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J. Differential Equations 215 (2005) 52–107

Journal of Differential Equations

www.elsevier.com/locate/jde

Boundedness vs. blow-up in a chemotaxis system

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Received 24 May 2004; revised 19 October 2004

Available online 15 December 2004

Abstract

We determine the critical blow-up exponent for a Keller–Segel-type chemotaxis model, where the chemotactic sensitivity equals some nonlinear function of the particle density. Assuming some growth conditions for the chemotactic sensitivity function we establish an a priori estimate for the solution of the problem considered and conclude the global existence and boundedness of the solution. Furthermore, we prove the existence of solutions that become unbounded in finite or infinite time in that situation where this a priori estimate fails.

MSC: 35K50; 35K57; 35K60; 92C17

Keywords: Chemotaxis; Global existence; Blow-up in finite or infinite time; Critical blow-up exponent

1. Introduction

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. The movement towards a higher concentration of the chemical substance is termed positive chemotaxis and the movement towards regions of lower chemical concentration is called negative chemotactical movement. Chemotaxis is an important means for cellular communication. Communication

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doi:10.1016/j.jde.2004.10.022

by chemical signals determines how cells arrange and organize themselves, like for instance in development or in living tissues.

In the present paper we consider the problem

$$\left.\begin{aligned} u_t &= \Delta u - \nabla\cdot(f(u)\nabla v), && \text{in} \quad \Omega\times(0,T),\\ v_t &= \Delta v - v + u, && \text{in } \Omega\times(0,T),\\ \frac{\partial}{\partial N}u|_{\partial\Omega} &= 0, \quad \frac{\partial}{\partial N}v|_{\partial\Omega} = 0, &&\\ u|_{t=0} &= u_0, \quad v|_{t=0} = v_0 && \end{aligned}\right\} \tag{1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, where $f\in C^{1+\theta}([0,\infty))$ (for some $\theta>0$) satisfies $f(0)=0$; the initial data u_0 and v_0 are assumed to be nonnegative, where $u_0\in C^0(\bar{\Omega})$ with mass

$$\lambda := \int_\Omega u_0,$$

and $v_0\in\bigcup_{q>n}W^{1,q}(\Omega)$. The symbol $\frac{\partial}{\partial N}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. This problem is a version of the well-known Keller–Segel model in chemotaxis. The function $u(x,t)$ describes the particle density at time t, at position $x\in\Omega$; $v(x,t)$ is the density of the external chemical substance.

The classical chemotaxis model — the so-called Keller–Segel model — has been extensively studied in the last few years (see [17,18] for a recent survey article). The function $f(u)$ denotes a chemotactic sensitivity function. In general, this function depends on the particle density and the external signal. In the present paper, however, we will assume that it only depends on the particle density u. For $f(u)=u$ system (1) equals the most common formulation of the Keller–Segel model. One interesting question in connection with this version of the model is the possibility that the solution of the Keller–Segel model might become unbounded in finite or infinite time for $n=2$ or $n\geqslant 3$ (see [12,15,17,19,26,36] and the references therein).

As mentioned, the chemotactic sensitivity function, in general, may depend on the particle density u and the chemoattractant v, and it is known that it plays a crucial role in the asymptotic behavior of the solution. There have been several attempts to introduce certain reasonable effects into the Keller–Segel equations that might prevent blow-up like volume-filling and quorum sensing aspects. The volume filling aspect is reflected as a certain dependence of the chemotactic sensitivity function on the particle density u, which leads to bounded global-in-time solutions of (1). This has been done for example by Hillen and Painter in [13,33].

However, to our knowledge it has never been analyzed whether the solution of system (1) might become unbounded if $f(u)$ equals other powers of u, i.e. $f(u)=u^\alpha$ with some $\alpha>0$. Of course, this question is more motivated from the mathematical point of view than from the biological one, but it will help to get more insights in the understanding of the blow-up mechanism of the problem. Furthermore, the functional forms in the most common version of the Keller–Segel model are based on simplifying assumptions made by Nanjundiah in [31]. The original paper by Keller and Segel [21] allows more general functional forms. In the present paper we will look at this aspect more carefully and we will determine the critical exponent α which decides whether unbounded solutions can exist or not in dependence of the spatial dimension. Of course, according to the known results, it seems to be clear that for $n = 2$ or $n \geqslant 3$ there exist solutions of (1) that become unbounded for $\alpha > 1$. It is known that for $n \geqslant 3$ and Ω is a sphere there exist radially symmetric solutions of a simplified parabolic–elliptic version of (1) that blow up in finite time if $\alpha = 1$ (see [2,9–12,19,24,25,36]). For the full system (1) no such results are known. However, what happens if $n \geqslant 2$ and $\alpha < 1$?

While for $\alpha = 1$ and $n = 1$ there is no possibility that the solution of this simplified parabolic–elliptic version of (1) blows up, there exists a threshold value for the initial data in spacial dimension $n = 2$ that decides whether the solution can blow up or exists globally in time (see for instance [20]). In case $n \geqslant 3$ and Ω is a sphere there is no such threshold. Thus one wonders whether the existence of unbounded solutions of (1) with $\alpha \in \mathbb{R}_+$ depends on the exponent α. Furthermore, one might expect that the exponent for which unbounded solutions might exist will depend on the underlying space dimension. Therefore we ask, motivated from the mathematical point of view, whether one can determine the “right” blow-up exponent in dependence of the underlying space dimension.

Our main results in connection with this question are the following:

- If $f(s) \leqslant cs^\alpha$ for all $s \geqslant 1$ and some $\alpha < \frac{2}{n}$ then all solutions are global and uniformly bounded. Furthermore, for given $\Lambda > 0$ and $\tau \in (0,1)$ there exists a constant $c(\Lambda, \tau) > 0$ such that the solution satisfies the a priori estimate

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + \bar{K}^m(\tau)\, e^{-\nu t}\Big) \quad \forall t \geqslant \tau,$$

  where $\bar{K}(\tau) := \max_{t \in [\frac{\tau}{4}, \tau]} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)}\Big)$ and ν is some positive constant (cf. Theorem 4.1).
- If $f(s) \geqslant cs^\alpha$ for all $s \geqslant 1$ and some $\alpha > \frac{2}{n}$ (and $n \geqslant 2$) then this a priori estimate fails to be true (Theorem 5.1).

As a conclusion we remark that $\alpha = \frac{2}{n}$ is critical with respect to the validity of this estimate. However, if Ω is a ball in $\mathbb{R}^n$ we can go even further. In this situation we have the following blow-up result:

- If $f(s) \geqslant cs^\alpha$ for some $\alpha > \frac{2}{n}$ and Ω is a ball in $\mathbb{R}^n$, $n \geqslant 2$, then (1) possesses unbounded solutions, provided that one of the following — technical — assumptions is satisfied:
  - $\alpha > 2$ (and nothing else, cf. Theorem 6.1),
  - $\alpha \in (1, 2)$, $n \in \{2, 3\}$ and f fulfills an additional *upper* growth estimate (Theorem 6.2),
  - $\alpha \in (\frac{2}{n}, 1)$ if $n \in \{2, 3\}$ and $\alpha \in (\frac{2}{n}, \frac{2}{n-2})$ if $n \geqslant 4$; in both cases also an upper growth condition has to be imposed on f (Theorem 6.3).

Therefore, we see that $\alpha = \frac{2}{n}$ has been uniquely detected to be the critical blow-up exponent for $n \geqslant 2$. The proof of this blow-up result generalizes some ideas that have been used in [19] to establish the existence of unbounded solutions of system (1) for $\alpha = 1$ and a simply connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. An alternative proof of the blow-up result presented in [19] has been given in [36].

Beside these blow-up results for $n \geqslant 2$, we will see that for $n = 1$ the function v is uniformly bounded in $W^{1,2}(\Omega)$ for all times — a fact that follows from analyzing a Lyapunov functional available for system (1) (see the remark following Lemma 5.1). Accordingly, for $n = 1$ the solution exists globally in time (and remains uniformly bounded) independent of the choice of α.

2. Preliminaries

Let us first collect some tools that will frequently be used in the sequel (see, for instance, [4,5,8,23,37]).

In several places we shall need the following derivate of Poincaré's inequality:

$$\|u\|_{W^{1,p}(\Omega)} \leqslant c\Big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}\Big) \quad \forall u \in W^{1,p}(\Omega)$$

with arbitrary $p > 1$ and $q > 0$. Also, an essential role will be played by the Gagliardo–Nirenberg interpolation inequality

$$\|u\|_{L^p(\Omega)} \leqslant c\|u\|^a_{W^{1,q}(\Omega)} \cdot \|u\|^{1-a}_{L^r(\Omega)} \quad \forall u \in W^{1,q}(\Omega),$$

which holds for all $p, q \geqslant 1$ satisfying $p(n-q) < nq$ and all $r \in (0, p)$ with

$$a = \frac{\frac{n}{r} - \frac{n}{p}}{1 - \frac{n}{q} + \frac{n}{r}} \quad \in (0, 1).$$

(In fact, the classical version in Theorem I.10.1 in [5] is stated only for $r \geqslant 1$, but this restriction can easily be removed upon an application of Hölder's inequality.)

For $p \in (1, \infty)$, let $A := A_p$ denote the sectorial operator defined by

$$A_p u := -\Delta u \text{ for } u \in D(A_p) := \left\{\varphi \in W^{2,p}(\Omega) \;\middle|\; \frac{\partial}{\partial N}\varphi|_{\partial\Omega} = 0\right\}.$$

The fact that the spectrum of A is a p-independent countable set of positive real numbers $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ entails the following consequences:

(i) The operator $A + 1$ possesses fractional powers $(A+1)^\beta$, $\beta \geqslant 0$, the domains of which have the embedding properties

$$D((A_p + 1)^\beta) \hookrightarrow W^{1,p}(\Omega) \qquad \text{if } \beta > \frac{1}{2}$$

and

$$D((A_p+1)^\beta) \hookrightarrow C^\delta(\bar{\Omega}) \qquad \text{if } 2\beta - \frac{n}{p} > \delta \geqslant 0. \tag{2}$$

(ii) The analytic semigroup $(e^{-tA})_{t\geqslant 0}$ (which is independent of p in the sense that

$$e^{-tA_p}u = e^{-tA_q}u$$

whenever $u \in L^p(\Omega) \cap L^q(\Omega)$ satisfies

$$\|(A+1)^\beta e^{-t(A+1)}u\|_{L^p(\Omega)} \leqslant ct^{-\beta}e^{-\nu_1 t}\|u\|_{L^p(\Omega)}$$

for all $u \in L^p(\Omega)$, any $t > 0$ and some $\nu_1 > 0$.

(iii) For each $t > 0$ the operator e^{-tA} maps $L^p(\Omega)$ into $L^q(\Omega)$, with norm controlled according to

$$\|e^{-tA}u\|_{L^q(\Omega)} \leqslant ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\Omega)}$$

for all $t \in (0,1)$ and $1 \leqslant p < q < \infty$. (For $p > 1$ this actually is implied by (ii) via a standard interpolation argument; in the non-standard borderline case $p = 1$ this requires a pointwise estimate on the corresponding Green's function which is provided by Theorem 2.2 in [23].)

(iv) When restricted to the orthogonal complement of the null space of A, e^{-tA} decays exponentially with time in the sense that for all

$$u \in L^p_\perp(\Omega) := \left\{ \varphi \in L^p(\Omega) \;\middle|\; \int_\Omega \varphi = 0 \right\},$$

we have $\|e^{-tA}u\|_{L^p(\Omega)} \leqslant ce^{-\nu_2 t}\|u\|_{L^p(\Omega)}$ for any $t > 0$ and some $\nu_2 > 0$.

As a consequence of (ii) and (iii), we have for all $1 \leqslant p < q < \infty$ and $u \in L^p(\Omega)$ the general $L^p - L^q$ estimate

$$\|(A+1)^\beta e^{-tA}u\|_{L^q(\Omega)} \leqslant ct^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{(1-\mu)t}\|u\|_{L^p(\Omega)}, \tag{3}$$

for any $t > 0$ and $\beta \geqslant 0$ with some $\mu > 0$. After diminishing μ if necessary, from (ii)–(iv) we obtain for all $1 \leqslant p < q < \infty$ and $u \in L^p_\perp(\Omega)$ the restricted counterpart

$$\|(A+1)^\beta e^{-tA}u\|_{L^q(\Omega)} \leqslant ct^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\mu t}\|u\|_{L^p(\Omega)} \tag{4}$$

for $t > 0$ and $\beta \geqslant 0$.

Unlike $(A+1)^\beta$, the divergence operator $\nabla\cdot$ does not commute with e^{-tA}. However, in estimates for expressions like $\|e^{-tA}\nabla\cdot w\|$, this operator does not behave much worse than $(A+1)^{\frac{1}{2}}$, as stated by the following:

Lemma 2.1. *Let $\beta \geqslant 0$ and $p \in (1,\infty)$. Then for all $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for all $w \in C_0^\infty(\Omega)$ we have*

$$\begin{aligned}\|(A+1)^\beta e^{-tA}\nabla\cdot w\|_{L^p(\Omega)} &\leqslant c(\varepsilon)t^{-\beta-\frac{1}{2}-\varepsilon}e^{-\mu t}\|w\|_{L^p(\Omega)} \\ &\leqslant c(\varepsilon)t^{-\beta-\frac{1}{2}-\varepsilon}\|w\|_{L^p(\Omega)} \quad \forall t>0. \qquad (5)\end{aligned}$$

Accordingly, for all $t>0$ the operator $(A+1)^\beta e^{-tA}\nabla\cdot$ admits a unique extension to all of $L^p(\Omega)$ which, again denoted by $(A+1)^\beta e^{-tA}\nabla\cdot$, satisfies (5) *for all $w \in L^p(\Omega)$.*

Proof. Writing

$$\bar\varphi := \frac{1}{|\Omega|}\int_\Omega \varphi$$

for $\varphi \in L^1(\Omega)$, we have

$$\|\varphi-\bar\varphi\|_{L^{p'}(\Omega)} \leqslant 2\|\varphi\|_{L^{p'}(\Omega)}$$

for all $\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$. Consequently, employing the notation

$$C^\infty_{\perp,N}(\bar\Omega) := \left\{\psi \in C^\infty(\bar\Omega) \;\middle|\; \int_\Omega \psi = 0 \text{ and } \frac{\partial}{\partial N}\psi|_{\partial\Omega} = 0\right\}$$

we find that

$$\begin{aligned}&\|(A+1)^\beta e^{-(t-s)A}\nabla\cdot w\|_{L^p(\Omega)} \\ &= \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\varphi\|_{L^{p'}(\Omega)}\leqslant 1}} \left|\int_\Omega (A+1)^\beta e^{-tA}(\nabla\cdot w)\cdot(\varphi-\bar\varphi) + \bar\varphi\cdot\int_\Omega (A+1)^\beta e^{-tA}\nabla\cdot w\right| \\ &= \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\varphi\|_{L^{p'}(\Omega)}\leqslant 1}} \left|\int_\Omega (A+1)^\beta e^{-tA}(\nabla\cdot w)\cdot(\varphi-\bar\varphi)\right| \\ &\leqslant \sup_{\substack{\psi\in C^\infty_{\perp,N}(\bar\Omega)\\ \|\psi\|_{L^{p'}(\Omega)}\leqslant 2}} \left|\int_\Omega (A+1)^\beta e^{-tA}(\nabla\cdot w)\psi\right| = \sup_{\substack{\psi\in C^\infty_{\perp,N}(\bar\Omega)\\ \|\psi\|_{L^{p'}(\Omega)}\leqslant 2}} \left|\int_\Omega w\nabla(A+1)^\beta e^{-tA}\psi\right|\end{aligned}$$

$$\leqslant c\|w\|_{L^p(\Omega)} \cdot \sup_{\substack{\psi\in C^\infty_{\perp,N}(\bar\Omega)\\ \|\psi\|_{L^{p'}(\Omega)}\leqslant 2}} \left\|(A+1)^{\beta+\frac12+\varepsilon}e^{-tA}\psi\right\|_{L^{p'}(\Omega)}$$

$$\leqslant ct^{-(\beta+\frac12+\varepsilon)}e^{-\mu t}\cdot\|w\|_{L^p(\Omega)}$$

by (4). Here we have tacitly used the facts that $A_p\chi$ and $A_2\chi$ coincide for $\chi\in C^\infty(\bar\Omega)$, that A_2 is self-adjoint in $L^2(\Omega)$, and that

$$\|\nabla\chi\|_{L^p(\Omega)}\leqslant c(\varepsilon)\|(A+1)^{\frac12+\varepsilon}\chi\|_{L^p(\Omega)}$$

for all $\varepsilon>0$ and any $\chi\in D(A_p)$ (Lemma ii.17.1 in [5]). This proves the lemma. □

3. Local existence and uniqueness of classical solutions

Let us first establish the existence of a local-in-time smooth solution by employing Banach's fixed point theorem. The proof that the solution is *classical* is the only place in this work where Hölder regularity of f' is required.

Before we state our result let us briefly mention, that the existence of local-in-time smooth solutions for a quite general version of the Keller–Segel model has been established by Yagi in [39]. However, Yagi does not considered chemotactic sensitivity functions which depend on powers of the particle density. Therefore, we cannot apply his results and have to present our own local existence result.

Theorem 3.1. *Suppose $q>n$, and that $u_0\in C^0(\bar\Omega)$ and $v_0\in W^{1,q}(\Omega)$ are nonnegative in Ω. Then there exists $T_{\max}\leqslant\infty$ (depending on $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,q}(\Omega)}$ only) and exactly one pair (u,v) of nonnegative functions*

$$u\in C^0([0,T_{\max});C^0(\bar\Omega))\cap C^{2,1}(\bar\Omega\times(0,T_{\max})),$$
$$v\in C^0([0,T_{\max});C^0(\bar\Omega))\cap L^\infty_{loc}([0,T_{\max});W^{1,q}(\Omega))\cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$$

that solves (1) *in the classical sense. If $T_{\max}<\infty$ then*

$$\lim_{t\to T_{\max}}\left(\|u(t)\|_{L^\infty(\Omega)}+\|v(t)\|_{W^{1,q}(\Omega)}\right)=\infty. \tag{6}$$

Moreover, the solution (u,v) satisfies the mass identities

$$\int_\Omega u(t)=\int_\Omega u_0\quad\forall t\in(0,T_{\max}) \tag{7}$$

and

$$\int_\Omega v(t) = \int_\Omega u_0 + \Big(\int_\Omega v_0 - \int_\Omega u_0\Big)e^{-t} \quad \forall t \in (0, T_{\max}). \tag{8}$$

Proof. *Existence*: The existence proof follows a standard contraction argument. We extend f to all of $\mathbb{R}$ by defining $f(s) := f'(0) \cdot s$ for $s < 0$, whereby f becomes an element of $C^{1+\theta}(\mathbb{R})$. With numbers $T \in (0, 1)$ and $R > 0$ to be fixed below, in the Banach space

$$X := C^0([0, T]; C^0(\bar{\Omega})) \times L^\infty((0, T); W^{1,q}(\Omega))$$

we consider the closed set

$$S := \Big\{(u, v) \in X \mid \|(u, v)\|_X \leqslant R\Big\}$$

and claim that for R sufficiently large and T small enough, the map

$$\Psi(u, v)(t) := \begin{pmatrix} \Psi_1(u, v)(t) \\ \Psi_2(u, v)(t) \end{pmatrix} := \begin{pmatrix} e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\nabla \cdot (f(u(s))\nabla v(s))\,ds \\ e^{-t(A+1)}v_0 + \int_0^t e^{-(t-s)(A+1)}u(s)\,ds \end{pmatrix},$$

for $t \in [0, T]$, is a contraction from S into itself.

