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Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups

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ABSTRACT

The paper contains a generalization of known properties of Chebyshev polynomials of the second kind in one variable to polynomials of n variables based on the root lattices of compact simple Lie groups G of any type and of any rank n. The results, inspired by work of H. Li and Y. Xu where they derived cubature formulae from A-type lattices, yield Gaussian cubature formulae for each simple Lie group G based on nodes (interpolation points) that arise from regular elements of finite order in G. The polynomials arise from the irreducible characters of G and the nodes as common zeros of certain finite subsets of these characters. The consistent use of Lie theoretical methods reveals the central ideas clearly and allows for a simple uniform development of the subject. Furthermore it points to genuine and perhaps far reaching Lie theoretical connections.
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1. Introduction

During most of the century and half long history of Chebyshev polynomials, only polynomials of one variable were studied [17]. In recent years a considerable overlap of the subject can be found in the emerging and practically important field of cosine and sine transforms [16,18]. In the absence of additional constraints, truly higher-dimensional generalizations of Chebyshev polynomials are hidden in the vast number of possibilities of defining orthogonal polynomials of more than one variable [19,10]. In 2D [9] such constraints were provided by requiring that the polynomials of two variables be simultaneous eigenvectors of two differential operators.

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Our motivation here comes from two directions:

(i) The method of constructing Chebyshev-like polynomials for any simple Lie group [14], and particularly from the recognition of the basic role played by the group characters.
(ii) The work of H. Li and Y. Xu in which they derived cubature formulae based on the symmetries of A-type lattices [8].

From [14] we understood that a general formulation, uniform over all the types and ranks of simple Lie algebras should be possible, and from [8] we saw what possibilities, beyond the construction of the polynomials, might be achievable in a general formulation guided by the theory of compact simple Lie groups.

In this paper the characters of the irreducible representations of a compact simply-connected simple Lie group G of rank n play the role of the Chebyshev polynomials (of the second kind) and elements of finite order [11] in G give rise to nodes through which we arrive at Gaussian cubature formulae. The ring generated by the characters of G has a $\mathbb{Z}$-basis consisting of the irreducible characters and it is a polynomial ring in terms of the n irreducible characters of the fundamental representations. These fundamental characters then serve as new variables for functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$. This domain Ω is derived from the fundamental domain of the affine Weyl group of G and the kernel function used is the absolute value of the denominator of Weyl's character formula [20].

There are two technical ingredients in the paper which are indispensable for the uniformity of our approach to simple Lie groups of all types and thus for generality of our conclusions.

(i) The 'natural' grading of polynomials by their total degree is replaced by new m-degree grading. It is based on a set of Lie theoretical invariants, called the marks, which are unique for simple Lie group of each type. The two gradings coincide only in the case of A_n.
(ii) The set of nodes is uniformly specified for each simple Lie group as a finite set of lattice points, characterising all conjugacy classes of elements of a certain finite order in the underlying Lie group.

The cubature formula (see Theorem 7.2) is then a formula that equates a weighted integral of a general polynomial P, say of m-degree $\leqslant 2M+1$, to weighted finite sums of the values of P sampled at the nodes which are common zeros of the polynomials of m-degree $M+1$. So the cubature formula is a type of interpolation formula with the nodes as the points of interpolation. The cubature is *Gaussian* [21,22] in the sense that the number nodes coincides with the dimension of the space of polynomials of m-degree $\leqslant M$. It turns out that the nodes are the elements whose adjoint order divides $M+h$ where h is the Coxeter number.

In more detail, having fixed a specific m-degree M, we are interested in the properties of the set of all polynomials of m-degree at most M. There are three main results:

(1) Theorem 6.1: The nodes are common zeros for the set of polynomials of m-degree $M+1$. These nodes are directly related to regular elements of finite adjoint order $M+h$ in G, where h is the Coxeter number of G, and their number is precisely the dimension of the space of polynomials of m-degree less than or equal to M.
(2) Theorem 7.2: There is a cubature formula that equates weighted integrals $\int_\Omega f K^{1/2}$ of each polynomial f of m-degree $\leqslant 2M+1$ with K-weighted linear combinations of its values at the nodes. This formula also appears in Theorem 7.2 in the guise of a discrete formula for a $K^{1/2}$-weighted inner product of polynomials with m-degree at most $M+1$.
(3) Proposition 8.2: The expansions of functions via irreducible characters (or rather these characters interpreted as polynomials) using the nodes yield the best approximations in the $K^{1/2}$-weighted L^2-norm on the functions on Ω.

These results and their proofs are natural and completely uniform within the framework of the character theory of simple Lie groups and their elements of finite order. In fact it is this naturalness and perfection of fit that suggests that there are a deeper Lie theoretical implications to all of this that are still to be discovered. The potential role of the finite reflection groups in the theory of orthogonal polynomials was recognized already in [3], although only a limited use of these groups is made there, the objects of interest being group invariant differential and difference-differential operators. The present paper can be understood as a new contribution to the fulfilment of that potential, much closer to the properties of the simple Lie groups which give rise to the finite reflection groups.

Our discussion requires a certain familiarity with root and weight lattices, their corresponding co-root and co-weight lattices, the Weyl group and its affine extension, and particularly the structure of the natural fundamental domain for the affine Weyl group. We use Section 2, which sets up the notation, to briefly review the key points of this theory, while Section 3 and Section 4 contain preparatory extensions of the standard theory. The paper proper begins at Section 5.

2. Simple compact Lie groups

This section is a brief review of the material that we need for this paper and establishes the notation that we shall be using. The main facts about simple Lie groups and their representations are classical and can be found in many places. One source that uses the same notation as in this paper is [5, vol. 1]. Material on the fundamental domain, and in particular information about the stabilizers of its various points, can be found in Ch. V of [1] and material on both the fundamental domain and the elements of finite order can be found in [12].

Let G be a simply connected simple Lie group with Lie algebra $\mathfrak{g}$. We let c_G denote the order of its centre and let $\mathbb{T}$ be a maximal torus of G. We let $i\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$, so that we have the exact sequence

$$0 \to \check{Q} \to \mathfrak{t} \xrightarrow{\exp(2\pi i(\cdot))} \mathbb{T} \to 1$$

via the exponential map. Using $\mathfrak{t}$ instead of the actual Lie algebra $i\mathfrak{t}$ of $\mathbb{T}$ has the advantages that the Killing form $(\cdot \mid \cdot)$ of G restricted to $\mathfrak{t}$ is positive definite and $e^{2\pi i(\cdot|\cdot)}$ is perfect for the Fourier analysis to follow. Let $n := \dim_{\mathbb{R}} \mathfrak{t}$, the *rank* of G.

The kernel of $e^{2\pi i(\cdot)}$ on $\mathfrak{t}$ is the *co-root lattice* $\check{Q}$, and $\mathfrak{t}/\check{Q} \simeq \mathbb{T}$ naturally expresses $\mathbb{T}$ as a real space factored by a lattice. We denote by $\mathfrak{t}^*$ the dual space of $\mathfrak{t}$ and let $\langle \cdot,\cdot \rangle$ be the natural pairing of $\mathfrak{t}^*$ and $\mathfrak{t}$.

Let $\theta_{\mathbb{T}}$ be the Haar measure on $\mathbb{T}$ that gives it volume equal to 1. In practice we most often write integration over $\mathbb{T}$ in the form of integration over some fundamental region FR for $\check{Q}$ in $\mathfrak{t}$:

$$\int_{\mathbb{T}} f \, d\theta_{\mathbb{T}} = \int_{FR} f\left(e^{2\pi i x}\right) d\theta_{\mathfrak{t}} \tag{1}$$

where $\theta_{\mathfrak{t}}$ is ordinary Lebesgue measure in $\mathfrak{t}$ normalized so that FR has volume equal to 1. In the sequel no fundamental domain FR ever makes an appearance, but rather we work with a smaller fundamental domain F° of the affine Weyl group, see below.

Given any finite-dimensional complex representation V of the Lie group G, the action of the elements of $\mathbb{T}$ on V can be simultaneously diagonalized, the resulting eigenspaces in V being the *weight spaces*. The corresponding action of $\mathfrak{t}$ on these weight spaces then affords elements $\lambda \in \mathfrak{t}^*$ for which $x \in \mathfrak{t}$ acts as $\exp(2\pi i \langle \lambda, x \rangle)$. Naturally $\langle \lambda, \check{Q} \rangle \subset \mathbb{Z}$ and the set of weights taken over all finite-dimensional representations is the subgroup $P \subset \mathfrak{t}^*$ which is the $\mathbb{Z}$-dual of the co-root lattice relative to our pairing. P is the *weight lattice* of G relative to $\mathbb{T}$. The adjoint representation, the representation of G on its own Lie algebra, produces its own set of weights. Apart from the weight 0, which is of multiplicity n, the remaining weights of the adjoint representation are the *roots* of G and they all occur with multiplicity 1. They generate (as a group) the root lattice Q of G, and we have $Q \subset P$ with index equal to c_G defined above. The $\mathbb{Z}$-dual of Q in $\mathfrak{t}$ is the co-weight lattice $\check{P} \supset \check{Q}$ with the index of $\check{Q}$ in $\check{P}$ being c_G, again.

Let W be the Weyl group associated with $\mathbb{T}$, that is, the quotient by $\mathbb{T}$ of the normalizer of $\mathbb{T}$ in G. Then W acts as a finite group of isometries of $\mathfrak{t}$ relative to $(\cdot \mid \cdot)$, and W stabilizes $\check{Q}$. It also stabilizes $\check{P}$, and if we pull its action over to $\mathfrak{t}^*$ by duality, then W also stabilizes the weight lattice P and the root lattice Q.

The relationships between the lattices and between the various root and weight bases and their co-equivalents described below are summarized in:

$$\begin{array}{ccccccc} \{\alpha_1,\dots,\alpha_n\} & \subset & Q & & \check{Q} & & \\ & & \cap & \times & \cap & & \\ & & P & & \check{P} & \supset & \{\check{\omega}_1,\dots,\check{\omega}_n\}. \\ & & \cap & & \cap & & \\ & & \mathfrak{t}^* & & \mathfrak{t} & & \end{array} \tag{2}$$

The times symbol is meant to indicate that Q and $\check{P}$, as well as P and $\check{Q}$, are in $\mathbb{Z}$-duality with each other. The indicated bases are also in $\mathbb{Z}$-duality (see below).

We have the semi-direct product $W_{\mathrm{aff}} = W \ltimes \check{Q}$ acting on $\mathfrak{t}$, with $\check{Q}$ acting as translations and the Weyl group as point symmetries. W_{aff} is called the *affine Weyl group*.

A fundamental region F for $\mathfrak{t}$ under the action of W_{aff} can be given as follows. Let Δ be the set of all the roots and let $\Pi = \{\alpha_1,\dots,\alpha_n\}$ be a simple system of roots for Δ. Let Δ_+ (resp. Δ_-) denote the corresponding set of positive (resp. negative) roots, and let

$$\alpha_0 = -(m_1\alpha_1 + \cdots + m_n\alpha_n) \tag{3}$$

be the lowest root[1] of Δ. The positive integers $m_1,\dots,m_n$ are called the *marks* of G. The marks and co-marks (introduced below) are shown on Fig. 1. They are independent of the choice of $\mathbb{T}$ and the choice of Π (but depend on the choice of ordering of $\alpha_1,\dots,\alpha_n$). It is convenient to define one more mark, $m_0 := 1$, so one has $\sum_{j=0}^n m_j\alpha_j = 0$. One then has the *Coxeter number* of G

$$h := \sum_{j=0}^n m_j.$$

This constant appears in many places and in many guises in Lie theory, and plays an important role in what follows.

