On the dual rigidity matrix

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Abstract

A novel characterization of bar-and-joint framework rigidity was introduced in [A.Y. Alfakih, Graph rigidity via Euclidean distance matrices, Linear Algebra Appl. 310 (2000) 149–165; A.Y. Alfakih, On rigidity and realizability of weighted graphs, Linear Algebra Appl. 325 (2001) 57–70]. This characterization uses the notion of normal cones of convex sets to define a matrix $\bar{R}$ whose rank determines whether or not a given generic framework is rigid. Furthermore, this characterization was derived under the assumption that the framework of interest $G(p)$ has an equivalent framework $G(q)$ in $\mathbb{R}^{n-1}$, where $n$ is the number of vertices of $G(p)$. In this paper we show that the matrix $\bar{R}$ corresponding to a framework $G(p)$ contains the same information as the well-known rigidity matrix $R$. Whereas the entries of $R$ are a function of the positions of the vertices of $G(p)$, the entries of $\bar{R}$ are a function of the Gale matrix corresponding to $G(p)$. Furthermore, while the number of rows of $R$ is equal to the number of edges of $G(p)$, the number of columns of $\bar{R}$ is equal to the number of missing edges of $G(p)$. We also show that the assumption of the existence of an equivalent framework $G(q)$ in $\mathbb{R}^{n-1}$ can be dropped and we give the precise relation between the left-nullspaces, and consequently the nullspaces, of $R$ and $\bar{R}$.

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1. Introduction

A configuration $p$ in $\mathbb{R}^r$ is a finite collection of $n$ points $p^1, \ldots, p^n$ which span $\mathbb{R}^r$. Let $G = (V, E)$ be a simple graph on the vertices $1, 2, \ldots, n$. A bar-and-joint framework (or simply a framework), denoted by $G(p)$, in $\mathbb{R}^r$ is graph $G$ together with a configuration $p$ in $\mathbb{R}^r$, where each vertex $i$ of $G$ is located at $p^i$. With a slight abuse of notation, sometimes we will refer to the vertices and the edges of graph $G$ as the vertices and the edges of the framework $G(p)$. Also, to avoid trivialisations, we assume that $G$ is not the complete graph and $p^1, \ldots, p^n$ are not affinely independent.

Two frameworks $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be equivalent if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i, j) \in E$, where $\|\cdot\|$ denotes the Euclidean norm. The term bar is used to describe such frameworks because in any two equivalent frameworks $G(p)$ and $G(q)$, every two adjacent vertices $i$ and $j$ must stay the same distance apart. Thus edges of $G$ can be thought of as stiff rods. See Fig. 1 for an example of three frameworks in the plane.

Two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be congruent if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \ldots, n$. That is, $G(p)$ and $G(q)$ are congruent if configuration $q$ can be obtained from configuration $p$ by applying a rigid motion such as a translation or a rotation in $\mathbb{R}^r$. A framework $G(p)$ in $\mathbb{R}^r$ is said to be rigid if there exists an $\epsilon > 0$ such that if framework $G(q)$ in $\mathbb{R}^r$ is equivalent to $G(p)$ and $\|q^i - p^i\| \leq \epsilon$ for all $i = 1, \ldots, n$, then $G(q)$ is congruent to $G(p)$. If a framework is not rigid we say it is flexible. For other equivalent definitions of rigidity, and consequently of flexibility, see [13,14]. In this paper we do not distinguish between congruent frameworks since our formulation is rigid-motion independent.

Consider the process of continuously twisting a framework $G(p)$ into another equivalent framework $G(q)$. Configuration $q$ can then be thought of as a function of time $q(t)$ where $q(0) = p$. The quantity $(q^i(t) - q^j(t))^T(q^i(t) - q^j(t))$ for each edge $(i, j)$ must remain constant under such a process. Differentiating with respect to time $t$ and setting $t = 0$ we get

$$
(p^i - p^j)^T(p^i - p^j) = 0 \quad \text{for all } (i, j) \in E. \tag{1}
$$

Any $p' = (p'^1, \ldots, p'^n)$ that satisfies (1) is called an infinitesimal flex of $G(p)$. We say that an infinitesimal flex is trivial if it results from a rigid motion of $G(p)$. A framework $G(p)$ is said to be infinitesimally rigid if it has only trivial infinitesimal flexes. Otherwise, $G(p)$ is said to be infinitesimally flexible [10,9,11,14,16].

As the following theorem shows, the notion of infinitesimal rigidity of a framework is stronger than that of rigidity.