To see this, we first observe that, for $(u, v) \in S$, $\Psi_1(u, v)$ is continuous on $[0, T]$ with values in $C^0(\bar{\Omega})$, because $u_0 \in C^0(\bar{\Omega})$ and e^{-tA} is strongly continuous in $C^0(\bar{\Omega})$ due to the maximum principle. Also, $\Psi_2(u, v)$ is bounded on $(0, T)$ as a $W^{1,q}(\Omega)$-valued function. This is a consequence of the fact that $\|e^{-t(A+1)}v_0\|_{W^{1,q}(\Omega)} \leqslant c\|v_0\|_{W^{1,q}(\Omega)}$ which is valid for $q = 2$ (by a simple energy argument) and $q = \infty$ (cf. [22, pp. 478 ff.]) and thus, via a standard interpolation technique, also for $q \in (2, \infty)$ (see e.g. Theorem 9.8 in [7]). In the case $n = 1$ a differentiation of the heat equation with respect to x (involving zero Dirichlet boundary data) shows that the same estimate even holds for all $q > 1$.

Next, we let $M(R) := \|f\|_{L^\infty((-R,R))}$ and $L(R) > 0$ denote a Lipschitz constant for f on $(-R, R)$ and fix $\beta \in (\frac{n}{2q}, \frac{1}{2})$ as well as $\varepsilon \in (0, \frac{1}{2} - \beta)$.

Since $D((A_q + 1)^\beta) \hookrightarrow C^0(\bar{\Omega})$ in this case, we can estimate with the aid of Lemma 2.1

$$\begin{aligned} \|\Psi_1(u, v)(t)\|_{C^0(\bar{\Omega})} \leqslant\ & \|e^{-tA}u_0\|_{C^0(\bar{\Omega})} \\ & + c\int_0^t \Big\|(A + 1)^\beta e^{-(t-s)A}\nabla \cdot (f(u(s))\nabla v(s))\Big\|_{L^q(\Omega)} ds \end{aligned}$$

$$\leqslant \|u_0\|_{C^0(\bar\Omega)} + c\int_0^t (t-s)^{-\beta-\frac{1}{2}-\varepsilon}\|f(u(s))\nabla v(s)\|_{L^q(\Omega)}\,ds$$

$$\leqslant \|u_0\|_{C^0(\bar\Omega)} + cM(R)RT^{\frac{1}{2}-\beta-\varepsilon} \quad \forall t\in[0,T]. \tag{9}$$

Moreover, picking $\gamma\in(\frac{1}{2},1)$ we have, using (2),

$$\begin{aligned}\|\Psi_2(u,v)(t)\|_{W^{1,q}(\Omega)} &\leqslant \|e^{-t(A+1)}v_0\|_{W^{1,q}(\Omega)}\\ &\quad + c\int_0^t \|(A+1)^\gamma e^{-(t-s)(A+1)}u(s)\|_{L^q(\Omega)}\,ds\\ &\leqslant c\|v_0\|_{W^{1,q}(\Omega)} + c\int_0^t (t-s)^{-\gamma}\|u(s)\|_{L^q(\Omega)}\,ds\\ &\leqslant c\|v_0\|_{W^{1,q}(\Omega)} + cRT^{1-\gamma} \quad \forall t\in[0,T].\end{aligned} \tag{10}$$

From (9) and (10) it results that $\Psi S\subset S$ if we choose first R large and then T small. With this value of R fixed (but T still at our disposal), we proceed to check that for all $(u,v),(\bar u,\bar v)\in S$,

$$\begin{aligned}&\|\Psi_1(u,v)(t)-\Psi_1(\bar u,\bar v)(t)\|_{C^0(\bar\Omega)}\\ &\leqslant c\int_0^t \left\|(A+1)^\beta e^{-(t-s)A}\nabla\cdot[f(u(s))\nabla v(s)-f(\bar u(s))\nabla\bar v(s)]\right\|_{L^q(\Omega)}ds\\ &\leqslant c\int_0^t (t-s)^{-\beta-\frac{1}{2}-\varepsilon}\|f(u(s))\nabla v(s)-f(\bar u(s))\nabla\bar v(s)\|_{L^q(\Omega)}\,ds\\ &\leqslant c\Big(L(R)R+M(R)\Big)T^{\frac{1}{2}-\beta-\varepsilon}\|(u,v)-(\bar u,\bar v)\|_X \quad \forall t\in[0,T]\end{aligned}$$

and

$$\begin{aligned}&\|\Psi_2(u,v)(t)-\Psi_2(\bar u,\bar v)(t)\|_{W^{1,q}(\Omega)}\\ &\leqslant c\int_0^t \|(A+1)^\beta e^{-(t-s)(A+1)}(u(s)-\bar u(s))\|_{L^q(\Omega)}\,ds\\ &\leqslant c\int_0^t (t-s)^{-\gamma}\|u(s)-\bar u(s)\|_{L^q(\Omega)}\,ds\\ &\leqslant cT^{1-\gamma}\|(u,v)-(\bar u,\bar v)\|_X \quad \forall t\in[0,T],\end{aligned}$$

so that Ψ is shown to be a contraction if T is sufficiently small. From Banach's fixed point theorem we therefore obtain the existence of $(u,v)\in X$ satisfying $(u,v) = \Psi(u,v)$.

Since the above choice of T depends only on $\|u_0\|_{L^\infty(\Omega)}+\|v_0\|_{W^{1,q}(\Omega)}$, it is clear by a standard argument that (u,v) can be extended up to some $T_{\max}\leqslant\infty$, where necessarily (6) holds in case of $T_{\max}<\infty$. Clearly, u and v are weak solutions — in the natural sense defined in [22, p. 136] — of their respective equations in (1).

Regularity: Since $v_0\in W^{1,q}(\Omega)\hookrightarrow C^0(\bar\Omega)$, the relation $v=\Psi_2(u,v)$ immediately shows that $v\in C^0([0,T_{\max});C^0(\bar\Omega))$. Relying on this, the inclusions $u,v\in C^{2,1}(\bar\Omega\times(0,T_{\max}))$ result from straightforward regularity arguments including standard semigroup techniques, parabolic Schauder estimates (Theorem IV.5.3 in [22]) and Lemma 2.1.

We now can apply the comparison principle for classical sub- and supersolutions of scalar parabolic equations to conclude first that $u\geqslant 0$ (because $\underline{u}\equiv 0$ is a subsolution of the first in (1) due to $f(0)=0$) and then that $v\geqslant 0$ (since we know that $u\geqslant 0$, whence $\underline{v}\equiv 0$ is a subsolution of the second in (1)).

Properties (7) and (8) easily follow by integrating the PDEs in (1) in space.

Uniqueness: Let us finally prove uniqueness of solutions in the indicated class by assuming there were two different solutions (u,v) and $(\bar u,\bar v)$ on some interval $[0,T]$. Letting $w:=u-\bar u$ and $z:=v-\bar v$, for $t\in(0,T)$ we obtain upon subtracting the respective equations in (1) and performing obvious testing procedures the identities

$$\int_\Omega z_t^2+\frac{d}{dt}\Big[\frac12\int_\Omega|\nabla z|^2+\frac12\int_\Omega z^2\Big]=\int_\Omega wz_t$$
$$=-\int_\Omega\nabla w\cdot\nabla z-\int_\Omega wz+\int_\Omega w^2 \qquad (11)$$

and

$$\frac12\frac{d}{dt}\int_\Omega w^2+\int_\Omega|\nabla w|^2=\int_\Omega[f(u)\nabla v-f(\bar u)\nabla\bar v]\cdot\nabla w. \qquad (12)$$

Since u and $\bar u$ are bounded on $\Omega\times[0,T]$, we have $|f(u)-f(\bar u)|\leqslant L|w|$ and $f(\bar u)\leqslant M$ in this region with some positive L and M, whence

$$\Big|\int_\Omega[f(u)\nabla v-f(\bar u)\nabla\bar v]\cdot\nabla w\Big|\leqslant\frac14\int_\Omega|\nabla w|^2+c\Big(L^2\int_\Omega w^2|\nabla v|^2+M^2\int_\Omega|\nabla z|^2\Big). \qquad (13)$$

As $\int_\Omega u(t)=\int_\Omega\bar u(t)\equiv\int_\Omega u_0$ for all t by (7), we have $\int_\Omega w(t)\equiv 0$ and hence the standard Poincaré inequality ensures that $\|w\|_{W^{1,2}(\Omega)}\leqslant c\|\nabla w\|_{L^2(\Omega)}$. Therefore, once again relying on the fact that $q>n$, we can estimate with the help of the Hölder and the Gagliardo–Nirenberg inequality

$$\int_\Omega w^2|\nabla v|^2\leqslant\Big(\int_\Omega|\nabla v|^q\Big)^{\frac2q}\cdot\Big(\int_\Omega|w|^{\frac{2q}{q-2}}\Big)^{\frac{q-2}{q}}\leqslant c\|\nabla z\|_{L^2(\Omega)}^{\frac{2n}{q}}\cdot\|w\|_{L^2(\Omega)}^{\frac{2(q-n)}{q}}$$
$$\leqslant\varepsilon\|\nabla z\|_{L^2(\Omega)}^2+c(\varepsilon)\|w\|_{L^2(\Omega)}^2, \qquad (14)$$

where $\varepsilon > 0$ is arbitrary. Moreover,

$$-\int_\Omega \nabla w \cdot \nabla z \leqslant \frac{1}{4}\int_\Omega |\nabla w|^2 + \int_\Omega |\nabla z|^2 \tag{15}$$

and

$$-\int_\Omega wz \leqslant \frac{1}{2}\int_\Omega w^2 + \frac{1}{2}\int_\Omega z^2, \tag{16}$$

so that adding (11) to (12) yields, taking into account (12)–(16) and omitting positive terms,

$$\frac{d}{dt}\Big(\int_\Omega |\nabla z|^2 + \int_\Omega z^2 + \int_\Omega w^2\Big) \leqslant c\Big(\int_\Omega |\nabla z|^2 + \int_\Omega z^2 + \int_\Omega w^2\Big) \quad \forall t \in (0,T).$$

Now Gronwall's lemma says that $z \equiv w \equiv 0$, as desired. □

The local-in-time existence and uniqueness of a solution for system (1) with $n = 2$ and $\alpha = 1$ has also been established by Gajewski and Zacharias in [6] and — as already mentioned — by Yagi in [39]. However their results cannot be applied to our generalized system.

4. Boundedness in case of subcritical growth

Let us first look a little bit closer at that situation that we will later call the case of subcritical growth for the chemotactic sensitivity $f(u)$. Therefore, we now assume that f satisfies the one-sided growth condition

$$f(s) \leqslant c_0 s^\alpha \quad \forall s \in (1,\infty) \tag{17}$$

for some $c_0 > 0$ and some $\alpha > 0$ (which will actually throughout this section be supposed to fulfill $\alpha < \frac{2}{n}$). Since f is continuous, we of course may equivalently — and more conveniently for our proofs — require

$$f(s) \leqslant c_0(s+1)^\alpha \quad \forall s > 0 \quad \text{with } \alpha \in \Big(0, \frac{2}{n}\Big) \tag{18}$$

for some $c_0 > 0$.

Also for convenience in notation, let us abbreviate

$$\Lambda := \max\Big\{\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}\Big\}.$$

The main result of this section, the a priori estimate in Theorem 4.1, will be obtained as the final in a series of steps. The basic idea is to use the $L^1(\Omega)$-bounds (8) and — mainly — (7) as the initializing information in an iterative bootstrap procedure, which at its starting point uses both equations in (1) (see Lemma 4.3), but then alternately exploits the second (Lemma 4.1) and the first equation (Lemma 4.4) in (1) to successively establish estimates in higher L^p spaces. The complete iteration is carried out in Lemma 4.5 which will reach all $p < \infty$, while the final step towards L^∞ is accomplished in Theorem 4.1.

The first auxiliary lemma asserts that an a bound for u in $L^\gamma(\Omega)$ for $t \geqslant \tau$ implies an estimate for v in some $W^{1,q}(\Omega)$ for all t bounded away from τ. The proof exclusively uses the second equation in (1). In this lemma, as throughout this section, all appearing constants are independent of $T_{\max}$.

Lemma 4.1. *Assume that there exist* $\tau \in (0, \min\{1, T_{\max}\})$ *and* $\gamma \in [1, n]$ *such that*

$$\|u(t)\|_{L^\gamma(\Omega)} \leqslant c_1 \quad \forall t \in [\tau, T_{\max}).$$

Then for any $\eta \in (0, T_{\max} - \tau)$,

$$\|v(t)\|_{W^{1,q}(\Omega)} \leqslant c(q, \Lambda, \tau, \eta)(1 + c_1) \quad \forall t \in [\tau + \eta, T_{\max})$$

holds for all $q > 1$ *satisfying*

$$q < \frac{n\gamma}{n - \gamma}.$$

Proof. We fix $q < \frac{n\gamma}{n-\gamma} = \frac{1}{\frac{1}{\gamma} - \frac{1}{n}}$ and choose some $\beta > \frac{1}{2}$ such that

$$q < \frac{1}{\frac{1}{\gamma} - \frac{1}{n} + \frac{2}{n}(\beta - \frac{1}{2})}. \tag{19}$$

Applying $(A + 1)^\beta$ to both sides of the representation formula

$$v(t) = e^{-(t-\tau)(A+1)} v(\tau) + \int_\tau^t e^{-(t-s)(A+1)} u(s)\, ds, \quad t \in [\tau, T_{\max}),$$

we obtain in the case $q \geqslant 2$, using (3) and (8),

$$\begin{aligned} \|(A + 1)^\beta v(t)\|_{L^q(\Omega)} \leqslant\ & c(q) \int_\tau^t (t - s)^{-\beta - \frac{n}{2}(\frac{1}{\gamma} - \frac{1}{q})} e^{-\mu(t-s)} \|u(s)\|_{L^\gamma(\Omega)}\, ds \\ & + c(t - \tau)^{-\beta - \frac{n}{2}(1 - \frac{1}{q})} \|v(\tau)\|_{L^1(\Omega)} \end{aligned}$$

$$\leqslant c\eta^{-\beta-\frac{n}{2}(1-\frac{1}{q})}\|v(\tau)\|_{L^1(\Omega)} + c\cdot c_1\int_0^\infty \sigma^{-\beta-\frac{n}{2}(\frac{1}{\gamma}-\frac{1}{q})}e^{-\mu\sigma}\,d\sigma$$

$$\leqslant c(q,\Lambda,\tau,\eta)(1+c_1) \quad \forall t\in[\tau+\eta,T_{\max}),$$

because $\beta+\frac{n}{2}(\frac{1}{\gamma}-\frac{1}{q})<1$ due to (19). As $\beta>\frac{1}{2}$ entails $D((A_q+1)^\beta)\hookrightarrow W^{1,q}(\Omega)$ by (2), the claim follows. □

In the proof of Lemma 4.3 we need the following elementary variant of Young's inequality, the proof of which is left to the reader.

Lemma 4.2. *Let r and s be nonnegative real numbers satisfying $r+s<2$. Then for any $\varepsilon>0$ there exists a constant $c_\varepsilon>0$ such that*

$$a^r b^s \leqslant \varepsilon(a^2+b^2)+c_\varepsilon \quad \forall a,b>0.$$

We now see how the L^1-bound (7) can be improved to an L^γ-estimate for some $\gamma>1$ by using both equations in (1) simultaneously. Here the condition $\alpha<\frac{2}{n}$ plays an essential role. A simplified variant of our procedure was performed in [29].

Lemma 4.3. *Suppose $n\geqslant 2$ and f satisfies* (18) *with some $\alpha<\frac{2}{n}$. Then there exist*

$$\gamma>\max\left\{2-\frac{2}{n},2-2\alpha\right\}$$

and $\nu>0$ such that for all $\tau\in(0,\min\{1,T_{\max}\})$ we have

$$\|u(t)\|_{L^\gamma(\Omega)}\leqslant c(\Lambda,\tau)\left(1+\left(\|u(\tau)\|_{L^\gamma(\Omega)}+\|\nabla v(\tau)\|^{\frac{2}{\gamma}}_{L^2(\Omega)}\right)e^{-\nu t}\right) \quad \forall t\in[\tau,T_{\max}). \tag{20}$$

Proof. With $\gamma>\max\{1,2-2\alpha\}$ to be fixed below, we multiply the first in (1) by $(u+1)^{\gamma-1}$ and the second in (1) by $(-\Delta v)$ to obtain for $t\in[\tau,T_{\max})$, using (18),

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega (u+1)^\gamma+(\gamma-1)\int_\Omega(u+1)^{\gamma-2}|\nabla u|^2=(\gamma-1)\int_\Omega(u+1)^{\gamma-2}f(u)\nabla u\nabla v.$$

Now we see that

$$\int_\Omega(u+1)^{\gamma-2}f(u)\nabla u\nabla v \leqslant c_0\int_\Omega(u+1)^{\alpha+\gamma-2}|\nabla u\nabla v|$$

$$\leqslant \frac{1}{2}\int_\Omega(u+1)^{\gamma-2}|\nabla u|^2+\frac{c_0^2}{2}\int_\Omega(u+1)^{2\alpha+\gamma-2}|\nabla v|^2$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega u\Delta v \leqslant \frac{1}{2}\int_\Omega |\Delta v|^2 + \frac{1}{2}\int_\Omega u^2. \tag{21}$$

Writing $w := (u+1)^{\frac{\gamma}{2}}$, we conclude that

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega w^2 + \frac{2(\gamma-1)}{\gamma^2}\int_\Omega |\nabla w|^2 \leqslant \frac{(\gamma-1)c_0^2}{2}\int_\Omega w^{\frac{2(2\alpha+\gamma-2)}{\gamma}}|\nabla v|^2. \tag{22}$$

By Hölder's inequality, we have

$$\int_\Omega w^{\frac{2(2\alpha+\gamma-2)}{\gamma}}|\nabla v|^2 \leqslant \|w\|_{L^{\frac{2p(2\alpha+\gamma-2)}{\gamma}}(\Omega)}^{\frac{2(2\alpha+\gamma-2)}{\gamma}} \cdot \|\nabla v\|_{L^{2p'}(\Omega)}^2$$

for any $p>1$, where $\frac{1}{p}+\frac{1}{p'}=1$. Now if

$$-\frac{n\gamma}{2p(2\alpha+\gamma-2)} < 1-\frac{n}{2} \tag{23}$$

and

$$p(2\alpha+\gamma-2) > 1, \tag{24}$$

we can use the Gagliardo–Nirenberg inequality and the Poincaré inequality to estimate

$$\begin{aligned}\|w(t)\|_{L^{\frac{2p(2\alpha+\gamma-2)}{\gamma}}(\Omega)}^{\frac{2(2\alpha+\gamma-2)}{\gamma}} &\leqslant c\|w(t)\|_{W^{1,2}(\Omega)}^{\frac{2(2\alpha+\gamma-2)}{\gamma}a} \cdot \|w(t)\|_{L^{\frac{2}{\gamma}}(\Omega)}^{\frac{2(2\alpha+\gamma-2)}{\gamma}(1-a)} \\ &\leqslant c(\Lambda)\Big(\|\nabla w(t)\|_{L^2(\Omega)}^{\frac{2(2\alpha+\gamma-2)}{\gamma}a}+1\Big) \quad \forall\, t\in(0,T_{\max})\end{aligned}$$

with

$$a = \frac{\frac{n\gamma}{2}-\frac{n\gamma}{2p(2\alpha+\gamma-2)}}{1-\frac{n}{2}+\frac{n\gamma}{2}} \in (0,1),$$

where we observe that

$$\|w(t)\|_{L^{\frac{2}{\gamma}}(\Omega)}^{\frac{2}{\gamma}} = \|u_0\|_{L^1(\Omega)} + |\Omega| \quad \forall t\in(0,T_{\max}) \tag{25}$$

due to the mass conservation property (7). Next, from Lemma 4.1 (applied to $\gamma := 1$) we have

$$\|\nabla v(t)\|_{L^q(\Omega)} \leqslant c(q, \Lambda, \tau) \quad \forall t \in [\tau, T_{\max})$$

for any $q < \frac{n}{n-1}$ and thus, if

$$p' < \frac{n}{n-2}, \tag{26}$$

we can again employ the Gagliardo–Nirenberg inequality to obtain

$$\begin{aligned} \|\nabla v(t)\|^2_{L^{2p'}(\Omega)} &\leqslant c\|\nabla v(t)\|^{2b}_{W^{1,2}(\Omega)} \cdot \|\nabla v(t)\|^{2(1-b)}_{L^q(\Omega)} \\ &\leqslant c(\Lambda, \tau)\|\Delta v(t)\|^{2b}_{L^2(\Omega)} \quad \forall t \in [\tau, T_{\max}) \end{aligned} \tag{27}$$

with

$$b = \frac{\frac{n}{q} - \frac{n}{2p'}}{1 - \frac{n}{2} + \frac{n}{q}} \in (0, 1).$$