Now we define

$$F := \bigl\{x \in \mathfrak{t} \bigm| \langle \alpha_j, x\rangle \geqslant 0 \text{ for all } j = 1,\dots,n,\ \langle -\alpha_0, x\rangle \leqslant 1\bigr\}. \tag{4}$$

This is a simplex in $\mathfrak{t}$. The Weyl group W contains the so-called *simple* reflections $r_1,\dots,r_n$ in the walls of F (the hyperplanes generated by the $(n-1)$-faces of F).

$$H_j := \bigl\{x \in \mathfrak{t} \bigm| \langle \alpha_j, x\rangle = 0;\ j = 1,\dots,n\bigr\} \tag{5}$$

and W is generated by these reflections. Thus any $w \in W$ is expressible as a product of the simple reflections, $w = r_{i_1}\dots r_{i_k}$; but such a writing of w is not unique, and in particular the length $l(w) = k$ is not unique. However its parity $(-1)^{l(w)}$ is well defined. It is convenient to write this parity function in the form $(-1)^{l(w)}$ even though $l(w)$ depends entirely on the way in which we choose to write w as a product of simple (or in fact any) reflections. The mapping

[1] In what follows, the use of the lowest root rather than the highest root, which might seem more natural, is in keeping with notation from the theory of affine root systems.

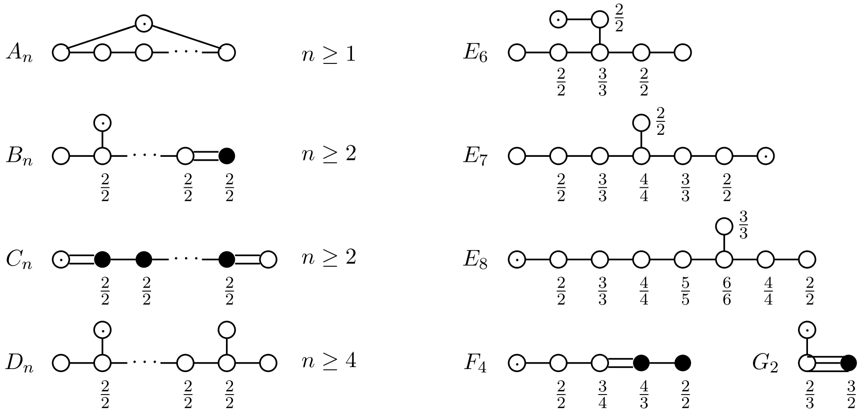

Fig. 1. The usual Coxeter–Dynkin diagrams or graphs, which are simply a visually efficient way to encode the Cartan matrices, are shown, along with information giving the corresponding marks and co-marks. The circular vertices, ignoring those with a dot in them, stand for the simple roots with the convention that open (resp. filled) circles indicate long (resp. short) roots. When there are both types of vertices, interchanging open and filled results in the co-diagram. Thus types *B* and *C* interchange and types *F* and *G* end up as the same type but with the numbering of the roots permuted. The dotted vertex stands for the root α_0 (which is linearly dependent on the simple roots with the negatives of the marks as its coefficients). Under the duality operation of roots to co-roots, it passes from being the lowest long root to being the lowest short co-root, a fact that we do not explicitly have to use here.
The links between roots occur only when the roots are not orthogonal to one another. The marks and co-marks are shown as the fractions $m/\check{m}$ attached to the corresponding vertices of the diagram. When both are equal to one, they are not shown. We refer the reader to [2] for more details. The numbering of the simple roots goes from the left to right, the undotted vertex above the main line (when it exists) carrying the highest value. The dotted vertex has number 0.

$$l : W \to \{\pm 1\}, \quad w \mapsto (-1)^{l(w)}$$

is a homomorphism.

There is a unique element $w_{\mathrm{opp}} \in W$ which maps Δ_+ into Δ_-. It is an involution and is minimally represented by the product of exactly $|\Delta_+|$ of the simple reflections.

W_{aff} is also a reflection group and is obtained by adding to $r_1, \dots, r_n$ the additional generator r_0 which is reflection in the remaining wall of F

$$H_0 := \{x \in \mathfrak{t} \mid \langle -\alpha_0, x\rangle = 1\}. \tag{6}$$

There is a similar length function for W_{aff}. The action of W_{aff} on F tiles the entire space $\mathfrak{t}$ with copies of F, and in this way F serves as a fundamental domain for it. See Fig. 2 for an example.

The way in which F is the fundamental region is rather beautiful:

- Every element of G is conjugate to $e^{2\pi i x}$ for a unique element $x \in F$.
- Each W_{aff} orbit in $\mathfrak{t}$ has a unique element in F.
- For $x \in F$ the stabilizer of x in W_{aff} is the subgroup of W_{aff} generated by the reflections r_j, $j = 0, 1, \dots, n$, for which x is on the wall H_j. In particular, all $x \in F^\circ$ have trivial stabilizer in W_{aff}.

The *fundamental co-weights* are the elements $\check{\omega}_k \in \mathfrak{t}$ dual to the simple roots: $\langle \alpha_j, \check{\omega}_k\rangle = \delta_{jk}$. They form a $\mathbb{Z}$-basis of $\check{P}$. In terms of them we can write F as the convex hull of

$$\left\{0, \frac{\check{\omega}_1}{m_1}, \frac{\check{\omega}_2}{m_2}, \dots, \frac{\check{\omega}_n}{m_n}\right\}.$$

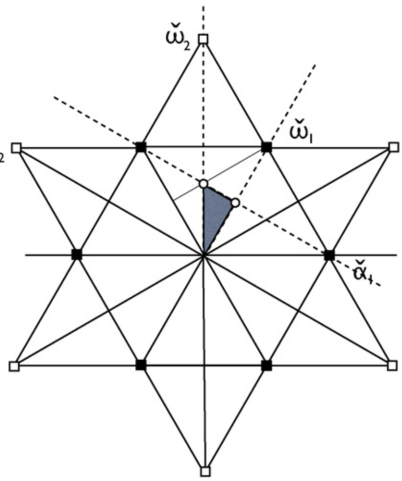

Fig. 2. A schematic view of the co-root system of G_2. The shaded triangle is the fundamental region F. The dotted lines are the mirrors which define its boundaries, the reflections in which generate the affine Weyl group. The action of the affine Weyl group on F tiles the plane. A few tiles of this tiling are shown. Filled (resp. open) squares are the short (resp. long) co-roots of G_2.

The Cartan matrix of G is the integer $n \times n$ matrix

$$A = (A_{ij}) = \left(\frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)}\right), \quad 1 \leqslant i, j \leqslant n.$$

It is unique up to the numbering of simple roots. The Cartan matrices classify the compact simply-connected simple Lie groups into the well-known A, B, C, D series and the five exceptional groups E_6, E_7, E_8, F_4, and G_2.

The simple *co-roots* $\{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \subset \check{Q}$ corresponding to $\{\alpha_1, \ldots, \alpha_n\} \subset Q$ are defined by

$$\langle \alpha_i, \check{\alpha}_j \rangle = A_{ij} = \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)} \quad \text{for all } i, j.$$

They form a $\mathbb{Z}$-basis of $\check{Q}$ and their W-translates in $\check{Q}$ form the set of co-roots $\check{\Delta}$. Actually $\check{\Delta}$ and the simple co-roots are a root system and simple roots for a simply-connected group $\check{G}$ whose Cartan matrix is A^T. We do not need this group directly in what follows but we occasionally use information about co-objects that we know is true from the fact that they have such an interpretation. In particular we have the co-marks $\check{m}_1, \ldots, \check{m}_n$ arising from the lowest co-root written in terms of the simple co-roots $\check{\alpha}_1, \ldots, \check{\alpha}_n$.

Dual to the co-roots we have the *fundamental weights* $\omega_j \in P$ defined by $\langle \omega_j, \check{\alpha}_k \rangle = \delta_{jk}$. The fundamental weights form a $\mathbb{Z}$-basis of P.

Each finite-dimensional irreducible representation L of G has a unique 1-dimensional weight space L^λ, $\lambda \in P$, with the property that all other weights of L are of the form $\lambda - \beta$, where β is a sum of positive roots. We have

$$\lambda \in P^+ := \left\{\mu \in \mathfrak{t}^* \,\middle|\, \langle \mu, \check{\alpha}_i \rangle \in \mathbb{Z}^{\geqslant 0},\ i = 1, \ldots, n\right\}.$$

Here $P^+ \subset P$ is the set of *dominant weights*. The dominant weight λ is called the *highest weight* of L. If we designate L now by $L(\lambda)$, then the correspondence

$$\lambda \in P^+ \leftrightarrow L(\lambda)$$

classifies all the irreducible finite-dimensional representations of G up to isomorphism.

We shall also need the set of *strictly dominant weights*

$$P^{++} := \{\mu \in \mathfrak{t}^* \mid \langle \mu, \check{\alpha}_i \rangle \in \mathbb{Z}^{>0},\ i = 1, \ldots, n\}.$$

The 'simplest' strictly dominant weight is ρ defined by $\langle \rho, \check{\alpha}_i \rangle = 1$ for $i = 1, \ldots, n$. The element ρ plays an important role in what follows. We also know that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{k=1}^{n} \omega_k. \tag{7}$$

The character of a finite-dimensional representation L of G is the mapping

$$g \mapsto \mathrm{tr}_L\, g, \quad g \in G.$$

Since a character is unaffected by conjugation by group elements, we can always restrict it to $\mathbb{T}$ without any loss of information. We then further consider it as a function χ_L on $\mathfrak{t}$ by

$$\chi_L : x \mapsto \mathrm{tr}_L \exp(2\pi i x) = \sum_{\mu} \dim L^{\mu} e^{2\pi i \langle \mu, x \rangle},$$

where μ runs over the weights of L. In particular, we have the characters $\chi_\lambda = \chi_{L(\lambda)}$, $\lambda \in P^+$.

Weyl's character formula is

$$\chi_\lambda(x) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle w(\lambda+\rho), x \rangle}}{\sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle w\rho, x \rangle}} =: \frac{S_{\lambda+\rho}(x)}{S_\rho(x)}, \quad \text{for all } x \in \mathfrak{t}. \tag{8}$$

Also there is the special product formula for the denominator:

$$S_\rho(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle w\rho, x \rangle} = \prod_{\alpha \in \Delta_+} \left(e^{\pi i \langle \alpha, x \rangle} - e^{-\pi i \langle \alpha, x \rangle} \right). \tag{9}$$

The functions $S_{\lambda+\rho}(x)$ and $S_\rho(x)$ are W-skew invariant whereas their quotient is W-invariant. The functions $S_{\lambda+\rho}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle w(\lambda+\rho), x \rangle}$ are called S-functions in [7,13,14] due to their similarity in form to the sine function, which they are in the case of A_1. In any case the values of the characters and of the S-functions are determined by their values on the fundamental domain F, and it is this fact that becomes the centre of our attention in the sequel.

Note that none of the reflecting hyperplanes (5) or (6), nor indeed any reflecting hyperplane of W_{aff}, meets the interior F° of F and so $S_\rho(x)$ is never 0 in F°. Thus $|S_\rho^2(x)|^2$ is positive on the interior F° of F. On the other hand, $S_{\lambda+\rho}(x)$ and $S_\rho(x)$ vanish on the boundary of F since $x \in H_j$ implies that $r_j x = x$ while replacing x by $r_j x$ in any S-function changes its sign.

3. *W*-invariant and *W*-skew invariant functions on $\mathbb{T}$

3.1. The algebra of formal exponentials

Starting with the weight lattice P one may form the algebra $\mathbb{C}[P]$ of formal exponentials, which has a $\mathbb{C}$-basis of symbols e^{λ}, $\lambda \in P$, together with a multiplication defined by bilinear extension of the rule

$$e^{\lambda}e^{\mu} = e^{\lambda+\mu}.$$

Thus typical elements of $\mathbb{C}[P]$ are finite complex linear combinations $\sum c_{\lambda}e^{\lambda}$. $\mathbb{C}[P]$ is an unique factorization domain and its group of invertible elements are the elements ce^{λ}, where $c \neq 0$, $\lambda \in P$.