**Theorem 1.1** [13]. If a framework $G(p)$ is infinitesimally rigid, then it is rigid.
Fig. 2. A framework in $\mathbb{R}^2$ which is both rigid and infinitesimally flexible. A non-trivial infinitesimal flex is $p'_1 = p'_2 = p'_3 = p'_4 = (0, 0)^T$, $p'_5 = (0, -1)^T$.

The converse of the previous theorem is false. Fig. 2 shows a framework which is both rigid and infinitesimally flexible.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be generic if all the coordinates of $p_1, \ldots, p_n$ are algebraically independent over the integers. That is, $G(p)$ is generic if there does not exist a polynomial $f$ with integer coefficients such $f(p_1, \ldots, p_r, \ldots, p_n) = 0$. It is well known [13,8] that framework rigidity is a generic property. i.e., if a generic framework $G(q)$ in $\mathbb{R}^r$ is rigid, then all generic frameworks $G(q)$ in $\mathbb{R}^r$ are also rigid. Furthermore, Asimow and Roth [7] showed that the notions of rigidity and infinitesimal rigidity coincide for generic frameworks.

Given a framework $G(p)$ in $\mathbb{R}^r$ with $n$ vertices and $m$ edges, let $R$ be the $m \times nr$ matrix whose rows and columns are indexed, respectively, by the edges and the vertices of $G$ such that the $(i, j)$th row of $R$ is given by

$$[0 \cdots 0 \overline{(p'_i - p'_j)^T} 0 \cdots 0 \overline{(p'_i - p'_j)^T} 0 \cdots 0].$$

(2)

$R$ is called the rigidity matrix of $G(p)$ and obviously, the space of infinitesimal flexes of a framework is the nullspace of its rigidity matrix $R$. i.e., an infinitesimal flex of $G(p)$ is just a linear dependency among the columns of $R$.

**Theorem 1.2** [7]. Let $R$ be the rigidity matrix of a generic framework $G(p)$ of $n$ vertices in $\mathbb{R}^r$. Then $G(p)$ is rigid if and only if

$$\text{rank } R = nr - \frac{r(r + 1)}{2}.$$  

(3)

A novel characterization of generic framework rigidity was introduced in [1,2]. This characterization uses the notion of normal cones of convex sets to define a matrix $\overline{R}$ whose rank determines whether or not a given generic framework is rigid. Furthermore, this characterization was derived under the assumption that the framework of interest $G(p)$ has an equivalent framework $G(q)$ in $\mathbb{R}^{n-1}$.

Let $G(p)$ be a framework in $\mathbb{R}^r$ with $n$ vertices and $m$ edges. Then matrix $\overline{R}$ is $\bar{r}(\bar{r} + 1)/2 \times m$ where $\bar{r} = n - 1 - r$ and $\bar{m}$ is the number of missing edges of $G(p)$, i.e., $\bar{m} = n(n - 1)/2 - m$. Recall that the rigidity matrix $R$ is $m \times nr$. Furthermore, whereas the entries of $R$ are a function of the positions $p_1, \ldots, p_n$ of the vertices of $G(p)$, the entries of $\overline{R}$ are a function of the Gale matrix corresponding to $G(p)$.

In this paper we present the precise relationship between the left-nullspaces, and consequently the nullspaces, of $R$ and $\overline{R}$. In particular we show that the left-nullspaces of $R$ and $\overline{R}$ are isomorphic. In other words, we show that matrix $\overline{R}$ contains the same information as the rigidity matrix $R$. 


Thus with a slight abuse of terminology, we will call $\mathcal{R}$ the dual rigidity matrix. We also show that the assumption of the existence of an equivalent framework $G(q)$ in $\mathbb{R}^{n-1}$ can be dropped.

2. An alternative approach to infinitesimal rigidity

In this section we present an alternative approach to infinitesimal rigidity based on Gram matrices. Given a framework $G(p)$ in $\mathbb{R}^r$, we first characterize the set of all frameworks $G(q)$ in $\mathbb{R}^r$ such that $G(q)$ is equivalent to $G(p)$ and configuration $q$ is arbitrarily close to configuration $p$.

