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Simple regression in view of elliptical models

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ABSTRACT

For the simple linear model $\boldsymbol{Y} = \theta\mathbf{1} + \beta\boldsymbol{x} + \boldsymbol{\varepsilon}$ where the error vector follows the elliptically contoured distribution, we consider the unrestricted, restricted, preliminary test and shrinkage estimators for the intercept parameter, θ when it is suspected that the slope parameter β may be β_o. The exact bias and MSE expressions are derived and the mean-square relative efficiency is taken to determine the relative dominance properties of the proposed estimators in comparison. In the continuation, the optimal level of significance of the preliminary test estimator is tabulated and some graphical result are also displayed.

1. Introduction

Consider a simple linear model

$$\boldsymbol{Y} = \theta\mathbf{1} + \beta\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{A}\eta + \boldsymbol{\varepsilon}, \quad \boldsymbol{A} = [\mathbf{1}, \boldsymbol{x}], \quad \eta = (\theta, \beta)', \tag{1.1}$$

where $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$ is the response vector and $\boldsymbol{x} = (x_1, \ldots, x_n)'$ is a fixed vector of known constants, while $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)'$ is the n-vector of random errors distributed according to the law belonging to the class of elliptically contoured distributions (ECDs), $EC_n(\mathbf{0}, \sigma^2\mathbf{V}, \psi)$ for $\sigma \in \mathbb{R}^+$ and un-structured known matrix $\mathbf{V} \in S(n)$, where $S(n)$ denotes the set of all positive definite matrices of order $(n \times n)$ with the following characteristic function

$$\phi_{\boldsymbol{\varepsilon}}(\boldsymbol{t}) = \psi\left(\sigma^2\boldsymbol{t}'\mathbf{V}\boldsymbol{t}\right) \tag{1.2}$$

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for some functions $\psi : [0, \infty) \to \mathbb{R}$ say characteristic generator [9].

If $\boldsymbol{\varepsilon}$ has a density, then it is of the form

$$f(\boldsymbol{\varepsilon}) \propto |\sigma^2 \mathbf{V}|^{-\frac{1}{2}} g\left(\frac{1}{\sigma^2}\, \boldsymbol{\varepsilon}' \mathbf{V}^{-1} \boldsymbol{\varepsilon}\right), \tag{1.3}$$

where $g(.)$ is a non-negative function over $\mathbb{R}^+$ such that $f(.)$ is a density function w.r.t (with respect to) a σ-finite measure μ on $\mathbb{R}^p$. In this case, notation $\boldsymbol{\varepsilon} \sim EC_n(\mathbf{0}, \sigma^2\mathbf{V}, g)$ would probably be used.

Some of the well-known members of the class of ECDs are the multivariate normal, Kotz Type, Pearson Type II and VII, multivariate Student's t, multivariate Cauchy, Logistic, Bessel and generalized slash distributions. Dating back to Kelker [14], there are many known results concerning ECDs, in particular the mathematical properties and its application to statistical inference. These results have been put forward by Muirhead [21] and Fang et al. [9] among others .

It is sometimes difficult to have complete analysis of the regression model with ECD errors of the type (1.2) or (1.3). To overcome such difficulties, one may consider any of the three sub-classes of ECDs, namely,

(i) scale mixture of normal distributions,
(ii) Laplace class of mixture of normal distributions, and
(iii) signed measure mixture of normal distributions.

General formula for the above mixture of distributions is given by

$$f_{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \int_0^\infty \mathcal{W}(t)\phi_{\mathcal{N}_n(\mathbf{0}, t^{-1}\sigma^2\mathbf{V})}(\boldsymbol{x})dt, \tag{1.4}$$

where $\phi_{N_n(\mathbf{0}, t^{-1}\sigma^2\mathbf{V})}(.)$ is the pdf (probability density function) of $N_n(\mathbf{0}, t^{-1}\sigma^2\mathbf{V})$.

(a) If

$$\mathcal{W}(\tau) = 2\left(\Gamma(\gamma/2)\right)^{-1}\left(\frac{\gamma\sigma^2}{2}\right)^{\gamma/2} \tau^{-(\gamma+1)} e^{-\frac{\gamma\sigma^2}{2\tau^2}}, \quad 0 < \gamma, \sigma^2, \tau < \infty \tag{1.5}$$

then we have

$$f(\boldsymbol{\varepsilon}) = \frac{\Gamma\left(\frac{n+\gamma}{2}\right)|\mathbf{V}|^{-\frac{1}{2}}}{(\pi\gamma)^{n/2}\Gamma\left(\gamma/2\right)\sigma^n}\left(1 + \frac{\boldsymbol{\varepsilon}'\mathbf{V}^{-1}\boldsymbol{\varepsilon}}{\gamma\sigma^2}\right)^{-\frac{1}{2}(n+\gamma)}, \tag{1.6}$$

where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \frac{n\gamma\sigma^2}{\gamma-2}\mathbf{V} = \sigma_e^2$ for $\gamma > 2$.

(b) Chu [7] considered

$$\mathcal{W}(t) = (2\pi)^{\frac{n}{2}}|\sigma^2\mathbf{V}|^{\frac{1}{2}} t^{-\frac{p}{2}}\mathcal{L}^{-1}[f(s)], \tag{1.7}$$

$\mathcal{L}^{-1}[f(s)]$ denotes the inverse Laplace transform of $f(s)$ with $s = [\boldsymbol{x}'(\sigma^2\mathbf{V})^{-1}\boldsymbol{x}/2]$. For some examples of $f(.)$ and $\mathcal{W}(.)$ see Arashi and Tabatabaey [5].

The inverse Laplace transform of $f(.)$ exists provided that the following conditions are satisfied.
(i) $f(t)$ is differentiable when t is sufficiently large.
(ii) $f(t) = o(t^{-m})$ as $t \to \infty$, $m > 1$.
Although, it is rather difficult to derive the inverse Laplace transform of some functions, we are able to handle it for many density generators of elliptical densities. We refer the readers to Debnath and Batta [8] for more specific details.

The mean of $\boldsymbol{\varepsilon}$ is the zero-vector and the covariance-matrix of $\boldsymbol{\varepsilon}$ is

$$\begin{aligned}\mathbf{\Sigma}_{\boldsymbol{\varepsilon}} = Cov(\boldsymbol{\varepsilon}) &= \int_0^\infty Cov(\boldsymbol{\varepsilon}|t)\mathcal{W}(t)dt \\ &= \int_0^\infty \mathcal{W}(t)Cov\left\{N_p(\mathbf{0}, t^{-1}\sigma^2\mathbf{V})\right\} dt \\ &= \left(\int_0^\infty t^{-1}\mathcal{W}(t)dt\right)\sigma^2\mathbf{V} \end{aligned} \tag{1.8}$$

provided the above integral exists.
Comparing the models (1.3) and (1.4), since $\mathbf{\Sigma}_{\boldsymbol{\varepsilon}} = Cov(\boldsymbol{\varepsilon}) = -2\psi'(0)\sigma^2\mathbf{V}$, it can be concluded that

$$-2\psi'(0) = \int_0^\infty t^{-1}\mathcal{W}(t)dt.$$

Now suppose that $\boldsymbol{X} \sim EC_n(\boldsymbol{\mu}, \mathbf{V}, g)$. Then it is important to point out that since $\int_{\boldsymbol{x}} f(\boldsymbol{x})d\boldsymbol{x} = 1$, using Fubini's theorem we have

$$\begin{aligned}1 &= \int_{\boldsymbol{x}}\int_0^\infty \mathcal{W}(t)\phi_{N_n(\boldsymbol{\mu},t^{-1}\mathbf{V})}(\boldsymbol{x})dtd\boldsymbol{x} \\ &= \int_0^\infty \mathcal{W}(t)\int_{\boldsymbol{x}} \phi_{\mathcal{N}_n(\boldsymbol{\mu},t^{-1}\mathbf{V})}(\boldsymbol{x})d\boldsymbol{x}dt \\ &= \int_0^\infty \mathcal{W}(t)dt.\end{aligned}$$

Thus for nonnegative function $\mathcal{W}(.)$, it is a density. For nonnegative function $W(.)$, the elliptical models can be interpreted as a scale mixture of normal distributions.