W acts on P and hence as a linear operator and even an automorphism on $\mathbb{C}[P]$ by $w.e^{\lambda} = e^{w\lambda}$. There is a partial order on P with $\lambda \geqslant \mu$ if and only if $\mu = \lambda - \beta$ where β is a sum (possibly empty) of positive roots. This is important because in the weight systems of irreducible representations of G the highest weight is highest in this sense. We will use this notion of highest below.

There are some advantages to introducing formal exponentials at times since they often clarify the mathematics. However, we are really interested in their manifestations as functions on $\mathfrak{t}$ and on $\mathbb{T}$ which arise from

$$x \mapsto e^{2\pi i x} \mapsto e^{2\pi i\langle\lambda,x\rangle}. \tag{10}$$

These mappings are the characters $\phi_{\lambda} : x \mapsto \exp(2\pi i\langle\lambda, x\rangle)$ of the torus, so $\mathbb{C}[P]$, as functions, is its character ring. This relates directly back to the previous section where we have defined the G-characters and S-functions, all of which may be viewed as arising from corresponding elements $\chi_{\lambda}, S_{\lambda+\rho}$ of $\mathbb{C}[P]$.

At the level of functions the action of W is completely consistent:

$$w.\phi_{\lambda}(x) = \phi_{\lambda}\big(w^{-1}x\big) = \exp\big(2\pi i\lambda.w^{-1}x\big) = \exp 2\pi i w\lambda.x = \phi_{w\lambda}(x).$$

We are most interested in the subring $\mathbb{C}[P]^W$ of W-invariant elements. The simplest forms of W-invariant functions are the orbit sums $\sum_{w\in W}\phi_{w\lambda}$, called C-functions in [15,6,13,14] due to their similarity in form to the cosine function (which they are in type A_1). More relevant here are the characters χ_{λ} of G (restricted to $\mathbb{T}$) already introduced in Section 2:

$$\chi_{\lambda} = \sum_{\mu\in P}\dim L(\lambda)^{\mu}e^{2\pi i\mu} \tag{11}$$

where $L(\lambda)$ is the irreducible representation of highest weight $\lambda \in P^{+}$ and $\dim L(\lambda)^{\mu}$ is the dimension of its μ-weight space. These are W-invariant and they form a basis for $\mathbb{C}[P]^W$.

Proposition 3.1. *(See [1, Ch. VI].) $\mathbb{C}[P]^W$ is a polynomial ring with the fundamental characters $\chi_{\omega_1}, \ldots, \chi_{\omega_n}$ as the generators.*[2]

[2] The orbit sums over the various dominant weights also form a basis for $\mathbb{C}[P]^W$ and those for $\omega_1, \ldots, \omega_n$ also form a set of generators for it as a polynomial ring. The relationship between orbit sums and corresponding characters is a triangular matrix of integers with 1s down the diagonal [2].

This result really underlies the results of this paper. It says that the fundamental characters of G can be used as new variables by which the algebra of invariant elements of $\mathbb{C}[P]$ becomes a polynomial ring in these variables. It is in working out the Fourier and functional analysis implied by this statement that the cubature formulae arise.

In the sequel we prefer to reserve the word *character* for the characters of G (as opposed to the characters of $\mathbb{T}$) since they are of fundamental importance to the paper.

3.2. Skew invariants elements of $\mathbb{C}[P]$

Elements $\xi \in \mathbb{C}[P]$ for which $w.\xi = (-1)^{l(w)}\xi$ for all $w \in W$ are called *skew-invariants*. They play a vital role in the paper. The simplest example is S_ρ, and it is the foundation for all the skew-invariant elements.

Proposition 3.2. *(See [1, Ch. VI].) $S_\rho \mathbb{C}[P]^W$ is the set of all W-skew-invariant elements of $\mathbb{C}[P]$.*

Later on, when we use the basic characters χ_{ω_j} as new variables X_j and have polynomial functions of the X_j, we shall have need of a Jacobian for the switch of variables from the X_j back to the variables that parameterize $\mathbb{T}$. We establish the key result here.

For each $\check{\alpha} \in \check{Q}$ there is a unique derivation[3] $D_{\check{\alpha}}$ on $\mathbb{C}[P]$ satisfying

$$D_{\check{\alpha}} e^\lambda := D_{\check{\alpha}}\big(e^\lambda\big) = \langle \lambda, \check{\alpha}\rangle e^\lambda$$

for all λ. $D_{\check{\alpha}}$ is linear in $\check{\alpha}$.

Let $\chi_{\omega_1}, \dots, \chi_{\omega_n}$ be the basic characters and let $\check{\alpha}_1, \dots, \check{\alpha}_n$ be the standard basis of $\check{Q}$ dual to $\{\omega_1, \dots, \omega_n\}$. Let J be the matrix with entries $J_{jk} = D_{\check{\alpha}_j}\chi_{\omega_k}$.

Proposition 3.3 *(Steinberg[4]).*

$$\det(J) = S_\rho.$$

Proof. All the exponentials in χ_{ω_k} are of the form $e^{\omega_k - \beta_k}$ where β_k is a sum of positive roots, and the highest term is e^{ω_j}. Now $D_{\check{\alpha}_j}\chi_{\omega_k}$ is a sum of exponentials of the same form, but since $\langle \omega_j, \check{\alpha}_k\rangle = \delta_{jk}$, the highest terms only survive along the diagonal of J. Thus when we compute the determinant we obtain a sum of signed products $e^{\omega_1-\beta_1} \dots e^{\omega_n - \beta_n}$ and only the term from the diagonal can contribute an exponential of the form $e^{\omega_1} \dots e^{\omega_n}$, and its coefficient is 1. Thus $\det(J)$ is an element of $\mathbb{C}[P]$ whose highest term is $\omega_1 + \dots + \omega_n = \rho$ and this occurs with coefficient equal to 1.

We shall prove that $\det(J)$ is W-skew invariant. Then by Proposition 3.2 it is a multiple of S_ρ. Because all the weights in the expansion of $\det(J)$ are less than or equal ρ, this multiple can only be a scalar. Since the leading coefficient is 1 in both cases, $\det(J) = S_\rho$, as we wish to prove.

A simple computation shows that $wD_{\check{\alpha}}w^{-1} = D_{w\check{\alpha}}$ for all $\check{\alpha}$. Fix any $l = 1, \dots, n$ and let $r_l = r_{\check{\alpha}_l}$. We have $r_l\check{\alpha}_j = \check{\alpha}_j - A_{lj}\check{\alpha}_l$. Then,

$$\begin{aligned}
r_l \det(J) &= \det\big(r_l(J)\big) = \det\big(\big(r_l(D_{\check{\alpha}_j}\chi_{\omega_k})\big)\big) \\
&= \det\big(\big(r_l D_{\check{\alpha}_j} r_l(r_l\chi_{\omega_k})\big)\big) = \det((D_{r_l\check{\alpha}_j}\chi_{\omega_k})) \\
&= \det\big(\big(D_{\check{\alpha}_j - A_{lj}\check{\alpha}_l}(\chi_{\omega_k})\big)\big) = \det\big(\big((D_{\check{\alpha}_j} - A_{lj}D_{\check{\alpha}_l})(\chi_{\omega_k})\big)\big) \\
&= \det\big(\big(D_{\check{\alpha}_j}(\chi_{\omega_k}) - A_{lj}D_{\check{\alpha}_l}(\chi_{\omega_k})\big)\big),
\end{aligned}$$

[3] A derivation of $\mathbb{C}[P]$ is a linear mapping $D : \mathbb{C}[P] \to \mathbb{C}[P]$ satisfying $D(\phi\xi) = D(\phi)\xi + \phi D(\xi)$ for all $\phi, \xi \in \mathbb{C}[P]$.

[4] This result is an Exercise to Ch. VI in [1] and attributed to R. Steinberg there.

where in the second line we used the W-invariance of the characters. The operation has resulted in altering all rows except the lth row by a multiple of the lth row (which does not alter the determinant) and replacing the lth row by its negative, which changes the sign of the determinant. Thus $r_l \det(J) = -\det(J)$, which gives the desired skew-symmetry. □

3.3. An inner product on $\mathbb{C}[P]^W$

The natural inner product on $\mathbb{C}[P]$ is $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ defined by

$$\langle f, g\rangle_{\mathbb{T}} = \int_{\mathbb{T}} f\bar{g}\, d\theta_{\mathbb{T}},$$

where the $\theta_{\mathbb{T}}$ is the normalized Haar measure of the torus. Relative to this the functions ϕ_λ form an orthonormal basis. Using this inner product we can complete $\mathbb{C}[P]$ in the corresponding L^2-norm to the Hilbert space $L^2(\mathbb{T}, \theta_{\mathbb{T}})$ with the normalized ϕ_λ forming an orthonormal basis, in the sense of Hilbert spaces. Of course we can look at the closure of $\mathbb{C}[P]^W$ in $L^2(\mathbb{T}, \theta_{\mathbb{T}})$, which is in fact the subspace of W-invariant elements $L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$ of $L^2(\mathbb{T}, \theta_{\mathbb{T}})$.

However, the inner product $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ is not ideal for this subspace, and rather we would like to find one with respect to which the characters χ_λ form an orthonormal base.

We note that for any $f \in L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$, $f S_\rho \in L^2(\mathbb{T}, \theta_{\mathbb{T}})$, and it is skew-invariant with respect to W. Form its Fourier expansion

$$f S_\rho = \sum_{\mu} \langle f S_\rho, \phi_\mu\rangle_{\mathbb{T}} \phi_\mu,$$

equality being in the L^2 sense. The summands can be gathered together into W-orbits, and on each orbit the coefficients $\langle f S_\rho, \phi_\mu\rangle_{\mathbb{T}}$ are equal in absolute value and alternate in sign according to the parity of the Weyl group elements. The only orbits that do not vanish are those containing a weight $\mu \in P^{++}$, and we get

$$f S_\rho = \sum_{\lambda \in P^+} \langle f S_\rho, \phi_{\lambda+\rho}\rangle_{\mathbb{T}} S_{\lambda+\rho}.$$

Dividing out the function S_ρ, which is valid as long as the functions are restricted to the interior of the fundamental chamber F°, we obtain

$$f = \sum_{\lambda \in P^+} \langle f S_\rho, \phi_{\lambda+\rho}\rangle_{\mathbb{T}} \chi_\lambda. \tag{12}$$

Now, using the W-invariance of $\theta_{\mathbb{T}}$ and the skew-invariance of $f S_\rho$, we have

$$\begin{aligned}\langle f S_\rho, \phi_{\lambda+\rho}\rangle_{\mathbb{T}} &= \int_{\mathbb{T}} f S_\rho \overline{\phi_{\lambda+\rho}}\, d\theta_{\mathbb{T}} = \frac{1}{|W|}\int_{\mathbb{T}} \sum_{w\in W} (-1)^w f S_\rho \overline{\phi_{w(\lambda+\rho)}}\, d\theta_{\mathbb{T}} \\ &= \frac{1}{|W|}\int_{\mathbb{T}} f S_\rho \overline{S_{\lambda+\rho}}\, d\theta_{\mathbb{T}} = \frac{1}{|W|}\int_{\mathbb{T}} f\,\overline{\chi_\lambda} S_\rho \overline{S_\rho}\, d\theta_{\mathbb{T}} = \int_{F^\circ} f\,\overline{\chi_\lambda} S_\rho \overline{S_\rho}\, d\theta_{\mathbb{T}}. \end{aligned} \tag{13}$$

Thus (12) and (13) show that $f \in L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$ has a Fourier expansion in terms of the characters χ_λ, $\lambda \in P^+$, with coefficients given by a new inner product defined on $L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$ by

$$(f, g) \mapsto \int_{F^\circ} f \bar{g} S_\rho \overline{S_\rho}\, d\theta_{\mathbb{T}}. \tag{14}$$

We can then rewrite (12) as

$$f = \sum_{\lambda \in P^+} (f, \chi_\lambda) \chi_\lambda. \tag{15}$$

We shall use these results in Section 8.