Let us represent a configuration $p^1, \ldots, p^n$ of a framework $G(p)$ in $\mathbb{R}^r$ by the following $n \times r$ matrix:

$$P = \begin{bmatrix} p^1^T \\ \vdots \\ p^n^T \end{bmatrix}. $$

Since we do not distinguish between congruent frameworks, we can assume without loss of generality that the centroid of the points $p^1, \ldots, p^n$ coincides with the origin. i.e., $P^T e = 0$, where $e$ is the vector of all 1’s in $\mathbb{R}^n$. Let $B$ denote the Gram matrix of the points $p^1, \ldots, p^n$, i.e., $B = P P^T$. Let $V$ be any $n \times (n-1)$ matrix such that

$$V^T e = 0, \quad V^T V = I_{n-1},$$

(4)

where $I_{n-1}$ is the identity matrix of order $n-1$. For the purposes of this paper, we will find it convenient to represent a configuration of a framework $G(p)$ in $\mathbb{R}^r$ by the $(n-1) \times (n-1)$ projected Gram matrix $X$ defined by

$$X := V^T B V := V^T P P^T V.$$  

(5)

Clearly $X$, which is invariant under rigid motions, is positive semidefinite with rank $r$. Furthermore, since we do not distinguish between congruent frameworks, and in particular between $P$ and $PQ$ where $Q$ is an $r \times r$ orthogonal matrix, it follows that $P$ and $X$ uniquely determine each other [4]. Thus, we will use $G(p)$ and $G(X)$ interchangeably.

Let $E_{ij}$ denote the $n \times n$ matrix with 1’s in the $(i, j)$th and $(j, i)$th entries and zeros elsewhere and let

$$M_{ij} := -\frac{1}{2} V^T E_{ij} V.$$  

(6)

Given a framework $G(p_1)$ in $\mathbb{R}^r$, let $X_1$ be the projected Gram matrix corresponding to configuration $p_1$, i.e., $X_1 = V^T P_1 P_1^T V$, and let

$$\mathcal{M}(y) := \sum_{(i, j) \notin E} y_{ij} M_{ij}. $$  

(7)

Further, let

$$\Omega = \left\{ y \in \mathbb{R}^m : X(y) := X_1 + \mathcal{M}(y) := X_1 + \sum_{(i, j) \notin E} y_{ij} M_{ij} \geq 0 \right\},$$  

(8)

where $A \succeq 0 (A \succ 0)$ means that matrix $A$ is symmetric positive semidefinite (symmetric positive definite). Then it was shown in [1] that the set of all frameworks $G(q)$ in $\mathbb{R}^r$ that are equivalent to $G(X_1)$ is given by
\[ \{ G(X(y)) : y \in \Omega \text{ and rank } X(y) = r \}; \] (9)

and that the set of all frameworks \( G(q) \) in \( \mathbb{R}^s \), equivalent to \( G(X_1) \), for some \( s, 1 \leq s \leq n - 1 \), is given by
\[ \{ G(X(y)) : y \in \Omega \}. \] (10)

For more details on set \( \Omega \) see [3].

Let \( W \) and \( U \) be the matrices whose columns form orthonormal bases of the rangespace and the nullspace of \( X_1 \) respectively. Then
\[ \begin{bmatrix} W^T & U^T \end{bmatrix} X(y) \begin{bmatrix} W & U \end{bmatrix} = \begin{bmatrix} A + \overset{\rightarrow}{M}(y)W & W^T\overset{\rightarrow}{M}(y)U \\ U^T\overset{\rightarrow}{M}(y)W & U^T\overset{\rightarrow}{M}(y)U \end{bmatrix}, \] (11)

where \( A \) is the \( r \times r \) diagonal matrix consisting of the positive eigenvalues of \( X_1 \).

The following lemma, which follows from Schur complement, is well known.

**Lemma 2.1.** Let
\[ M = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \]
be a symmetric matrix, where \( A_1 \) is an \( r \times r \) positive definite matrix. Then matrix \( M \) is positive semi-definite with rank \( r \) if and only if \( A_3 - A_2^T A_1^{-1} A_2 = 0 \).

Note that on a sufficiently small neighborhood \( \zeta \) of zero in \( \mathbb{R}^\tilde{m} \), \( A + W^T\overset{\rightarrow}{M}(y)W \succ 0 \). Therefore, it follows from Lemma 2.1 that for \( y \in \zeta \), \( X(y) \) is positive semidefinite with rank \( r \) if and only if
\[ \Phi(y) = U^T\overset{\rightarrow}{M}(y)U - U^T\overset{\rightarrow}{M}(y)W(A + W^T\overset{\rightarrow}{M}(y)W)^{-1}W^T\overset{\rightarrow}{M}(y)U = 0. \] (12)

Thus
\[ \{ G(X(y)) : \Phi(y) = 0 \} \]
is the set of all frameworks in \( \mathbb{R}^r \) that are both equivalent to, and arbitrarily close to \( G(p_1) \). Hence, the linearization of \( \Phi(y) \) near \( y = 0 \) is given by
\[ U^T\overset{\rightarrow}{M}(y)U = 0. \] (13)

Therefore, framework \( G(p_1) \) is infinitesimally flexible if and only if there exists a non-zero \( y \) satisfying (13). Next we express Eq. (13) in terms of the Gale matrix corresponding to \( G(p_1) \).