(Note here that if $q < \frac{n}{n-1}$ then $q < 2$ and hence $2p' > q$.) In deriving the second inequality in (27) we have used that $-\Delta$ acts as an isomorphism from

$$D := \left\{ \varphi \in W^{2,2}(\Omega) \;\middle|\; \frac{\partial \varphi}{\partial N}|_{\partial\Omega} = 0 \text{ and } \int_\Omega \varphi = 0 \right\}$$

to $L^2(\Omega)$; therefore, since $v(t) - \bar{v}(t)$ is in D with $\bar{v}(t) := \frac{1}{|\Omega|}\int_\Omega v(t)$, it follows that

$$\begin{aligned} \|\nabla v(t)\|_{W^{1,2}(\Omega)} &= \|\nabla(v(t) - \bar{v}(t))\|_{W^{1,2}(\Omega)} \leqslant \|v(t) - \bar{v}(t)\|_{W^{2,2}(\Omega)} \\ &\leqslant c\|\Delta(v(t) - \bar{v}(t))\|_{L^2(\Omega)} = c\|\Delta v(t)\|_{L^2(\Omega)}. \end{aligned}$$

As a result, we see that

$$\int_\Omega w^{\frac{2(2\alpha+\gamma-2)}{\gamma}}(t)|\nabla v(t)|^2 \leqslant c(\Lambda, \tau)(\|\nabla w(t)\|^r_{L^2(\Omega)} + 1) \cdot \|\Delta v(t)\|^s_{L^2(\Omega)}$$

for all $t \in [\tau, T_{\max})$, where

$$r = \frac{2(2\alpha + \gamma - 2)}{\gamma} a = \frac{(2\alpha + \gamma - 2)n - \frac{n}{p}}{1 + (\gamma - 1)\frac{n}{2}}$$

and

$$s = 2b = \frac{\frac{2n}{q} - \frac{n}{p'}}{1 - \frac{n}{2} + \frac{n}{q}}.$$

Thus, if $r + s < 2$, that is, if

$$\kappa(\alpha, \gamma, p, q) := \frac{(2\alpha + \gamma - 2)n - \frac{n}{p}}{1 + (\gamma - 1)\frac{n}{2}} + \frac{\frac{2n}{q} - \frac{n}{p'}}{1 - \frac{n}{2} + \frac{n}{q}} < 2, \tag{28}$$

then Lemma 4.2 says that

$$\int_\Omega w^{\frac{2(2\alpha+\gamma-2)}{\gamma}}(t)|\nabla v(t)|^2 \leqslant \frac{\gamma - 1}{2\gamma^2}\int_\Omega |\nabla w(t)|^2 + \frac{1}{4}\int_\Omega |\Delta v(t)|^2 + c(\Lambda, \tau) \tag{29}$$

for all $t \in [\tau, T_{\max})$. As to the right-hand side of (21), we interpolate similarly and recall (25) to obtain

$$\begin{aligned}\int_\Omega u^2(t) \leqslant \|w(t)\|^{\frac{4}{\gamma}}_{L^{\frac{4}{\gamma}}(\Omega)} &\leqslant c\|w(t)\|^{\frac{4}{\gamma}d}_{W^{1,2}(\Omega)}\|w(t)\|^{\frac{4}{\gamma}(1-d)}_{L^{\frac{2}{\gamma}}(\Omega)} \\ &\leqslant c(\Lambda)\Big(\|\nabla w(t)\|^{\frac{4}{\gamma}d}_{L^2(\Omega)} + 1\Big) \quad \forall t \in (0, , T_{\max})\end{aligned}$$

with

$$d = \frac{n\gamma}{4(1 - \frac{n}{2} + \frac{n\gamma}{2})} \in (0, 1),$$

provided that $\gamma > \frac{2n-4}{n}$. If even

$$\gamma > 2 - \frac{2}{n} \tag{30}$$

then $\frac{4}{\gamma}d < 2$ and therefore by Young's inequality

$$\int_\Omega u^2(t) \leqslant \frac{\gamma - 1}{2\gamma^2}\int_\Omega |\nabla w(t)|^2 + c(\Lambda) \quad \forall t \in (0, T_{\max}). \tag{31}$$

Let us summarize: Adding (21)–(22) and using (29) and (31), we conclude that if (23), (24), (26), (28) and (30) are satisfied then

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{\gamma}\int_\Omega w^2 + \frac{1}{2}\int_\Omega |\nabla v|^2\Big) + \Big(\frac{\gamma - 1}{\gamma^2}\int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\Delta v|^2\Big) \\ &\quad \leqslant c(\Lambda, \tau) \quad \forall t \in [\tau, T_{\max}).\end{aligned}$$

Since

$$\int_\Omega |\nabla w|^2 \geqslant c\left(\int_\Omega w^2 - 1\right)$$

by the Poincaré inequality and

$$\int_\Omega |\Delta v|^2 \geqslant c\int_\Omega |\nabla v|^2$$

(see the remark following (27)), Gronwall's lemma yields

$$\int_\Omega w^2(t) + \int_\Omega |\nabla v|^2(t) \leqslant c(\Lambda,\tau)\left(1 + \left(\int_\Omega w^2(\tau) + \int_\Omega |\nabla v|^2(\tau)\right) e^{-\tilde{v}(t-\tau)}\right) \quad \forall\, t \in [\tau, T_{\max})$$

for some $\tilde{v} > 0$. In particular, in this case

$$\int_\Omega (u+1)^\gamma(t) \leqslant c(\Lambda,\tau)\left(1 + \left(\int_\Omega (u+1)^\gamma(\tau) + \int_\Omega |\nabla v|^2(\tau)\right) e^{-\tilde{v}t}\right) \quad \forall t \in [\tau, T_{\max})$$

holds. Since this implies the desired estimate, all that remains to be shown is that (23), (24), (26), (28) and (30) can be fulfilled simultaneously.

To this end, we observe that (23) is equivalent to

$$p < \frac{n\gamma}{(n-2)(2\alpha+\gamma-2)},$$

while (24) and (26) mean

$$p > \frac{n}{2} \quad \text{and} \quad p > \frac{1}{2\alpha+\gamma-2},$$

so that (23), (24) and (26) can be achieved for some $p \in (1,\infty)$ (that will be fixed henceforth) if and only if $\frac{n\gamma}{n-2} > 1$ – which is trivial for $\gamma > 1$ — and

$$(n-4)\gamma < 2(n-2)(1-\alpha).$$

Since $\alpha < \frac{2}{n}$, this is satisfied whenever either $n \leqslant 4$ or $\gamma < 2\frac{(n-2)^2}{n(n-4)}$ and thus particularly if $\gamma < 2$. Accordingly, we need to verify that (28) holds with some $q < \frac{n}{n-1}$ and some $\gamma \in (\max\{2-\frac{2}{n}, 2-2\alpha\}, 2)$. To see this, we first assume $2-2\alpha \leqslant 2-\frac{2}{n}$ and consider the limit $q \to \frac{n}{n-1}$ and $\gamma \to 2 - \frac{2}{n}$ to obtain for the left-hand side in (28)

$$\kappa\left(\alpha, 2-\frac{2}{n}, p, \frac{n}{n-1}\right) = \frac{(2\alpha - \frac{2}{n})n - \frac{n}{p}}{1 + \frac{n-2}{n} \cdot \frac{n}{2}} + \frac{2(n-1) - n + \frac{n}{p}}{1 - \frac{n}{2} + n - 1}$$
$$= 4\alpha - \frac{8}{n} + 2.$$

Since $\alpha < \frac{2}{n}$, we thus have $\kappa(\alpha, \gamma, p, q) < 2$ for γ and q sufficiently close to $2 - \frac{2}{n}$ and $\frac{n}{n-1}$, respectively. If $2 - 2\alpha > 2 - \frac{2}{n}$, however, then similarly

$$\kappa(\alpha, 2-2\alpha, p, \frac{n}{n-1}) = 2\left(1 - \frac{2}{n} + \frac{1}{p}\right) < 2$$

due to $p > \frac{n}{2}$, whence we conclude that $\kappa(\alpha, \gamma, p, q) < 2$ for γ close to $2-2\alpha > 2-\frac{2}{n}$ and q near $\frac{n}{n-1}$. Upon these respective choices of γ, estimate (20) is thereby proved for any value of $\alpha \in (0, \frac{2}{n})$. $\square$

The next lemma uses only the first equation in (1) to derive from given bounds for u and v a better one for u.

Lemma 4.4. *Suppose u and v satisfy the estimates*

$$\|u(t)\|_{L^{\gamma_0}(\Omega)} \leqslant c_1 \quad \textit{and} \quad \|\nabla v(t)\|_{L^{q_0}(\Omega)} \leqslant c_1 \quad \forall t \in [\tau, T_{\max}) \tag{32}$$

for some $\tau \in (0, \min\{1, T_{\max}\})$, $c_1 > 0$ and numbers $\gamma_0 \geqslant 1$ and $q_0 > 2$ satisfying

$$\left(\frac{n}{q_0} - 1\right)\gamma_0 < n(1-\alpha). \tag{33}$$

Then for any $\gamma > \max\{\gamma_0, 2-2\alpha\}$ which fulfills

$$\left(\frac{n}{q_0} - 1\right)\gamma < (n-2)(1-\alpha), \tag{34}$$

there exist positive constants $c(\gamma), m = m(\gamma)$ and $\nu = \nu(\gamma)$ such that

$$\|u(t)\|_{L^{\gamma}(\Omega)} \leqslant c(\gamma)\left(1 + c_1^m + \|u(\tau)\|_{L^{\gamma}(\Omega)} e^{-\nu t}\right) \quad \forall t \in [\tau, T_{\max})$$

holds.

Proof. Throughout the proof, by m we denote a generic positive constant which may vary from line to line and which depends only on γ. Similar to the proof of

Lemma 4.3, we test the first in (1) by $(u+1)^{\gamma-1}$ and write $w := (u+1)^{\frac{\gamma}{2}}$ to see that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{\gamma}\int_\Omega w^2 + \frac{2}{\gamma-1}\gamma^2\int_\Omega |\nabla w|^2 \\
&\quad\leqslant \frac{(\gamma-1)c_0^2}{2}\int_\Omega w^{\frac{2(2\alpha+\gamma-2)}{\gamma}}|\nabla v|^2 \\
&\quad\leqslant \frac{(\gamma-1)c_0^2}{2}\left(\int_\Omega |\nabla v|^{q_0}\right)^{\frac{2}{q_0}}\left(\int_\Omega w^{\frac{2q_0(2\alpha+\gamma-2)}{(q_0-2)\gamma}}\right)^{\frac{q_0-2}{q_0}} \\
&\quad\leqslant c(\gamma)(1+c_1^m)\|w\|^{\frac{2(2\alpha+\gamma-2)}{\gamma}}_{L^{\frac{2q_0(2\alpha+\gamma-2)}{(q_0-2)\gamma}}(\Omega)} \qquad \forall\, t\in[\tau, T_{\max}).
\end{aligned} \tag{35}$$

As (34) is equivalent to

$$-\frac{(q_0-2)n\gamma}{2q_0(2\alpha+\gamma-2)} < 1-\frac{n}{2},$$

we may apply the Gagliardo–Nirenberg and the Poincaré inequality in estimating

$$\begin{aligned}
\|w(t)\|^{\frac{2(2\alpha+\gamma-2)}{\gamma}}_{L^{\frac{2q_0(2\alpha+\gamma-2)}{(q_0-2)\gamma}}(\Omega)} &\leqslant c(\gamma)\|w(t)\|^{\frac{2(2\alpha+\gamma-2)}{\gamma}a}_{W^{1,2}(\Omega)}\cdot\|w(t)\|^{\frac{2(2\alpha+\gamma-2)}{\gamma}(1-a)}_{L^{\frac{2\gamma_0}{\gamma}}(\Omega)} \\
&\leqslant c(\gamma)(1+c_1^m)\left(\|\nabla w(t)\|^{\frac{2(2\alpha+\gamma-2)}{\gamma}a}_{L^2(\Omega)}+1\right) \quad \forall\, t\in[\tau, T_{\max}),
\end{aligned} \tag{36}$$

where we have used that

$$\|w(t)\|^{\frac{2\gamma_0}{\gamma}}_{L^{\frac{2\gamma_0}{\gamma}}(\Omega)} \leqslant c(1+c_1^m)$$

for $t\in[\tau, T_{\max})$ by (32), and have set

$$a = \frac{\frac{n\gamma}{2\gamma_0}-\frac{(q_0-2)n\gamma}{2q_0(2\alpha+\gamma-2)}}{1-\frac{n}{2}+\frac{n\gamma}{2\gamma_0}} \in (0,1).$$

Since (33) means that

$$(2\alpha+\gamma-2)\frac{n}{\gamma_0}-\frac{q_0-2}{q_0}n < 2-n+\frac{n\gamma}{\gamma_0},$$

we have

$$\frac{2(2\alpha+\gamma-2)}{\gamma}a = \frac{(2\alpha+\gamma-2)\frac{n}{\gamma_0} - \frac{q_0-2}{q_0}n}{1-\frac{n}{2}+\frac{n\gamma}{2\gamma_0}} < 2,$$

so that Young's inequality applied to (36) yields

$$\|w(t)\|_{L^{\frac{2q_0(2\alpha+\gamma-2)}{(q_0-2)\gamma}}(\Omega)}^{\frac{2(2\alpha+\gamma-2}{\gamma}} \leqslant \frac{\gamma-1}{\gamma^2}\int_\Omega |\nabla w(t)|^2 + c(\gamma)(1+c_1^m) \quad \forall\, t \in [\tau, T_{\max}).$$

Inserted into (35), this entails, again by the Poincaré inequality,

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_\Omega w^2 &\leqslant -\frac{\gamma-1}{\gamma^2}\int_\Omega |\nabla w|^2 + c(\gamma)(1+c_1^m)\\ &\leqslant -\delta\int_\Omega w^2 + c(\gamma)(1+c_1^m) \quad \forall t\in[\tau, T_{\max})\end{aligned}$$

with some $\delta > 0$. In view of Gronwall's lemma, this shows that

$$\int_\Omega (u+1)^\gamma(t) \leqslant \left(\int_\Omega (u+1)^\gamma(\tau)\right)e^{-\delta\gamma(t-\tau)} + c(\gamma)(1+c_1^m) \quad \forall\, t\in[\tau, T_{\max})$$

and thereby proves the lemma. □

We have now collected all the elements for the announced iteration process.

Lemma 4.5. *Suppose $n \geqslant 1$ and f satisfies* (18) *with some $\alpha < \frac{2}{n}$. Then for any $\gamma > 2-\frac{2}{n}$ and all $\tau > 0$ there exist $c(\gamma, \Lambda, \tau) > 0$, $m = m(\gamma) > 0$ and $\nu = \nu(\gamma) > 0$ such that*

$$\|u(t)\|_{L^\gamma(\Omega)} \leqslant c(\gamma,\Lambda,\tau)\left(1+K^m(\tau)e^{-\nu t}\right) \quad \forall t\in[\tau, T_{\max}), \tag{37}$$

where $K(\tau) := \max_{t\in[\frac{\tau}{2},\tau]}\left(\|u(t)\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)}\right)$.

Proof. Let us fix γ and first consider the case $n = 1$. Since $\alpha < 2$ and we may assume that $\gamma > 1$, there exists $q_0 > 2$ such that

$$(\frac{1}{q_0}-1)\gamma_0 < 1-\alpha \text{ and } (\frac{1}{q_0}-1)\gamma < -(\alpha-1)$$

hold with $\gamma_0 := 1$. As

$$\|u(t)\|_{L^{\gamma_0}(\Omega)} = \int_\Omega u_0 = \lambda \text{ for } t \in [0, T_{\max})$$

and

$$\|v(t)\|_{W^{1,q_0}(\Omega)} \leqslant c(\Lambda, \tau) \text{ for } t \in [\tau, T_{\max})$$

by Lemma 4.1, Lemma 4.4 implies that

$$\|u(t)\|_{L^\gamma(\Omega)} \leqslant c(\gamma, \Lambda, \tau) \text{ for } t \in [\tau, T_{\max})$$

holds, which is obviously sharper than (37).

If $n \geqslant 2$, however, we start by applying Lemma 4.3 to obtain some $\gamma_0 > 2 - \frac{2}{n}$ and $v_0 > 0$ such that

$$\|u(t)\|_{L^{\gamma_0}(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + K^{\frac{2}{\gamma_0}}(\tau)e^{-v_0\tau}\Big) \quad \forall t \in [\frac{\tau}{2}, T_{\max}), \tag{38}$$

where we have estimated

$$\|u(\frac{\tau}{2})\|_{L^{\gamma_0}(\Omega)} + \|\nabla v(\frac{\tau}{2})\|_{L^2(\Omega)}^{\frac{2}{\gamma_0}} \leqslant cK^{\frac{2}{\gamma_0}}(\tau).$$

In the case $n = 2$ we then employ Lemma 4.1 to achieve

$$\|\nabla v(t)\|_{L^{q_0}(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + K^{\frac{2}{\gamma_0}}(\tau)e^{-v_0\tau}\Big) \text{ for all } t \in [\tau, T_{\max})$$

and some $q_0 > 2$–in fact, we may choose q_0 close to $\frac{2\gamma_0}{2-\gamma_0} > 2$. Then hypotheses (33) and (34) of Lemma 4.4 are trivially fulfilled for arbitrarily large γ and hence

$$\begin{aligned}\|u(t)\|_{L^\gamma(\Omega)} &\leqslant c(\gamma, \Lambda, \tau)\Big(1 + \Big(K^{\frac{2}{\gamma_0}}(\tau)e^{-v_0\tau}\Big)^{\tilde{m}} + \|u(\tau)\|_{L^\gamma(\Omega)}e^{-\tilde{v}t}\Big) \\ &\leqslant c(\gamma, \Lambda, \tau)\Big(1 + K^m(\tau)e^{-vt}\Big) \quad \forall t \in [\tau, T_{\max})\end{aligned}$$

holds with suitable m and v.

