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Note

Axiom of choice and chromatic number of the plane

Saharon Shelah[a,b,1] and Alexander Soifer[c,d,e,2]

[a] The Hebrew University, Jerusalem, Israel
[b] Rutgers University, Piscataway, NJ, USA
[c] DIMACS, Rutgers University, USA
[d] Princeton University, Mathematics, Fine Hall, Princeton, NJ 08544, USA
[e] University of Colorado at Colorado Springs, USA

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1. Question

Define a graph U^2 on the set of all points of the plane R^2 as its vertex set, with two points adjacent iff they are distance 1 apart. The graph U^2 ought to be called *unit distance plane*, and its chromatic number χ is called *chromatic number of the plane*.[3] Finite subgraphs of U^2 are called *finite unit distance plane graphs*.

In 1950 the 18-year old Edward Nelson posed the problem of finding χ (see the problem's history in [Soi1]). A number of relevant results were obtained under additional restrictions on monochromatic sets (see surveys in [CFG,KW,Soi2,Soi3]). Falconer, for example, showed [F] that χ is at least 5 if monochromatic sets are Lebesgue measurable. Amazingly though, the problem has withstood all assaults in the general case, leaving us with an embarrassingly wide range for χ being 4, 5, 6 or 7.

In their fundamental 1951 paper [EB], Erdös and de Bruijn have shown that the chromatic number of the plane is attained on some finite subgraph. This result has naturally channeled much of research in the direction of finite unit distance graphs. One limitation of the Erdös–de-Bruijn result, however, has remained a low key: they

E-mail address: asoifer@uccs.edu (A. Soifer).

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[3] Chromatic number $\chi(G)$ of a graph G is the smallest number of colors required for coloring the vertices, so that no two vertices of the same color are connected by an edge.

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used quite essentially the axiom of choice. So, it is natural to ask, what if we have no choice? Absence of choice—in mathematics as in life—may affect outcome.

We will present here an example of a distance graph on the line R, whose chromatic number depends upon the system of axioms we choose for set theory. While the setting of our example differs from that of the chromatic number of the plane problem, the example illuminates how the value of chromatic number can be dramatically affected by the inclusion or the exclusion of the axiom of choice in the system of axioms for sets.

Finally, we formulate a *Conditional Chromatic Number Theorem*, which specifically describes a setting in which the chromatic number of the plane takes on two different values depending upon the axioms for set theory.

2. Preliminaries

Let us recall basic set-theoretic definitions and notations. In 1904 Zermelo [Z] formalized the axiom of choice that had previously been used informally:

Axiom of choice (AC). Every family Φ of nonempty sets has a choice function, i.e., there is a function f such that $f(S) \in S$ for every S from Φ.

Many results in mathematics really need just a countable version of choice:

Countable axiom of choice ($AC_{\aleph_0}$). Every countable family of nonempty sets has a choice function.

In 1942 Bernays [B] introduced the following axiom:

Principle of dependent choices (DC). If E is a binary relation on a nonempty set A, and for every $a \in A$ there exists $b \in A$ with aEb, then there is a sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

AC implies DC (see Theorem 8.2 in [J], for example), but not conversely. In turn, DC implies $AC_{\aleph_0}$, but not conversely. DC is a weak form of AC and is sufficient for the classical theory of Lebesgue measure. We observe that, in particular, DC is sufficient for Falconer's result [F] formulated in Question above.

We will make use of the following axiom:

(LM) Every set of real numbers is Lebesgue measurable.

As always, ZF stands for Zermelo–Fraenkel system of axioms for sets, and ZFC for Zermelo–Fraenkel with the addition of the axiom of choice.

Assuming the existence of an inaccessible cardinal, Solovay constructed in 1964 (and published in 1970) a model that proved the following consistency result [Sol]:

Solovay Theorem. *The system of axioms ZF + DC + LM is consistent.*

As Jech [J] observes, in the Solovay model, every set of reals differs from a Borel set by a set of measure zero.

Finally, we say a set $X \subseteq R$ has the *Baire property* if there is an open set U such that $X \Delta U$ (symmetric difference) is *meager*, (or *of first category*), i.e., a countable union of nowhere dense sets.