Let \( G(p) \) be a framework in \( \mathbb{R}^r \). Then it immediately follows that the following \( (r + 1) \times n \) matrix
\[ \begin{bmatrix} p^1 & p^2 & \cdots & p^n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \] (14)
has full row rank since \( p^1, \ldots, p^n \) span \( \mathbb{R}^r \). Note that \( r \leq n - 1 \) where \( r = n - 1 \) corresponds to the case where \( p^1, \ldots, p^n \) are affinely independent. For \( r \leq n - 2 \), let \( \tilde{r} = n - 1 - r \) and let \( \Gamma \) be the \( n \times \tilde{r} \) matrix, whose columns form a basis for the nullspace of the matrix in (14). \( \Gamma \) is called a Gale matrix corresponding to \( G(p) \); and the \( i \)th row of \( \Gamma \), considered as a vector in \( \mathbb{R}^{\tilde{r}} \), is called a Gale transform of \( p^i \) [12]. Gale transform is a well-known technique in the theory of polytopes [15]. We will exploit the fact that \( \Gamma \) is not unique to define a special Gale matrix \( Z \) which is more sparse than \( \Gamma \) and more convenient for our purposes.
Let us write $\Gamma$ in block form as
\[
\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix},
\]
where $\Gamma_1$ is $\bar{r} \times \bar{r}$ and $\Gamma_2$ is $(r + 1) \times \bar{r}$. Since $\Gamma$ has full column rank, we can assume without loss of generality that $\Gamma_1$ is non-singular. Then $Z$ is defined by
\[
Z := \Gamma_1^{-1} \begin{bmatrix} I_{\bar{r}} \\ \Gamma_2 \end{bmatrix} \Gamma_1^{-1}.
\]  
(15)

The next lemma allows us to express (13) in terms of the Gale matrix $Z$.

**Lemma 2.2** [2]. Let $Z$ be the Gale matrix corresponding to a framework $G(p)$ in $\mathbb{R}^{r}$. Let $U$ be the matrix whose columns form an orthonormal basis of the nullspace of the projected Gram matrix $X = V^T P P^T V$. Then $VU = ZQ$ for some non-singular matrix $Q$. i.e., $VU$ is a Gale matrix corresponding to $G(p)$, where $V$ is defined in (4).

Thus, the next theorem follows from (6), (13) and Lemma 2.2.

**Theorem 2.1.** Let $Z$ be the Gale matrix corresponding to a framework $G(p)$ in $\mathbb{R}^{r}$. Then $G(p)$ is infinitesimally flexible if and only if there exists a non-zero $y$ such that
\[
Z^T \delta(y) Z = 0,
\]
where $\delta(y) = \sum_{(i, j) \notin E} y_{ij} E_{ij}$.

Recall that $E_{ij}$ is the symmetric matrix of order $n$ with 1’s in the $(i, j)$th and the $(j, i)$th entries and zeros elsewhere. Using Theorem 2.1, we derive next what we call the dual rigidity matrix $\overline{R}$.

### 3. The dual rigidity matrix $\overline{R}$

The dual rigidity matrix $\overline{R}$ is derived using equation (16). We first start with some definitions.

Given an $n \times n$ symmetric matrix $A$, let $\text{svec}(A)$ denote the $\frac{n(n+1)}{2}$ vector formed by stacking the columns of $A$ from the principle diagonal downwards after having multiplied the off-diagonal entries of $A$ by $\sqrt{2}$. For example, if $A$ is a $3 \times 3$ matrix, then
\[
\text{svec}(A) = \begin{bmatrix} a_{11} \\ \sqrt{2}a_{21} \\ \sqrt{2}a_{31} \\ a_{22} \\ \sqrt{2}a_{32} \\ a_{33} \end{bmatrix}.
\]  
(17)

Let $B$ be an $m \times n$ matrix and let $A$ be an $n \times n$ symmetric matrix. The symmetric Kronecker product between $B$ and itself, denoted by $B \otimes_s B$, is defined such that
\[
(B \otimes_s B)\text{svec}(A) = \text{svec}(BAB^T).
\]  
(18)

For more details on the symmetric Kronecker product see [6].

