Operator

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Each surjective derivation on the algebra of formal power series can be given in a simple form. An operator on the vector space of polynomials in a single variable is constructed such that the Sheffer polynomials are its eigenvectors; furthermore, some known examples are quoted. © 1990 Academic Press, Inc

1. INTRODUCTION

Let K be a field of characteristic zero. The vector space $K[x]$ of all polynomials in the variable x with coefficients in K is denoted by P. It is shown in [2] that there is an isomorphism from P^* (the dual of P) onto the algebra of formal power series $K[[t]]$ with coefficient in K. The algebra $K[[t]]$ will be denoted by $\mathscr{F}$. The formal power series in the exponential form (see, e.g., [1])

$$f(t)=\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \tag{1}$$

defines a linear functional and a linear operator on P by setting

$$\langle f(t) \mid x^n \rangle = a_n, \qquad f(t)\, x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k},$$

respectively. Two important examples are $tp(x)=p'(x)$, $\exp(yt)\, p(x)= p(x+y)$, $y \in K$.

The order $o(f(t))$ of the series (1) is the smallest integer k for which $a_k \neq 0$. We define $o(0)=\infty$. The series (1) is invertible in $\mathscr{F}$ if and only if $o(f(t))=0$; its inverse is denoted by $f(t)^{-1}$. If $o(f(t))=1$ then there exists the compositional inverse $\bar{f}(t)$ such that $\bar{f}(f(t))=t$; such a series is called a *delta series*.

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For every ordered pair $(g(t), f(t))$, where $g(t)$ is an invertible series in P^* and $f(t)$ a delta series in P^*, we have a unique sequence of polynomials $\{s_n(x)\}_{n=0}^{\infty}$, such that the degree of s_n equals n and such that the following condition is satisfied: $\langle g(t)\, f(t)^k \mid s_n(x)\} = n!\, \delta_{k,n}$ for all non-negative integers k, n ($\delta_{k,n}$ is the Kronecker delta). The polynomials $s_n(x)$ are called *Sheffer* for the pair $(g(t), f(t))$. In the case $g(t) = 1$ we write $p_n(x)$ instead of $s_n(x)$, and we say that the sequence $\{p_n(x)\}_{n=0}^{\infty}$ is *associated* to the delta series $f(t)$.

The order of formal power series defines a topology on $\mathscr{F}$. The sequence $\{f_k(t)\}_{k=0}^{\infty}$ in $\mathscr{F}$ converges to $f(t)$ in $\mathscr{F}$ if and only if $\lim_{k\to\infty} o(f_k(t) - f(t)) = \infty$. The linear operator T on $\mathscr{F}$ is continuous if and only if $\lim_{k\to\infty} o(Tf_k(t)) = \infty$ whenever $\lim_{k\to\infty} o(f_k(t)) = \infty$. The adjoint λ^* on $P^* = \mathscr{F}$ of a linear operator λ on P is defined by the relation $\langle \lambda^* f(t) \mid p(x)\rangle = \langle f(t) \mid \lambda p(x)\rangle$, $f(t) \in P^*$, $p(x) \in P$. For a given linear operator λ on P the adjoint λ^* exist and is given by

$$\lambda^* f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid \lambda x^k\rangle}{k!}\, t^k.$$

It is shown in [2] that an operator T on P^* is the adjoint of a linear operator λ on P if and only if T is continuous, moreover, all automorphisms and surjective derivations on P^* are continuous. We shall here amend the Theorem 3.6.1 in [2].

2. Results

Let $\{p_n(x)\}_{n=0}^{\infty}$ be the associated sequence of polynomials for the delta series $f(t)$. The umbral shift θ_f on P is defined by the relation $\theta_f\, p_n(x) = p_{n+1}(x)$. The formal derivative of the series $h(t)$ is denoted by $h'(t)$.