Finally, if $n \geqslant 3$ we use the same basic idea, but this time we have to apply Lemmas 4.1 and 4.4 several times to obtain (37) after a finite number of steps. In order to prepare our bootstrapping procedure, we let $a_0, a_1, a_2, \ldots \in \mathbb{R} \cup \{+\infty\}$ be defined by

$$a_0 := \gamma_0 \quad \text{and} \quad a_k := \begin{cases} \frac{(n-2)(1-\alpha)}{n-2a_{k-1}} & \text{if } a_{k-1} < \frac{n}{2}, \\ +\infty & \text{else,} \end{cases}$$

and claim that there exists $k_0 \in \mathbb{N}$ such that

$$a_0 < a_1 < \cdots < a_{k_0} = +\infty.$$

Indeed, suppose $a_{k-1} < \frac{n}{2}$ for all $k \in \{1, \ldots, k_1\}$ and some $k_1 \in \mathbb{N}$. Then, since $\alpha < \frac{2}{n}$,

$$\frac{a_k}{a_{k-1}} > \frac{(n-2)(1-\frac{2}{n})}{n-2a_{k-1}} = \frac{(n-2)^2}{n^2-2na_{k-1}} > 1,$$

provided that $(n-2)^2 > n^2 - 2a_{k-1}$ or, equivalently, $a_{k-1} > 2 - \frac{2}{n}$. As this is true for $k = 1$, it follows by induction that $a_0 < a_1 < \cdots < a_{k_1}$. Hence, if a_k were finite for all $k \in \mathbb{N}$, we would have $a_k \nearrow a_\infty \leqslant \frac{n}{2}$ as $k \to \infty$ and thus

$$a_\infty = \frac{(n-2)(1-\alpha)}{n-2a_\infty},$$

that is,

$$a_\infty = \frac{n-(n-2)(1-\alpha)}{2} < \frac{n-(n-2)(1-\frac{2}{n})}{2} = 2 - \frac{2}{n} < a_0,$$

contradicting the monotonicity of $(a_k)_{k\in\mathbb{N}}$. Therefore we must have $a_{k_0} = +\infty$ for some $k_0 \in \mathbb{N}$.

By a continuity argument, it is thus possible to choose positive $\varepsilon_1, \ldots, \varepsilon_{k_0}$ such that the numbers $\gamma_1, \ldots, \gamma_{k_0} \in \mathbb{R}$ defined by

$$\gamma_k := \frac{(n-2)(1-\alpha)}{2-2\gamma_{k-1}} - \varepsilon_k, \qquad k = 1, \ldots, k_0,$$

satisfy $\gamma_0 < \gamma_1 < \ldots < \gamma_{k_0-1} < \frac{n}{2}$ and $\gamma_{k_0} > \gamma$.

Now for $k = 1, \ldots, k_0$ we let

$$\bar{q}_{k-1} := \frac{n\gamma_{k-1}}{n-\gamma_{k-1}}.$$

Then

$$\bar{q}_{k-1} > 2 \quad \forall k = 1, ..., k_0, \tag{39}$$

because $\gamma_{k-1} \geqslant \gamma_0 > 2 - \frac{2}{n}$ and therefore

$$\bar{q}_{k-1} - 2 = \frac{(n+2)\gamma_{k-1} - 2n}{n - \gamma_{k-1}} > \frac{(n+2) \cdot \frac{2(n-1)}{n} - 2n}{n - \gamma_{k-1}} = \frac{2(n-2)}{n \cdot (n - \gamma_{k-1})} \quad \geqslant 0.$$

Furthermore,

$$\Big(\frac{n}{\bar{q}_{k-1}} - 1\Big)\gamma_{k-1} < n(1-\alpha) \quad \forall k = 1, \ldots, k_0, \tag{40}$$

for $\gamma_{k-1} > 2 - \frac{2}{n}$ and $\alpha < \frac{2}{n}$ imply

$$\Big(\frac{n}{\bar{q}_{k-1}} - 1\Big) = n - 2\gamma_{k-1} < n - \frac{4(n-1)}{n} < n - n\alpha.$$

Finally,

$$\Big(\frac{n}{\bar{q}_{k-1}} - 1\Big)\gamma_k < (n-2)(1-\alpha) \quad \forall k = 1, \ldots, k_0, \tag{41}$$

since by construction of γ_k,

$$\gamma_k < \frac{(n-2)(1-\alpha)}{n - 2\gamma_{k-1}}\gamma_{k-1} = \frac{(n-2)(1-\alpha)}{\frac{n}{\bar{q}_{k-1}} - 1}.$$

Due to (39) – (41) it is possible to fix $q_0, \ldots, q_{k_0-1}$ such that

$$2 < q_{k-1} < \bar{q}_{k-1}, \ \Big(\frac{n}{q_{k-1}} - 1\Big)\gamma_{k-1} < n(1-\alpha) \tag{42}$$

and

$$\Big(\frac{n}{q_{k-1}} - 1\Big)\gamma_k < (n-2)(1-\alpha) \tag{43}$$

for all $k = 1, \ldots, k_0$. Furthermore, we choose any sequence of numbers $\tau_0, \ldots, \tau_{k_0}$ satisfying $\frac{\tau}{2} = \tau_0 < \tau_1 < \ldots < \tau_{k_0} = \tau$. We now claim that for any $k = 0, \ldots, k_0$ we have

$$\|u(t)\|_{L^{\gamma_k}(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + K^{m_k}(\tau)\, e^{-\nu_k t}\Big) \quad \forall t \in [\tau_k, T_{\max}) \tag{44}$$

for appropriate $m_k > 0$ and $\nu_k > 0$, which will entail

$$\|u(t)\|_{L^\gamma(\Omega)} \leqslant c(\gamma, \Lambda, \tau)\Big(1 + K^{m_{k_0}}(\tau)\, e^{-\nu_{k_0} t}\Big)$$

for all $t \in [\tau, T_{\max})$, because $\gamma < \gamma_{k_0}$. In the case $k = 0$, (44) is implied by (38). However, if

$$\|u(t)\|_{L^{\gamma_{k-1}}(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + K^{m_{k-1}}(\tau)\, e^{-\nu_{k-1} t}\Big) \quad \forall\, t \in [\tau_{k-1}, T_{\max})$$

holds for some $k \in \{1, \ldots, k_0\}$ and suitable m_{k-1} and ν_{k-1} then, since $\gamma_{k-1} < n$ and

$$q_{k-1} < \bar{q}_{k-1} = \frac{n\gamma_{k-1}}{n - \gamma_{k-1}},$$

Lemma 4.1 (with $\eta := \tau_k - \tau_{k-1}$) yields

$$\|\nabla v(t)\|_{L^{q_{k-1}}(\Omega)} \leqslant c(\Lambda, \tau)\Big(1 + K^{m_{k-1}}(\tau)\, e^{-\nu_{k-1} t}\Big)$$

for all $t \in [\tau_k, T_{\max})$. Therefore, in view of (42) and (43), Lemma 4.4 provides some $\tilde{m}_k$ and $\tilde{\nu}_k$ such that

$$\begin{aligned} \|u(t)\|_{L^{\gamma_k}(\Omega)} &\leqslant c(\Lambda, \tau)\Big(1 + \Big(K^{m_{k-1}}(\tau)\, e^{-\nu_{k-1} t}\Big)^{\tilde{m}_k} + \|u(\tau_k)\|_{L^{\gamma_k}(\Omega)}\, e^{-\tilde{\nu}_k}\Big) \\ &\leqslant c(\Lambda, \tau)\Big(1 + K^{m_k}(\tau)\, e^{-\nu_k t}\Big) \quad \forall t \in [\tau_k, T_{\max}) \end{aligned}$$

is valid with certain constants m_k and ν_k, so that (44) has been proved. □

After the main work has been done now, the final step to L^∞ (and even to C^δ spaces) is now straightforward. Let us mention that the pure information '(u, v) is uniformly bounded' could alternatively obtained from the previous lemma and another iterative procedure introduced in [1]. This iterative procedure has been used most commonly in the literature related to the Keller–Segel chemotaxis system to establish the uniformly boundedness of (u, v) for the case where $\alpha = 1$ or where other chemotactic sensitivity functions have been considered. We will mention some of this results in the concluding section of the present paper.

Theorem 4.1. *If* $n \geqslant 1$ *and* f *satisfies* (17) *for some* $\alpha < \frac{2}{n}$ *then all solutions of* (1) *are global in time and uniformly bounded. Moreover, given* $\Lambda > 0$ *and* $\tau \in (0, 1)$ *there exist* $c(\Lambda, \tau) > 0$, $m > 0$ *and* $\nu > 0$ *such that*

$$\|u_0\|_{L^1(\Omega)} \leqslant \Lambda \quad \textit{and} \quad \|v_0\|_{L^1(\Omega)} \leqslant \Lambda$$

implies

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \leqslant c(\Lambda,\tau)\Big(1+\bar{K}^m(\tau)\,e^{-\nu t}\Big) \quad \forall t \geqslant \tau; \tag{45}$$

actually, we even have

$$\|u(t)\|_{C^\delta(\bar\Omega)} + \|v(t)\|_{C^{2+\delta}(\bar\Omega)} \leqslant c(\delta,\Lambda,\tau)\Big(1+\bar{K}^m(\tau)\,e^{-\nu t}\Big) \quad \forall t \geqslant \tau \tag{46}$$

for any $\delta \in (0,1)$, *with* $m = m(\delta)$ *and* $\nu = \nu(\delta)$. *Here we have set*

$$\bar{K}(\tau) := \max_{t\in[\frac{\tau}{4},\tau]} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)}\Big).$$

Proof. Since $D((A+1)^\beta) \hookrightarrow C^\delta(\bar\Omega)$ and $D((A+1)^{1+\beta}) \hookrightarrow C^{2+\delta}(\bar\Omega)$ for $\beta \in (0,\frac12)$ and $p>1$ satisfying $2\beta - \frac{n}{p} > \delta$, the proof of (46) will be accomplished if we can show that

$$\|(A+1)^\beta u(t)\|_{L^p(\Omega)} \leqslant c(\beta,p,\Lambda,\tau)\Big(1+\bar{K}^m(\tau)\,e^{-\nu t}\Big) \tag{47}$$

for all $t \in [\frac{3\tau}{4}, T_{\max})$, $\beta \in (0,\frac12)$, $p>1$ and

$$\|(A+1)^{1+\beta} v(t)\|_{L^p(\Omega)} \leqslant c(\beta,p,\Lambda,\tau)\Big(1+\bar{K}^m(\tau)\,e^{-\nu t}\Big) \tag{48}$$

for all $t \in [\tau, T_{\max})$, $\beta \in (0,\frac12)$, $p>1$. (Note here that this particularly entails $T_{\max}=\infty$ by Theorem 3.1.) The constants m and ν, which depend on β and p only, may vary from line to line.

To see (47), we let $\lambda = \int_\Omega u_0$, fix β and $p>1$ and apply $(A+1)^\beta$ to both sides of the formula

$$u(t)-\lambda = e^{-(t-\frac{\tau}{2})A}(u(\tau/2)-\lambda) - \int_{\frac{\tau}{2}}^t e^{-(t-s)A}\nabla\cdot(f(u(s))\nabla v(s))\,ds \quad \forall t \in (\frac{\tau}{2}, T_{\max}),$$

which is valid because of the fact that $e^{-tA}\lambda = \lambda$ for all $t>0$. By Lemmas 4.5 and 4.1, we have

$$\|(u+1)(t)\|_{L^{2p\alpha}(\Omega)} + \|\nabla v(t)\|_{L^{2p}(\Omega)} \leqslant c(p,\Lambda,\tau)\Big(1+\bar{K}^m(\tau)\,e^{-\nu t}\Big) \quad \forall t \in [\frac{\tau}{2}, T_{\max}).$$

Using this, (4), Lemma 2.1 (with any fixed $\varepsilon \in (0, \frac{1}{2} - \beta)$) and the Hölder inequality, we obtain

$$\begin{aligned}
&\Big\|(A+1)^{\beta}\Big(u(t)-\lambda\Big)\Big\|_{L^p(\Omega)}\\
&\leqslant \Big\|(A+1)^{\beta}e^{-(t-\frac{\tau}{2})A}(u(\frac{\tau}{2})-\lambda)\Big\|_{L^p(\Omega)}\\
&\quad+\int_{\frac{\tau}{2}}^{t}\Big\|(A+1)^{\beta}e^{-(t-s)A}\nabla\cdot(f(u(s))\nabla v(s))\Big\|_{L^p(\Omega)}\,ds\\
&\leqslant c(\beta,p)\big(t-\tfrac{\tau}{2}\big)^{-\beta-\frac{n}{2}(1-\frac{1}{p})}\|u(\tfrac{\tau}{2})-\lambda\|_{L^1(\Omega)}\\
&\quad+c(\beta,p)\int_{\frac{\tau}{2}}^{t}(t-s)^{-\beta-\frac{1}{2}-\varepsilon}e^{-\mu(t-s)}\|(u+1)^{\alpha}(s)\nabla v(s)\|_{L^p(\Omega)}\,ds\\
&\leqslant c(\beta,p)\lambda\tau^{-\beta-\frac{n}{2}(1-\frac{1}{p})}\\
&\quad+\int_{\frac{\tau}{2}}^{t}(t-s)^{\beta-\frac{1}{2}-\varepsilon}e^{-\mu(t-s)}\|(u+1)(s)\|^{\alpha}_{L^{2p\alpha}(\Omega)}\cdot\|\nabla v(s)\|_{L^{2p}(\Omega)}\,ds\\
&\leqslant c(\beta,p)\lambda\tau^{-\beta-\frac{n}{2}(1-\frac{1}{p})}\\
&\quad+c(\beta,p,\Lambda,\tau)\bigg(1+\bar{K}^{m}(\tau)\cdot\int_{\frac{\tau}{2}}^{t}(t-s)^{-\beta-\frac{1}{2}-\varepsilon}e^{-\mu(t-s)}e^{-\nu s}\,ds\bigg)\\
&\leqslant c(\beta,p,\Lambda,\tau)\Big(1+\bar{K}^{m}(\tau)\,e^{-\nu t}\Big)\quad\forall t\in[\tfrac{3\tau}{4},T_{\max})
\end{aligned}$$

with $\nu > 0$ small enough. This easily yields (47).

For the proof of (48) we use the result just obtained in applying $(A+1)^{\beta+1}$ to both sides of

$$v(t)=e^{-(t-\frac{3\tau}{4})(A+1)}v(\frac{3\tau}{4})+\int_{\frac{3\tau}{4}}^{t}e^{-(t-s)(A+1)}u(s)\,ds,\quad t\in[\frac{3\tau}{4},T_{\max}).$$

From (3), (8) and (47), for any fixed $\varepsilon \in (0, \frac{1}{2} - \beta)$ we infer that

$$\begin{aligned}
\|(A+1)^{\beta+1}v(t)\|_{L^p(\Omega)} \leqslant\ &\Big\|(A+1)^{\beta+1}e^{-(t-\frac{3\tau}{4})(A+1)}v(\tfrac{3\tau}{4})\Big\|_{L^p(\Omega)}\\
&+\int_{\frac{3\tau}{4}}^{t}\Big\|(A+1)^{\beta+1}e^{-(t-s)(A+1)}u(s)\Big\|_{L^p(\Omega)}ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c(\beta,p)\big(t-\frac{3\tau}{4}\big)^{-1-\beta-\frac{n}{2}(1-\frac{1}{p})}\|v(\frac{3\tau}{4})\|_{L^1(\Omega)}\\
&\quad+c(\beta,p)\int_{\frac{3\tau}{4}}^{t}(t-s)^{1-\varepsilon}e^{-\mu(t-s)}\|(A+1)^{\beta+\varepsilon}u(s)\|_{L^p(\Omega)}\,ds\\
&\leqslant c(\beta,p,\tau)\Lambda\\
&\quad+c(\beta,p,\Lambda,\tau)\Bigg(1+\bar{K}^m(\tau)\int_{\frac{3\tau}{4}}^{t}(t-s)^{-1+\varepsilon}e^{-\mu(t-s)}e^{-\nu s}\,ds\Bigg)\\
&\leqslant c(\beta,p,\Lambda,\tau)\Big(1+\bar{K}^m\,e^{-\nu t}\Big)\quad\forall t\in[\tau,T_{\max}),
\end{aligned}
$$

so that (48) follows and the proof is complete. □

As an interesting by-product of (46) we obtain some information on the ω-limit sets of solutions.

Corollary 4.1. *Suppose f satisfies* (17) *with some $\alpha<\frac{2}{n}$. Then for all $\Lambda>0$ and any $\delta\in(0,1)$ there exists a ball $B_{R(\delta,\Lambda)}$ in $C^{\delta}(\bar{\Omega})\times C^{2+\delta}(\bar{\Omega})$ centered at zero, with radius $R(\delta,\Lambda)$ depending on δ and Λ only, that has the following property: If $u_0\in C^0(\bar{\Omega})$ and $v_0\in\bigcup_{q>n}W^{1,q}(\Omega)$ are such that*

$$
\|u_0\|_{L^1(\Omega)}<\Lambda
$$

then the ω-limit set

$$
\begin{aligned}
\omega(u_0,v_0):=\Big\{&(u_\infty,v_\infty)\in(L^1(\Omega))^2\mid\exists\,t_k\to\infty\ \text{such that}\\
&u(t_k)\to u_\infty\ \text{and}\ v(t_k)\to v_\infty\ \text{a.e. in}\ \Omega\Big\}
\end{aligned}
$$

of the unique global solution (u,v) emanating from (u_0,v_0) satisfies

$$
\emptyset\neq\omega(u_0,v_0)\subset B_{R(\delta,\Lambda)}.
$$

Remark. Note particularly that the asymptotic bound $R=R(\delta,\Lambda)$ does not in any way depend on v_0 (which is actually due to the absorption term $-v$ in the second equation of (1)). Also, the dependence on u_0 is only through its $L^1(\Omega)$ norm. A result of this type is (for any fixed δ) the best that can be expected in the sense that R must depend at least on $\|u_0\|_{L^1(\Omega)}$ since $\|u_\infty\|_{L^1(\Omega)}=\|u_0\|_{L^1(\Omega)}$ holds for all elements $(u_\infty,v_\infty)\in\omega(u_0,v_0)$ due to (7) and, for instance, the equicontinuity property (46). Particularly, there is no hope for a global attractor or only a uniformly absorbing bounded set (in the L^1 topology). For a result on the existence of a finite dimensional attractor for $n=1$ and $\alpha=1$ we refer the interested reader to [32].

Proof. We fix an arbitrary $\delta \in (0,1)$ and set $\tau := \frac{1}{2}$ in (46) to obtain a constant $R(\delta, \Lambda)$ such that

$$\limsup_{t\to\infty} \Big(\|\hat{u}(t)\|_{C^{\delta}(\bar{\Omega})} + \|\hat{v}(t)\|_{C^{2+\delta}(\bar{\Omega})} \Big) \leqslant R(\delta, \Lambda) \tag{49}$$

holds for all solutions $(\hat{u}, \hat{v})$ of (1) evolving from initial data $(\hat{u}_0, \hat{v}_0)$ with the property that $\max\{\|\hat{u}_0\|_{L^1(\Omega)}, \|\hat{v}_0\|_{L^1(\Omega)}\} \leqslant \Lambda$.