**Definition 3.1.** Let $Z$ be the Gale matrix of a framework $G(p)$ in $\mathbb{R}^{r}$ and let $\overline{R}$ be the sub-matrix of $Z \otimes_s Z$ obtained by keeping only rows that correspond to missing edges of $G$. Then $\overline{R}$ is called the dual rigidity matrix corresponding to $G(p)$.
Let $\bar{m}$ be the number of missing edges of $G$ and let $\bar{r} = n - 1 - r$. Further, let $z_i^T$ denote the $i$th row of $Z$. Then the dual rigidity matrix $\bar{R}$ is the $\frac{\bar{r}(\bar{r}+1)}{2} \times \bar{m}$ matrix whose columns are indexed by the missing edges of $G$, where the $(i, j)$th column is equal to $\frac{\sqrt{2}}{\sqrt{2}} \text{svec}(z_i^T z_j^T + z_j^T z_i^T)$. For example, if the missing edges of $G$ are $(i_1, j_1), (i_2, j_2), \ldots, (i_{\bar{m}}, j_{\bar{m}})$, then

$$\bar{R} = \frac{1}{\sqrt{2}} \left[ \text{svec}(z_{i_1}^T z_{j_1}^T + z_{j_1}^T z_{i_1}^T) \ldots \text{svec}(z_{i_{\bar{m}}}^T z_{j_{\bar{m}}}^T + z_{j_{\bar{m}}}^T z_{i_{\bar{m}}}^T) \right].$$

That is

$$\bar{R} = \begin{bmatrix}
\sqrt{2} z_{i_1}^T z_{j_1} & \sqrt{2} z_{i_2}^T z_{j_2} & \ldots & \sqrt{2} z_{i_{\bar{m}}}^T z_{j_{\bar{m}}} \\
\sqrt{2} z_{j_1}^T z_{i_1} & \sqrt{2} z_{j_2}^T z_{i_2} & \ldots & \sqrt{2} z_{j_{\bar{m}}}^T z_{i_{\bar{m}}} \\
z_{i_1}^T z_{i_1} + z_{j_1}^T z_{j_1} & z_{i_2}^T z_{i_2} + z_{j_2}^T z_{j_2} & \ldots & z_{i_{\bar{m}}}^T z_{i_{\bar{m}}} + z_{j_{\bar{m}}}^T z_{j_{\bar{m}}} \\
z_{i_2}^T z_{i_2} + z_{j_2}^T z_{j_2} & \ldots & \ldots & \ldots \\
z_{i_{\bar{m}}}^T z_{i_{\bar{m}}} + z_{j_{\bar{m}}}^T z_{j_{\bar{m}}}
\end{bmatrix},$$

where $z_k^T$ denotes the $k$th coordinate of vector $z_i^T$. The next theorem justifies calling $\bar{R}$ the dual rigidity matrix.

**Theorem 3.1.** Let $\bar{R}$ be the dual rigidity matrix of a framework $G(p)$ in $\mathbb{R}^r$. Then $G(p)$ is infinitesimally rigid if and only if $\bar{R}$ has a trivial nullspace, i.e., if and only if

$$\text{rank } \bar{R} = \bar{m}. \quad (21)$$

**Proof.** This follows from (16) and the definition of $\bar{R}$ since $Z^T \dot{\theta}(y) Z = 0$ if and only if $\bar{R} y = 0$. □

Three remarks are in order here. First, the dual rigidity matrix $\bar{R}$ is invariant under rigid motions. Hence, in Eq. (21) there is no need to account for the trivial flexes as was the case in (3). Second, the dual rigidity matrix $\bar{R}$ is in general sparse since the Gale matrix $Z$ is sparse. Third, dropping the factors of $\sqrt{2}$ from the definition of $\bar{R}$ in (20), which is advantageous from a theoretic computational point of view, would not change the rank of $\bar{R}$. These factors are kept in order to make the definition of $\bar{R}$ in terms of the symmetric Kronecker product simple.