Theorem 1. *A linear operator D on P^* is a surjective derivation on P^* if and only if $D = \theta_f^*$ for some delta series $f(t)$ in P^*, moreover, the operator D is given by*

$$Dh(t) = f'(t)^{-1} h'(t). \tag{2}$$

Proof. Let D be a surjective derivation on P^*. By the Theorem 3.2.3 in [2] there is a dalta series $f(t)$ in P^* such that $Df(t) = 1$. Since

$$\begin{aligned}\langle Df(t)^k \mid p_n(x)\rangle &= k\langle f(t)^{k-1} Df(t) \mid p_n(x)\rangle\\ &= k\langle f(t)^{k-1} \mid p_n(x)\rangle = \langle f(t)^k \mid p_{n+1}(x)\rangle\\ &= \langle f(t)^k \mid \theta_f\, p_n(x)) = \langle \theta_f^* f(t)^k \mid p_n(x)\rangle\end{aligned}$$

for all non-negative integers k and n, we conclude that $Df(t)^k = \theta_f^* f(t)^k = kf(t)^{k-1}$. Since any series $h(t)$ in P^* can be written in the form $h(t) = \sum_{k=0}^{\infty} a_k f(t)^k$, the continuity of D implies $Dh(t) = \sum_{k=1}^{\infty} a_k kf(t)^{k-1}$. Since the map $f(t) \to f'(t)$ is a surjective derivation on P^* we get from $h'(t) = \sum_{k=1}^{\infty} a_k kf(t)^{k-1} f'(t)$ finally the result $Dh(t) = f'(t)^{-1} h'(t)$.

For the proof in the opposite direction suppose that $D = \theta_f^*$ for some delta series $f(t)$. Let $\{p_n(x)\}_{n=0}^{\infty}$ be the sequence of polynomials associated to $f(t)$. By the same argument as that above we obtain $Dh(t) = f'(t)^{-1} h'(t)$ for every $h(t)$ in P^*. Since $D(a(t)\, b(t)) = (Da(t))\, b(t) + a(t)(Db(t))$ for all $a(t)$ and $b(t)$ in P^*, the operator D is a derivation on P^*. For each given series $h(t)$ the equation $f'(t)\, x(t) = h(t)$ has a solution in P^* given by the formal integral $x(t) = \int_0^t f'(t)\, h(t)\, dt$, thus shows that the operator D is a surjective derivation on P^*.

The operator of multiplication of polynomials in P by x is denoted by x. Since $\langle h(t) \mid xp(x) \rangle = \langle h'(t) \mid p(x) \rangle$ for every $h(t)$ in P^* and every $p(x)$ in P, then it is clear that $x^* h(t) = h'(t)$ for all $h(t)$ in P^*. For a delta deries $f(t)$ and the associated polynomials $p_n(x)$ we have

$$\begin{aligned}
&\langle h(t) \mid \theta_f p_n(x) \rangle \\
&\quad = \langle \theta_f^* h(t) \mid p_n(x) \rangle = \langle Dh(t) \mid p_n(x) \rangle = \langle f'(t)^{-1} h'(t) \mid p_n(x) \rangle \\
&\quad = \langle h'(t) \mid f'(t)^{-1} p_n(x) \rangle = \langle h(t) \mid xf'(t)^{-1} p_n(x) \rangle
\end{aligned}$$

for every series $h(t)$ in P^*. Thus $\theta_f = xf'(t)^{-1}$. As a consequence of this result we get the recurrence formula

$$p_{n+1}(x) = xf'(t)^{-1} p_n(x). \tag{3}$$

THEOREM 2. *The associated polynomials $p_n(x)$ of a delta series $f(t)$ in P^* are eigenvectors of the operator $T_f = \theta_f f(t)$ with the eigenvalues n, $n = 0, 1, 2, \ldots$,*

$$T_f p_n(x) = np_n(x). \tag{4}$$

Proof. For $n = 0$ the proof is trivial.