Now let (u_0, v_0) be given with $\|u_0\|_{L^1(\Omega)} < \Lambda$, and let (u, v) denote the corresponding solution. From (8) we know that $\|v(t)\|_{L^1(\Omega)} \to \|u_0\|_{L^1(\Omega)}$ as $t \to \infty$, whence there exists $t_0 \geqslant 0$ such that $\|v(t_0)\|_{L^1(\Omega)} \leqslant \Lambda$. Setting $\hat{u}(t) := u(t_0+t)$ and $\hat{v}(t) := v(t_0+t)$ for $t \geqslant 0$, we see that $(\hat{u}, \hat{v})$ solves (1) with initial data $(\hat{u}_0, \hat{v}_0) := (u(t_0), v(t_0))$ satisfying $\max\{\|\hat{u}_0\|_{L^1(\Omega)}, \|\hat{v}_0\|_{L^1(\Omega)}\} \leqslant \Lambda$. Therefore (49) yields the claim. □

5. The supercritical case: absence of the a priori estimate

Let us now turn to the case of supercritical growth of $f(u)$. The goal of the present section is twofold: First, it explicitly shows that an apriori estimate as in Theorem 4.1 is not available if $f(s)$ grows faster than s^{α} for some $\alpha > \frac{2}{n}$ when $n \geqslant 2$. Secondly, at the same time it provides some useful preparations for the blow-up results to follow in the subsequent sections.

Our method will strongly rely on the fact that (1) possesses a natural Lyapunov functional (see [6,14,34] for more informations about Lyapunov functionals for Keller–Segel-type models). Its definition involves the nonnegative function $\Phi : (0,\infty) \to \mathbb{R}$ given by

$$\Phi(s) := \int_1^s \int_1^{\sigma} \frac{d\tau}{f(\tau)}, \qquad s > 0.$$

To be more precise, in the next lemma we shall see that

$$F(u,v) := \frac{1}{2}\int_{\Omega} |\nabla v|^2 + \frac{1}{2}\int_{\Omega} v^2 - \int_{\Omega} uv + \int_{\Omega} \Phi(u), \;\; 0 \leqslant u \in C^0(\bar{\Omega}), v \in W^{1,2}(\Omega),$$

acts as a Lyapunov functional for (1) in the following manner.

Lemma 5.1. *If (u, v) is a classical solution of (1) in $\Omega \times (0, T)$ for some $T \leqslant \infty$ then we have*

$$\int_s^t \int_{\Omega} v_t^2 + \int_s^t \int_{\Omega} f(u) \cdot \left| \frac{1}{f(u)} \nabla u - \nabla v \right|^2 + F(u(t), v(t)) = F(u(s), v(s)) \tag{50}$$

for all $0 \leqslant s < t < T$, provided that the initial data satisfy $\inf_{x\in\Omega} u_0(x) > 0$.

Proof. Since u_0 is strictly positive in $\bar{\Omega}$, the strong maximum principle guarantees that u is positive in $\bar{\Omega} \times [0, T)$ and hence $\frac{1}{f(u)}$ and $\Phi(u)$ are continuous functions in $\bar{\Omega} \times [0, T)$. Multiplying the second equation in (1) by v_t and integrating by parts yields

$$\int_s^t \int_\Omega v_t^2 + \left(\frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 \right) \Bigg|_s^t = \int_s^t \int_\Omega u v_t = \int_\Omega u v \Bigg|_s^t - \int_s^t \int_\Omega u_t v.$$

We now use the first equation in (1) to calculate

$$\int_s^t \int_\Omega u_t v = - \int_s^t \int_\Omega (\Delta u - \nabla \cdot (f(u) \nabla v)) \cdot v = \int_s^t \int_\Omega \nabla u \cdot \nabla v - \int_s^t \int_\Omega f(u) |\nabla v|^2.$$

Since

$$f(u) \left| \frac{1}{f(u)} \nabla u - \nabla v \right|^2 = \frac{1}{f(u)} |\nabla u|^2 - 2 \nabla u \cdot \nabla v + f(u) |\nabla u|^2$$

and

$$\begin{aligned} \int_\Omega \Phi(u) \Bigg|_s^t &= \int_s^t \int_\Omega \Phi'(u) u_t \;\; = \;\; \int_s^t \int_\Omega \Phi'(u)(\Delta u - \nabla \cdot (f(u) \nabla v)) \\ &= - \int_s^t \int_\Omega \Phi''(u) \nabla u \cdot (\nabla u - f(u) \nabla v) \\ &= - \int_s^t \int_\Omega \frac{1}{f(u)} |\nabla u|^2 + \int_s^t \int_\Omega \nabla u \cdot \nabla v, \end{aligned}$$

this gives

$$\begin{aligned} - \int_s^t \int_\Omega u_t v &= - \int_s^t \int_\Omega \nabla u \cdot \nabla v + \int_s^t \int_\Omega \frac{1}{f(u)} |\nabla u|^2 - \int_s^t \int_\Omega f(u) \left| \frac{1}{f(u)} \nabla u - \nabla v \right|^2 \\ &= \int_\Omega \Phi(u) \Bigg|_s^t - \int_s^t \int_\Omega f(u) \left| \frac{1}{f(u)} \nabla u - \nabla v \right|^2, \end{aligned}$$

so that (50) even holds with equality. □

Remark. We notice that in the subcritical case when $f(s) \leqslant c_1 s^\alpha \; \forall s \geqslant 1$ with $\alpha < \frac{2}{n}$, for any $\Lambda > 0$ and $\tau > 0$ we can find $c(\Lambda, \tau) > 0$ such that

$$F(u(t), v(t)) \geqslant -c(\Lambda, \tau) \quad \forall t \geqslant \tau \tag{51}$$

holds for all solutions (u, v) with initial data fulfilling

$$\max\{\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}\} \leqslant \Lambda.$$

For $n \geqslant 2$ this can be seen as follows (the proof in the case $n = 1$ is even simpler): From the growth condition on f we gain $\Phi(s) \geqslant c_2 s^{2-\alpha} - c_3$ for all $s \geqslant 1$ with positive c_2 and c_3. Thus, by Young's inequality, we can find $\varepsilon > 0$ small such that

$$\int_\Omega uv \leqslant \varepsilon \int_\Omega u^{2-\alpha} + c(\varepsilon) \int_\Omega v^{\frac{2-\alpha}{1-\alpha}} \leqslant \int_\Omega \Phi(u) + c + c \int_\Omega v^{\frac{2-\alpha}{1-\alpha}} \text{ for all } t > 0.$$

Now Lemma 4.1 and the Sobolev embedding theorem tell us that

$$\|v(t)\|_{L^q(\Omega)} \leqslant c(q, \Lambda, \tau)$$

for all $t \geqslant \tau$ and any $q < \frac{n}{n-2}$. Therefore we can apply the Gagliardo–Nirenberg and the Poincaré inequality to estimate

$$\begin{aligned}\int_\Omega v^{\frac{2-\alpha}{1-\alpha}} &\leqslant c\Big(\|v\|_{W^{1,2}(\Omega)}^{\frac{2-\alpha}{1-\alpha}a(q)} + 1\Big)\|v\|_{L^q(\Omega)}^{\frac{2-\alpha}{1-\alpha}(1-a(q))} \\ &\leqslant c(q, \Lambda, \tau)\Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{2-\alpha}{1-\alpha}a(q)} + 1\Big) \quad \forall\, t \geqslant \tau,\end{aligned}$$

where $a(q) = \frac{\frac{n}{q} - \frac{1-\alpha}{2-\alpha}n}{1 - \frac{n}{2} + \frac{n}{q}}$. Since $\frac{2-\alpha}{1-\alpha}a(q) \to \frac{n-2(2-\alpha)}{(\frac{n}{2}-1)(1-\alpha)} < 2$ as $q \to \frac{n}{n-2}$, we can pick q close to $\frac{n}{n-2}$ so as to achieve

$$\int_\Omega v^{\frac{2-\alpha}{1-\alpha}} \leqslant \varepsilon \int_\Omega |\nabla v|^2 + c(\Lambda, \tau, \varepsilon) \quad \forall t \geqslant \tau$$

for any $\varepsilon > 0$. Upon an appropriate choice of ε this yields (51).

The key to our results on nonexistence of a priori bounds and on blow-up solutions is the observation that for supercritical growth of f, the functional F is unbounded from below in the following sense.

Lemma 5.2. *Suppose $n \geqslant 2$ and*

$$f(s) \geqslant c_0 s^\alpha \quad \forall s \geqslant 1 \tag{52}$$

holds with some $c_0 > 0$ and some $\alpha > \frac{2}{n}$. Then for any fixed $\lambda > 0$ there exist $\varepsilon_0 > 0$ and families $(u_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)} \subset W^{1,\infty}(\Omega)$ and $(v_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)} \subset W^{1,\infty}(\Omega)$ such that $u_\varepsilon > 0$

and $v_\varepsilon > 0$ *in* $\bar{\Omega}$,

$$\int_\Omega u_\varepsilon = \lambda \quad \forall \varepsilon \in (0, \varepsilon_0) \quad \textit{and} \quad \int_\Omega v_\varepsilon \to 0 \quad \textit{as } \varepsilon \to 0,$$

but

$$F(u_\varepsilon, v_\varepsilon) \to -\infty \quad \textit{as } \varepsilon \to 0. \tag{53}$$

Actually, it is even possible to construct v_ε *such that*

$$\int_\Omega |\nabla v_\varepsilon|^2 \to +\infty \textit{ as } \varepsilon \to 0.$$

If Ω *is a ball then* u_ε *and* v_ε *can be chosen to be radially symmetric.*

Remark. (1) In the one dimensional case, (53) cannot occur for any choice of *f*. Then, namely, the Sobolev and the Young inequality yield

$$\int_\Omega uv \leqslant \Lambda \|v\|_{L^\infty(\Omega)} \leqslant \frac{1}{4} \int_\Omega |\nabla v|^2 + c\Lambda^2$$

for all $(u, v) \in L^1(\Omega) \times W^{1,2}(\Omega)$ with $\int_\Omega u$ and $\int_\Omega v$ not exceeding Λ; thus,

$$F(u, v) \geqslant \frac{1}{4} \int_\Omega |\nabla v|^2 - c\Lambda^2$$

holds for all (u, v) of this type, because $\Phi \geqslant 0$.

(2) A result related to Lemma 5.2 for the critical case $n = 2$ and $\alpha = 1$ is already known. One can find it in [15]. In that critical case one has to assume that the L^1-norm of u_0 is sufficiently large to guarantee the existence of sequences $(u_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)} \subset L^\infty(\Omega)$ and $(v_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)} \subset W^{1,\infty}(\Omega)$ such that

$$F(u_\varepsilon, v_\varepsilon) \to -\infty \text{ and } \int_\Omega |\nabla v_\varepsilon|^2 \to +\infty \text{ as } \varepsilon \to 0.$$

Proof. After a translation of the coordinate axes we may assume $B_{R_0} \subset \Omega \subset B_{R_1}$ with certain radii $0 < R_0 < R_1$. Also, it is sufficient to consider the case when $\alpha \notin \{1, 2\}$. Then, namely, we have

$$\Phi(s) \leqslant \frac{1}{c_0} \int_1^s \int_1^\sigma \tau^{-\alpha} \, d\tau \;=\; \frac{(s^{2-\alpha} - 1)}{c_0(1-\alpha)(2-\alpha)} - \frac{(s-1)}{c_0(1-\alpha)} \leqslant c\Big(s^{2-\alpha} + s\Big) \tag{54}$$

for all $s \geqslant 1$ due to (52).

First, if $n \geqslant 3$ we may additionally suppose $\alpha < 1$ and then define

$$u_\varepsilon(x) := \begin{cases} \frac{\lambda}{2|\Omega|} + a_\varepsilon \varepsilon^{-n}, & |x| \leqslant \varepsilon, \\ \frac{\lambda}{2|\Omega|} + a_\varepsilon \varepsilon^{\beta-n} |x|^{-\beta}, & x \in \Omega \setminus B_\varepsilon \end{cases}$$

and

$$v_\varepsilon(x) := \begin{cases} \varepsilon^{-\gamma}, & |x| \leqslant \varepsilon, \\ \varepsilon^{\delta-\gamma} |x|^{-\delta}, & x \in \Omega \setminus B_\varepsilon, \end{cases}$$

for $\varepsilon < \varepsilon_0 := R_0$, where we fix

$$\beta > n, \qquad \gamma \in \left(\max\{\tfrac{n-2}{2}, (1-\alpha)n\},\ n-2 \right) \qquad \text{and} \qquad \delta > n,$$

which is possible since $n \geqslant 3$ and $\alpha > \frac{2}{n}$. Moreover, we set

$$a_\varepsilon := \frac{\frac{\lambda}{2}}{\frac{\omega_n}{n} + \frac{\omega_n}{\beta-n}\left(1 - (\frac{\varepsilon}{R_0})^{\beta-n}\right) + \varepsilon^{\beta-n} \int_{\Omega \setminus B_{R_0}} |x|^{-\beta}},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$.

The choice of a_ε was done in such a way that

$$\begin{aligned} \int_\Omega u_\varepsilon &= \frac{\lambda}{2} + a_\varepsilon \varepsilon^{-n} |B_\varepsilon| + a_\varepsilon \varepsilon^{\beta-n} \omega_n \int_\varepsilon^{R_0} r^{n-1-\beta}\, dr + a_\varepsilon \varepsilon^{\beta-n} \int_{\Omega \setminus B_{R_0}} |x|^{-\beta} \\ &= \lambda \quad \forall \varepsilon \in (0, \varepsilon_0). \end{aligned}$$

Observe that $a_\varepsilon \to \frac{n(\beta-n)}{4\beta\omega_n}$ as $\varepsilon \to 0$ and hence a_ε is bounded above and below by positive constants. Clearly, u_ε and v_ε belong to $W^{1,\infty}(\Omega)$ and are positive in $\bar{\Omega}$.

We now estimate the terms making up $F(u_\varepsilon, v_\varepsilon)$ according to

$$\begin{aligned} \int_\Omega |\nabla v_\varepsilon|^2 &\leqslant \omega_n \int_\varepsilon^{R_1} r^{n-1} \left(\delta \varepsilon^{\delta-\gamma} r^{-\delta-1} \right)^2 dr \\ &= \frac{\delta^2 \omega_n}{2\delta + 2 - n} \varepsilon^{2\delta-2\gamma} \left(\varepsilon^{n-2\delta-2} - R_1^{n-2\delta-2} \right) \leqslant c\varepsilon^{n-2\gamma-2}, \end{aligned} \tag{55}$$

$$\int_\Omega v_\varepsilon^2 \leqslant \varepsilon^{-2\gamma} \frac{\varepsilon^n \omega_n}{n} + \omega_n \int_\varepsilon^{R_1} r^{n-1} \left(\varepsilon^{\delta-\gamma} r^{-\delta} \right)^2 dr$$

$$= \frac{\omega_n}{n}\varepsilon^{n-2\gamma} + \frac{\omega_n}{2\delta - n}\varepsilon^{2\delta-2\gamma}\Big(\varepsilon^{n-2\delta} - R_1^{n-2\delta}\Big) \leqslant c\varepsilon^{n-2\gamma},$$

$$\int_\Omega u_\varepsilon v_\varepsilon \geqslant a_\varepsilon \varepsilon^{-n-\gamma}\frac{\varepsilon^n \omega_n}{n} \geqslant c\varepsilon^{-\gamma}$$

and, using (54),

$$\int_\Omega \Phi(u_\varepsilon) = \int_{\{u_\varepsilon<1\}} \Phi(u_\varepsilon) + \int_{\{u_\varepsilon \geqslant 1\}} \Phi(u_\varepsilon) \leqslant |\Omega|\Phi\Big(\frac{\lambda}{2|\Omega|}\Big) + c\Big(\int_\Omega u_\varepsilon^{2-\alpha} + \lambda\Big), \tag{56}$$

because $\Phi'(s)$ is negative whenever $s < 1$. Since $\alpha < 1 < 2 - \frac{n}{\beta}$, we have

$$\begin{aligned}\int_\Omega \Big(u_\varepsilon - \frac{\lambda}{2|\Omega|}\Big)^{2-\alpha} &= a_\varepsilon^{2-\alpha}\varepsilon^{-(2-\alpha)n}\frac{\varepsilon^n\omega_n}{n} + \omega_n \int_\varepsilon^{R_0} r^{n-1}\Big(a_\varepsilon \varepsilon^{\beta-n} r^{-\beta}\Big)^{2-\alpha} dr \\ &\quad + \int_{\Omega\setminus B_{R_0}} \Big(a_\varepsilon \varepsilon^{\beta-n}|x|^{-\beta}\Big)^{2-\alpha} \\ &= \frac{a_\varepsilon^{2-\alpha}\omega_n}{n}\varepsilon^{-(1-\alpha)n} + a_\varepsilon^{2-\alpha}\varepsilon^{(\beta-n)(2-\alpha)}\int_{\Omega\setminus B_{R_0}} |x|^{-(2-\alpha)\beta} \\ &\quad + \frac{\omega_n a_\varepsilon^{2-\alpha}\varepsilon^{(\beta-n)(2-\alpha)}}{n-(2-\alpha)\beta}\Big(R_0^{n-(2-\alpha)\beta} - \varepsilon^{n-(2-\alpha)\beta}\Big)\end{aligned}$$

and thus

$$\begin{aligned}\int_\Omega u_\varepsilon^{2-\alpha} &\leqslant c\int_\Omega \Big[\Big(\frac{\lambda}{2|\Omega|}\Big)^{2-\alpha} + \Big(u_\varepsilon - \frac{\lambda}{2|\Omega|}\Big)^{2-\alpha}\Big] \\ &\leqslant c\Big(1 + \varepsilon^{-(1-\alpha)n} + \varepsilon^{(\beta-n)(2-\alpha)+n-(2-\alpha)\beta}\Big) \leqslant c\varepsilon^{-(1-\alpha)n}\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. We therefore obtain

$$F(u_\varepsilon, v_\varepsilon) \leqslant c\Big(\varepsilon^{n-2\gamma-2} + \varepsilon^{-(1-\alpha)n}\Big) - \bar{c}\varepsilon^{-\gamma}$$

with positive constants c and $\bar{c}$, and hence $F(u_\varepsilon, v_\varepsilon) \to -\infty$ as $\varepsilon \to 0$, provided that

$$\gamma > \max\{0, -(n-2\gamma-2), (1-\alpha)n\},$$

which however is guaranteed by our original choice of γ. Moreover, we have

$$\begin{aligned}\int_\Omega v_\varepsilon &\leqslant \frac{\omega_n}{n}\varepsilon^{n-\gamma} + \omega_n\varepsilon^{\delta-\gamma}\int_\varepsilon^{R_1} r^{n-1-\delta}\,dr = \frac{\omega_n}{n}\varepsilon^{n-\gamma} + \frac{\omega_n}{\delta-n}\varepsilon^{\delta-\gamma}\Big(\varepsilon^{n-\delta} - R_1^{n-\delta}\Big) \\ &\leqslant c\varepsilon^{n-\gamma} \to 0\end{aligned}$$

as $\varepsilon \to 0$, since $\delta > n$ and $\gamma < n-2 < n$, and (cf. (55))

$$\int_\Omega |\nabla v_\varepsilon|^2 \geqslant \omega_n \int_\varepsilon^{R_0} r^{n-1}\Big(\delta\varepsilon^{\delta-\gamma} r^{-\delta-1}\Big)^2 dr \ \geqslant c(\varepsilon^{n-2\gamma-2}-1) \to +\infty$$

as $\varepsilon \to 0$, because we have chosen $\gamma > \frac{n-2}{2}$. In the case $n=2$ the construction is similar, using u_ε in the same form as before with arbitrary $\beta > n = 2$, but setting

$$v_\varepsilon(x) := \begin{cases} \Big(\ln \frac{R_1}{\varepsilon}\Big)^{1-\kappa}, & |x| \leqslant \varepsilon, \\ \Big(\ln \frac{R_1}{\varepsilon}\Big)^{-\kappa} \ln \frac{R_1}{r}, & x \in \Omega \setminus B_\varepsilon, \end{cases}$$

this time, where $\kappa \in (0, \frac{1}{2})$. Then

$$\int_\Omega |\nabla v_\varepsilon|^2 \ \leqslant\ 2\pi \int_\varepsilon^{R_1} r\bigg(\Big(\ln \frac{R_1}{\varepsilon}\Big)^{-\kappa}\frac{1}{r}\bigg)^2 dr = 2\pi\Big(\ln\frac{R_1}{\varepsilon}\Big)^{1-2\kappa},$$

$$\int_\Omega v_\varepsilon^2 \ \leqslant\ 2\pi \int_0^{R_1} r\bigg(\Big(\ln \frac{R_1}{\varepsilon}\Big)^{-\kappa}\ln\frac{R_1}{r}\bigg)^2 dr \to 0 \qquad \text{as } \varepsilon \to 0,$$

$$\int_\Omega u_\varepsilon v_\varepsilon \ \geqslant\ \pi a_\varepsilon \Big(\ln\frac{R_1}{\varepsilon}\Big)^{1-\kappa},$$

and since $\alpha > 1$, (56) directly yields

$$\int_\Omega \Phi(u_\varepsilon) \leqslant c.$$

Therefore $\kappa > 0$ implies $F(u_\varepsilon, v_\varepsilon) \to -\infty$ as $\varepsilon \to 0$, while $\int_\Omega v_\varepsilon \to 0$ as $\varepsilon \to 0$ is obvious now and the additional property

$$\int_\Omega |\nabla v_\varepsilon|^2 \geqslant 2\pi \int_\varepsilon^{R_0} r\bigg(\Big(\ln\frac{R_1}{\varepsilon}\Big)^{-\kappa}\frac{1}{r}\bigg)^2 dr = 2\pi\Big(\ln\frac{R_1}{\varepsilon}\Big)^{-2\kappa}\ln\frac{R_0}{\varepsilon} \to +\infty$$

as $\varepsilon \to 0$ is fulfilled in virtue of $\kappa < \frac{1}{2}$. □

Now we are in the position to prove the absence of an a priori bound in the style of that given by Theorem 4.1, provided that the growth of f at infinity is supercritical.