**Example 3.1.** The framework in Fig. 2 has

$$P = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ -2 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 1 & -2 \end{bmatrix}.$$ 

Thus the dual rigidity matrix of this framework is

$$\bar{R} = \begin{bmatrix} \sqrt{2} & 0 & -2\sqrt{2} \\ -2 & 0 & 4 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Note that the rigidity matrix $R$ of this framework is $7 \times 10$. Also note that $y = (2, 0, 1)^T$ is a basis of the nullspace of $\bar{R}$ and $x = (2, \sqrt{2}, 0)^T$ is a basis of the left-nullspace of $\bar{R}$. 
4. Relations between $R$ and $\overline{R}$

The stress matrix of a framework plays a critical role in establishing the relation between the left-nullspaces, and hence the nullspaces, of $R$ and $\overline{R}$. An equilibrium stress of a framework $G(p)$ is a real valued function $\omega$ on $E$, the set of edges of $G$, such that

$$\sum_{j: (i,j) \in E} \omega_{ij} (p^i - p^j) = 0 \quad \text{for all } i = 1, \ldots, n.$$  \hfill (22)

It readily follows, then, that the space of the equilibrium stresses of a framework $G(p)$ is the left-nullspace of the rigidity matrix $R$ of $G(p)$. That is, an equilibrium stress of $G(p)$ is just a linear dependency among the rows of $R$.

Let $\omega$ be an equilibrium stress for $G(p)$. Define the following $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E, \\ \sum_k \omega_{ik} & \text{if } i = j. \end{cases}$$  \hfill (23)

$S$ is called a stress matrix of $G(p)$. The following theorem establishes the relation between $S$ and the Gale matrix $Z$.

**Theorem 4.1** [5]. Let $Z$ and $S$ be, respectively, the Gale matrix and a stress matrix of a framework $G(p)$ in $\mathbb{R}^r$. Then there exists an $\bar{r} \times \bar{r}$ symmetric matrix $\Psi$ such that

$$S = Z \Psi Z^T.$$  \hfill (24)

On the other hand, let $\Psi'$ be any $\bar{r} \times \bar{r}$ symmetric matrix such that $z^T \Psi' z = 0$ for all $(i, j) \notin E$, where $z^T$ denotes the $i$th row of $Z$. Then $S' = Z \Psi' Z^T$ is a stress matrix of $G(p)$.

**Example 4.1.** The framework in Fig. 2 has an equilibrium stress

$$\omega = (\omega_{12} = -1, \omega_{13} = -1, \omega_{14} = 4, \omega_{24} = 2, \omega_{25} = -1, \omega_{34} = 2, \omega_{35} = -1),$$

and a stress matrix

$$S = \begin{bmatrix} 2 & 1 & 1 & -4 & 0 \\ 1 & 0 & 0 & -2 & 1 \\ -4 & -2 & -2 & 8 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{bmatrix} = Z \Psi Z^T,$$

where $\Psi = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and the Gale matrix $Z$ was given in Example 3.1. Note that $\text{svec}(\Psi) = (2, \sqrt{2}, 0)^T$ belongs to the left-nullspace of the dual rigidity matrix $\overline{R}$.

The following theorem is the main result of this section.

**Theorem 4.2.** Let $R$ and $\overline{R}$ be, respectively, the rigidity and the dual rigidity matrices of a framework $G(p)$ in $\mathbb{R}^r$. Then

1. the left-nullspace of $\overline{R}$ is isomorphic to the left-nullspace of $R$;
2. dimension of nullspace of $\overline{R} = \text{dimension of nullspace of } R - \frac{r(r+1)}{2}$. 
Let \( \mathcal{C} \) denote the subspace of \( \mathcal{R} \times \mathcal{R} \) matrices \( A \) such that
\[
(ZAZ^T)_{ij} = 0 \quad \text{for all } (i, j) \notin E; \tag{25}
\]
and let \( \mathcal{S} \) denote the subspace of stress matrices of \( G(p) \). Then it follows from Theorem 4.1 that the linear map \( f : \mathcal{C} \rightarrow \mathcal{S} \) defined by \( f(A) = ZAZ^T \) is both one-to-one and onto. Therefore, the left-nullspace of \( R \) is isomorphic to \( \mathcal{C} \). Furthermore, it follows from the definition of \( \overline{R} \) that \( \Psi \) belongs to \( \mathcal{C} \) if and only if \( (svec(\Psi))^T \overline{R} = 0 \). Hence, \( \mathcal{C} \) is isomorphic to the left-nullspace of \( \overline{R} \) and thus statement 1 of the theorem follows.

To prove statement 2, note that
\[
\text{dim nullspace of } R = \text{dim left-nullspace of } R + nr - m, \n\text{dim nullspace of } \overline{R} = \text{dim left-nullspace of } \overline{R} + m - \frac{\tilde{r}(\tilde{r} + 1)}{2}. \n\]
But since left-nullspace of \( R \) is isomorphic to the left-nullspace of \( \overline{R} \), it follows that
\[
\text{dim nullspace of } \overline{R} = \text{dim nullspace of } R - nr + m + m - \frac{\tilde{r}(\tilde{r} + 1)}{2} = \text{dim nullspace of } R - \frac{r(r + 1)}{2}. \n\]
Thus the result follows. \( \square \)

Two remarks are in order here. First, statement 2 in the above theorem should come as no surprise since as we remarked earlier, the dual rigidity matrix \( \overline{R} \) is invariant under a rigid motion and the term \( \frac{r(r+1)}{2} \) is exactly the dimension of rigid motions in \( \mathbb{R}^r \). Second, it follows from Theorem 4.1 that \( S = Z\Psi Z^T \) is a stress matrix of \( G(p) \) if and only if \( svec(\Psi) \) is in the left-nullspace of \( \overline{R} \) as is the case in Example 4.1.