Example. We define a graph G as follows: the set R of real numbers serves as the vertex set, and the set of edges is $\{(s,t)\colon\ s-t-\sqrt{2}\in Q\}$.

Claim 1. *In ZFC the chromatic number of G is equal to* 2.

Proof. Let $S=\{q+n\sqrt{2};q\in Q,n\in Z\}$. We define an equivalence relation E on R as follows: $sEt\Leftrightarrow s-t\in S$,

Let Y be a set of representatives for E. For $t\in R$ let $y(t)\in Y$ be such that $tEy(t)$. We define a 2-coloring $c(t)$ as follows: $c(t)=l,\ l=0,\ 1$ iff there is $n\in Z$ such that $t-y(t)-2n\sqrt{2}-l\sqrt{2}\in Q$. $\square$

Without AC the chromatic situation changes dramatically:

Claim 2. *In* $ZF+AC_{\aleph_0}+LM$ *the chromatic number of the graph G cannot be equal to any positive integer n nor even to* $\aleph_0$.

The proof of Claim 2 immediately follows from the first of the following two statements:

1. If $A_1,\dots,A_n,\dots$ are measurable subsets of R and $\bigcup_{n<\omega}A_n\supseteq[0,1)$, then at least one set A_n contains two adjacent vertices of the graph G.

2. If $A\subseteq[0,1)$ and A contains no pair of adjacent vertices of G then A is null (of Lebesgue measure zero).

Proof. We start with the proof of statement 2. Assume to the contrary that A contains no pair of adjacent vertices of G yet A has positive measure. Then there is an interval I such that

$$\frac{\mu(A\cap I)}{\mu(I)}>\frac{9}{10}. \tag{2.1}$$

Choose $q\in Q$ such that $\sqrt{2}<q<\sqrt{2}+\frac{1}{10}$.

Let $B=A-(q-\sqrt{2})=\{x-q+\sqrt{2}:x\in A\}$. Then

$$\frac{\mu(B\cap I)}{\mu(I)}>\frac{8}{10}. \tag{2.2}$$

Inequalities (2.1) and (2.2) imply that there is $x\in I\cap A\cap B$. As $x\in B$, we have $y=x+(q-\sqrt{2})\in A$. So, we have $x,y\in A$ and $x-y-\sqrt{2}=-q\in Q$. Thus, $\{x,y\}$ is an edge of the graph G with both endpoints in A, which is the desired contradiction.

The proof of the statement 1 is now obvious. Since $\bigcup_{n<\omega}A_n\supseteq[0,1)$ and Lebesgue measure is a countably additive function in $\mathrm{AC}_{\aleph_0}$, there is a positive integer n such that A_n is a non-null set of reals. By statement 2, A_n contains a pair of adjacent vertices of G as required. $\square$

Remark. We can replace ZF + LM by ZF + "every set of real numbers has the property of Baire."

3. Epilogue

Is AC relevant to the problem of chromatic number χ of the plane? The answer depends upon the value of χ which we, of course, do not know yet. However, the presented here example points out circumstances in which AC would be quite relevant. We have the following conditional result to report.

Conditional Theorem. [4] *Assume that any finite unit distance plane graph has chromatic number not exceeding* 4. *Then*:

(*) *In ZFC the chromatic number of the plane is* 4.
(**) *In* $ZF + DC + LM$ *the chromatic number of the plane is* 5, 6 *or* 7.

Proof. Claim (*) is true due to [EB].

The system ZF + DC + LM implies that every subset S of the plane R^2 is Lebesgue measurable. Indeed, S is measurable iff there is a Borel set B such that the symmetric difference $S\Delta B$ is null. Thus, every plane set differs from a Borel set by a null set. We can think of a unit segment $I = [0, 1]$ as a set of infinite binary fractions and observe that the bijection $I \to I^2$ defined as $0.a_1a_2 \ldots a_n \ldots \mapsto (0.a_1a_3 \ldots ; 0.a_2a_4 \ldots)$ preserves null sets. Due to Falconer result [F] formulated in the Question above, we can now conclude that the chromatic number of the plane is at least 5. $\square$

Perhaps, the problem of finding the chromatic number of the plane has withstood all assaults in the general case, leaving us with a wide range for χ being 4, 5, 6 or 7 precisely because the answer depends upon the system of axioms we choose for set theory?

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