4. Elements of finite order

Elements of finite order [11] (EFOs) in G are used to create the nodes for the discrete Fourier analysis [12] and cubature formulae to follow.

We have seen in Section 2 that every element of G is conjugate to one of the form $g = \exp(2\pi i x)$ where $x \in F$. The element g is called *regular* if its centralizer is of dimension n, the rank of G. Since each element lies in an n-torus, n is the smallest possible dimension for a centralizer. Regularity is a property of the entire conjugacy class of an element and for $x \in F$ it is equivalent to saying that $x \in F^\circ$.

The condition that g has finite order dividing M is the equivalent to the condition that $\exp(2\pi i x)^M = \exp(2\pi i M x)$ acts trivially on every irreducible representation, and for this all we need is that it acts trivially on every weight space $L(\lambda)^\mu$. In turn this requires precisely that $\langle \mu, Mx\rangle \in \mathbb{Z}$ for all weights of P, and finally it is equivalent to $Mx \in \check{Q}$, since $\check{Q}$ is the $\mathbb{Z}$-dual of P.

In fact what we are going to need here is not that $x \in \frac{1}{M}\check{Q}$ but rather that

$$x \in \frac{1}{M}\check{P},$$

a statement that is equivalent to saying that $Ad(g)^M = 1$, i.e. g^M acts trivially in the adjoint representation. In this case we shall say that g has *Ad-order* M or *adjoint order* M, even though the actual adjoint order, which we shall call the *strict* adjoint order, namely the *least* N for which $Ad(g)^N = 1$, may be some proper divisor of M. We also say that $x \in \mathfrak{t}$ is an element of adjoint order M if $\exp(2\pi i x)$ is of adjoint order M.

Given the definition above, the conjugacy classes of elements of adjoint order M are represented by the points x of the form

$$x = \frac{1}{M}\sum_{j=1}^{n} s_j \check{\omega}_j \tag{16}$$

where

$$s_j \in \mathbb{Z}^{\geqslant 0} \quad \text{for all } j \quad \text{and} \quad \langle -\alpha_0, x\rangle = \frac{1}{M}\sum_{j=1}^{n} m_j s_j \leqslant 1. \tag{17}$$

The regular conjugacy classes of adjoint order M are represented by (17) where the inequalities are made strict. We write F_M (resp. F_M°) for the elements of F (resp. F°) of Ad-order M.

Using $m_0 = 1$ defined in Section 2 we can define $s_0 \in \mathbb{Z}^{\geqslant 0}$ so that

$$\sum_{j=0}^{n} m_j s_j = M. \tag{18}$$

Listing all the elements of F_M (resp. F_M°) is then just a question of finding all non-negative (resp. positive) integer solutions $[s_0, s_1, \ldots, s_n]$ to (18). We call $[s_0, s_1, \ldots, s_n]$ the *Kac coordinates* of x.

We will be particularly interested in the set F_{M+h} of elements $x \in F$ of *Ad*-order $M + h$ for some non-negative integer M:

$$x = \frac{1}{M+h}(s_1\check{\omega}_1 + s_2\check{\omega}_2 + \cdots + s_n\check{\omega}_n) \in \frac{1}{M+h}\check{P},$$
$$\text{where } s_1, \ldots, s_n \in \mathbb{Z}^{\geqslant 0} \text{ and } \sum_{j=1}^{n} m_j s_j \leqslant M + h. \tag{19}$$

Alternatively we have the Kac coordinates $[s_0, s_1, \ldots, s_n]$.

Each of the following three conditions assures that x of (19) is in F_{M+h}°:

$$s_j > 0, \quad j = 0, 1, \ldots, n;$$
$$\sum_{j=0}^{n} m_j t_j = M, \quad t_j := s_j - 1 \geqslant 0;$$
$$\sum_{j=1}^{n} m_j t_j \leqslant M \quad (\text{so } t_0 \text{ completes the sum to } M). \tag{20}$$

When $M = 0$, it contains only the element given by $s_0 = s_1 = \cdots = s_n = 1$. For $M \geqslant 0$, it clearly contains $|F_{M+h}^\circ| = |F_M|$ points. Formulae for the cardinality of $|F_{M+h}|$ have been worked out for all M and for all simple G in [4].

5. Points of F_{M+h}° as zeros of S-functions

We are now at a point where we begin the main development of the paper. We fix, once and for all a non-negative integer M. The first step is to show that the points of F_{M+h}° are common zeros of a certain set of S-functions. These points are the nodes for the cubature formulae to follow.

Consider a dominant weight $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$. We want to find points $x \in F_{M+h}^\circ$ at which the S-function $S_{\lambda+\rho}(x)$ vanishes:

$$S_{\lambda+\rho}(x) = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle w(\lambda+\rho), x\rangle} = \sum_{w \in W} (-1)^{l(w)} e^{2\pi i \langle \lambda+\rho, w^{-1}x\rangle} = 0.$$

One way to make this happen is to have

$$\langle \lambda + \rho, x\rangle - \langle \lambda + \rho, rx\rangle \in \mathbb{Z}, \quad \text{for all } x \in \frac{1}{M+h}\check{P}, \tag{21}$$

where r is the reflection in the highest coroot, for if this is the case then the sum collapses in pairs adding up to zero. Via $\langle w\alpha, \check{\beta}\rangle = \langle \alpha, w^{-1}\check{\beta}\rangle$ we obtain that r appears as the reflection in some root

Table 1
The m-degrees of the polynomial variables $X_1, \dots, X_n$ and of the functions S_ρ and K. The Coxeter number $h = 1 + \check{m}_1 + \cdots + \check{m}_n$.

variable	X_1	X_2	$\cdots$	X_n	S_ρ	K
m-degree	$\check{m}_1$	$\check{m}_2$	$\cdots$	$\check{m}_n$	$h-1$	$2h-2$

γ on the root/weight side of the picture: $rx = x - \langle \gamma, x\rangle \check{\alpha}_0$. The condition (21) is equivalent to

$$\langle \gamma, x\rangle \langle \lambda + \rho, \check{\alpha}_0\rangle \in \mathbb{Z}, \quad \text{for all } x \in \frac{1}{M+h}\check{P}.$$

Note that $\langle \gamma, x\rangle \in \frac{1}{M+h}\mathbb{Z}$ since $\langle \gamma, \check{P}\rangle \subset \mathbb{Z}$, so we only need $\langle \lambda + \rho, \check{\alpha}_0\rangle \in (M+h)\mathbb{Z}$.

The simplest case is to look for $\langle \lambda + \rho, -\check{\alpha}_0\rangle = M + h$, that is,

$$\sum_{j=1}^{n} (\lambda_j + 1)\langle \omega_j, -\check{\alpha}_0\rangle = \sum_{j=1}^{n} (\lambda_j + 1)\check{m}_j = M + h,$$

or equivalently,

$$\sum_{j=1}^{n} \lambda_j \check{m}_j = M + 1. \tag{22}$$

All solutions $\lambda = (\lambda_1, \dots, \lambda_n)$ to (22), where the $\lambda_j \in \mathbb{Z}^{\geqslant 0}$, lead to S-functions $S_{\lambda+\rho}$ that are zero at all EFOs of Ad-order $M + h$ in the interior of the fundamental domain.

6. Introducing the polynomial functions

Following [14], assign variables X_j to the characters for weights ω_j. Thus we have polynomial variables

$$X_1, X_2, \dots, X_n, \quad \text{where } X_j := \chi_{\omega_j}(x),\ x \in F^\circ. \tag{23}$$

With these we can introduce the domain

$$\Omega := \big\{\big(X_1(x), \dots, X_n(x)\big)\colon x \in F^\circ\big\} \subset \mathbb{C}^n. \tag{24}$$

We shall soon see that this is actually an open subset of a real n-dimensional space and eventually it will be the natural domain of the real-valued functions of the variables $X_1, \dots, X_n$ that we wish to study.

Define the *m-degree*[5] of the variables by assigning degree $\check{m}_j$ to X_j (see Table 1).

Then the monomials $X_1^{\lambda_1} \dots X_n^{\lambda_n}$ of m-degree $\leqslant M$ are those satisfying

$$\lambda_1 \check{m}_1 + \cdots + \lambda_n \check{m}_n \leqslant M \tag{25}$$

where $\lambda_1 \geqslant 0, \dots, \lambda_n \geqslant 0$. Although the marks and co-marks are not necessarily identical, see Fig. 1, they are at worst simply permutations of each other. Thus (25) has the same number of solutions

[5] This might be more properly called the $\check{m}$-degree, but this seems a bit cumbersome.

$(\lambda_1, \ldots, \lambda_n)$ as we saw before in (20), namely $|F^\circ_{M+h}| = |F_M|$. The constant polynomials are those of m-degree 0.

In keeping with this notation, we will say also that $\lambda = (\lambda_1, \ldots, \lambda_n)$ has m-degree equal to

$$\lambda_1 \check{m}_1 + \cdots + \lambda_n \check{m}_n = \langle \lambda, -\check{\alpha}_0 \rangle. \tag{26}$$

Theorem 6.1. *The number of monomials $\mathbb{C}[X_1, \ldots, X_n]$ of m-degree $\leqslant M$ is equal to the number of regular EFOs of Ad-order $M + h$ in the fundamental chamber. Each of the regular EFOs of Ad-order $M + h$ in the fundamental chamber is a common zero of all the S-functions $S_{\lambda+\rho}$ and all the character functions χ_λ for which λ has m-degree equal to $M + 1$.*

The trick that we have used above of using the internal reflective anti-symmetry to construct common zeros is taken from [8]. It is remarkable that in the case of type A_n root lattices it actually finds *all* the common zeros. The proof of this makes essential use of the fact that the new variables $X_1, \ldots, X_n$ are all of degree 1, something that is true only for type A_n. In fact our example of type G_2 in Section 9 indicates that this result does not hold in general.

The 'smallest' S-function is the one defined by the strictly dominant weight of lowest m-degree, namely ρ of (7) with m-degree $h - 1$. Writing S_ρ in its well-known form (9), we note that $\overline{S_\rho} = (-1)^{|\Delta_+|} S_\rho$. Thus

$$|S_\rho|^2 = S_\rho \bar{S}_\rho = (-1)^{|\Delta_+|} S_\rho^2,$$

and we note that this function is positive on all of F° and vanishes on its boundary. $S_\rho \bar{S}_\rho$ is a W-invariant function and so is expressible as a polynomial in the basic characters $\chi_{\omega_j} = X_j$. We then have the corresponding strictly positive function K on Ω:

$$K(X_1, \ldots, X_n) = S_\rho \bar{S}_\rho(x), \quad x \in F^\circ. \tag{27}$$

We note here that if $\mu \in P^{++}$ then

$$\bar{S}_\mu = \sum_{w \in W} (-1)^{l(w)} e^{-w\mu} = \sum_{w \in W} (-1)^{l(w)} (-1)^{l(w_{\mathrm{opp}})} e^{-w w_{\mathrm{opp}} \mu} = (-1)^{|\Delta_+|} S_{-w_{\mathrm{opp}} \mu},$$

where w_{opp} is the opposite involution in W (since w_{opp} is a product of $|\Delta_+|$ reflections).