Next we show that for each infinitesimal flex of \( G(p) \), i.e., for each vector in the nullspace of \( R \), there corresponds a vector \( y \) in the nullspace of \( \overline{R} \) whose value can be found explicitly. For any matrix \( A \), let \( \text{diag}(A) \) denote the vector consisting of the diagonal entries of \( A \). Also recall that \( E_{ij} \) is the symmetric \( n \times n \) matrix with 1’s in the \( (i, j) \)th and \( (j, i) \)th entries and zeros elsewhere.

**Theorem 4.3.** Let \( p' \) be an infinitesimal flex of \( G(p) \). Then there exists a vector \( y \in \mathbb{R}^n \) in the nullspace of \( \overline{R} \) such that
\[
\varepsilon(y) = \text{diag}(PP'T + P'P'T)e^T + e(\text{diag}(PP'T + P'P'T))^T - 2(PP'T + P'P'T), \tag{26}
\]
where the matrices \( P^T = [p^1 \ p^2 \ \ldots \ p^n], P'^T = [p'^1 \ p'^2 \ \ldots \ p'^n] \) and \( \varepsilon(y) = \sum_{(i, j) \notin E} y_{ij} E_{ij} \).

**Proof.** It is easy to verify that
\[
2(p^i - p'^i)(p'^i - p^j) = (PP'^T + P'P'^T)_{ii} + (PP'^T + P'P'^T)_{jj} - 2(PP'T + P'P'T)_{ij}. \n\]
Let \( \mathcal{L} \) denote the space of \( n \times n \) symmetric matrices \( A = (a_{ij}) \) such that \( a_{ij} = 0 \) for all \( (i, j) \in E \). Then since \( (p^i - p'^i)(p'^i - p^j) = 0 \) for all \( (i, j) \in E \), it follows that the right hand side of Eq. (26) belongs to \( \mathcal{L} \). Therefore, there exists \( y \in \mathbb{R}^m \) that satisfies (26) since matrices \( E_{ij} \)’s form a basis for \( \mathcal{L} \). Now by multiplying Eq. (26) from left and right by \( Z^T \) and \( Z \) respectively we get \( Z^T \varepsilon(y) Z = 0 \). Thus \( y \) belongs to the nullspace of \( \overline{R} \). \( \square \)
Note that if \( p' = (p'^1 \ldots p'^m) \) is a trivial infinitesimal flex resulting from a rigid motion then the right hand side of Eq. (26) is identically zero. Hence \( y = 0 \) in this case. Therefore, if \( y \) in Eq. (26) is non-zero, then the corresponding flex is non-trivial.

5. Geometric interpretation of \( \overline{R} \)

We end this paper by presenting a geometric interpretation of the rows of the dual rigidity matrix \( \overline{R} \) of a framework \( G(p) \) in \( \mathbb{R}^r \) under the assumption that there exists a framework \( G(q) \), equivalent to \( G(p) \), in \( \mathbb{R}^{n-1} \). As was mentioned earlier, this interpretation of the rows of \( \overline{R} \) in terms of the normal cone of set \( \Omega \) (defined in (8)) at the origin, was the basis for deriving \( \overline{R} \) in [1,2].

Let \( G(p_1) \) be a given framework in \( \mathbb{R}^r \). Recall that \( \{ Z(X(y)) : y \in \Omega \} \) is the set of all frameworks \( G(q) \) in \( \mathbb{R}^s \) which are equivalent to \( G(p) \), for all integers \( s \) between 1 and \( n - 1 \).

A point \( \gamma \in \Omega \) is said to be an extreme point of \( \Omega \) if \( \gamma \) can’t be represented as a proper convex combination of two distinct points \( y^1 \) and \( y^2 \) in \( \Omega \). Given an extreme point \( \gamma \) of \( \Omega \), the normal cone \( N_\Omega(\gamma) \), is defined by

\[
N_\Omega(\gamma) = \{ c \in \mathbb{R}^m : c^T \gamma \geq c^T y \text{ for all } y \in \Omega \}. \tag{27}
\]

The proofs of the next two lemmas are given in [1] and [2] respectively.