(c) Srivastava and Bilodeau [27] considered the signed measure, $W(t)$ such that

$$\begin{aligned}&(i)\ \int_0^\infty t^{-1}\mathcal{W}^+(dt) < \infty, \\ &(ii)\ \int_0^\infty t^{-1}\mathcal{W}^-(dt) < \infty,\end{aligned} \tag{1.9}$$

where $\mathcal{W}^+ - \mathcal{W}^-$ is the Jordan decomposition of W in positive and negative parts. Note that from $(i) - (ii)$ of (1.9),

$$\int_0^\infty t^{-1}\mathcal{W}(dt) < \infty \tag{1.10}$$

and thus, $Cov(\boldsymbol{\varepsilon})$ exists under the sub-class defined above.
This subclass contains the subclass defined by (b).

Remark 1.1. Regarding the above classifications, we should take the following notes:

1. In all the above classes we have
$$\mathbf{\Sigma}_{\boldsymbol{\varepsilon}} = -2\psi'(0)\sigma^2\mathbf{V} = \left(\int_0^\infty t^{-1}\mathcal{W}(t)dt\right)\sigma^2\mathbf{V}$$
resulting in $-2\psi'(0) = \int_0^\infty t^{-1}\mathcal{W}(t)dt$.
2. The subclass (a) is neither contained in subclass (b) nor in the subclass (c). However, subclass (b) in contained in the subclass(c). Thus, all the implications about the subclass (c) can be used for the subclass (b).
3. For the subclass (c) we can assure that $-2\psi'(0) = \int_0^\infty t^{-1}\mathcal{W}(t)dt$ exists. However it may not exist for the subclass (b).

Throughout the paper, we assume that

$$\sigma_{\boldsymbol{\varepsilon}}^2 = -2\psi'(0)\sigma^2. \tag{1.11}$$

There have been many studies in the area of the 'improved' estimation following the seminal work of Bancroft [6] and later Han and Bancroft [10]. They developed the preliminary test estimator that uses uncertain non-sample prior information (not in the form of prior distributions), in addition to the sample information. Stein [29] elegant approach dominates the usual maximum likelihood estimators under the squared error loss function. In a series of papers Saleh and Sen [25,26] explored the preliminary test approach to Stein-rule estimation. Many authors have contributed to this area, notably Judge and Bock [13], Stein [28], Khan [15–17], Kibria [18], Kibria and Saleh [19,20], Ahmed et al. [1,2], Saleh and Kibria [23,24], Hassanzadeh Bashtian et al. [11,12], Arashi et al. [4] and Arashi [3]. The recent book of Saleh [22] presents a comprehensive discussion of this area.

2. Estimation and testing

For convenience we express some notations due to the rest of the work. Let

$$\begin{aligned} K_1 &= \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}, \\ K_2 &= \boldsymbol{x}'\mathbf{V}^{-1}\boldsymbol{x}, \\ K_3 &= \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{x} = \boldsymbol{x}'\mathbf{V}^{-1}\mathbf{1}, \\ \boldsymbol{K} &= \boldsymbol{A}'\mathbf{V}^{-1}\boldsymbol{A}. \end{aligned} \tag{2.1}$$

2.1. Estimator of η

Based on the LS/ML principle, the *unrestricted* estimator of $\eta = (\theta, \beta)$ is given by

$$\begin{aligned} \tilde{\boldsymbol{\eta}} &= \left(\boldsymbol{A}'\mathbf{V}^{-1}\boldsymbol{A}\right)^{-1}\left(\boldsymbol{A}'\mathbf{V}^{-1}\boldsymbol{Y}\right) \\ &= \begin{pmatrix} K_1 & K_3 \\ K_3 & K_2 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{Y} \\ \boldsymbol{x}'\mathbf{V}^{-1}\boldsymbol{Y} \end{bmatrix} = \begin{pmatrix} \tilde{\theta}_n \\ \tilde{\beta}_n \end{pmatrix}. \end{aligned} \tag{2.2}$$

Theorem 2.1. *Assume in the simple linear model* (1.1)*,* $\boldsymbol{Y}|\theta, \beta, \sigma^2 \sim EC_n(\eta, \sigma^2\mathbf{V}, f)$*; then we have*

$$\tilde{\boldsymbol{\eta}} \sim EC_2(\eta, \sigma^2\boldsymbol{K}^{-1}, f).$$

Proof. Under the assumption $\boldsymbol{Y}|\theta, \beta, \sigma^2 \sim \mathcal{N}_n(\eta, \sigma^2\tau^{-1}\mathbf{V}, f)$, the exact distribution of $\tilde{\boldsymbol{\eta}}$ follows $\mathcal{N}_2(\eta, \sigma^2\tau^{-1}\boldsymbol{K}^{-1})$, where

$$\begin{aligned} \boldsymbol{K}^{-1} = (\boldsymbol{A}'\mathbf{V}^{-1}\boldsymbol{A})^{-1} &= \begin{pmatrix} K_1 & K_3 \\ K_3 & K_2 \end{pmatrix}^{-1} \\ &= \frac{1}{K_1K_2 - K_3^2}\begin{pmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{pmatrix}. \end{aligned}$$

Thus we get

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \int_0^\infty \mathcal{W}(\tau)\mathcal{N}_2\left(\eta, \sigma^2\tau^{-1}\boldsymbol{K}^{-1}\right)\, d\tau,$$

which completes the proof. □

Also the unbiased estimator of $\sigma_{\boldsymbol{\varepsilon}}^2$ is S_u^2 given by

$$S_u^2 = m^{-1}(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}})'\mathbf{V}^{-1}(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}}) \; ; \qquad (m = n-2). \tag{2.3}$$

2.2. Test of intercept parameter

At this step, first we propose test statistic of the parameter η, and then we focus on the problem of estimation of the intercept parameter in a more precise setup.

Theorem 2.2. *Let*

$$\Omega = \{(\eta, \sigma, \mathbf{V}) : \eta \in \mathbb{R}^2, \sigma \in \mathbb{R}^+, \mathbf{V} > 0\}, \quad \textit{and}$$
$$\omega = \{(\eta, \sigma, \mathbf{V}) : \eta = \eta_o = (\theta_o, \beta_o)', \eta_o \in \mathbb{R}^2, \sigma \in \mathbb{R}^+, \mathbf{V} > 0\}.$$

Moreover, suppose $y^{\frac{n}{2}} f(y)$ has a finite positive maximum y_f. Then the LR criterion for testing the hypothesis $H_o : \eta = \eta_o$ is given by

$$\mathcal{L}_n^{**} = S_u^{-2}\left[\frac{1}{2}(\tilde{\boldsymbol{\eta}} - \eta_o)'\boldsymbol{K}(\tilde{\boldsymbol{\eta}} - \eta_o)\right]$$

and it has the following modified generalized non-central F distribution

$$g_{2,m}^*(\mathcal{L}_n) = \sum_{r \geqslant 0} \frac{\left(\frac{2}{m}\right)^{\frac{1}{2}(2+2r)} \mathcal{L}_n^{\frac{1}{2}(2r)} K_r^{(0)}(\Delta^2)}{r!\, B\left(\frac{2r+2}{2}, \frac{m}{2}\right)\left(1 + \frac{2}{m}\mathcal{L}_n\right)^{\frac{1}{2}(2+2r+m)}},$$

where $\Delta^2 = \xi/\sigma_{\boldsymbol{\varepsilon}}^2$ for $\xi = (\eta - \eta_o)'\boldsymbol{K}(\eta - \eta_o)$, and

$$K_r^{(h)}(\Delta^2) = \left(\frac{\Delta^2}{2}\right)^r \int_0^\infty t^{r-h} e^{\frac{-t\Delta^2}{2}} \mathcal{W}(t)dt. \tag{2.4}$$