Theorem 5.1. *If $n \geqslant 2$ and there exists $c_0 > 0$ such that*

$$f(s) \geqslant c_0 s^\alpha \quad \forall s \geqslant 1$$

holds with some $\alpha > \frac{2}{n}$ *then there does not exist an a priori estimate in the sense of* (45). *More precisely, to any* $\lambda > 0$ *there corresponds a sequence of solutions* (u_j, v_j) *of solutions to* (1) *with initial data* $(u_{0,j}, v_{0,j})$ *satisfying*

$$\|u_{0,j}\|_{L^1(\Omega)} = \lambda \quad \forall j \in \mathbb{N} \qquad \textit{and} \qquad \|v_{0,j}\|_{L^1(\Omega)} \to 0 \quad \textit{as} \ j \to \infty, \tag{57}$$

such that

$$\limsup_{t \to T_{\max,j}} \|u_j(t)\|_{L^\infty(\Omega)} \to \infty \qquad \textit{as} \ j \to \infty, \tag{58}$$

where $T_{\max,j} \leqslant \infty$ *denotes the maximum existence time for* (u_j, v_j).

Proof. From Lemma 5.2 we know that there exist sequences $(u_{0,j})_{j\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ and $(v_{0,j})_{j\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ of strictly positive functions fulfilling (57) and

$$F(u_{0,j}, v_{0,j}) \to -\infty \qquad \text{as } j \to \infty.$$

Let (u_j, v_j) denote the (classical) solution of (1) emanating from $(u_{0,j}, v_{0,j})$, defined in the maximal time interval $(0, T_{\max,j})$.

If (58) was false, we could pass to a subsequence to obtain

$$\limsup_{t \to T_{\max,j}} \|u_j(t)\|_{L^\infty(\Omega)} \leqslant c_1 \quad \forall j \in \mathbb{N}$$

with some $c_1 > 0$. In particular, this implies $T_{\max,j} = \infty$ for all j, because then $u_j \leqslant 2c_1$ holds in $\Omega \times (T_{0,j}, T_{\max,j})$ with some $T_{0,j}$ sufficiently close to $T_{\max,j}$. We therefore may modify $f(s)$ beyond $s = 2c_1$ so as to be constant for large s without touching system (1) in $\Omega \times (T_{0,j}, T_{\max,j})$. Applying the results of Theorem 4.1 for sufficiently small $\tau = \tau_j > 0$ to $(u_j(\cdot - T_{0,j}), v_j(\cdot - T_{0,j}))$, we infer that

$$\limsup_{t \to T_{\max,j}} (\|u_j(t)\|_{C^\delta(\bar{\Omega})} + \|v_j(t)\|_{C^{2+\delta}(\bar{\Omega})}) < \infty$$

for each $j \in \mathbb{N}$ and some $\delta > 0$, whence all (u_j, v_j) must be global and, clearly, bounded solutions of (1).

Repeating the latter argument with $T_{\max,j} = \infty$ and $\tau_j = 1$ this time, we obtain a sequence $(T_j)_{j\in\mathbb{N}} \subset (1, \infty)$ and constants $\delta > 0$ and $c_2 > 0$ independent of j such that

$$\|u_j(t)\|_{C^\delta(\bar{\Omega})} + \|v_j(t)\|_{C^{2+\delta}(\Omega)} \leqslant c_2 \quad \forall t \in (T_j, \infty). \tag{59}$$

By the Arzelà–Ascoli theorem, for all j we can therefore extract a sequence $(t_{j,k})_{k\in\mathbb{N}} \subset (0, \infty)$ such that

$$u_j(t_{j,k}) \to u_{j,\infty} \text{ in } C^0(\bar{\Omega}) \quad \text{and} \quad v_j(t_{j,k}) \to v_{j,\infty} \text{ in } C^2(\bar{\Omega}) \tag{60}$$

as $k \to \infty$, where $u_{j,\infty} \not\equiv 0$ because $\int_\Omega u_j(t_{j,k}) = \lambda > 0$ for all k. As

$$\|u_{j,\infty}\|_{C^\delta(\bar\Omega)} + \|v_{j,\infty}\|_{C^{2+\delta}(\Omega)} \leqslant c_2$$

by (59) and an elementary argument, we may now take $j \to \infty$ along a suitable subsequence to achieve

$$u_{j,\infty} \to u_{\infty,\infty} \text{ in } C^0(\bar\Omega) \quad \text{and} \quad v_{j,\infty} \to v_{\infty,\infty} \text{ in } C^2(\bar\Omega),$$

again with $u_{\infty,\infty} \not\equiv 0$ due to $\int_\Omega u_{j,\infty} \equiv \lambda$. In virtue of (60) and the Lyapunov property of F (see Lemma 5.1), we infer that

$$F(u_{0,j}, v_{0,j}) \geqslant F(u_j(t_{j,k}), v_j(t_{j,k})) \to F(u_{j,\infty}, v_{j,\infty}) \qquad \text{as } k \to \infty$$

and hence, letting $j \to \infty$,

$$F(u_{\infty,\infty}, v_{\infty,\infty}) \leqslant \liminf_{j\to\infty} F(u_{0,j}, v_{0,j}) = -\infty,$$

which is absurd because $\int_\Omega u_{\infty,\infty} v_{\infty,\infty}$ must be finite. $\square$

6. Blow-up

We shall now proceed to prove the existence of unbounded solutions under some additional assumptions. Our approach is basically indirect: For certain sensitivity functions f, we shall find some initial data (using Lemma 5.2) for which it will be impossible that the corresponding solution of (1) remains bounded for all times, so that it will have to blow up either in finite or in infinite time. The contradiction to the boundedness hypothesis for such solutions will mainly be gained by an energy argument, 'energy' here being measured in terms of the functional F introduced in the previous section.

To become more concrete, the existence of the Lyapunov functional F encourages us to suspect a connection between the ω-limit set of a supposedly bounded solution of (1) and some kind of steady state solutions of (1). Here we also refer to related results by Osaki and Yagi [32] for $n = 1$ and $\alpha = 1$. Furthermore, some comments and results on the convergence to steady-state solutions for general Keller–Segel-type models have been established for example in [14,34]. However, here we follow a slightly different approach. Part of such a connection between the ω-limit set of a suppoesedly bounded solution of (1) and some kind of steady state solutions is established by the following lemma in which we use the strictly decreasing function φ defined by

$$\varphi(s) := \int_s^1 \frac{d\sigma}{f(\sigma)}, \qquad s > 0.$$

Lemma 6.1. *Suppose* (u,v) *is a global bounded solution of* (1) *with initial data* (u_0,v_0) *satisfying* $u_0>0$ *in* $\bar\Omega$, *and set* $\lambda:=\int_\Omega u_0$. *Then there exist* $u_\infty\in C^0(\bar\Omega)$, $v_\infty\in C^2(\Omega)$, $\Gamma\in\mathbb{R}$ *and a sequence of times* $t_k\to\infty$ *such that*

$$u(t_k)\to u_\infty \quad in\ C^0(\bar\Omega), \qquad v(t_k)\to v_\infty \quad in\ C^2(\bar\Omega), \tag{61}$$

and $(u_\infty,v_\infty,\Gamma)$ *is a solution of the stationary problem*

$$(\mathrm{S}_\lambda)\qquad \begin{cases} -\Delta v_\infty+v_\infty=u_\infty & in\ \Omega,\\ \varphi(u_\infty)+v_\infty=\Gamma & in\ \Omega,\\ \frac{\partial}{\partial N}v_\infty|_{\partial\Omega}=0,\\ \int_\Omega u_\infty=\int_\Omega v_\infty=\lambda. \end{cases}$$

Proof. It is easy to see that it is sufficient to prove the claim with φ replaced by

$$\hat\varphi(s):=\int_s^{s_0}\frac{d\sigma}{f(\sigma)},$$

where $s_0:=\|u\|_{L^\infty(\Omega\times(0,\infty))}$. In this case, $\hat\varphi(u)$ is positive in $\Omega\times(0,\infty)$.

Since (u,v) is a global bounded solution and $u_0>0$ in $\bar\Omega$, $F(u,v)$ is uniformly bounded from below for all times, whence Lemma 5.1 says that

$$\int_0^\infty\int_\Omega v_t^2+\int_0^\infty\int_\Omega f(u)\left|\frac{1}{f(u)}\nabla u-\nabla v\right|^2<\infty$$

and thus

$$\int_\Omega v_t(t_k)\to 0 \qquad \text{as } k\to\infty \tag{62}$$

as well as

$$\int_\Omega f(u(t_k))\left|\frac{1}{f(u(t_k))}\nabla u(t_k)-\nabla v(t_k)\right|^2\to 0 \qquad \text{as } k\to\infty \tag{63}$$

are valid for a suitable sequence $t_k\to\infty$. Again manipulating $f(s)$ for $s>2s_0$ to be constant for large s, we may apply Theorem 4.1 to extract a subsequence for which (61) holds. In order to gain further information from (63), let us first construct a positive nonincreasing $\rho\in W^{1,\infty}((0,\infty))$ that fulfills

$$\rho(\varphi(s))\leqslant f(s) \quad \forall s\in(0,s_0).$$

This can, for instance, be achieved by defining

$$\rho(\sigma) := f_0(\varphi^{-1}(\sigma)), \qquad \sigma \in (0,\infty),$$

where $f_0 \in W^{1,\infty}((0,s_0))$ is any nondecreasing minorant of f on $[0,s_0]$ which is positive in $(0,s_0]$; for example, we may take $f_0(s) := \min_{\sigma\in[s,s_0]} f(\sigma)$. Then, in fact, ρ is positive on $(0,\infty)$ and we have

$$\rho'(\sigma) = -f_0'(\varphi^{-1}(\sigma))\cdot f(\varphi^{-1}(\sigma)) \leqslant 0 \text{ as well as } \rho(\varphi(s)) = f_0(s) \leqslant f(s)$$

for all $s \in (0,s_0)$, as desired.

Using this function ρ, we write

$$P(s) := \int_0^s \sqrt{\rho(\sigma)}\, d\sigma$$

for $s > 0$ and calculate

$$\begin{aligned} \Big|\nabla P(\varphi(u)+v)\Big|^2 &= \Big|P'(\varphi(u)+v)\Big|^2 \Big|\nabla(\varphi(u)+v)\Big|^2 = \rho\big(\varphi(u)+v\big)\Big|\frac{1}{f(u)}\nabla u - \nabla v\Big|^2 \\ &\leqslant \rho\big(\varphi(u)\big)\Big|\frac{1}{f(u)}\nabla u - \nabla v\Big|^2 \leqslant f(u)\Big|\frac{1}{f(u)}\nabla u - \nabla v\Big|^2, \end{aligned}$$

because v is nonnegative. Therefore (63) implies

$$\int_\Omega \Big|\nabla P(\varphi(u(t_k)) + v(t_k))\Big|^2 \to 0 \qquad \text{as } k\to\infty,$$

whence

$$\int_\Omega \Big|P(\varphi(u(t_k)) + v(t_k)) - m_k\Big|^2 \to 0 \qquad \text{as } k\to\infty \tag{64}$$

by the Poincaré inequality, where m_k is the real number defined by

$$m_k := \frac{1}{|\Omega|}\int_\Omega P(\varphi(u(t_k)) + v(t_k)).$$

Extracting further subsequences, we may assume that the integrand in (64) tends to zero a.e. in Ω, and that $m_k \to m_\infty \in [0,\infty]$ as $k\to\infty$. Thus,

$$P(\varphi(u(t_k)) + v(t_k)) \to m_\infty \qquad \text{a.e. in } \Omega \quad \text{as } k\to\infty$$

and accordingly

$$\varphi(u(t_k)) + v(t_k) \to \Gamma := P^{-1}(m_\infty) \in [0,\infty] \qquad \text{a.e. in } \Omega \quad \text{as } k \to \infty.$$

But $\Gamma = +\infty$ actually is impossible since in such a case (61) would show that $u(t_k) \to 0$ uniformly in Ω which contradicts the fact that $\int_\Omega u(t_k) = \lambda$ for all k.

Now the validity of (S_λ) results from this, (61), (62) and (8). □

Combining the Lemmata 5.2, 6.1 and 5.1, we immediately obtain the following

Corollary 6.1. *Let $n \geqslant 2$ and f satisfy $f(s) \geqslant c_0 s^\alpha$ for all $s \geqslant 1$ with some $c_0 > 0$ and $\alpha > \frac{2}{n}$. If there exists $\lambda > 0$ and a constant c such that*

$$F(u,v) \geqslant -c \tag{65}$$

is valid for all for all solutions (u,v,Γ) of (S_λ) *then there exists a solution of* (1) *which blows up. The same is true if Ω is a ball and* (65) *holds only for all radially symmetric solutions of* (S_λ).

In the sequel we shall derive from this some results on the existence of radial blow-up solutions, assuming throughout that f satisfies the supercriticality condition

$$f(s) \geqslant c_0 s^\alpha \quad \forall s \geqslant 1$$

with some $c_0 > 0$ and $\alpha > \frac{2}{n}$. More precisely, we shall show that in some cases, under relatively mild additional conditions on f (which will be stated when required) there exists a (λ-dependent) a priori bound from below for $F(u,v)$ for all radially symmetric solutions (u,v,Γ) of (S_λ).

For technical reasons, we shall treat the three cases $\alpha > 2$, $\alpha \in (1,2)$ and $\alpha \in (\frac{2}{n},1)$ separately. Before going into detail, let us state an easily obtained but rather helpful information for the component v of solutions of (S_λ). Although the result is standard (cf. [3,16,19,36,38]), we include a short proof for the sake of completeness.

Lemma 6.2. *Let $n \geqslant 2$.*

(i) *For all $s \in (1, \frac{n}{n-1})$ there exists $c = c(s) > 0$ such that*

$$\|v\|_{W^{1,s}(\Omega)} \leqslant c(s)\lambda \tag{66}$$

holds for all solutions (u,v,Γ) of (S_λ).

(ii) *For all $q \in (1, \frac{n}{n-2})$ there is $c = c(q) > 0$ with the property that any solution (u,v,Γ) of* (S_λ) *satisfies*

$$\|v\|_{L^q(\Omega)} \leqslant c(q)\lambda.$$

Proof. (i) Let $s' := \frac{s}{s-1}$. Then $s' > n$, so that $W^{1,s'}(\Omega) \hookrightarrow L^\infty(\Omega)$. Therefore testing (S_λ) with arbitrary $\psi \in W^{1,s'}(\Omega)$ gives

$$\int_\Omega \nabla v \cdot \nabla \psi = \int_\Omega u\psi - \int_\Omega v\psi \leqslant 2\lambda \|\psi\|_{L^\infty(\Omega)} \leqslant c\lambda \|\psi\|_{W^{1,s'}(\Omega)}.$$

Together with the Poincaré inequality this implies (66).

(ii) This is an easy application of the Sobolev embedding theorem to (66). □

6.1. *The case* $\alpha > 2$

Let us start with the case $\alpha > 2$ which is the easiest one and which requires no further condition on *f*; that is, we suppose in this section that

$$f(s) \geqslant c_0 s^\alpha \quad \forall s \geqslant 1 \tag{67}$$

holds with some $c_0 > 0$ and $\alpha > 2$. We assume throughout that $\Omega = B_R$ is a ball in $\mathbb{R}^n$, $n \geqslant 2$, centered at zero, and we shall prove our first result on existence of blow-up solutions which reads as follows.

Theorem 6.1. *Suppose* $\Omega = B_R$ *is a ball in* $\mathbb{R}^n$, $n \geqslant 2$, *and f satisfies* (67) *with some* $c_0 > 0$ *and* $\alpha > 2$. *Then for any* $\lambda > 0$ *there exist radially symmetric solutions* (u, v) *of* (1) *which blow up and have mass* $\int_\Omega u(t) \equiv \lambda$.

For the proof we need the following lemma asserts that the component *v* of a solution (u, v, Γ) of (S_λ) has values of the same order as Γ in a set of uniformly positive measure.

Lemma 6.3. *If f satisfies* (67) *with some* $\alpha > 2$ *then there exist* $\kappa \in \mathbb{R}$ *and* $c > 0$ *such that any radially symmetric solution* (u, v, Γ) *of* (S_λ) *satisfies*

$$\Big|\{v \geqslant \Gamma - 2\kappa\}\Big| \geqslant c. \tag{68}$$

Proof. To make the proof more transparent, let us use

$$\bar{\varphi}(s) := \int_s^\infty \frac{d\tau}{f(\tau)} \equiv \varphi(s) + c_1$$

with

$$c_1 := \int_1^\infty \frac{d\tau}{f(\tau)}.$$

Note that c_1 is finite due to $\alpha > 2$. Then $\bar{\varphi}$ is positive and, by (S_λ),

$$\bar{\varphi}(u) + v \equiv \bar{\Gamma} := \Gamma + c_1 \text{ in } \Omega.$$

Clearly, the claim of the lemma will follow as soon as we have shown that there exist $\bar{\kappa} > 0$ and $c > 0$ such that

$$\left|\{v \geqslant \bar{\Gamma} - 2\bar{\kappa}\}\right| \geqslant c.$$

holds for all radial solutions of (B_λ). For convenience in notation, throughout this proof we shall omit the bars and thus again write φ, Γ and κ in place of $\bar{\varphi}, \bar{\Gamma}$ and $\bar{\kappa}$. We define

$$\kappa := \varphi\left(\frac{\lambda}{2|\Omega|}\right) > 0$$

and first observe that

$$\sup_{x\in\Omega} v(x) > \Gamma - \kappa, \tag{69}$$

since otherwise we would have $\varphi(u) = \Gamma - v \geqslant \kappa$ in Ω and hence

$$\int_\Omega u \leqslant \varphi^{-1}(\kappa) \cdot |\Omega| = \frac{\lambda}{2} < \lambda,$$

a contradiction.