For $n \geqslant 1$ the relation $\langle g(t) f(t)^k \mid s_n(x) \rangle = n!\, \delta_{k,n}$ implies the recurrence formula $f(t)\, s_n(x) = ns_{n-1}(x)$ and we have

$$T_f p_n(x) = \theta_f f(t)\, p_n(x) = \theta_f np_{n-1}(x) = np_n(x),$$

as was to be proved. See also [3].

EXAMPLE 1. The Mittag-Leffler polynomials $M_n(x)$ are associated to the delta series $f(t) = (e^t - 1)(e^t + 1)^{-1}$. Since $T_f = f'(t)^{-1} f(t) = (e^t - e^{-t})/2$ we get from (4)

$$x(M_n(x+1) - M_n(x-1)) = 2nM_n(x).$$

It is easy to see that for all $a(t)$ and $b(t)$ and each $p(x)$ in P the following relation holds (see, e.g., [2]):

$$a'(t) = a(t)x - xa(t). \tag{5}$$

For a given pair $(g(t), f(t))$ the polynomials $p_n(x)$ and $s_n(x)$ are related by the equation $g(t)\, s_n(x) = p_n(x)$ for all non-negative integers n. The Sheffer shift $\theta_{g,f}$ on P is defined by

$$\theta_{g,f} s_n(x) = s_{n+1}(x). \tag{6}$$

From (5) and (6) we derive

$$\begin{aligned}\theta_{g,f} &= g(t)^{-1}\theta_f g(t) = g(t)^{-1} x f'(t)^{-1} g(t)\\ &= (xg(t)^{-1} + (g(t)^{-1})')\, f'(t)^{-1} g(t)\\ &= xf'(t)^{-1} - g(t)^{-2} g'(t)\, f'(t)^{-1} g(t) = (x - g'(t)\, g(t)^{-1})\, f'(t)^{-1}.\end{aligned}$$

The relation (6) thus becomes

$$s_{n+1}(x) = (x - g'(t)\, g(t)^{-1})\, f'(t)^{-1} s_n(x), \tag{7}$$

the recurrence formula for Sheffer polynomials.

THEOREM 3. *The Scheffer polynomials $s_n(x)$ of the pair $(g(t), f(t))$ are eigenvectors of the operator $T_{g,f} = \theta_{g,f} f(t)$ with the corresponding eigenvalues $n = 0, 1, 2, \ldots$*

$$T_{g,f} s_n(x) = n s_n(x). \tag{8}$$

Proof (Compare [3]). For $n = 0$, (8) is trivial. For $n \geqslant 1$ we get by a simple calculation

$$\begin{aligned}T_{g,f} s_n(x) &= \theta_{g,f} f(t)\, s_n(x) = \theta_{g,f}(n s_{n-1}(x))\\ &= n\theta_{g,f} s_{n-1}(x) = n s_n(x),\end{aligned}$$

as was to be proved.

EXAMPLE 2. The Hermite polynomials $H_n^{(v)}(x)$ of variance v are Sheffer for $g(t) = \exp(vt^2/2)$ and $f(t) = t$.

Indeed, in this case $f'(t) = 1$ and $g'(t)\, g(t)^{-1} = (\log g(t))' = vt$ and from (8) we find $T_{g,f} = (x - vt)\, t$.

From (8) we get

$$\nu \frac{d^2}{dx^2} H_n^{(\nu)}(x) - x \frac{d}{dx} H_n^{(\nu)}(x) + nH_n^{(\nu)} = 0,$$

which is the differential equation of the Hermite polynomials.

EXAMPLE 3. The Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α are Sheffer for $g(t) = (1-t)^{-\alpha-1}$ and $f(t) = t(t-1)^{-1}$.

We have now $f'(t) = -(1-t)^{-2}$, $g'(t)\, g(t)^{-1} = (\alpha+1)(1-t)^{-1}$, and $T_{g,f} = -xt^2 + (x-\alpha-1)t$. From (8) we get the differential equation for the Laguerre polynomials:

$$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0.$$

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