Proof. For the test of the null hypothesis $H_o : \eta = \eta_o$ vs $H_A : \eta \neq \eta_o$, let

$$\tilde{\sigma}_{\boldsymbol{\varepsilon}}^2 = (\boldsymbol{Y} - \boldsymbol{A}\eta_o)'\mathbf{V}^{-1}(\boldsymbol{Y} - \boldsymbol{A}\eta_o).$$

Then using Theorem 2.1 we have

$$\begin{aligned}
\Lambda &= \frac{\max_\omega L(\boldsymbol{y})}{\max_\Omega L(\boldsymbol{y})} = \left(\frac{S_u}{\tilde{\sigma}_{\boldsymbol{\varepsilon}}}\right)^n \frac{f\ (y_f)}{f\ (y_f)} \\
&= \left[\frac{(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}})'\mathbf{V}^{-1}(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}})}{(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}}_o)'\mathbf{V}^{-1}(\boldsymbol{Y} - \boldsymbol{A}\tilde{\boldsymbol{\eta}}_o)}\right]^n = \left(\frac{mS_u^2}{mS_u^2 + (\eta - \eta_o)'\boldsymbol{K}(\eta - \eta_o)}\right)^n \\
&= \left(\frac{1}{1 + \frac{1}{m}\mathcal{L}_n^{**}}\right)^n.
\end{aligned}$$

Hence, $\mathcal{L}_n^{**}$ is the LR test for testing the underlying null hypothesis. For its non-null distribution, we note that under normality $\mathcal{L}_n$ follows the non-central F-distribution with $(1, m)$ d.f. and non-centrality parameter $\Delta_t^2 = \frac{(\eta-\eta_o)'\boldsymbol{K}(\eta-\eta_o)}{t^{-1}\sigma^2}$. Then integrating over t w.r.t. the signed measure $\mathcal{W}$, the proof is completed. □

Accordingly, we have

Corollary 2.2.1. *Under H_o, the pdf of $\mathcal{L}_n^{**}$ is given by*

$$\mathbf{g}^*_{2,m}(\mathcal{L}_n^{**}) = \frac{\left(\frac{2}{m}\right)}{B\left(1, \frac{m}{2}\right)\left(1 + \frac{2}{m}\mathcal{L}_n\right)^{\frac{1}{2}(m+2)}},$$

which is the central F-distribution with $(2, m)$ d.f.

Corollary 2.2.2. *The power function at γ-level of significance of $\mathcal{L}_n^{**}$, say, modified generalized non-central F cumulative distribution function of the statistic $\mathcal{L}_n^{**}$ is given by*

$$\mathcal{G}_{p,m}(l_\gamma; \Delta^2) = \sum_{r \geqslant 0} \frac{1}{r!} K_r^{(0)}(\Delta^2) I_x\left[\frac{1}{2}(p+2r), \frac{m}{2}\right],$$

where $I_x(.,.)$ is the incomplete Beta function, $x = \frac{l_\gamma}{m+l_\gamma}$ and $l_\gamma = F_{1,m}(\gamma)$.

Straightforward consequences of Theorem 2.2, gain the test statistics for individuals $H_o : \theta = \theta_0$ and $H_o : \beta = \beta_o$. In order to test the null hypothesis $H_o : \beta = \beta_o$, against an alternative $H_A : \beta \neq \beta_o$, one uses the test statistic $\mathcal{L}_n^*$, defined by

$$\mathcal{L}_n^* = \frac{(\tilde{\beta}_n - \beta_o)^2 K_4}{S_u^2}; \qquad K_4 = \left(\frac{K_1}{K_1K_2 - K_3^2}\right)^{-1}.$$

Then the exact distribution of $\mathcal{L}_n$ under H_o has the central F-distribution with $(1, m)$ d.f. Similarly, for the test of $H_o : \theta = \theta_o$ against $H_A : \theta \neq \theta_o$ one uses the test-statistic

$$\mathcal{L}_n = \frac{(\tilde{\theta}_n - \theta_o)^2 K_5}{S_u^2}; \quad K_5 = \left(\frac{K_2}{K_1K_2 - K_3^2}\right)^{-1}. \tag{2.5}$$

The exact distribution of $\mathcal{L}_n$ under H_o is central F-distribution with $(1, m)$ d.f. Note that based on the virtue of (2.5), one can directly conclude the following result.

Lemma 2.1. *The LR criterion $\mathcal{L}_n$ for testing the hypothesis $H_o : \theta = \theta_o$ has the following distribution*

$$g^*_{1,m}(\mathcal{L}_n) = \sum_{r \geqslant 0} \frac{\left(\frac{1}{m}\right)^{\frac{1}{2}(1+2r)} \mathcal{L}_n^{\frac{1}{2}(2r-1)} K_r^{(0)}(\Delta^2)}{r!\, B\left(\frac{2r+1}{2}, \frac{m}{2}\right)\left(1 + \frac{1}{m}\mathcal{L}_n\right)^{\frac{1}{2}(1+2r+m)}},$$

where $\Delta^2 = \xi/\sigma_{\boldsymbol{\varepsilon}}^2$ for $\xi = K_5(\theta - \theta_o)^2$.

Now we turn our attention to estimation of the **intercept parameter** θ when it is suspected that the slope parameter β may be β_o. As a special case it covers the two-sample problem of estimating one mean when it is suspected that the two means may be equal. Also, one-sample estimation of mean is obtained by letting $\boldsymbol{x} = \mathbf{0}$ and prior information $\theta = \theta_o$

2.3. Estimators of θ

In addition to $\tilde{\theta}_n$ and S_u^2, we present a few more estimators of θ and $\sigma_{\boldsymbol{\varepsilon}}^2$. First of all note that we have

$$\begin{aligned}\tilde{\theta}_n &= K_1^{-1}\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y} - K_1^{-1}K_3\tilde{\beta}_n \\ &= K_1^*\mathbf{Y} - K_2^*\tilde{\beta}_n, \quad K_1^* = K_1^{-1}\mathbf{1}'\mathbf{V}^{-1}, \quad K_2^* = K_1^{-1}K_3.\end{aligned} \tag{2.6}$$

Replacing $\mathbf{V}$ by $\boldsymbol{I}_n$ in (2.6), results $\tilde{\theta}_n = \bar{Y} - \bar{x}\tilde{\beta}_n$ as in Saleh [22, p. 56].
If we suspect β to be β_o, then the restricted estimator (RE) of θ is given by

$$\hat{\theta}_n = K_1^* \boldsymbol{Y} - K_2^* \beta_o. \tag{2.7}$$

Now following Saleh [22], we define the estimators given below:

Preliminary test estimator (PTE) of θ is given by

$$\begin{aligned} \hat{\theta}_n^{PT} &= \hat{\theta}_n \, I(\mathcal{L}_n^* < F_{1,m}(\alpha)) + \tilde{\theta}_n \, I(\mathcal{L}_n^* \geqslant F_{1,m}(\alpha)) \\ &= \tilde{\theta}_n + (\tilde{\beta}_n - \beta_o) K_2^* I(\mathcal{L}_n^* < F_{1,m}(\alpha)), \end{aligned} \tag{2.8}$$

where $F_{1,m}(\alpha)$ is the α-level upper critical value of a central F-distribution with $(1, m)$ d.f. and $I(A)$ is the indicator function of the set A.

Shrinkage type estimator (SE) of θ is given by

$$\hat{\theta}_n^S = \tilde{\theta}_n + c(\tilde{\beta}_n - \beta_o) K_2^* \left| \mathcal{L}_n^{*\frac{1}{2}} \right|^{-1}, \quad c > 0 \tag{2.9}$$

3. Properties of intercept parameter

In this section, we derive the exact bias and MSE expressions for the proposed estimators of the intercept parameter.