The opposite involution interchanges the positive roots (resp. positive coroots) with the negative ones, and, since μ is dominant so is $-w_{\mathrm{opp}}\mu$. Of course w_{opp} is not always simply the negation operator, see Table 2. Still, it does simply change the sign of the highest positive root (resp. highest coroot). Thus we have the important little equation

$$m\text{-}\deg(\mu) = \langle \mu, -\check{\alpha}_0 \rangle = \langle w_{\mathrm{opp}}\mu, -w_{\mathrm{opp}}\check{\alpha}_0 \rangle = \langle w_{\mathrm{opp}}\mu, \check{\alpha}_0 \rangle = m\text{-}\deg(-w_{\mathrm{opp}}\mu). \tag{28}$$

This is useful because it means that $(-1)^{|\Delta_+|}\overline{S_\mu}$ and $\overline{\chi_\mu}$ are just another S-function and another group character respectively, and the highest weight involved in each case has the same m-degree as before conjugation. In particular conjugation of the characters χ_{ω_j} can at worst permute some of them, say $\chi_{\omega_j} \mapsto \chi_{\sigma(\omega_j)}$ by some permutation σ of order 2 of the indices $\{1, \ldots, n\}$. Table 2 shows what happens in the cases when σ is not just the identity permutation.

If we let

$$\mathfrak{R} = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \bar{z} = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \}, \tag{29}$$

Table 2
Correspondence of the variables X_j to $\overline{X}_j$ $(j = 1, \dots, n)$ produced by the action of $-w_{\mathrm{opp}}$. In all other cases $\overline{X_j} = X_j$.

A_n $(n > 1)$	X	X_1	X_2		$\cdots$	X_{n-1}	X_n	
	$\overline{X}$	X_n	X_{n-1}		$\cdots$	X_2	X_1	
D_{2n+1} $(n > 1)$	X	X_1	X_2	$\cdots$	X_{2n-1}	X_{2n}	X_{2n+1}	
	$\overline{X}$	X_1	X_2	$\cdots$	X_{2n-1}	X_{2n+1}	X_{2n}	
E_6	X	X_1	X_2	X_3	X_4	X_5	X_6	
	$\overline{X}$	X_5	X_4	X_3	X_2	X_1	X_6	

then $\mathfrak{R}$ is a real space of dimension n and $\Omega \subset \mathfrak{R}$ is an n-dimensional subdomain, as follows from the non-vanishing of the Jacobian S_ρ on F° (32). This is the space on which we shall think of $X_1, \dots, X_n$ as real variables.

Define

$$\begin{aligned} \xi : \mathbb{T} &\to \mathfrak{R}, \\ x &\mapsto X(x) := \big(X_1(x), \dots, X_n(x)\big), \end{aligned} \tag{30}$$

so we have $\xi(F^\circ) = \Omega$. We define

$$\mathfrak{F}_{M+h} = \xi\big(F^\circ_{M+h}\big).$$

Remark 1. As we have seen, conjugation actually permutes some of the basic variables X_j. We shall use the overline symbol to indicate this form of conjugation. Thus one should understand the conjugation symbol as having this dual meaning of actual complex conjugation when the X_j are treated as functions on $\mathbb{T}$ and as the permutation σ when treated as the coordinate variables of $\mathfrak{R}$. Thus we shall write $\overline{cX_{j_1} \dots X_{j_r}}$ (where $c \in \mathbb{C}$) to mean $\bar{c}\overline{X_{j_1}} \dots \overline{X_{j_r}}$, understanding that the $\overline{X}_j$ has this dual meaning. For a polynomial $g(X_1, \dots, X_n) = \sum c_{j_1,\dots,j_n} X_1^{j_1} \dots X_n^{j_n}$, $\overline{g(X_1, \dots, X_n)} := \sum \overline{c_{j_1,\dots,j_n}}\, \overline{X_1}^{j_1} \dots \overline{X_n}^{j_n} = \bar{g}(\overline{X_1}^{j_1} \dots \overline{X_n}^{j_n})$.

Notice that since

$$X_j \overline{X_k} K \leftrightarrow \chi_{\omega_j} \overline{\chi_{\omega_k}} S_\rho \overline{S_\rho} = S_{\omega_j+\rho} \overline{S_{\omega_k+\rho}}$$

and

$$\langle \omega_j + \rho, -\check{\alpha}_0 \rangle = \check{m}_j + h - 1 \quad \text{and} \quad \big\langle -w_{\mathrm{opp}}(\omega_k + \rho), -\check{\alpha}_0 \big\rangle = \check{m}_k + h - 1,$$

we understand that K has m-degree equal to $2h - 2$. This is indicated in Table 1.

7. The integration formula

We wish to study weighted integrals of the form

$$\int_\Omega f \bar{g} K^{1/2} \, dX_1 \dots dX_n,$$

where f, g are functions of the variables $X_1, \dots, X_n$ defined on Ω. These are related back to $\mathfrak{t}$ (more specifically to F°) and the torus $\mathbb{T}$ via the defining Eqs. (23).

7.1. The key integration formula

Natural variables for $\mathfrak{t}$ are $x = (x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j \check{\alpha}_j$ where the x_j run over $[0,1) \times \cdots \times [0,1)$.

The derivation $D_{\check{\alpha}_j}$ on $\mathbb{C}[P]$, $D_{\check{\alpha}_j} e^{\lambda} = \langle \lambda, \check{\alpha}_j \rangle e^{\lambda}$ introduced in Section 3.2 is, when $\mathbb{C}[P]$ is treated as an algebra of functions on $\mathbb{T}$, the mapping

$$\begin{aligned} D_{\check{\alpha}_j} e^{\langle \lambda, 2\pi i x\rangle} &= D_{\check{\alpha}_j} e^{\langle \lambda, 2\pi i \sum x_k \check{\alpha}_k\rangle} \\ &= \langle \lambda, \check{\alpha}_j \rangle e^{\langle \lambda, 2\pi i x\rangle} = \frac{1}{2\pi i} \frac{d}{dx_j} e^{\langle \lambda, 2\pi i x\rangle}. \end{aligned} \tag{31}$$

Using Proposition 3.3 we then see that the Jacobian of the transformation of the variables x to variables X is

$$\left|(2\pi i)^n S_{\rho}(x)\right| = (2\pi)^n \left|S_{\rho}(x)\right|. \tag{32}$$

Thus from the definition of K we have

$$\begin{aligned} &\int_{\Omega} f \bar{g} K^{1/2} \, dX_1 \ldots dX_n \\ &\quad = \int_{\Omega} f(X_1, \ldots, X_n) \overline{g(X_1, \ldots, X_n)} K^{1/2}(X_1, \ldots, X_n) \, dX_1 \ldots dX_n \\ &\quad = (2\pi)^n \int_{F^\circ} f\big(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\big) \bar{g}\big(\overline{\chi_{\omega_1}}(x), \ldots, \overline{\chi_{\omega_n}}(x)\big) S_{\rho}(x) \overline{S_{\rho}}(x) \, dx_1 \ldots dx_n \end{aligned} \tag{33}$$

for all functions f, g are in the variables $X_1, \ldots, X_n$ on Ω.

Theorem 7.1. *Let M be a positive integer. Then for all polynomials $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ with $m\text{-}\deg(f) \leqslant M+1$ and $m\text{-}\deg(g) \leqslant M$ we have*

$$\begin{aligned} &\int_{\Omega} f \bar{g} K^{1/2} \, dX_1 \ldots dX_n \\ &\quad = (2\pi)^n \int_{F^\circ} f\big(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\big) \overline{g\big(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\big)} S_{\rho}(x) \overline{S_{\rho}}(x) \, dx_1 \ldots dx_n \\ &\quad = \frac{1}{c_G} \left(\frac{2\pi}{M+h}\right)^n \sum_{x \in F^\circ_{M+h}} f\big(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\big) \overline{g\big(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\big)} S_{\rho}(x) \overline{S_{\rho}}(x). \end{aligned} \tag{34}$$

This theorem is proved in Section 7.5.

7.2. The cubature formula

For any function f defined on Ω, let $\widetilde{f}$ be defined by $\widetilde{f}(x) = f(\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x))$.

Theorem 7.2. *Let M be a non-negative integer. Then for all polynomials $f \in \mathbb{C}[X_1, \ldots, X_n]$ with $m\text{-}\deg(f) \leqslant 2M+1$ we have*

$$\int_{\Omega} f K^{1/2}\, dX_1 \ldots dX_n = \frac{1}{c_G}\left(\frac{2\pi}{M+h}\right)^n \sum_{X \in \mathfrak{F}_{M+h}} f(X)K(X)$$
$$= \frac{1}{c_G}\left(\frac{2\pi}{M+h}\right)^n \sum_{x \in F^{\circ}_{M+h}} \widetilde{f}(x)\widetilde{K}(x). \tag{35}$$

Eq. (35) is the **cubature formula**. The points $x \in F^{\circ}_{M+h} = (\frac{1}{M+h}\check{P}) \cap F^{\circ}$, or more precisely their images under ξ, are the nodes. These points are common zeros of the character functions of the m-degree $M+1$. We shall also refer to the elements of F°_{M+h} as the nodes. The coefficients of the interpolation are the values of K or $\widetilde{K} = |S_\rho|^2$ at the nodes. Theorem 7.2 is proved in exactly the same way as Theorem 7.1, with λ now satisfying $m\text{-}\deg(\lambda) \leqslant 2M+1$ and μ replaced by 0.

We first prove a key lemma.

7.3. Separation lemma

Lemma 7.3. *If $\phi = \sum_{j=1}^n \phi_j \omega_j \in P^+$ and $\phi \neq 0$, and if $m\text{-}\deg(\phi) < 2(M+h)$, then $\phi \notin (M+h)Q$.*

Proof. Suppose by way of contradiction that $\phi \in (M+h)Q$. We have

$$0 < \langle \phi, -\check{\alpha}_0 \rangle < 2(M+h)$$

from our assumption on the m-degree of ϕ and since $\langle \phi, -\check{\alpha}_0 \rangle = \sum \phi_j \check{m}_j > 0$. However, since $\langle Q, -\check{\alpha}_0 \rangle \subset \mathbb{Z}$, our assumption on ϕ forces $\sum \phi_j \check{m}_j = \langle \phi, -\check{\alpha}_0 \rangle \in (M+h)\mathbb{Z}$ and hence $\sum \phi_j \check{m}_j = M+h$.

In the same way, applying ϕ to each simple coroot $\check{\alpha}_j$ in turn, we obtain $\phi_j = \langle \phi, \check{\alpha}_j \rangle = (M+h)a_j$ for some $a_j \in \mathbb{Z}^{\geqslant 0}$, $j = 1, \ldots, n$. Thus $M+h = \sum \phi_j \check{m}_j = (M+h)\sum_{j=1}^n a_j \check{m}_j$. This implies that exactly one $a_j \neq 0$ and for this j, $a_j = 1$ and $\check{m}_j = 1$. Thus $\phi = (M+h)\omega_j \in (M+h)Q$, i.e. $\omega_j \in Q$.

In fact this can't happen. One way to see this is to use a well-known fact that the non-trivial elements of the centre of the simply-connected simple Lie group G are given by the elements $\check{\omega}_j$ over the $j > 0$ which correspond the places where $m_j = 1$ (these are certain vertices, different from the vertex 0, of the fundamental chamber). Of course these elements $\check{\omega}_j \notin \check{Q}$ for such an element would be a representative of the identity element. Our case here is the same situation except it is for the simply-connected simple Lie group $\check{G}$ based on the dual root system: we have a j for which $\check{m}_j = 1$, and hence $\omega_j \notin Q$.