With this value of κ fixed henceforth, we observe that (67) (together with the positivity of $f(s)$ for $s > 0$) implies

$$\varphi(s) = \int_s^\infty \frac{d\tau}{f(\tau)} \leqslant \frac{1}{c_0(\alpha - 1)} s^{1-\alpha}$$

for all $s \geqslant \varphi^{-1}(2\kappa)$. Thus,

$$\varphi^{-1}(\sigma) \leqslant c_6 \sigma^{-\frac{1}{\alpha-1}} \quad \forall \sigma \leqslant 2\kappa \tag{70}$$

holds with a suitable $c_6 > 0$. Let us set $w := \Gamma - v \equiv \varphi(u)$. Then $w + w(r)$ is a positive radial function, and (69) says that if w takes its minimum at $r_0 \in [0, R]$, we have $w(r_0) \leqslant \kappa$. We first consider the case $r_0 \leqslant \frac{R}{2}$ and claim that then

$$w(r) \leqslant 2\kappa \quad \forall r \in [r_0, r_+] \tag{71}$$

holds with $r_+ := r_0 + r_{R,\alpha,\kappa}$, where

$$r_{R,\alpha,\kappa} := \min\left\{\frac{R}{2}, \sqrt{\frac{(\alpha-2)\kappa^{\frac{\alpha}{\alpha-1}}}{2^{\frac{2\alpha-3}{\alpha-1}}c_6(\alpha-1)}}\right\}.$$

In fact, if $w < 2\kappa$ throughout $[r_0, R]$ we are done. Otherwise there exists $r_1 \in (r_0, R]$ such that $w < 2\kappa$ on (r_0, r_1) and $w(r_1) = 2\kappa$. We will show that

$$w_r(r) \leqslant \sqrt{\frac{2c_6(\alpha-1)(2\kappa)^{\frac{\alpha-2}{\alpha-1}}}{\alpha-2}} \qquad \forall\, r \in (r_0, r_1), \tag{72}$$

from which it will result that

$$2\kappa = w(r_1) \leqslant \kappa + \sqrt{\frac{2c_6(\alpha-1)(2\kappa)^{\frac{\alpha-2}{\alpha-1}}}{\alpha-2}}(r_1 - r_0).$$

This in turn will imply

$$r_1 - r_0 \geqslant \sqrt{\frac{\alpha-2}{2c_6(\alpha-1)(2\kappa)^{\frac{\alpha-2}{\alpha-1}}}}\,\kappa \geqslant r_{R,\alpha,\kappa}$$

and thereby prove (71).

To see (72), we fix $r \in (r_0, r_1)$ and may assume $w_r(r) > 0$. Then $w_r > 0$ on $(\tilde{r}, r)$, where $\tilde{r} := \max\{\rho < r \mid w_r(\rho) = 0\}$, and therefore

$$\begin{aligned} w_r r \;&=\; -v_{rr} = u - v + \frac{n-1}{r}v_r = \varphi^{-1}(w) - v - \frac{n-1}{r}w_r \leqslant \varphi^{-1}(w) \\ &\leqslant\; c_6 w^{-\frac{1}{\alpha-1}} \qquad \text{on } (\tilde{r}, r), \end{aligned}$$

where we have used (70) and the fact that v is nonnegative. Consequently, after multiplying by $w_r \geqslant 0$ we obtain

$$\frac{1}{2}\Big(w_r^2(\rho) - w_r^2(\tilde{r})\Big) \leqslant \frac{c_6(\alpha-1)}{\alpha-2}\Big(w^{\frac{\alpha-2}{\alpha-1}}(\rho) - w^{\frac{\alpha-2}{\alpha-1}}(\tilde{r})\Big) \quad \forall \rho \in (\tilde{r}, r).$$

As $\alpha > 2$ and $w_r(\tilde{r}) = 0$, this yields

$$\frac{1}{2}w_r^2(r) \leqslant \frac{c_6(\alpha-1)}{\alpha-2}\, w^{\frac{\alpha-2}{\alpha-1}}(r) \leqslant \frac{c_6(\alpha-1)}{\alpha-2}(2\kappa)^{\frac{\alpha-2}{\alpha-1}}$$

and hence completes the proof of (72). Having thus shown (71), we now obtain

$$|\{w \leqslant 2\kappa\}| \geqslant |B_{r_+} \setminus B_{r_0}| \ \geqslant \ |B_{r_+ - r_0}| \geqslant |B_{r_{R,\alpha,\kappa}}| = \frac{\omega_n}{n} r^n_{R,\alpha,\kappa},$$

which yields the desired estimate in the case $r_0 \leqslant \frac{R}{2}$.

If $r_0 > \frac{R}{2}$, however, we proceed similarly, claiming that instead of (71), $w(r) \leqslant 2\kappa$ holds for $r \in [r_-, r_0]$, where $r_- := r_0 - r_{R,\alpha,\kappa}$, and replacing $w_r(r)$ by $-w_r(r)$ in (72). □

Combining the above lemma with Corollary 6.1 we can now prove Theorem 6.1.

Proof of Theorem 6.1. In view of Corollary 6.1 it is sufficient to show that for any $\lambda > 0$ there exists $c_\lambda > 0$ such that

$$F(u, v) \geqslant -c_\lambda \qquad \text{holds for all radial solutions} (u, v, \Gamma) \text{ of } (\mathrm{S}_\lambda). \tag{73}$$

To this end, we multiply the first in (S_λ) by v to obtain

$$\int_\Omega |\nabla v|^2 + \int_\Omega v^2 = \int_\Omega uv,$$

so that

$$F(u, v) = -\frac{1}{2} \int_\Omega uv + \int_\Omega \Phi(u).$$

Since Φ is nonnegative, $v = \Gamma - \varphi(u)$ and $\int_\Omega u = \lambda$, this gives

$$F(u, v) = -\frac{\Gamma}{2} \int_\Omega u + \frac{1}{2} \int_\Omega u\varphi(u) + \int_\Omega \Phi(u) \geqslant -\frac{\lambda\Gamma}{2} - \frac{\lambda}{2} \int_1^\infty \frac{d\tau}{f(\tau)}.$$

for all solutions of (S_λ) — no matter whether radial or not. So if (73) were false, there would exist a sequence of radial solutions (u_k, v_k, Γ_k) of (S_λ) such that $\Gamma_k \to +\infty$ as $k \to \infty$. But then Lemma 6.3 states that for some $\kappa > 0$,

$$\int_\Omega v_k \geqslant |\{v_k \geqslant \Gamma_k - 2\kappa\}| (\Gamma_k - 2\kappa) \to \infty \text{ as } k \to \infty,$$

which contradicts $\int_\Omega v_k = \lambda$ for all k. Therefore (73) must be true. □

6.2. The case $1 < \alpha < 2$

In this section we shall derive some blow-up results in space dimensions $n \in \{2, 3\}$ under less restrictive growth conditions on *f*. However, for technical reasons we shall need that (67) be supplemented with an estimate from *below* for $f(s)$ for large *s*. To be more precise, throughout this section we will assume that

$$c_0 s^{\alpha} \leqslant f(s) \leqslant c_1 s^{\alpha_+} \quad \forall s \geqslant 1 \tag{74}$$

holds with

$$\alpha \in \begin{cases} (1,2] & \text{if } n = 2, \\ (1,2) & \text{if } n = 3 \end{cases} \quad \text{and } \alpha_+ \in \left[\alpha, \tfrac{1}{2-\alpha}\right]. \tag{75}$$

Note that, particularly, this admits the choice $\alpha_+ = \alpha$ and thereby covers the homogeneous case $f(s) = s^{\alpha}$ with α as indicated in (75). However, also rather strong oscillations of *f* are allowed.

Actually our method would apply to any $\alpha \in (1, \infty)$ in the two dimensional case, but in view of the previous section this would not provide any progress.
Our main result will be

Theorem 6.2. *Assume that* $\Omega = B_R$ *is a ball in* $\mathbb{R}^n$, *where* $n = 2$ *or* $n = 3$, *and that f satisfies* (74) *with* α *and* α_+ *fulfilling* (75). *Then for any* $\lambda > 0$ *there exist radially symmetric solutions* (u, v) *of* (1) *which blow up and have mass* $\int_\Omega u(t) \equiv \lambda$.

The proof of this theorem will be given in the end of this section; it will be prepared by three lemmata for which we need some preliminaries.

As in the last section, we shall use the function

$$\bar\varphi(s) := \int_s^\infty \frac{d\tau}{f(\tau)} = \varphi(s) + \Sigma, \text{ where } \Sigma := \int_1^\infty \frac{d\tau}{f(\tau)}$$

is finite due to $\alpha > 1$. Then (74) implies that

$$c s^{1-\alpha_+} \leqslant \bar\varphi(s) \leqslant C s^{1-\alpha} \quad \forall s \geqslant 1,$$

whence its inverse $\bar\varphi^{-1}$ fulfills

$$a\sigma^{-\frac{1}{\alpha_+ - 1}} \leqslant \bar\varphi^{-1}(\sigma) \leqslant b\sigma^{-\frac{1}{\alpha-1}} \quad \forall \sigma \leqslant 1 \tag{76}$$

with certain positive *a* and *b*.

If (u, v, Γ) is any radial solution of (S_λ) then $w := \bar\varphi(u)$ is a positive solution of

$$\Delta w = \bar\varphi^{-1}(w) - v \qquad \text{in } \Omega, \tag{77}$$

satisfying

$$\int_\Omega \bar{\varphi}^{-1}(w) = \lambda \tag{78}$$

and

$$\int_\Omega |\nabla w|^s = \int_\Omega |\nabla v|^s \leqslant c(s,\lambda) \quad \forall s < \frac{n}{n-1} \tag{79}$$

due to Lemma 6.2. In contrast with the previous section, we now treat v as a nonnegative *perturbation* of the equation $\Delta w = \bar{\varphi}^{-1}(w)$ which is small in the sense that

$$\int_\Omega v^q \leqslant c(q,\lambda) \quad \forall q < \frac{n}{n-2}, \tag{80}$$

also by Lemma 6.2.

Our strategy roughly is as follows: As in the proof of Theorem 6.1, we only need to show that the possible values of Γ in radial solutions (u, v, Γ) are bounded above. It is sufficient for this purpose (cf. the proof of Theorem 6.2 below for details) to exclude the possibility that $w_k \to +\infty$ as $k \to \infty$ a.e. in Ω for a sequence of correspondingly transformed solutions w_k of (77). As a starting point we may employ (78) which shows that w must be 'bounded' at least at some point in Ω, so we will be successful if we can control the growth of w near such a point. In doing this, we shall go along a remarkable indirection: We first prove a *lower* bound for such a growth (using the bound for f from *above* in (74)) to derive from this (and the left inequality in (74)) the desired upper bound.

Lemma 6.4. *Let $\Omega = B_R$ be a ball in $\mathbb{R}^n$, $n \in \{2,3\}$, and suppose f and α, α_+ satisfy* (74) *and* (75). *Then there exist $R_0 \in (0,R)$ and $C_0 > 0$ such that for any radial solution (u,v,Γ) of* (S_λ), *the function $w = \bar{\varphi}(u)$ satisfies*

$$w(r) \geqslant C_0 r^{\frac{2(\alpha_+-1)}{\alpha_+}} \qquad \forall r \in (0, R_0). \tag{81}$$

Remark. Actually, this lemma only requires the right inequality in (74) and that w and v satisfy (77) and (80).

Proof. The proof will consist of three steps. Throughout, let us write $p := \frac{1}{\alpha_+-1}$ and $\beta := \frac{2}{p+1} \equiv \frac{2(\alpha_+-1)}{\alpha_+}$.

Step 1: We first claim that there exists $C_1 > 0$ (depending on α_+, a, λ and R only) with the property that whenever $r_0 \in (0, \frac{R}{2})$ is such that $w(r_0) \geqslant C_1 r_0^\beta$ then there exists $r_1 \in (r_0, 2r_0)$ with $w(r_1) \geqslant C_1 r_1^\beta$.

This can be seen as follows. Suppose C_1 is a positive number such that our claim does not hold for C_1. Here we may assume that $C_1 R^\beta \leqslant 1$, so that, according to (76),

$$\bar{\varphi}^{-1}(w) \geqslant a w^{-p} \quad \text{whenever} \quad w(r) \leqslant C_1 r^\beta. \tag{82}$$

Then there exists $r_0 > 0$ with $w(r_0) = C_1 r_0^\beta$ but

$$w(r) < C_1 r^\beta \quad \forall r \in (r_0, 2r_0). \tag{83}$$

Integrating (77), that is,

$$\frac{1}{r^{n-1}}(r^{n-1} w_r)_r = \bar{\varphi}^{-1}(w) - v, \qquad r \in (0, R),$$

twice with respect to r, we successively obtain

$$r^{n-1} w_r(r) = \bar{r}_0^{n-1} w_r(\bar{r}_0) + \int_{\bar{r}_0}^r \rho^{n-1} \bar{\varphi}^{-1}(w(\rho))\, d\rho - \int_{\bar{r}_0}^r \rho^{n-1} v(\rho)\, d\rho, \tag{84}$$

for $0 \leqslant \bar{r}_0 < r \leqslant R$, and thus, using (82),

$$\begin{aligned} C_1\Big((2r_0)^\beta - r_0^\beta\Big) &\geqslant w(2r_0) - w(r_0) \\ &\geqslant r_0^{n-1} w_r(r_0) \cdot \int_{r_0}^{2r_0} \rho^{1-n}\, d\rho + a \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} w^{-p}(\sigma)\, d\sigma\, d\rho \\ &\quad - \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} v(\sigma)\, d\sigma\, d\rho \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{85}$$

In order to estimate from below the terms on the right, let us first fix some $\varepsilon \in (0, 4 - n - \beta)$, which is possible due to (75). (Indeed, the inequality $\beta \equiv 2 - \frac{2}{\alpha_+} < 4 - n$ is trivial if $n = 2$, while for $n = 3$ it is guaranteed by the fact that $\alpha_+ < 2$.) Using $q := \frac{n}{n-2+\varepsilon}$ in (80), we then infer that

$$\begin{aligned} \int_0^r \rho^{n-1} v(\rho) d\rho &\leqslant \Big(\int_0^r \rho^{n-1} v^q(\rho) d\rho \Big)^{\frac{1}{q}} \cdot \Big(\int_0^r \rho^{n-1} d\rho \Big)^{\frac{2-\varepsilon}{n}} \\ &\leqslant c r^{2-\varepsilon} \ \forall\, r \in (0, R), \end{aligned} \tag{86}$$

where c denotes a generic constant independent of w. Inserted into (84), this gives (with $\bar{r}_0 := 0$)

$$w_r(r) \geqslant -c r^{3-n-\varepsilon} \quad \forall r \in (0, R) \tag{87}$$

and thus

$$I_1 = r_0 w_r(r_0) \cdot \int_1^2 \sigma^{1-n} d\sigma \geqslant -cr_0^{4-n-\varepsilon}. \tag{88}$$

As to I_3, (86) entails

$$I_3 \geqslant -c \int_{r_0}^{2r_0} \rho^{3-n-\varepsilon}\, d\rho = -cr_0^{4-n-\varepsilon}.$$

We now use (83) to estimate

$$\begin{aligned} I_2 &\geqslant a \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} \cdot (C_1 \sigma^\beta)^{-p}\, d\sigma\, d\rho \\ &= \frac{aC_1^{-p_+}}{n-p\beta} \int_{r_0}^{2r_0} \rho^{1-n} (\rho^{n-p\beta} - r_0^{n-p\beta})\, d\rho \\ &= \frac{aC_1^{-p_+}}{n-p\beta} \Big(\frac{2^{2-p\beta}-1}{2-p\beta} - \int_1^2 \sigma^{1-n} d\sigma \Big) r_0^{2-p\beta}. \end{aligned} \tag{89}$$

Since $n \geqslant 2$ and $p\beta < 2$, the constant $\frac{2^{2-p\beta}-1}{2-p\beta} - \int_1^2 \sigma^{1-n} d\sigma$ is positive, whence from (85)–(89) we infer

$$C_1^{-p} r_0^\beta = C_1^{-p} r_0^{2-p\beta} \leqslant c\Big(C_1 r_0^\beta + r_0^{4-n-\varepsilon} \Big),$$

so that, since $\varepsilon < 4 - n - \beta$,

$$C_1^{-p} \leqslant c\Big(C_1 + r_0^{4-n-\varepsilon-\beta} \Big) \leqslant c\Big(C_1 + R^{4-n-\varepsilon-\beta} \Big),$$

which is impossible if C_1 is appropriately small. Thereby our claim has been proved.

Step 2: Let us set

$$S := \{ r \in [0, R] \mid w(r) \geqslant C_1 r^\beta \}$$

with C_1 as above. Since w is positive, S is not empty and $r_0 := \max\{r \in [0, R] \mid [0, r] \subset S\}$ is well-defined and positive. Therefore

$$r_{k+1} := \max \Big\{ r \in [r_k, 4r_k] \cap [0, R] \mid r \in S \Big\}, \qquad k = 0, 1, 2, \ldots,$$

defines an increasing sequence of numbers r_k. Let us make sure that

$$\exists\, k_0 \in \mathbb{N} \text{ such that } r_{k_0} \geqslant \frac{R}{2}.$$

In fact, if this were false then r_k would converge to some $r_\infty \leqslant \frac{R}{2}$ which, by continuity of w, would belong to S. By the outcome of Step 1, there would exist $\tilde{r} \in (r_\infty, 2r_\infty) \cap S$. Since $r_k \to r_\infty$, this implies that $\tilde{r} \in [r_k, 4r_k] \cap S$ for large k and hence $r_{k+1} \geqslant \tilde{r} > r_\infty$ for such k, a contradiction.

Step 3: In order to conclude that the lemma is true, we take $\varepsilon \in (0, 4-n-\beta)$ as in Step 1. Then (87) holds and thus

$$\int_{r_k}^{r} w_r(\rho)d\rho \geqslant -c\int_{r_k}^{r} \rho^{3-n-\varepsilon} d\rho \geqslant C_2 r_k^{4-n-\varepsilon} \quad \forall r \in [r_k, 4r_k] \cap [0, R]$$

is valid with some C_2 independent of w. We now fix $R_0 < \frac{R}{2}$ such that

$$R_0^{4-n-\varepsilon-\beta} \leqslant \frac{C_1}{2C_2}$$

and set $C_0 := \frac{C_1}{2\cdot 4^\beta} < C_1$. Then, while (81) trivially holds for $r < r_0$, for all $r \in [r_0, R_0]$ we can find (due to Step 2) some $k \in \mathbb{N}$ such that $r_k \leqslant r < 4r_k$ and hence

$$\begin{aligned} w(r) &= w(r_k) + \int_{r_k}^{r} w_r(\rho)d\rho \geqslant C_1 r_k^\beta - C_2 r_k^{4-n-\varepsilon} = r_k^\beta\Big(C_1 - C_2 r_k^{4-n-\varepsilon-\beta}\Big) \\ &\geqslant r_k^\beta\Big(C_1 - C_2 R_0^{4-n-\varepsilon-\beta}\Big) \geqslant \frac{C_1}{2} r_k^\beta \geqslant \frac{C_1}{2}\left(\frac{r}{4}\right)^\beta = C_0 r^\beta, \end{aligned}$$

whereby (81) has been shown. □

We can now prove a result which is in fact much sharper than needed: Namely, we can show that w will be locally uniformly bounded in $B_R \setminus \{0\}$.