Lemma 3.1 (Saleh, [22]). *If $Z \sim N(\Delta, 1)$, then*

$$E(|Z|) = \sqrt{\frac{2}{\pi}} e^{-\frac{\Delta^2}{2}} + \Delta\,(2\Phi(\Delta) - 1)$$

$$E\left[\frac{Z}{|Z|}\right] = 1 - 2\Phi(-\Delta),$$

where $\Phi(.)$ is the cdf of the standard normal distribution.

3.1. Bias expressions of the estimators

The biases of $\tilde{\theta}_n$ and $\hat{\theta}_n$ are obvious and given by

$$b_1(\tilde{\theta}_n) = 0, \quad b_2(\hat{\theta}_n) = K_2^*(\beta - \beta_o). \tag{3.1}$$

For the PTE, we have

$$\begin{aligned} b_3(\hat{\theta}_n^{PT}) &= E\left[\tilde{\theta}_n + (\tilde{\beta}_n - \beta_o) K_2^* I(\mathcal{L}_n^* < F_{1,m}(\alpha)) - \theta\right] \\ &= K_2^* E\left[(\tilde{\beta}_n - \beta_o) I(\mathcal{L}_n^* < F_{1,m}(\alpha))\right] \\ &= E_t\left\{E\left[\sqrt{\frac{t^{-1}\sigma_{\boldsymbol{\varepsilon}}^2}{K_4}} Z I\left(\frac{Z^2}{\chi_m^2/m} < F_{1,m}(\alpha)\right) | t\right]\right\} \\ &= K_2^* \sqrt{K_4} E_t\left[(\beta - \beta_0)\sqrt{K_4} I\left(\frac{\chi_3^2}{\chi_m^2/m}\right)\right] \\ &= K_2^* \sqrt{K_4} \sigma_{\boldsymbol{\varepsilon}} \Delta G_{3,m}^{(0)}\left(\frac{1}{3} F_{1,m}(\alpha);\, \Delta^2\right), \end{aligned} \tag{3.2}$$

where $\Delta_t^2 = t\Delta^2 = t\frac{(\beta-\beta_o)^2K_4}{\sigma_\varepsilon^2}$ and $G_{p,m}^{(h)}(.;.)$ is given by

$$G_{p,m}^{(h)}(q,\Delta^2) = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{p+m+2r}{2})}{\Gamma(\frac{p+2r}{2})\Gamma(m/2)} K_r^{(h)}(\Delta^2) I_x\left[\frac{p+2r}{2}, \frac{m}{2}\right];$$
$$x = \frac{pq}{m+pq}.$$

Finally for the bias expression of SE, taking $Z = \frac{(\tilde{\beta}_n-\beta_o)\sqrt{K_4}}{\sqrt{t^{-1}\sigma_\varepsilon^2}}$, we have

$$\begin{aligned} b_4(\hat{\theta}_n^S) &= E\left[\tilde{\theta}_n + c(\tilde{\beta}_n-\beta_o)K_2^*\left|\mathcal{L}_n^{*\frac{1}{2}}\right|^{-1} - \theta\right] \\ &= K_2^* E\left[c(\tilde{\beta}_n-\beta_o)\frac{S_u}{(\tilde{\beta}_n-\beta_o)\sqrt{K_4}}\right] \\ &= cK_2^*K_4^{-\frac{1}{2}} E_t\left\{E\left[Z\left|\frac{S_u}{Z}\right| |t\right]\right\}. \end{aligned} \tag{3.3}$$

Since $Z|t \sim N(\Delta_t, 1)$, $\Delta_t = \sqrt{\frac{(\beta-\beta_o)^2K_4}{t^{-1}\sigma^2}}$, $\frac{mS_u^2}{t^{-1}\sigma^2}|t \sim \chi_m^2$ and $Z|t$ is independent of $S_u^2|t$, using Lemma 3.1 the expression in (3.3) simplifies to

$$\begin{aligned} b_4(\hat{\theta}_n^S) &= cK_2^*K_4^{-\frac{1}{2}}\int_0^\infty \mathcal{W}(t)E\left[Z\left|\frac{S_u}{Z}\right| |t\right]dt \\ &= cK_2^*K_4^{-\frac{1}{2}}\int_0^\infty \mathcal{W}(t)E\left[\sqrt{\frac{mS_u^2}{t^{-1}\sigma^2}}\sqrt{\frac{t^{-1}\sigma^2}{m}}|t\right]E\left[\frac{Z}{|Z|}|t\right]dt \\ &= cK_2^*K_4^{-\frac{1}{2}}\frac{\Gamma(\frac{m+1}{2})}{\sqrt{2}\Gamma(\frac{m}{2})}\int_0^\infty \mathcal{W}(t)\sqrt{\frac{t^{-1}\sigma^2}{m}}E\left[\frac{Z}{|Z|}|t\right]dt \\ &= cK_2^*K_4^{-\frac{1}{2}}\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\sqrt{\frac{\sigma^2}{2m}}\int_0^\infty t^{-\frac{1}{2}}\mathcal{W}(t)(1-2\Phi(-\Delta_t))dt. \end{aligned} \tag{3.4}$$

3.2. MSE expressions of the estimators

Using Theorem 2.1 we get

$$M_1(\tilde{\theta}_n) = \sigma_\varepsilon^2 K_2(K_1K_2-K_3^2)^{-1}. \tag{3.5}$$

For the restricted estimator, applying Theorem 2.1 we have

$$\begin{aligned} M_2(\hat{\theta}_n) &= E\left[(\tilde{\theta}_n-\theta) + K_2^*(\tilde{\beta}_n-\beta_o)\right]^2 \\ &= M_1(\tilde{\theta}_n) + K_2^{*2}E(\tilde{\beta}_n-\beta_o)^2 + 2K_2^*E\left[(\tilde{\theta}_n-\theta)(\tilde{\beta}_n-\beta_o)\right] \\ &= \sigma_\varepsilon^2K_2(K_1K_2-K_3^2)^{-1} + K_2^{*2}\left[\frac{K_1\sigma_\varepsilon^2}{K_1K_2-K_3^2} + (\beta-\beta_o)^2\right] - 2K_2^*\frac{K_3\sigma_\varepsilon^2}{K_1K_2-K_3^2} \end{aligned}$$

$$= \frac{(K_2 - K_2^* K_3) + K_4{}^{-1}\Delta^2 (K_1K_2 - K_3^2)}{(K_1K_2 - K_3^2)}\ \sigma_{\boldsymbol{\varepsilon}}^2$$
$$= \left(K_1^{-1} + \Delta^2 K_4{}^{-1}\right)\sigma_{\boldsymbol{\varepsilon}}^2. \tag{3.6}$$