Thus $\phi = (M+h)\omega_j$ cannot lie in $(M+h)Q$. $\square$

7.4. Weyl integral formula and its consequences

We recall here the Weyl integral formula [20],

$$\int_G \mathcal{F}\, d\theta_G = \frac{1}{|W|}\int_{\mathbb{T}} \mathcal{F}|S_\rho|^2\, d\theta_{\mathbb{T}}$$

for all class functions $\mathcal{F}$ (functions which are invariant on conjugacy classes in G). Here the measures are normalized Haar measure on G and $\mathbb{T}$ respectively and the function $\mathcal{F}$ is simply being restricted to the maximal torus $\mathbb{T}$ in the second integral. In particular characters are class functions, and so for the irreducible characters χ_λ, χ_μ of the irreducible representations of G of highest weights λ, μ respectively, we have from the standard orthogonality relations [20]:

$$\delta_{\lambda,\mu} = \int_G \chi_\lambda \overline{\chi_\mu}\, d\theta_G = \frac{1}{|W|} \int_{\mathbb{T}} \chi_\lambda \overline{\chi_\mu} |S_\rho|^2\, d\theta_{\mathbb{T}}. \tag{36}$$

Here we are using the usual Kronecker delta.

We do not need all of (36), only the equality of the left- and right-hand sides; and that fact is not hard to see. We have

$$\chi_\lambda S_\rho \overline{\chi_\mu S_\rho} = S_{\lambda+\rho}\overline{S_{\mu+\rho}} = \sum_{w\in W}\sum_{v\in W} (-1)^{l(w)}(-1)^{l(v)} e^{2\pi i\langle w(\lambda+\rho),x\rangle} e^{-2\pi i\langle v(\mu+\rho),x\rangle}$$

by Weyl's character formula.[6] The integral over $\mathbb{T}$ of $e^{2\pi i\langle w(\lambda+\rho),x\rangle - 2\pi i\langle v(\mu+\rho),x\rangle}$ is 0 unless $w(\lambda+\rho) - v(\mu+\rho) = 0$, in which case it integrates to 1. Since $\lambda+\rho$ and $\mu+\rho$ are strictly dominant, this happens only if $\lambda = \mu$ and $w = v$. If indeed $\lambda = \mu$ then there are exactly $|W|$ times when $w(\lambda+\rho) - v(\mu+\rho) = 0$, and we see that the right-hand side of (36) is $\delta_{\lambda,\mu}$.

We also wish to recall a result from discrete Fourier analysis [13]. There it is proved that, using the notation established above,

$$\int_F S_{\lambda+\rho}\overline{S_{\mu+\rho}}\, d\theta_{\mathbb{T}} = \frac{1}{c_G(M+h)^n} \sum_{x\in F^\circ_{M+h}} S_{\lambda+\rho}(x)\overline{S_{\mu+\rho}(x)} = \delta_{\lambda,\mu}, \tag{37}$$

as long as when $\lambda \neq \mu$, the points of F°_{M+h}, can *separate* all the weights appearing in the W-orbit of $\lambda+\rho$ from all those appearing in the W-orbit of $\mu+\rho$. Explicitly, separation means that it never happens that $w(\lambda+\rho) - v(\mu+\rho)$ takes integer values on all the points of $\frac{1}{M+h}\check{P}$, or equivalently, it never happens that $w(\lambda+\rho) - v(\mu+\rho) \in (M+h)Q$, except when $\lambda = \mu$.

This proof of the last equality in (37) is actually a straightforward thing to see. First note that $S_{\lambda+\rho}\overline{S_{\mu+\rho}}$ is W-invariant, and hence its integral over all of $\mathbb{T}$ is $|W|$ times its integral over F. There is no need to worry about the boundary of F which has measure 0 and in any case the function takes the value 0 on all of its boundary. In the same way, the sum over F°_{M+h} can be extended by the operation of the Weyl group to obtain a full set of representatives of $\frac{1}{M+h}\check{P}/\check{Q}$, noting again that since the function is 0 on the boundaries of the chambers, adding in the extra boundary elements that may appear in $\frac{1}{M+h}\check{P}/\check{Q}$ makes no difference to the sum. This again increases the value of the sum by $|W|$ but has the benefit of turning the sum into a sum over a group. Then usual considerations of sums of exponentials over a group give the desired orthogonality as long as the separation condition is satisfied. The order of $\frac{1}{M+h}\check{P}/\check{Q}$ is $c_G(M+h)^n$, which explains the factor outside the sum part of the formula.

7.5. Demonstration of Theorem 7.1

With these facts, we now prove the integral formula (34):

Proof. Because of linearity, it is enough to prove (34) when f, g are monomials of the form

$$\chi_{\omega_1}^{\nu_1}\chi_{\omega_2}^{\nu_2}\cdots\chi_{\omega_n}^{\nu_n}, \quad \text{where } \nu_1 m_1 + \nu_2 m_2 + \cdots + \nu_n m_n \leqslant N,$$

where $N = M+1$ for f and $N = M$ for g.

[6] In [20] Weyl's character formula is derived from the integral formula, but algebraists usually use an algebraic proof of the character formula.

Since $\chi_{\omega_1}^{\nu_1}\chi_{\omega_2}^{\nu_2}\cdots\chi_{\omega_n}^{\nu_n}$ decomposes into a linear combination of characters $\sum a_\lambda \chi_{\lambda_1\omega_1+\cdots+\lambda_n\omega_n}$, where $\lambda_j \leqslant \nu_j$ for all j, and $a_\nu = 1$ (i.e. the exponent $\nu = \nu_1\omega_1 + \nu_2\omega_2 + \cdots + \nu_n\omega_n$ appears in the sum with multiplicity 1), we need only prove the theorem for $f(X_1,\ldots,X_n) = \chi_\lambda = \chi_{\lambda_1\omega_1+\cdots+\lambda_n\omega_n}$ and $g(X_1,\ldots,X_n) = \chi_\mu = \chi_{\mu_1\omega_1+\cdots+\mu_n\omega_n}$. Notice that it is quite possible for χ_0, which is the constant function $1_{\mathbb{T}}$, to appear here.

We have from (36) and (37),

$$\delta_{\lambda,\mu} = \frac{1}{|W|}\int_{\mathbb{T}} \chi_\lambda \overline{\chi_\mu}|S_\rho|^2\, d\theta_{\mathbb{T}} = \frac{1}{|W|}\int_{\mathbb{T}} S_{\lambda+\rho}\overline{S_{\mu+\rho}}\, d\theta_{\mathbb{T}} = \int_F S_{\lambda+\rho}\overline{S_{\mu+\rho}}\, d\theta_{\mathbb{T}}$$

$$= \frac{1}{c_G(M+h)^n}\sum_{x\in F^\circ_{M+h}} S_{\lambda+\rho}(x)\overline{S_{\mu+\rho}(x)} \tag{38}$$

(which is exactly what we have to prove) as long as when $\lambda \neq \mu$ there are no pairs $w, v \in W$ for which $w(\lambda+\rho)(x) - v(\mu+\rho)(x) \in \mathbb{Z}$ for all $x \in \frac{1}{M+h}\check{P}$, or, as we pointed out above,

$$w(\lambda+\rho) - v(\mu+\rho) \in (M+h)Q.$$

We will now show this cannot happen.

Consider the weights $w(\lambda+\rho)$, $w \in W$. These weights are all of the form $\lambda+\rho-\beta$ where β is a sum of positive roots (including the case when it is the empty sum, 0). Now $-\check{\alpha}$ is the highest co-root and so is actually a dominant co-weight. This gives us that $\langle\beta, -\check{\alpha}\rangle \geqslant 0$ and so

$$m\text{-}\deg\big(w(\lambda+\rho)\big) \leqslant m\text{-}\deg(\lambda+\rho) \leqslant M+h.$$

The lowest weight in the W-orbit of $\lambda+\rho$ is $w_{\mathrm{opp}}(\lambda+\rho)$ and its m-degree is the negative of the m-degree of $\lambda+\rho$ as we saw above. All the other weights in its orbit are of the form $w_{\mathrm{opp}}(\lambda+\rho)+\beta$ for some sum β of positive roots and this then gives us

$$-(M+h) \leqslant -m\text{-}\deg(\lambda+\rho) \leqslant m\text{-}\deg\big(w(\lambda+\rho)\big).$$

In short

$$-(M+h) \leqslant -m\text{-}\deg(\lambda+\rho) \leqslant m\text{-}\deg\big(w(\lambda+\rho)\big) \leqslant m\text{-}\deg(\lambda+\rho) \leqslant M+h.$$

Exactly the same holds for $v(\mu+\rho)$ except that the inequalities are now strict since the degree of g is at most M. Combining, we obtain

$$-2(M+h) < m\text{-}\deg\big(w(\lambda+\rho) - v(\mu+\rho)\big) < 2(M+h)$$

for all $w, v \in W$.

Now for any choice of w, v, each element in the W-orbit of $w(\lambda+\rho) - v(\mu+\rho)$ is another element of the same form, and so its m-degree is constrained in the same way. Thus if there is a pair w, v for which $w(\lambda+\rho) - v(\mu+\rho) \in (M+h)Q$ then, since $(M+h)Q$ is W-invariant, we can assume that $\phi := w(\lambda+\rho) - v(\mu+\rho) \in P^+ \cap (M+h)Q$, i.e. it is dominant. But then if $\phi \neq 0$ it contradicts Lemma 7.3. It follows that $\phi = 0$, and then due to the fact that $\lambda+\rho$ and $\mu+\rho$ are strictly dominant, we have $\lambda = \mu$.

This proves the separation condition holds, and finishes the proof of the theorem. □

7.6. Duality

The essence of the cubature formula of Theorem 7.2 is the duality between dominant weights of m-degree not exceeding M and the elements of finite order $M+h$ arising from the fundamental region F. If we use the fact that the m-degree of ρ is $h-1$ and the fact that any regular EFO in F can be expressed as $\check{\mu}+\check{\rho}$ (or $x_{\check{\mu}+\check{\rho}}$ in $\mathbb{T}$), where $\check{\mu}$ is co-dominant, then the cubature matrix is

$$(M_{\lambda\check{\mu}}) = \big(S_{\lambda+\rho}(x_{\check{\mu}+\check{\rho}})\big),$$

where $\lambda, \check{\mu}$ run over all solutions to the equations

$$\sum \check{m}_j(\lambda_j+1) = M+h,$$
$$\sum m_j(\check{\mu}_j+1) = M+h.$$

The sums run over $j=0,\dots,n$, and $\lambda_0 \geqslant 0$, $\check{\mu}_0 \geqslant 0$ are defined so as to make the equations valid. Since the marks and co-marks are simply permutations of each other, the symmetry of this pairing is completely manifest: if we order the co-ordinates on each side to take account of this permutation, the solutions to the two equations look identical and the matrix becomes symmetric. Moreover, formally the situation is the same for G and the corresponding group $\check{G}$, with the roles of characters and elements of finite order interchanged.

8. Approximating functions on Ω

In this section we show how polynomials f of m-degree at most M can be expanded as linear combinations of the basic polynomials X_λ arising from the characters λ of m-degree at most M. The coefficients are calculated using the values of f at the corresponding nodes. The same calculations applied to an arbitrary function $f \in L^2_K(\Omega)$ lead to a polynomial approximation of f as a linear combination of the these basic polynomials X_λ, and this approximation is shown to be the best possible in the L^2_K-norm by polynomials of m-degree at most M.

8.1. Polynomial expansion in terms of the X_λ

To simplify notation and the following discussion we introduce an inner product on the space $L^2_K(\Omega)$ of all complex-valued functions f on Ω for which $\int_\Omega |f|^2 K^{1/2} < \infty$. It is defined by

$$\begin{aligned}\langle f,g\rangle_K :&= (2\pi)^{-n}\int_\Omega f\overline{g}K^{1/2}\\ &= (2\pi)^{-n}\int_\Omega f(X_1,\dots,X_n)\overline{g(X_1,\dots,X_n)}K^{1/2}(X_1,\dots,X_n)\,dX_1\dots dX_n. \qquad (39)\end{aligned}$$

Note here that the definition of conjugation is made in terms of Remark 1.