**Lemma 5.1.** Let \( G(p_1) \) be a given framework with \( n \) vertices in \( \mathbb{R}^r \). Assume that there exists a framework \( G(q) \) in \( \mathbb{R}^{n-1} \), which is equivalent to \( G(p_1) \). Then the normal cone \( N_\Omega(\gamma) \) is given by

\[
N_\Omega(\gamma) = \{ c \in \mathbb{R}^m : c_{ij} = -\text{trace}(M_{ij} \Pi), \text{ for some } \Pi \succeq 0 : \text{trace}(X(\gamma) \Pi) = 0 \},
\]

where matrices \( M_{ij} \) are defined in (6).

**Lemma 5.2.** Let \( G(p_1) \) be a given framework with \( n \) vertices in \( \mathbb{R}^r \). Assume that there exists a framework \( G(q) \) in \( \mathbb{R}^{n-1} \), which is equivalent to \( G(p_1) \). Let \( Z \) be the Gale matrix for \( G(p_1) \). Then

\[
N_\Omega(0) = \{ c \in \mathbb{R}^m : c_{ij} = \text{trace}(Z^T E_{ij} Z \Psi) \}, \tag{28}
\]

for some \( \tilde{r} \times \tilde{r} \) symmetric positive semidefinite matrix \( \Psi \).

Let \( u_1, u_2, \ldots, u_{\tilde{r}} \) denote the standard unit vectors in \( \mathbb{R}^{\tilde{r}} \). Then the following \( \tilde{r}(\tilde{r} + 1)/2 \) matrices

\[
\psi^{11} = u_1 u_1^T,
\psi^{21} = \frac{1}{\sqrt{2}} (u_2 u_1^T + u_1 u_2^T) + u_1 u_1^T + u_2 u_2^T,
\ldots = \ldots,
\psi^{\tilde{r}1} = \frac{1}{\sqrt{2}} (u_{\tilde{r}} u_1^T + u_1 u_{\tilde{r}}^T) + u_1 u_1^T + u_{\tilde{r}} u_{\tilde{r}}^T,
\psi^{22} = u_2 u_2^T,
\ldots = \ldots,
\psi^{2\tilde{r}} = \frac{1}{\sqrt{2}} (u_{\tilde{r}} u_2^T + u_2 u_{\tilde{r}}^T) + u_2 u_2^T + u_{\tilde{r}} u_{\tilde{r}}^T,
\ldots = \ldots,
\psi^{\tilde{r}2\tilde{r}} = u_{\tilde{r}} u_{\tilde{r}}^T.
\]
are obviously symmetric positive semidefinite, and their conic hull is a full-dimensional subset of the cone of symmetric positive semidefinite matrices of order $\tilde{r}$. Therefore, the conic hull of the $\tilde{r}(\tilde{r} + 1)/2$ vectors $c^{kl}_{ij} \in \mathbb{R}^{\tilde{m}}$ where $c^{kl}_{ij} = \frac{1}{\sqrt{2}}\text{trace}(Z^{T}E^{ij}Z\psi^{kl})$ is a full-dimensional subset of $N_{\Omega}(0)$. But

$$c^{kl}_{ij} = \frac{1}{\sqrt{2}}\text{trace}(Z^{T}E^{ij}Z\psi^{kl}) = \begin{cases} \sqrt{2}z^{l}_{k}z^{l}_{k} & \text{if } k = l, \\ z^{l}_{k}z^{l}_{l} + z^{l}_{k}z^{l}_{l} + \sqrt{2}z^{l}_{k}z^{l}_{k} + \sqrt{2}z^{l}_{l}z^{l}_{l} & \text{if } k \neq l, \end{cases}$$

where $z^{l}_{k}$ denotes the $k$th coordinate of vector $z^{l}$. Hence, vector $c^{kl}$ is equal to the $(k, k)$th row of $\tilde{R}$ if $k = l$, and it is equal to the sum of the $(k, l)$th, $(k, k)$th and $(l, l)$th rows of $\tilde{R}$ if $k \neq l$. Therefore, a framework $G(p_{1})$, which has an equivalent framework $G(q)$ in $\mathbb{R}^{n-1}$, is infinitesimally rigid if and only if $\text{rank } R = \tilde{m}$ if and only if $N_{\Omega}(0)$ is full dimensional; i.e., $\dim N_{\Omega}(0) = \tilde{m}$.

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References