For the MSE of PTE, using equation (3.2.9b) of Saleh [22] we can obtain

$$\begin{aligned}
M_3(\hat{\theta}_n^{PT}) &= E\left[(\tilde{\theta}_n - \theta) + K_2^*(\tilde{\beta}_n - \beta_o) I(\mathcal{L}_n^* < F_{1,m}(\alpha))\right]^2 \\
&= M_1(\tilde{\theta}_n) + K_2^{*2} E\left[(\tilde{\beta}_n - \beta_o)^2 I(\mathcal{L}_n^* < F_{1,m}(\alpha))\right] \\
&\quad + 2K_2^* E[(\tilde{\theta}_n - \theta)(\tilde{\beta}_n - \beta_o) I(\mathcal{L}_n^* < F_{1,m}(\alpha))] \\
&= M_1(\tilde{\theta}_n) + K_2^{*2} K_4{}^{-1} E_t\left\{E\left[(t^{-1}\sigma_{\boldsymbol{\varepsilon}}^2) Z^2 I\left(\frac{Z^2}{\chi_m^2/m} < F_{1,m}(\alpha)\right)|t\right]\right\} \\
&\quad - 2K_2^{*2} K_4{}^{-1} E_t\left\{E\left[(t^{-1}\sigma_{\boldsymbol{\varepsilon}}^2) Z^2 I\left(\frac{Z^2}{\chi_m^2/m} < F_{1,m}(\alpha)\right)|t\right]\right\} \\
&\quad + 2K_2^{*2} K_4{}^{-1}\sigma_{\boldsymbol{\varepsilon}}\Delta E_t\left\{E\left[\sqrt{t^{-1}\sigma_{\boldsymbol{\varepsilon}}^2} Z I\left(\frac{Z^2}{\chi_m^2/m} < F_{1,m}(\alpha)\right)|t\right]\right\} \\
&= \sigma_{\boldsymbol{\varepsilon}}^2 K_2 (K_1K_2 - K_3^2)^{-1} + 2\sigma_{\boldsymbol{\varepsilon}}^2 \Delta^2 K_2^{*2} K_4{}^{-1}\left[G_{3,m}^{(0)}\left(\frac{1}{3}F_{1,m}(\alpha);\ \Delta^2\right)\right] \\
&\quad - \sigma_{\boldsymbol{\varepsilon}}^2 K_2^{*2} K_4{}^{-1}\left\{G_{3,m}^{(1)}\left(\frac{1}{3}F_{1,m}(\alpha);\ \Delta^2\right) + \Delta^2 G_{5,m}^{(0)}\left(\frac{1}{5}F_{1,m}(\alpha);\ \Delta^2\right)\right\}.
\end{aligned} \tag{3.7}$$

Finally, for the shrinkage estimator, using Lemma 3.1 we have

$$\begin{aligned}
M_4(\hat{\theta}_n^S) &= E\left[\tilde{\theta}_n + c(\tilde{\beta}_n - \beta_o) K_2^* |\mathcal{L}_n^{*\frac{1}{2}}|^{-1} - \theta\right]^2 \\
&= M_1(\tilde{\theta}_n) + c^2 K_2^{*2} E\left[(\tilde{\beta}_n - \beta_o)^2 |\mathcal{L}_n^{*\frac{1}{2}}|^{-2}\right] + 2cK_2^* E\left[(\tilde{\theta}_n - \theta)(\tilde{\beta}_n - \beta_o)|\mathcal{L}_n^{*\frac{1}{2}}|^{-1}\right] \\
&= M_1(\tilde{\theta}_n) + c^2 K_2^{*2} K_4{}^{-1} E\left(S_u^2\right) \\
&\quad - 2cK_2^{*2} K_4^{-1} E_t\left\{\sqrt{t^{-1}\sigma^2} E\left[S_u\left(\frac{Z^2}{|Z|} - \Delta_t \frac{Z}{|Z|}\right)\right]\right\} \\
&= \sigma^2 K_2 (K_1K_2 - K_3^2)^{-1} + c^2 k_2^{*2} k_4{}^{-1}\sigma^2 \\
&\quad - 2cK_2^{*2} K_4{}^{-1}\sigma E_t E[S_u|t] E_t\Bigg[t^{-\frac{1}{2}}\Bigg\{\sqrt{\frac{2}{\pi}} e^{\frac{-\Delta_t^2}{2}} + \Delta_t\{2\Phi(\Delta_t) - 1\} \\
&\quad - \Delta_t\{1 - 2\Phi(-\Delta_t)\}\Bigg\}\Bigg| t\Bigg],
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
E_t E\left[S_u|t\right] &= \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\sqrt{\frac{\sigma^2}{2m}}\int_0^\infty \mathcal{W}(t) t^{-\frac{1}{2}} dt \\
E_t &\left[t^{-\frac{1}{2}}\left\{\sqrt{\frac{2}{\pi}} e^{\frac{-\Delta_t^2}{2}} + \Delta_t\{2\Phi(\Delta_t) - 1\} - \Delta_t\{1 - 2\Phi(-\Delta_t)\}\right\}\right] \\
&= \int_0^\infty \sqrt{\frac{2}{\pi}} t^{-\frac{1}{2}}\ e^{\frac{-\Delta_t^2}{2}} \mathcal{W}(t) dt.
\end{aligned} \tag{3.9}$$

4. Comparison

In this section we compare the proposed estimators with respect to their MSE functions. The mean-square relative efficiency (MRE) of $\hat{\theta}_n$ compared to $\tilde{\theta}_n$ may be written as

$$\begin{aligned}\mathrm{MRE}(\hat{\theta}_n; \tilde{\theta}_n) &= \frac{M_1(\tilde{\theta}_n)}{M_2(\hat{\theta}_n)} \\ &= \frac{(K_1K_2 - K_3^2)^{-1}\sigma_\varepsilon^2}{(K_1^{-1} + \Delta^2 K_4^{-1})\sigma_\varepsilon^2} \\ &= \frac{K_1K_4K_2}{(K_4 + \Delta^2 K_1)(K_1K_2 - K_3^2)} \\ &= \frac{K_2}{K_4 + \Delta^2 K_1}. \end{aligned} \tag{4.1}$$

The efficiency is a decreasing function of Δ^2. Under $H_o : \beta = \beta_o$ it has the maximum

$$\mathrm{MRE}(\hat{\theta}_n; \tilde{\theta}_n) = \frac{K_2}{K_4}. \tag{4.2}$$

In general to compare $\hat{\theta}_n$ and $\tilde{\theta}_n$, using (4.1) $\mathrm{MRE}(\hat{\theta}_n; \tilde{\theta}_n) > 1$ whenever $\Delta^2 < (\frac{K_3}{K_1})^2$.

The relative efficiency of $\hat{\theta}_n^{PT}$ compared to $\tilde{\theta}_n$ is given by

$$\mathrm{MRE}(\hat{\theta}_n^{PT}; \tilde{\theta}_n) = [1 + g(\Delta^2)]^{-1}, \tag{4.3}$$

where

$$\begin{aligned} g(\Delta^2) = -\frac{K_2^{*2}K_1}{K_2}\Bigg\{ & G_{3,m}^{(1)}\left(\frac{1}{3}F_{1,m}(\alpha);\, \Delta^2\right) \\ & +\Delta^2\left(G_{5,m}^{(0)}\left(\frac{1}{5}F_{1,m}(\alpha);\, \Delta^2\right) - 2G_{3,m}^{(0)}\left(\frac{1}{3}F_{1,m}(\alpha);\, \Delta^2\right)\right)\Bigg\}. \end{aligned} \tag{4.4}$$

Under H_o, it has the maximum value

$$\mathrm{MRE}(\hat{\theta}_n^{PT}; \tilde{\theta}_n) = \left\{1 - \frac{K_2^{*2}K_1}{K_2} G_{3,m}^{(1)}\left(\frac{1}{3}F_{1,m}(\alpha);\, 0\right)\right\}^{-1}. \tag{4.5}$$

In general, $\mathrm{MRE}(\hat{\theta}_n^{PT}; \tilde{\theta}_n) \gtreqless 1$ according as

$$\Delta^2 \lesseqgtr \frac{G_{3,m}^{(1)}\left(\frac{1}{3}F_{1,m}(\alpha);\, \Delta^2\right)}{2G_{3,m}^{(0)}\left(\frac{1}{3}F_{1,m}(\alpha);\, \Delta^2\right) - G_{5,m}^{(0)}\left(\frac{1}{5}F_{1,m}(\alpha);\, \Delta^2\right)}. \tag{4.6}$$