Since $K^{1/2}$ is continuous and strictly positive on Ω, $\langle f,f\rangle_K \geqslant 0$ with equality if and only if f is zero almost everywhere in Ω (relative to Lebesgue measure). Relative to this inner product $L^2_K(\Omega)$ is a Hilbert space. We call $\langle f,f\rangle_K^{1/2}$ the L^2_K-norm of f.

The results of (33) and (38) show that the polynomials defined by

$$X_\lambda := \chi_\lambda(x), \quad x \in F^\circ,$$

where $\lambda = \sum \lambda_j \omega_j$ runs through the set of dominant weights P^+, form an orthonormal set:

$$\langle X_\lambda, X_\mu \rangle_K = \delta_{\lambda,\mu}.$$

These polynomials are just those that result from expanding χ_λ as a polynomial (of degree equal to the m-degree $|\lambda|_m$ of λ) in the fundamental characters $X_j = \chi_{\omega_j}$.

The set $\{X_\lambda\}$ actually forms a Hilbert basis for $L^2_K(\Omega)$ (the main point being that they actually span the entire space). This can be seen by relating functions f on Ω back to functions $\widetilde{f}$ on F° through $\widetilde{f}(x) := f(X_1, \dots, X_n) = f(\chi_{\omega_1}(x), \dots, \chi_{\omega_n}(x))$. Then using (33),

$$\infty > (2\pi)^{-n} \int_\Omega |f|^2 K^{1/2} = \int_{F^\circ} |\widetilde{f}|^2(x) S_\rho(x) \overline{S_\rho(x)}\, dx. \tag{40}$$

Formally $\widetilde{f}$ exists only on F°, but we can extend it to a function on all of $\mathbb{T}$ by W-symmetry if necessary. This might make it a bit easier to relate to Section 3, which we are now going to use.

We have already seen the integral on the right-hand side of (40) in (14), and according to (12) and (13) we have

$$\widetilde{f} = \sum_\lambda b_\lambda \chi_\lambda = \sum_\lambda b_\lambda \widetilde{X_\lambda},$$

with

$$b_\lambda = \frac{1}{|W|} \int_{\mathbb{T}} \widetilde{f}\, \overline{\chi_\lambda} |S_\rho|^2 \, d\theta_{\mathbb{T}} = \int_{F^\circ} \widetilde{f}\, \overline{\chi_\lambda} |S_\rho|^2 \, d\theta_{\mathbb{T}} = (2\pi)^{-n} \int_\Omega f \overline{X_\lambda} K^{1/2} = \langle f, X_\lambda \rangle_K.$$

Thus for each $f \in L^2_K(\Omega)$ we have its expansion

$$f \doteq \sum_\lambda \langle f, X_\lambda \rangle_K X_\lambda,$$

where the $\doteq$ means equality in the sense of equality Lebesgue-almost-everywhere.

The truncated sums

$$\sum_{|\lambda|_m \leqslant M} \langle f, X_\lambda \rangle_K X_\lambda,$$

where $|\lambda|_m$ stands for the m-degree of λ, are polynomials of m-degree at most M in the variables $X_1, \dots, X_n$.

If f is a polynomial in the variables $X_1, \dots, X_n$ of m-degree M then $\widetilde{f} := f \circ \xi$ is a polynomial function of $\chi_{\omega_1}, \dots, \chi_{\omega_n}$. However, the polynomials in $\chi_{\omega_1}, \dots, \chi_{\omega_n}$ of m-degree $\leqslant M$ span exactly the same space as the characters χ_λ where λ runs through all the weights of m-degree $\leqslant M$. As a consequence

$$\widetilde{f} = \sum_{|\lambda|_m \leqslant M} a_\lambda \chi_\lambda, \qquad f = \sum_{|\lambda|_m \leqslant M} a_\lambda X_\lambda$$

for some $a_\lambda \in \mathbb{C}$. From the $\langle \cdot, \cdot \rangle$-orthogonality relations it is clear that each $a_\lambda = \langle f, X_\lambda \rangle_K$. Since the latter expressions can be computed by simple summing over the nodes, we obtain:

Proposition 8.1. *Any polynomial f of m-degree at most M can be written in the form*

$$f=\sum_{|\lambda|_m\leqslant M}\langle f,X_\lambda\rangle_K X_\lambda$$

where

$$\begin{aligned}\langle f,X_\lambda\rangle_K&=\frac{1}{c_G(M+h)^n}\sum_{X\in\mathfrak{F}^\circ_{M+h}} f(X)K(X)^{1/2}\\&=\frac{1}{c_G(M+h)^n}\sum_{x\in F^\circ_{M+h}}\widetilde{f}(x)\widetilde{K}(x)^{1/2}.\end{aligned}\tag{41}$$

8.2. Optimality of the approximation

We now show that these polynomials are the best possible approximations for f in terms of the L^2_K-norm. This result is in fact a natural consequence that one would expect from the situation that we have created here, since it essentially relates back to the Fourier analysis of $\mathbb{T}$. See also [21].

Proposition 8.2. *Let $f\in L^2_K(\Omega)$. Amongst all polynomials $p(X_1,\dots,X_n)$ of m-degree less than or equal to M, the polynomial $\sum_{|\lambda|_m\leqslant M}\langle f,X_\lambda\rangle_K X_\lambda$ is the best approximation to f relative to the L^2_K-norm.*

Proof. Let $p=\sum_{|\lambda|_m\leqslant M}b_\lambda X_\lambda$ be any polynomial. Let $a_\lambda:=\langle f,X_\lambda\rangle_K$ for all λ and set $q:=\sum_{|\lambda|_m\leqslant M}a_\lambda X_\lambda$.

$$\begin{aligned}\langle f-p,f-p\rangle_K&=\langle f,f\rangle_K-\langle f,p\rangle_K-\langle p,f\rangle_K+\langle p,p\rangle_K\\&=\langle f,f\rangle_K-\sum_\lambda a_\lambda\overline{b_\lambda}-\sum_\lambda b_\lambda\overline{a_\lambda}+\sum_\lambda|b_\lambda|^2\\&=\langle f,f\rangle_K-\sum_\lambda|a_\lambda|^2+\sum_\lambda|a_\lambda|^2-\sum_\lambda a_\lambda\overline{b_\lambda}-\sum_\lambda b_\lambda\overline{a_\lambda}+\sum_\lambda|b_\lambda|^2\\&=\langle f-q,f-q\rangle_K+\sum_\lambda|b_\lambda-a_\lambda|^2\geqslant\langle f-q,f-q\rangle_K\end{aligned}$$

with equality if and only if $b_\lambda=a_\lambda$ for all λ. □

9. Example: A cubature formula for G_2

We illustrate the mathematics developed above by an example where the Lie group is the exceptional simple Lie group of type G_2 and the value of M is chosen to be $M=8$.

We begin with the pertinent information about G_2 and related objects. Then we can start to solve the problem of our example. There are three steps. Since $M+h=14$, the nodes will come from the EFOs of *Ad*-order 14. These appear in Section 9.2. These are supposed to be common zeros of the S-functions of m-degree $M+1=9$, which are described in Section 9.3. Finally we get to the cubature formula whose various components are worked out in Section 9.4.

9.1. Pertinent data about G_2

From the G_2 diagram in Fig. 1 we find the following:

$$\begin{aligned}
&\langle\alpha_1,\alpha_1\rangle:\langle\alpha_2,\alpha_2\rangle = 3:1, && A=\begin{pmatrix} 2 & -3 \\ -1 & 2\end{pmatrix};\\
&\langle\check\alpha_1,\check\alpha_1\rangle:\langle\check\alpha_2,\check\alpha_2\rangle = 1:3, && \check A=\begin{pmatrix} 2 & -1 \\ -3 & 2\end{pmatrix};\\
&-\alpha_0 = 2\alpha_1+3\alpha_2, && -\check\alpha_0 = 3\check\alpha_1+2\check\alpha_2;\\
&\text{marks } 2,3, && \text{co-marks } 3,2;\\
&\text{Coxeter number } h=6, && c_G = 1.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\alpha_1 = 2\omega_1-3\omega_2, && \alpha_2=-\omega_1+2\omega_2, && \omega_1 = 2\alpha_1+3\alpha_2, && \omega_2=\alpha_1+2\alpha_2,\\
&\check\alpha_1 = 2\check\omega_1-\check\omega_2, && \check\alpha_2=-3\check\omega_1+2\check\omega_2, && \check\omega_1 = 2\check\alpha_1+\check\alpha_2, && \check\omega_2=3\check\alpha_1+2\check\alpha_2.
\end{aligned}$$

The link between the bases is the $\mathbb{Z}$-duality requirement

$$\langle\alpha_j,\check\omega_k\rangle = \langle\omega_k,\check\alpha_j\rangle = \delta_{jk}, \quad j,k=1,2.$$

The set of positive roots Δ_+ and their half sum ρ are:

$$\Delta_+ = \{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2,\alpha_1+3\alpha_2,2\alpha_1+3\alpha_2\}, \qquad \rho = 3\alpha_1+5\alpha_2 = \omega_1+\omega_2.$$

The fundamental domain of G_2 is the convex hull of its three vertices $\{0, \frac{1}{2}\check\omega_1, \frac{1}{3}\check\omega_2\}$.

9.2. Finding the nodes

The nodes are the ξ-images of the 10 points of the set F°_{14} of EFO's of *Ad*-order $M+h=14$ which are in the interior of F. They must satisfy the sum rule $14 = s_0+2s_1+3s_2$. We write them in the Kac coordinates $[s_0,s_1,s_2]$ as well as in the $\check\omega$-basis:

$$\begin{aligned}
&[9,1,1] = \left(\frac{1}{14},\frac{1}{14}\right), && [7,2,1] = \left(\frac{1}{7},\frac{1}{14}\right), && [5,3,1] = \left(\frac{3}{14},\frac{1}{14}\right),\\
&[3,4,1] = \left(\frac{2}{7},\frac{1}{14}\right), && [1,5,1] = \left(\frac{5}{14},\frac{1}{14}\right), && [6,1,2] = \left(\frac{1}{14},\frac{1}{7}\right),\\
&[2,1,1] = \left(\frac{1}{7},\frac{1}{7}\right), && [2,3,2] = \left(\frac{3}{14},\frac{1}{7}\right), && [3,1,3] = \left(\frac{1}{14},\frac{3}{14}\right),\\
&[1,2,3] = \left(\frac{1}{14},\frac{3}{14}\right).
\end{aligned} \tag{42}$$

The strict adjoint order of all the EFO's but one is 14. It is 7 for $\frac{1}{7}\check\omega_1+\frac{1}{7}\check\omega_2$, and we have removed the common factor 2 from its original Kac coordinates $[4,2,2]$ to get the simpler form $[2,1,1]$. Other EFO's of adjoint order 14 are not shown in (42) because they are on the boundary of F. The boundary points are easily discarded because at least one of their Kac coordinates has to be zero. At the EFOs of *Ad*-order 14, the S-functions $S_{\lambda+\rho}$ for which the m-degree of λ is $M+1=9$ must simultaneously vanish.

Table 3
Values of λ (in the ω-basis) for the character χ_{ω_1}. All the weights are of multiplicity 1 except for $(0,0)$, which as multiplicity 2. This is the adjoint representation.

$(1,0)$	$(-1,3)$	$(0,1)$	$(1,-1)$	$(2,-3)$	$(-1,2)$
		$(0,0)$	$(0,0)$		
$(-2,3)$	$(1,-2)$	$(-1,1)$	$(0,-1)$	$(1,-3)$	$(-1,0)$

Table 4
Values of λ (in the ω-basis) for the character χ_{ω_2}. All the weights have multiplicity 1. This is the 7-dimensional irreducible representation of G_2.