Lemma 6.5. *Let $\Omega = B_R$ be a ball in $\mathbb{R}^n$, $n \in \{2, 3\}$, and suppose f and α, α_+ satisfy* (74) *and* (75). *Then for all $R_1 \in (0, R)$ there exists $c(R_1) > 0$ such that for any radial solution (u, v, Γ) of* (S_λ), *the function $w = \bar{\varphi}(u)$ satisfies*

$$w(r) \leqslant c(R_1) \quad \forall r \in [R_1, R].$$

Proof. If the lemma was false, there would exist $R_1 \in (0, R)$ and a sequence of solutions (u_k, v_k, Γ_k) of (S_λ) such that

$$\sup_{r \in (R_1, R)} w_k(r) \to \infty \quad \text{as } k \to \infty \tag{90}$$

holds with $w_k := \bar\varphi(u_k)$. Since $\int_\Omega \bar\varphi^{-1}(w_k) = \lambda$ by (78), for each k there exists $r_k \in [0, R]$ such that $w_k(r_k) \leqslant c_3 := \bar\varphi(\frac{\lambda}{|\Omega|})$ and $w_{kr}(r_k) = 0$. After extracting a subsequence, we may assume $r_k \to r_\infty \in [0, R]$ as $k \to \infty$. In the case $r_\infty > 0$ we would have, using Hölder's inequality and (79),

$$\begin{aligned} |w_k(r) - w_k(r_k)| &= \left| \int_{r_k}^r w_{kr}(\rho) d\rho \right| \leqslant \left| \int_{r_k}^r \rho^{n-1} |w_{kr}(\rho)|^s d\rho \right|^{\frac{1}{s}} \cdot \left| \int_{r_k}^r \rho^{-\frac{n-1}{s-1}} d\rho \right|^{\frac{s-1}{s}} \\ &\leqslant c \min\left\{ r_\infty^{-\frac{n-1}{s}}, R_1^{-\frac{n-1}{s}} \right\} \cdot |r - r_k|^{\frac{s-1}{s}} \leqslant c \quad \forall r \in [R_1, R] \end{aligned} \tag{91}$$

for any $s \in (1, \frac{n}{n-1})$ and large k, contradicting (90).

Thus, $r_k \to 0$ as $k \to \infty$. In this situation, however, Lemma 6.4 applies to tell us that the lower estimate $w(r) \geqslant C_0 r^{\frac{2(\alpha_+-1)}{\alpha_+}}$ $\forall\, r \in (r_k, R_0)$ holds with certain positive C_0 and R_0 satisfying $C_0 R^{\frac{2(\alpha_+-1)}{\alpha_+}} \leqslant 1$. Therefore, integrating (84) with $\bar r_0 := r_k$ and using the right inequality in (76) as well as the monotonicity of $\bar\varphi^{-1}$, we obtain

$$\begin{aligned} w_k(r) &= w_k(r_k) + \int_{r_k}^r \rho^{1-n} \int_{r_k}^\rho \sigma^{n-1} \bar\varphi^{-1}(w_k(\sigma))\, d\sigma\, d\rho - \int_{r_k}^r \rho^{1-n} \int_{r_k}^\rho \sigma^{n-1} v(\sigma)\, d\sigma\, d\rho \\ &\leqslant c_3 + \int_{r_k}^r \rho^{1-n} \int_{r_k}^\rho \sigma^{n-1} \bar\varphi^{-1}\Big(C_0 \sigma^{\frac{2(\alpha++-1)}{\alpha_+}} \Big)\, d\sigma\, d\rho \\ &\leqslant c_3 + bC_0^{-\frac{1}{\alpha-1}} \int_0^r \rho^{1-n} \int_0^\rho \sigma^{n-1-\frac{2(\alpha_+-1)}{\alpha_+(\alpha-1)}}\, d\sigma\, d\rho \\ &= c_3 + \frac{bC_0^{-\frac{1}{\alpha-1}}}{\Big(n - \frac{2(\alpha_+-1)}{\alpha_+(\alpha-1)}\Big)\Big(2 - \frac{2(\alpha_+-1)}{\alpha_+(\alpha-1)}\Big)} \cdot r^{2-\frac{2(\alpha_+-1)}{\alpha_+(\alpha-1)}} \quad \forall\, r \in [r_k, R_0], \end{aligned} \tag{92}$$

because $2 - \frac{2(\alpha_+-1)}{\alpha_+(\alpha-1)} > 0$ due to (75). Consequently, $w_k(R_0) \leqslant c_4$ holds for all k and some $c_4 > 0$. Repeating now the argument leading to (91) yields, with arbitrary $s \in \frac{n}{n-1}$,

$$|w_k(r) - w_k(R_0)| \leqslant c R_0^{-\frac{n-1}{s}} |r - R_0|^{\frac{s-1}{s}} \quad \forall r \in [R_0, R],$$

which together with (92) is incompatible with (90). $\square$

After these preparations the proof of Theorem 6.2 is comparatively simple now.

Proof of Theorem 6.2. As in the proof of Theorem 6.1, we first observe that all solutions (u, v, Γ) of (S_λ) satisfy

$$F(u, v) = -\frac{1}{2} \int_\Omega uv + \int_\Omega \Phi(u) = -\frac{\Gamma}{2} \int_\Omega u + \frac{1}{2} \int_\Omega u\varphi(u) + \int_\Omega \Phi(u) \geqslant -\frac{\lambda \bar\Gamma}{2}$$

with $\bar{\Gamma} = \Gamma + \int_1^\infty \frac{d\tau}{f(\tau)}$. In virtue of Corollary 6.1 it thus again remains to derive an upper bound for $\bar{\Gamma}$ (or, equivalently, for Γ).

So suppose there were a sequence of solutions (u_k, v_k, Γ_k) to (S_λ) such that $\bar{\Gamma}_k \to +\infty$ as $k \to \infty$. Due to (79), we may extract a subsequence along which v_k converges to some v_∞ in $L^1(\Omega)$ and a.e. in Ω. Therefore the functions $w_k = \bar{\varphi}(u_k) \equiv \bar{\Gamma}_k - v_k$ tend to $+\infty$ a.e. in Ω, which is absurd in virtue of Lemma 6.5. □

6.3. *The case* $\frac{2}{n} < \alpha < 1$

Let us now turn to the possibly most delicate question in respect of the criticality of the exponent $\alpha = \frac{2}{n}$: Does blow-up occur for $\alpha > \frac{2}{n}$ close to $\frac{2}{n}$? In space dimension two, this has been answered in the previous section already. But in the case $n = 3$, the above results leave a gap between the suspectedly critical exponent $\alpha = \frac{2}{n}$ and 1. It is the purpose of the present section to close this gap and, additionally, provide some blow-up results also for higher space dimensions and exponents $\alpha < 1$. Particularly, we shall find that blow-up indeed occurs for α arbitrarily close to $\frac{2}{n}$ in *any* space dimension $n \geqslant 3$ (and hence for any $n \geqslant 2$). This strongly underlines the role of $\alpha = \frac{2}{n}$ as a critical blow-up exponent.

In this section we assume $n \geqslant 3$ and the two-sided growth condition

$$c_0 s^\alpha \leqslant f(s) \leqslant c_1 s^\alpha \quad \forall s \geqslant s_0 \tag{93}$$

with some $s_0 \geqslant 1$. Here, we need to restrict α according to

$$\alpha \in \begin{cases} (\frac{2}{n}, 1) & \text{if } n = 3 \text{ or } n = 4, \\ (\frac{2}{n}, \frac{2}{n-2}) & \text{if } n \geqslant 5, \end{cases} \tag{94}$$

and suppose that the positive numbers c_0 and c_1 satisfy

$$c_1 \leqslant \frac{c_0}{1-\alpha}. \tag{95}$$

Note that these assumptions again include the homogeneous case $f(s) = s^\alpha$ (for large s) but also a wider class of f with possibly oscillatory behavior.

It is easy to see that since $f(s)$ is positive for any $s > 0$, (93) implies

$$a|\sigma|^{\frac{1}{1-\alpha}} \leqslant \varphi^{-1}(\sigma) \leqslant b|\sigma|^{\frac{1}{1-\alpha}} \quad \forall \sigma \leqslant -1 \tag{96}$$

with certain positive constants a and b.

Before proving two auxiliary lemmata, let us state our main result on radial blow-up for $\alpha < 1$.

Theorem 6.3. *Let $\Omega = B_R$ be a ball in $\mathbb{R}^n$, $n \geqslant 3$, and suppose that f obeys condition* (93) *with α, c_0 and c_1 satisfying* (94) *and* (95). *Then for any $\lambda > 0$ there exist radially symmetric solutions (u, v) of* (1) *which blow up and have mass $\int_\Omega u(t) \equiv \lambda$.*

The key to the proof of Theorem 6.3 is the following lemma, which has a lot in common with Lemma 6.4. Observe again that it not yet uses the left estimate in (93).

Lemma 6.6. *Let $\Omega = B_R$ be a ball in $\mathbb{R}^n$, $n \geqslant 3$, and suppose f and α satisfy* (93) *and* (94). *Then there exist $R_0 \in (0, R)$ and $C_0 > 0$ such that for any radial solution (u, v, Γ) of* (S_λ), *the function $w = \varphi(u)$ satisfies*

$$w(r) \geqslant -C_0 r^{-\frac{2(1-\alpha)}{\alpha}} \qquad \forall r \in (0, R_0).$$

Proof. Let us abbreviate $p := \frac{1}{1-\alpha}$ and $\gamma := \frac{2}{p-1}$.

Step 1: As in Step 1 of Lemma 6.4, we first claim that there exist $C_1 > 0$ such that whenever $w(r_0) \geqslant -C_1 r_0^{-\gamma}$ for some $r_0 \in (0, \frac{R}{\kappa})$ then there exists $r_1 \in (r_0, \kappa r_0)$ with $w(r_1) \geqslant -C_1 r_1^{-\gamma}$.

To see this, we suppose this were false for some C_1 which we may assume to be large such that $C_1 R^{-\gamma} \geqslant 1$, implying

$$\varphi^{-1}(w) \geqslant a|w|^p \quad \text{whenever} \quad w(r) \leqslant -C_1 r^{-\gamma}.$$

Then there exists $r_0 > 0$ with $w(r_0) = -C_1 r_0^{-\gamma}$ but

$$w(r) < -C_1 r^{-\gamma} \quad \forall r \in (r_0, 2r_0). \tag{97}$$

Thus, upon integrating (77) we obtain

$$\begin{aligned} -C_1\Big((2r_0)^{-\gamma} - r_0^{-\gamma}\Big) &\geqslant w(2r_0) - w(r_0) \\ &\geqslant r_0^{n-1} w_r(r_0) \int_{r_0}^{2r_0} \rho^{1-n}\, d\rho \\ &\quad + a \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} |w|^p(\sigma)\, d\sigma\, d\rho \\ &\quad - \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} v(\sigma)\, d\sigma\, d\rho \\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{98}$$

where it can be seen from the fact that $\int_\Omega v^q \leqslant c(q) \forall q < \frac{n}{n-2}$ that

$$w_r(r) \geqslant -c r^{3-n-\varepsilon} \quad \forall r \in (0, R) \tag{99}$$

and thus

$$I_1 + I_3 \geqslant -cr_0^{4-n-\varepsilon} \tag{100}$$

holds for any $\varepsilon > 0$. In what follows we fix $\varepsilon \in (0, 4-n+\gamma)$ which is possible due to (94). Next, (97) allows us to estimate

$$\begin{aligned} I_2 &\geqslant a \int_{r_0}^{2r_0} \rho^{1-n} \int_{r_0}^{\rho} \sigma^{n-1} (C_1 \sigma^{-\gamma})^p \, d\sigma \, d\rho = \frac{aC_1^p}{n-p\gamma} \int_{r_0}^{2r_0} \rho^{1-n} (\rho^{n-p\gamma} - r_0^{n-p\gamma}) \, d\rho \\ &= \frac{aC_1^p}{n-p\gamma} \Big(\frac{1-2^{2-p\gamma}}{p\gamma - 2} - \frac{1-2^{2-n}}{n-2} \Big) r_0^{2-p\gamma}. \end{aligned} \tag{101}$$

Since $p\gamma < n$ and therefore also $\frac{1-2^{2-p\gamma}}{p\gamma-2} - \frac{1-2^{2-n}}{n-2}$ is positive, from (98)–(101) we obtain

$$C_1^p r_0^{-\gamma} = C_1^p r_0^{2-p\gamma} \leqslant c \Big(C_1 r_0^{-\gamma} + r_0^{4-n-\varepsilon} \Big),$$

so that, since $\varepsilon < 4 - n + \gamma$,

$$C_1^p \leqslant c \Big(C_1 + r_0^{4-n-\varepsilon+\gamma} \Big) \leqslant c \Big(C_1 + R^{4-n-\varepsilon+\gamma} \Big).$$

This is absurd for large C_1.

Having thus proved our claim, we now can easily derive from this the assertion of the lemma, using slightly modified variants of Steps 2 and 3 from the proof of Lemma 6.4. □

The reason for the restriction on c_1 (as related to c_0) lies in the following lemma. It asserts that the term $\frac{1}{2} \int_\Omega u\varphi(u)$ appearing in $F(u, v)$ (cf. the proofs of Theorem 6.1 or 6.3 below), albeit being no longer bounded from below by a constant, at least may be compensated by $\int_\Omega \Phi(u)$. In fact, as compared to the previous two sections, this will be the first place where any growth properties of Φ are used, and where Φ is not trivially estimated from below by zero.

Lemma 6.7. *Suppose* $\alpha \in (0, 1)$ *and* f, c_0 *and* c_1 *satisfy* (93) *and* (95). *Then there exists* $c > 0$ *such that*

$$\tfrac{1}{2} s\varphi(s) + \Phi(s) \geqslant -c(1+s) \quad \forall s > 0. \tag{102}$$

Proof. Since $\varphi(s)$ and $\Phi(s)$ are positive for $s < 1$, the claim easily follows if we can show that the derivative of $g(s) := \frac{1}{2}\varphi(s) + \Phi(s)$ is bounded from below for $s > s_0$.

To see this, we differentiate and use (93) to obtain

$$\begin{aligned} g'(s) &= -\frac{1}{2}\varphi(s) - \frac{1}{2}\frac{s}{f(s)} \geqslant \frac{1}{2}\int_{s_0}^{s}\frac{d\tau}{f(\tau)} - \frac{1}{2}\frac{s}{f(s)} \\ &\geqslant \frac{1}{2c_0(1-\alpha)}\left(s^{1-\alpha} - s_0^{1-\alpha}\right) - \frac{1}{2c_1}s^{1-\alpha} \\ &= \frac{1}{2}\left(\frac{1}{c_0(1-\alpha)} - \frac{1}{c_1}\right)s^{1-\alpha} - \frac{s_0^{1-\alpha}}{2c_0(1-\alpha)} \quad \forall s > s_0, \end{aligned}$$

which is bounded from below due to (95). □

Although we do not have at hand a locally uniform estimate in the style of Lemma 6.5 now, we can nevertheless proceed to prove the main result.

Proof of Theorem 6.3. If (u, v, Γ) is a solution of (S_λ) then we have already seen that (cf. the proof of 6.1) that

$$F(u,v) = -\frac{1}{2}\int_\Omega uv + \int_\Omega \Phi(u) = -\frac{\Gamma}{2}\int_\Omega u + \frac{1}{2}\int_\Omega u\varphi(u) + \int_\Omega \Phi(u),$$

so that Lemma 6.7 yields

$$F(u,v) \geqslant -\frac{\lambda\Gamma}{2} - c(\lambda+1) \tag{103}$$

with some $c > 0$. In order to show the existence of an upper bound for all possible Γ, let us suppose on the contrary than (S_λ) has a sequence of radially symmetric solutions (u_k, v_k, Γ_k) for which $\Gamma_k \to +\infty$ holds as $k \to \infty$. Since a subsequence of $(v_k)_{k\in\mathbb{N}}$ converges strongly in $L^1(\Omega)$ by Lemma 6.2, the identity $\varphi(u_k) + v_k \equiv \Gamma_k$ implies that we may assume $\varphi(u_k) \to +\infty$ and hence

$$u_k \to 0 \qquad \text{a.e. in } \Omega \qquad \text{as } k \to \infty. \tag{104}$$

On the other hand, using Lemma 6.6 and the right estimate in (96) we see that

$$\begin{aligned} u_k(r) &= \varphi^{-1}(w_k(r)) \leqslant \varphi^{-1}\left(-C_0 r^{-\frac{2(1-\alpha)}{\alpha}}\right) \\ &\leqslant b\left(C_0 r^{-\frac{2(1-\alpha)}{\alpha}}\right)^{\frac{1}{1-\alpha}} = bC_0^{\frac{1}{1-\alpha}} r^{-\frac{2}{\alpha}} \quad \forall r \in (0,R),\ \forall k \in \mathbb{N}, \end{aligned} \tag{105}$$

where we have assumed C_0 to be so large that $C_0 R^{-\frac{2(1-\alpha)}{\alpha}} \geqslant 1$. Now (105) implies that

$$\int_\Omega u_k^q \leqslant c\int_0^R r^{n-1} \cdot r^{-\frac{2q}{\alpha}}\,dr \leqslant c$$

for arbitrary $q \in (1, \frac{n\alpha}{2})$ — note that such q exist since $\alpha > \frac{2}{n}$. But this means that u_k converges weakly in $L^q(\Omega)$ for a further subsequence. By (104) and Egorov's theorem, this weak limit must be zero a.e. in Ω. Therefore the lower semicontinuity of $\|\cdot\|_{L^q(\Omega)}$ with respect to weak convergence shows that $\|u_k\|_{L^q(\Omega)} \to 0$ as $k \to \infty$, which leads to the absurd conclusion

$$0 < \lambda = \|u_k\|_{L^1(\Omega)} \leqslant c\|u_k\|_{L^q(\Omega)} \to 0 \qquad \text{as } k \to \infty.$$

Thus, $\Gamma_k \to +\infty$ is impossible, so that now (103) in combination with Corollary 6.1 proves the theorem. □

7. Concluding remarks

The intention of present results is completely different from the approaches to blow-up in the classical chemotaxis model so far. Of course, there are the results for Keller–Segel-type model if the chemotatic sensitivity function depends only on the chemoattractant like those established by Nagai and Senba [27,28], Nagai et al. [30], Senba [35] and Post [34] for example. However, up to now no one has tried to get more insights in the determination of the right blow-up exponent. Thus our results are completely new and give more insights in the known results for parabolic–elliptic versions of (1) with $\alpha = 1$ and the expected behavior of the solution of the full system (1). Our results explain, why there is no blow-up for $n = 1$; there is the possibility of unbounded solutions for $n = 2$ if the initial data has sufficiently large mass; and why there are unbounded solutions without any restriction on the initial mass for $n \geqslant 3$.

Furthermore our approach is completely different from the attempts by Hillen and Painter (see [13,33]). The approach presented in the present paper allows some kind of "unified treatment" of all cases that exclude blow-up. Thus the results given here also include the existence results in [13]. The approach to our explanation of blow-up is also completely different from the perspective used by Herrero and Velázquez [12] and Herrero et al. [10,11] and Herrero [9].

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