The relative efficiency of $\hat{\theta}_n^S$ compared to $\tilde{\theta}_n$, is given by

$$\mathrm{MRE}(\hat{\theta}_n^S; \tilde{\theta}_n) = [1 + h(\Delta^2)]^{-1}, \tag{4.7}$$

where

$$h(\Delta^2) = M_1^{-1}(\tilde{\theta}_n)\left\{c^2 k_2^{*2} k_4{}^{-1}\sigma^2 - 2cK_2^{*2}K_4{}^{-1}\sigma \times \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\sqrt{\frac{\sigma^2}{\pi m}}\int_0^\infty t^{-1} e^{\frac{-\Delta t}{2}} \mathcal{W}(t)dt\right\}. \tag{4.8}$$

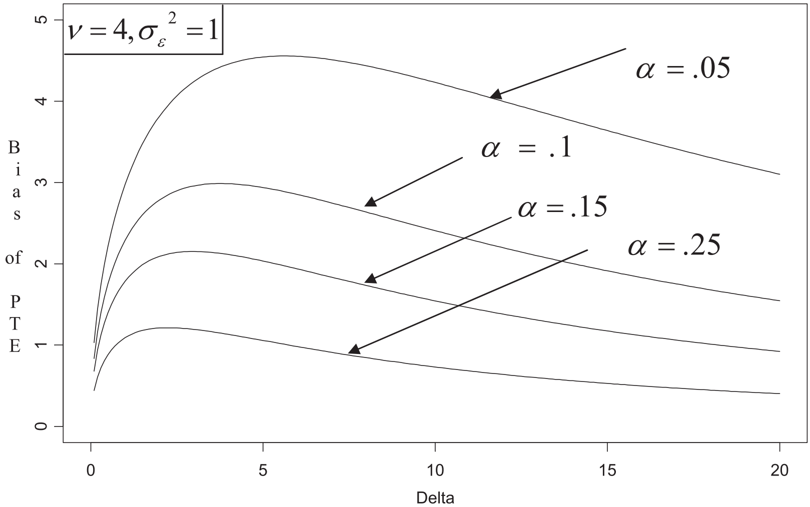

Fig. 1. Graph of bias function for PTE.

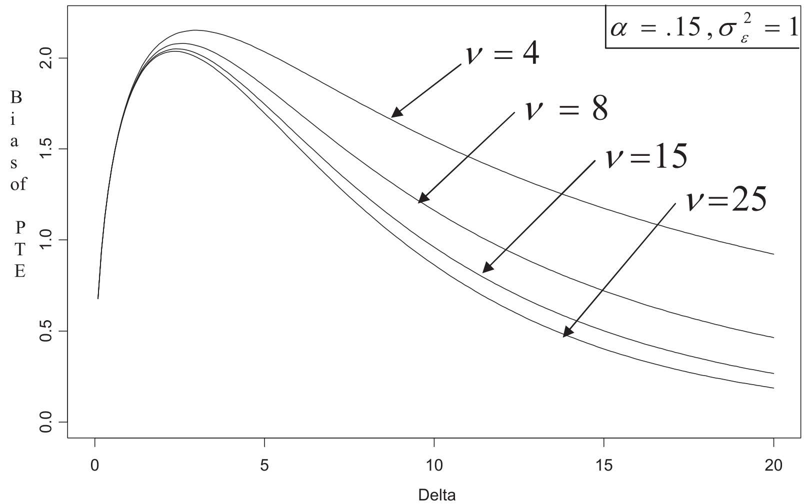

Fig. 2. Graph of bias function for PTE.

It is a decreasing function with respect to Δ^2. Under H_o, it simplifies to

$$\mathrm{MRE}(\hat{\theta}_n^S; \tilde{\theta}_n) = \left\{1 + M_1^{-1}(\tilde{\theta}_n)\left[c^2 k_2^{*2} k_4{}^{-1}\sigma^2 + 4\psi'(0) c K_2^{*2} K_4{}^{-1} \times \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})\sqrt{\pi m}}\right]\right\}^{-1} \geqslant 1 \tag{4.9}$$

whenever by Remark 1.1

$$0 < c \leqslant \frac{-4\Gamma(\frac{m+1}{2})}{\sqrt{\pi m}\,\Gamma(\frac{m}{2})}\psi'(0). \tag{4.10}$$

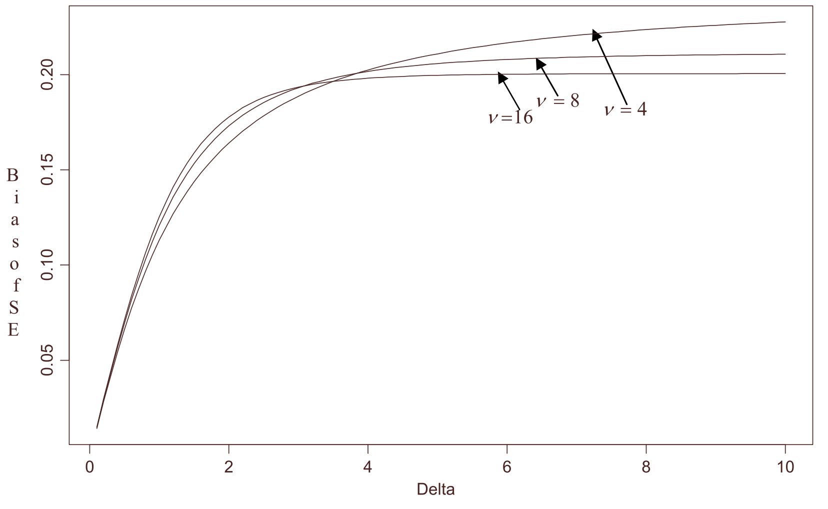

Fig. 3. Graph of bias function for SE.

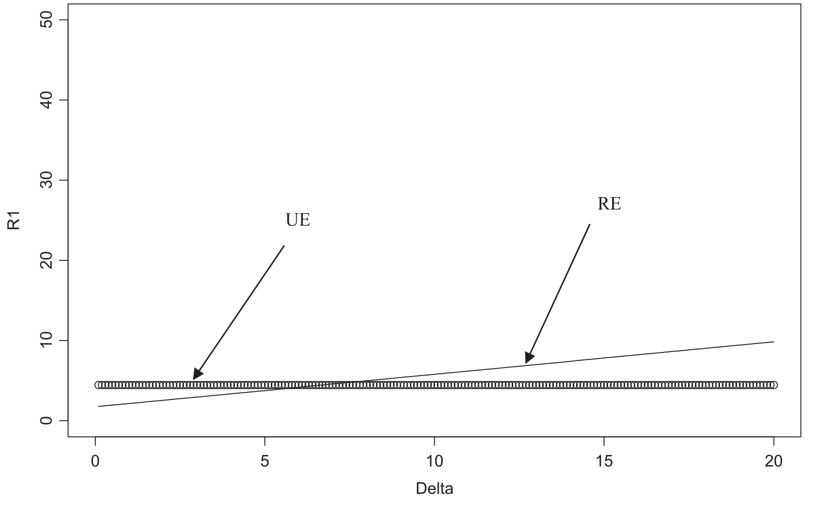

Fig. 4. Graph of risk function for UE and RE.