$(0,1)$	$(1,-1)$	$(-1,2)$	$(0,0)$	$(1,-2)$	$(-1,1)$	$(0,-1)$

9.3. Finding S-functions of m-degree 9

The m-degree of a G_2 weight $\lambda = (\lambda_1, \lambda_2) = \lambda_1\omega_1 + \lambda_2\omega_2$ is calculated (remember that we use the co-marks for this) as $\check{m}_1\lambda_1 + \check{m}_2\lambda_2 = 3\lambda_1 + 2\lambda_2$. We find first all the weights with m-degree $\leqslant 9$. They are the following,

$$(0,0),\ (0,1),\ (0,2),\ (0,3),\ (0,4),\ (1,0),\ (1,1),\ (1,2),\ (1,3),\ (2,0),\ (2,1),\ (3,0).$$

Among these there are just two weights with m-degree equal to 9, namely $(1,3)$ and $(3,0)$. Thus there are only two S-functions we need to consider, $S_{(2,4)}(x)$ and $S_{(4,1)}(x)$ (since the S functions are of the form $S_{\lambda+\rho}$ and $\rho = (1,1)$), and these that vanish simultaneously on all the EFOs found in Section 9.2.

9.4. Cubature formula

The cubature formula itself reads (see (35))

$$\int_{\Omega} f K^{1/2}\, dX_1\, dX_2 = \left(\frac{\pi}{7}\right)^2 \sum_{X\in\mathfrak{F}_{14}} f(X)K(X). \tag{43}$$

The points X run over the ξ-images of the EFO list (42). The corresponding values of K are given by Table 6. The formula is valid for all polynomials $f = f(X) = f(X_1, X_2)$ of m-degree less than or equal to $2M + 1 = 17$.

The mapping of the fundamental domain into $\mathbb{R}^2$ is $\xi : x \mapsto (X_1(x), X_2(x))$ where X_1, X_2 are given by the two fundamental characters: $X_1 = \chi_{\omega_1}$, $X_2 = \chi_{\omega_2}$. These characters are exponential functions of the form:

$$\sum_{\lambda} L^{\lambda} e^{2\pi i\langle\lambda, x\rangle}$$

where the λ run over the weight system of the irreducible modules L of highest weights ω_1 and ω_2 respectively, and the quantities L^{λ} are the multiplicities of the weights and x varies over the fundamental domain. For the two modules in question these quantities are given by Tables 3 and 4.

In G_2 irreducible characters are real-valued (as is clear since the Weyl group contains the central inversion $\lambda \mapsto -\lambda$ making the characters self-conjugate), and so the mapping $\xi : \mathbb{T} \to \mathbb{C}^2$ is actually into $\mathbb{R}^2$, and $\mathfrak{R} \subset \mathbb{R}^2$.

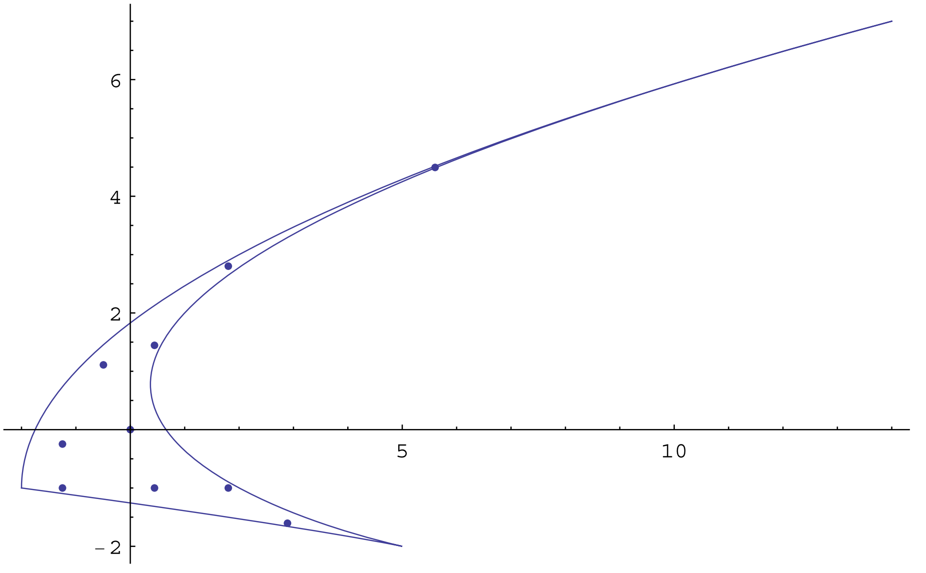

Fig. 3. The region Ω along with the 10 regular EFOs of the example.

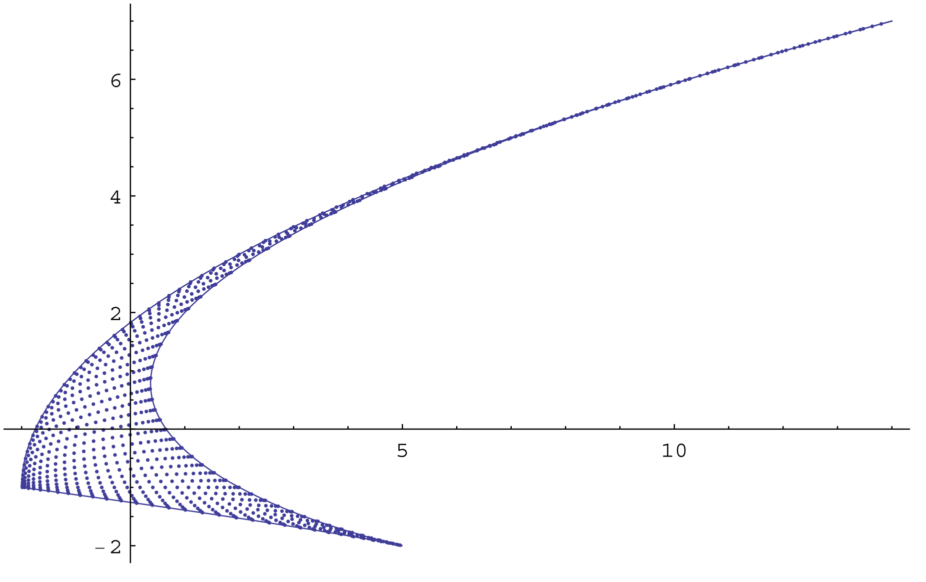

Fig. 4. The region Ω along with the 884 regular EFOs of *Ad*-order 106.

The image of the fundamental domain F under the mapping $x \mapsto (X_1(x), X_2(x))$ in $\mathbb{R}^2$ along with the images of the 10 nodes that we found in (42) are shown in Fig. 3. The way in which EFOs fill out the region Ω is made clearer by the Fig. 4 which shows the distribution of the EFOs of *Ad*-order 106. The actual coordinate values are given in Table 5.

The function K is most easily written down in terms of $\widetilde{K}$ on the torus side of the picture, and in its product form:

Table 5
A table of EFOs of Ad-order 14 and the corresponding nodes (rounded to 4 figures) in $\mathfrak{R}$. The positions of the nodes are shown in Fig. 3.

[9, 1, 1]	[7, 2, 1]	[5, 3, 1]	[3, 4, 1]	[1, 5, 1]
(5.604, 4.494)	(1.802, 2.802)	(−0.494, 1.11)	(−1.247, −0.247)	(−1.247, −1.)
[6, 1, 2]	[2, 1, 1]	[2, 3, 2]	[3, 1, 3]	[1, 2, 3]
(0.445, 1.445)	(0., 0.)	(0.445, −1.),	(1.802, −1.)	(2.89, −1.604)

Table 6
The values of the weighting function $\widetilde{K}$ at the 10 nodes.

[9, 1, 1]	[7, 2, 1]	[5, 3, 1]	[3, 4, 1]	[1, 5, 1]	[6, 1, 2]	[2, 1, 1]	[2, 3, 2]	[3, 1, 3]	[1, 2, 3]
0.364666	7.36467	30.1836	37.1836	7.36467	11.4517	49.	37.1836	11.4517	4.45175

Table 7
A table of values $S_{\lambda+\rho}(x_{\check{\mu}})$ (rounded to 4 figures), with the weights λ heading the rows and the EFOs heading the columns.

$\lambda\backslash x_{\check{\mu}}$	[9, 1, 1]	[7, 2, 1]	[5, 3, 1]	[3, 4, 1]	[1, 5, 1]	[6, 1, 2]	[2, 1, 1]	[2, 3, 2]	[3, 1, 3]	[1, 2, 3]
(0, 0)	−0.604	−2.714	−5.494	−6.098	−2.714	−3.384	−7.	−6.098	−3.384	−2.11
(0, 1)	−2.714	−7.604	−6.098	1.506	2.714	−4.89	0.	6.098	3.384	3.384
(0, 2)	−5.494	−6.098	2.11	−3.384	−6.098	2.714	7.	−3.384	2.714	−0.604
(0, 3)	−6.098	1.506	−3.384	−4.89	6.098	7.604	0.	3.384	−2.714	−2.714
(0, 4)	−2.714	2.714	−6.098	6.098	−7.604	3.384	0.	1.506	−4.89	3.384
(1, 0)	−3.384	−4.89	2.714	7.604	3.384	−1.506	0.	−2.714	−6.098	−6.098
(1, 1)	−7.	0.	7.	0.	0.	0.	−7.	0.	0.	7.
(1, 2)	−6.098	6.098	−3.384	3.384	1.506	−2.714	0.	−4.89	7.604	−2.714
(2, 0)	−3.384	3.384	2.714	−2.714	−4.89	−6.098	0.	7.604	−1.506	−6.098
(2, 1)	−2.11	3.384	−0.604	−2.714	3.384	−6.098	7.	−2.714	−6.098	5.494

$$\begin{aligned}\widetilde{K}(x) &= |S_\rho|^2(x)\\ &= \big(2-2\cos\langle\alpha_1,x\rangle\big)\big(2-2\cos\langle\alpha_2,x\rangle\big)\big(2-2\cos\langle\alpha_1+\alpha_2,x\rangle\big)\big(2-2\cos\langle\alpha_1+2\alpha_2,x\rangle\big)\\ &\quad\times\big(2-2\cos\langle\alpha_1+3\alpha_2,x\rangle\big)\big(2-2\cos\langle 2\alpha_1+3\alpha_2,x\rangle\big).\end{aligned}$$

The values of the weighting function $\widetilde{K}$ at the nodes are given in Table 6.

Underlying the cubature formula (for $M=8$ in G_2) is the fact that the matrix

$$X = X^{(8)} = \big(S_{\lambda+\rho}(x_{\check{\mu}})\big)$$

satisfies $\frac{1}{(8+6)^2}XX^T = I_{10\times 10}$.

Explicitly this works out as follows. The 10 weights of m-degree less than or equal to 8 are

$$(0,0),\ (0,1),\ (0,2),\ (0,3),\ (0,4),\ (1,0),\ (1,1),\ (1,2),\ (2,0),\ (2,1),$$

as we have seen. For each of these weights λ and for each EFO $x_{\check{\mu}}$ of (42) we compute $S_{\lambda+\rho}(x_{\check{\mu}})$. The results are displayed in Table 7.

Direct computation shows that $X^{(8)}(X^{(8)})^T = 14^2 I_{10\times 10}$, as it should be by (37).

Proposition 8.1 and Proposition 8.2 show how to use the elements of F°_{14} or $\mathfrak{F}^\circ_{14}$ to compute Fourier coefficients to expand polynomials exactly and to find best approximants in general.

Examination of the graphs of the two functions $S_{2,4}$ and $S_{4,1}$ reveals that they have (at least) 2 common zeros in F beyond the expected ones. These zeros do not seem to be related to EFOs and are somewhat of a mystery to us.

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