4.1. Optimum level of significance of $\hat{\theta}_n^{PT}$

Following Section 3.2.4 of Saleh [22], denote the relative efficiency of $\hat{\theta}_n^{PT}$ compared to $\tilde{\theta}_n$ by $\mathrm{MRE}(\alpha, \Delta^2)$. Its maximum value occurs at $\Delta^2 = 0$ as given in (4.5), i.e. $\max_{\Delta^2} \mathrm{MRE}(\alpha, \Delta^2) = \mathrm{MRE}(\alpha, 0)$. Subsequently, in order to obtain preliminary test estimator with a minimum guaranteed efficiency E_0 say, we adopt the following procedure: If $\Delta^2 \leqslant 1$, we always choose $\tilde{\theta}_n$. However, in general, Δ^2 is unknown, so there is no way to choose an estimator that is uniformly best. For this reason, we select an estimator with minimum guaranteed efficiency, such as E_0, and look for a suitable α from the set $A_0 = \left\{\alpha | \mathrm{MRE}(\alpha, \Delta^2) \geqslant E_0\right\}$. The estimator chosen maximizes $\mathrm{MRE}(\alpha, \Delta^2)$ over all

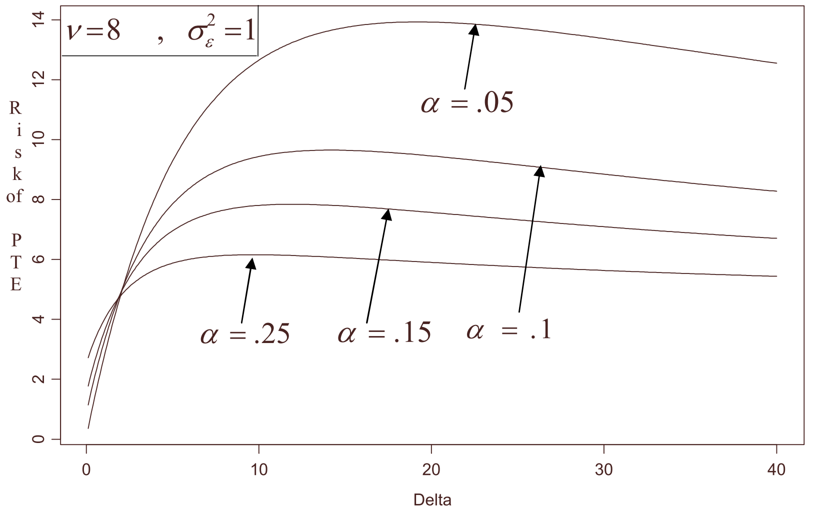

Fig. 5. Graph of risk function for PTE.

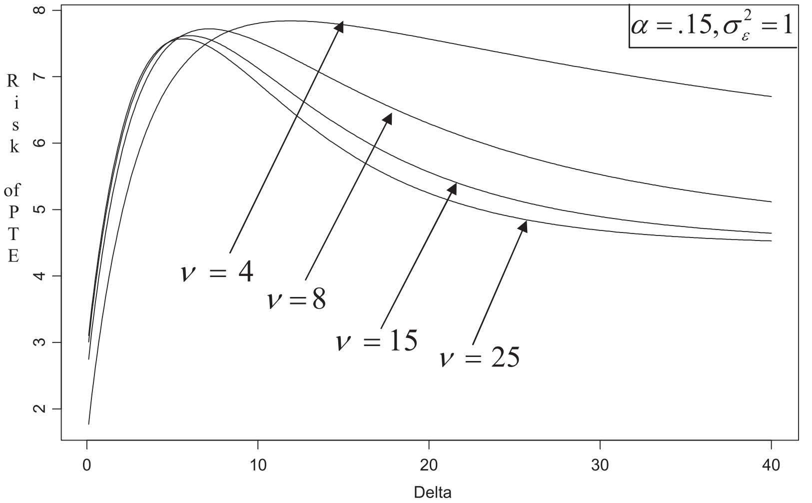

Fig. 6. Graph of risk function for PTE.

$\alpha \in A_0$ and Δ^2. Thus, we solve the following equation for the optimum α^*:

$$\min_{\Delta^2} \mathrm{MRE}(\alpha, \Delta^2) = E(\alpha, \Delta_0^2(\alpha)) = E_0. \tag{4.11}$$

The solution α^* obtained this way gives the PTE with minimum guaranteed efficiency E_0.

5. Numerical example

In this section, we proceed by a numerical example based on the multivariate Student's t (Mt) distribution, a well-known member of ECDs. First of all assume that $\boldsymbol{\varepsilon}$ in the model (1.1), follows a Mt distribution with the scale matrix

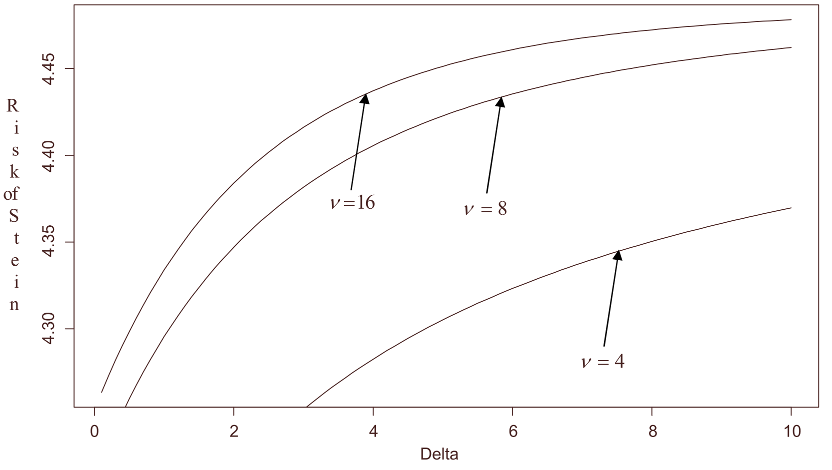

Fig. 7. Graph of risk function for SE.

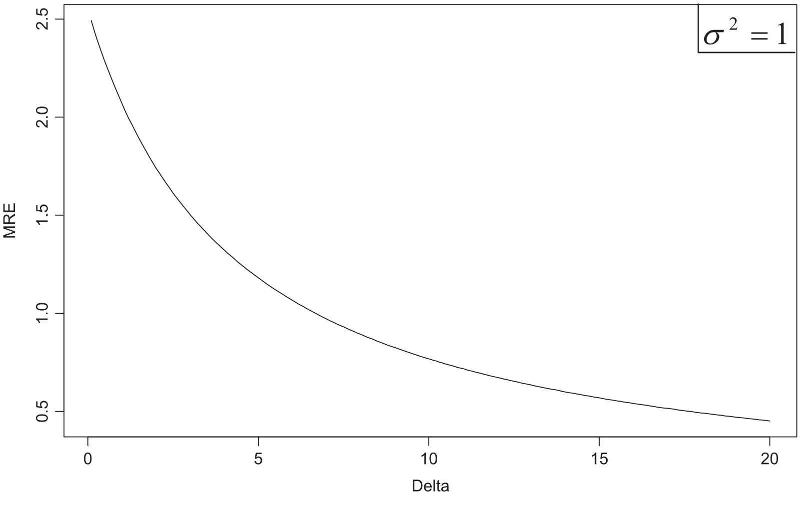

Fig. 8. Graph of MRE (RE vs UE).

$$\mathbf{V} = \begin{bmatrix} 2.57 & 0.85 & 1.56 & 1.79 & 1.33 & 0.42 \\ 0.85 & 37.00 & 3.34 & 13.47 & 7.59 & 0.52 \\ 1.56 & 3.34 & 8.44 & 5.77 & 2.00 & 0.50 \\ 1.79 & 13.47 & 5.77 & 34.01 & 10.50 & 1.77 \\ 1.33 & 7.59 & 2.00 & 10.50 & 23.01 & 3.43 \\ 0.42 & 0.52 & 0.50 & 1.77 & 3.43 & 4.59 \end{bmatrix}$$

and ν degrees of freedom with the pdf as in (1.6). Then we have

$$W(t) = \frac{\nu(\nu t/2)^{\nu/2-1}}{2e^{\nu t/2}\Gamma(\nu/2)}.$$

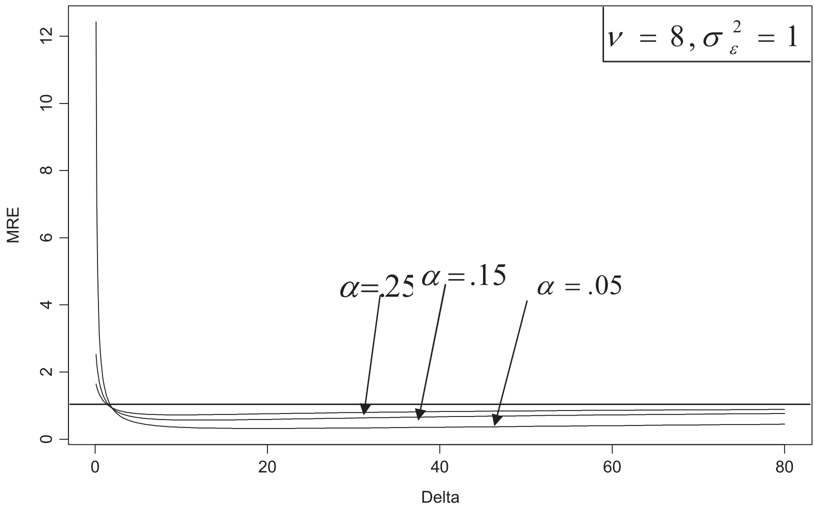

Fig. 9. Graph of MRE (PTE vs UE).

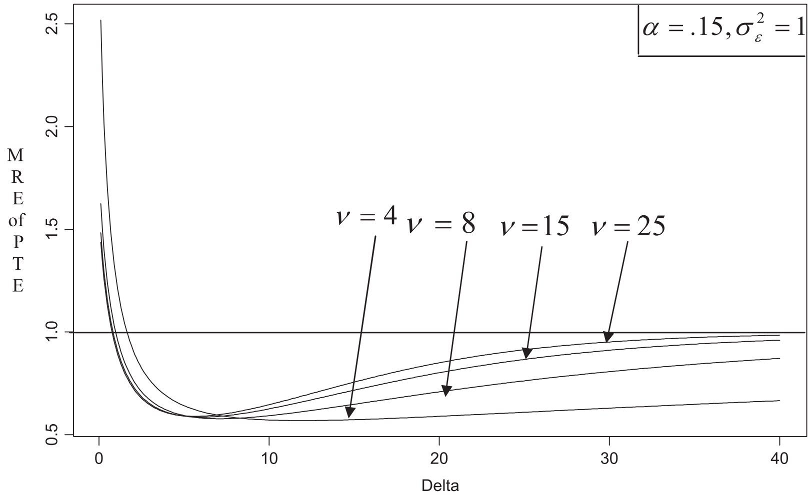

Fig. 10. Graph of MRE (PTE vs UE).

The respective expressions for $G_{p,m}^{(h)}(q, \Delta^2)$, $E^{(h)}\left[\chi_p^{-2}(\Delta^2)\right]$ and $E^{(h)}\left[\chi_p^{-4}(\Delta^2)\right]$ can be found in Khan [15].
Further assume that $\boldsymbol{x}' = (2\ 6\ 1\ 8\ 3\ 4)$.

According to the result of Section 3, the graphs of PTE and SE biases vs Δ are displayed in Figs. 1–3. As it can be realized, when the both level of significance α and degrees of freedom ν increase the bias of PTE decreases. The bias of SE performs the same as ν increases. Similar conclusions can be made for the MSE graphs in Figs. 4–7.

For the MRE graphs in Figs. 8–11, it can be concluded that the efficiency of $\hat{\theta}_n$ relative to $\tilde{\theta}_n$ is a decreasing function as discussed in Section 4. MRE$(\hat{\theta}_n^{PT}; \tilde{\theta}_n)$ is a decreasing function relative to Δ and also for small level of significance α, the UE performs better than the PTE. This scenario has a little bit

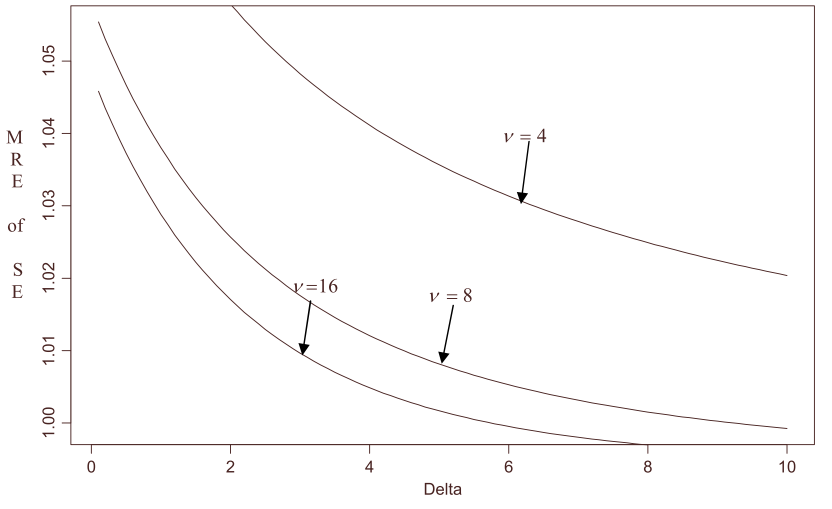

Fig. 11. Graph of MRE (SE vs UE).

Table 1
Maximum and minimum guaranteed efficiencies for $n = 6$.

α	ξ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	$E_{\max}$	1.12	1.27	1.47	1.75	2.16	2.82	4.06	7.23	32.83
	$E_{\min}$	0.75	0.60	0.50	0.43	0.38	0.34	0.30	0.27	0.25
	$\Delta^2_{\min}$	12.10	12.10	12.10	12.10	12.10	12.10	12.10	12.10	12.10
0.1	$E_{\max}$	1.09	1.21	1.35	1.54	1.78	2.11	2.60	3.38	4.81
	$E_{\min}$	0.84	0.73	0.64	0.57	0.52	0.47	0.44	0.40	0.38
	$\Delta^2_{\min}$	8.70	8.70	8.70	8.70	8.70	8.70	8.70	8.70	8.70
0.15	$E_{\max}$	1.07	1.16	1.27	1.40	1.56	1.75	2.01	2.35	2.83
	$E_{\min}$	0.89	0.80	0.73	0.67	0.62	0.58	0.54	0.50	0.40
	$\Delta^2_{\min}$	7.10	7.10	7.10	7.10	7.10	7.10	7.10	7.10	7.10
0.20	$E_{\max}$	1.06	1.13	1.21	1.30	1.41	1.53	1.69	1.87	2.10
	$E_{\min}$	0.92	0.85	0.79	0.74	0.70	0.66	0.62	0.59	0.56
	$\Delta^2_{\min}$	6.20	6.20	6.20	6.20	6.20	6.20	6.20	6.20	6.20
0.25	$E_{\max}$	1.04	1.10	1.16	1.23	1.30	1.39	1.49	1.60	1.73
	$E_{\min}$	0.94	0.88	0.84	0.80	0.76	0.72	0.69	0.66	0.64
	$\Delta^2_{\min}$	5.60	5.60	5.60	5.60	5.60	5.60	5.60	5.60	5.60
0.30	$E_{\max}$	1.03	1.08	1.12	1.17	1.23	1.29	1.35	1.42	1.51
	$E_{\min}$	0.95	0.91	0.87	0.84	0.81	0.78	0.75	0.73	0.70
	$\Delta^2_{\min}$	5.20	5.20	5.20	5.20	5.20	5.20	5.20	5.20	5.20
0.35	$E_{\max}$	1.03	1.06	1.09	1.13	1.17	1.21	1.26	1.30	1.36
	$E_{\min}$	0.95	0.93	0.90	0.88	0.85	0.83	0.80	0.78	0.76
	$\Delta^2_{\min}$	4.90	4.90	4.90	4.90	4.90	4.90	4.90	4.90	4.90

change for the degrees of freedom ν; its behavior can be verified from Fig. 10. Finally the shrinkage estimator performs better than the unrestricted estimator as ν increases.

To conclude this section, Table 5 gives selected values of $\xi = \frac{K_2^{*2}K_1}{K_2}$ and $\alpha = 0.05(0.05)0.35$ for the procedure of choosing the level α^* of significance.

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