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Stone's Theorem for Strongly Continuous Groups of Isometries in Hardy Spaces

W. RICKER*

School of Mathematics and Physics, Macquarie University, North Ryde, 2113, Australia

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For $1 < p < \infty$, $p \neq 2$, it is well known that a strongly continuous, one-parameter group of isometries in the Hardy space $H^p(\mathbb{D})$ does not satisfy the classical Stone theorem, that is, there is no σ-additive spectral measure on the Borel sets of the real line whose Fourier–Stieltjes transform is the given group. However, if we relax the topology of $H^p(\mathbb{D})$ by imbedding it into some larger space, then it is shown, for the case of $1 < p < 2$, that any such group does satisfy Stone's theorem when considered as acting in some appropriate larger space. The situation for $2 < p < \infty$ is more complicated. If the group of isometries is of elliptic type then it again satisfies Stone's theorem in a suitable larger space. However, for parabolic groups the situation is fundamentally different; such a group can never be interpreted as a Fourier–Stieltjes transform in any space containing $H^p(\mathbb{D})$. © 1986 Academic Press, Inc.

INTRODUCTION

The classical Stone theorem asserts that if $T: \mathbb{R} \to \mathscr{L}(X)$ is a weakly continuous group of unitary operators $\{T(t);\ t \in \mathbb{R}\}$ in a Hilbert space X, where $\mathscr{L}(X)$ denotes the space of all continuous linear operators on X equipped with the strong operator topology, then there exists a spectral measure $P: \mathscr{B} \to \mathscr{L}(X)$, where $\mathscr{B}$ is the σ-algebra of Borel subsets of the real line $\mathbb{R}$, such that T is its Fourier–Stieltjes transform. That is,

$$T(t) = \int_{\mathbb{R}} e^{its}\, dP(s), \qquad t \in \mathbb{R}. \tag{1}$$

It is important to note that the spectral measure P is *countably additive* with respect to the strong operator topology and that the integral in (1) is in the usual sense of integration with respect to a σ-additive vector measure, [9, Chap. II].

* This work was done while the author was visiting the University of Illinois (Urbana).

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On the other hand, if X is a Banach space and $T: \mathbb{R} \to \mathscr{L}(X)$ a group of surjective isometries in X, the natural analogues of unitary operators in Hilbert space, then it is well known that in general the classical formulation of Stone's theorem is no longer valid, even in the case when X is reflexive. However, if it is not required that the "spectral measure" P whose Fourier–Stieltjes transform is the group T be countably additive, then it is shown in [1, Theorem 4.20] that for certain types of groups $T: \mathbb{R} \to \mathscr{L}(X)$ there does exist a version of Stone's theorem. Namely, there exists a unique projection-valued function $P: \mathbb{R} \to \mathscr{L}(X)$, called a *spectral family* (cf., [1, Sect. 2] for the definition) such that in the strong operator topology,

$$T(t) = \lim_{a \to \infty} \int_{-a}^{a} e^{its}\, dP(s), \qquad t \in \mathbb{R}, \tag{2}$$

where, for each $a > 0$, the "integrals" over $[-a, a]$ exist as the strong limit of Riemann–Stieltjes sums. But, it should be stressed that in general the spectral family does *not* generate a countably additive measure in $\mathscr{L}(X)$. Accordingly, the associated functional calculus is somewhat limited.

However, as suggested in the note [10, Sect. 4], an alternative interpretation of (2) is possible. Namely, a group $T: \mathbb{R} \to \mathscr{L}(X)$ may fail to be the Fourier–Stieltjes transform of a spectral measure solely because the underlying domain space X is "too small" to accomodate the projections needed to form the spectral measure. Accordingly, if the group is interpreted as acting in some suitable space containing the space X, it happens often that the so extended group is the Fourier–Stieltjes transform of a σ-additive spectral measure in the larger space. This has the advantage that the group and its corresponding spectral measure have associated with them a rich functional calculus.

In this paper we are concerned with groups of isometries in certain Hardy spaces. In this setting, it is shown in [2, 3] that any strongly continuous one-parameter group $\{T(t); t \in \mathbb{R}\}$ of isometries in a Hardy space $H^p(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ and $1 < p < \infty$, *always* has a Stone-type representation of the form (2) for a suitable spectral family $P(\cdot)$. However, if $p \neq 2$, this (unique) spectral family does not generate a spectral measure in $H^p(\mathbb{D})$ and so the interpretation of (2) cannot be as in the classical Stone theorem. Accordingly, there naturally arises the question of whether the group does satisfy the classical Stone theorem in some larger space containing $H^p(\mathbb{D})$.

Our aim is to show that this is indeed the case if $1 < p < 2$; see Theorems 2.1, 3.1, and 4.1. Hence, the failure for the group $\{T(t); t \in \mathbb{R}\}$ being a Fourier–Stieltjes transform in $H^p(\mathbb{D})$, in the case of $1 < p < 2$, is only apparent. It is due, roughly speaking, to the fact that $H^p(\mathbb{D})$ is "too small" in some sense to accomodate the projections which ought to be present if the spectral family $P(\cdot)$ were to generate a spectral measure (this is made precise in Sect. 1). However, there does exist some larger space, containing a dense copy of $H^p(\mathbb{D})$, in which each of the operators forming the given group in $H^p(\mathbb{D})$ has a natural extension and such that the extended group does satisfy the classical Stone theorem.

In contrast, the situation for $2 < p < \infty$ is more complicated. If the group is of elliptic type (see Sect. 1 for the definition), then it is again a Fourier–Stieltjes transform in a suitable space containing $H^p(\mathbb{D})$; see Theorem 2.1. However, the situation for parabolic groups is fundamentally different. It is shown (cf., Theorem 3.4) that such a group can *never* be interpreted as a Fourier–Stieltjes transform in any space containing $H^p(\mathbb{D})$. In this case the reason that the group fails to satisfy the classical Stone theorem is genuine; no modification of the domain space $H^p(\mathbb{D})$ will save the situation. The proof of this statement requires a detailed knowledge of the associated spectral family $P(\cdot)$, [3, Sect. 3, Sect. 5].

Unfortunately, for the remaining case of hyperbolic groups, no such detailed description of the associated spectral family $P(\cdot)$ is available (except for the particular case of $p = 2$ [11, Chap. 4]). Accordingly, for hyperbolic groups we have been unable to determine (in the case of $2 < p < \infty$) whether the group is a Fourier Stieltjes transform in some space containing $H^p(\mathbb{D})$. There is some evidence, however, to suggest that this is not the case (cf., Sect. 4).

1. Preliminaries and Notation

Let X be a locally convex Hausdorff space, always assumed to be quasi-complete, X' its continuous dual space and $\mathscr{L}(X)$ the space of all continuous linear operators on X equipped with the topology of pointwise convergence on X. The identity operator is denoted by I.

A *spectral measure* in X is an $\mathscr{L}(X)$-valued, σ-additive, and multiplicative map $P: \mathscr{M} \to \mathscr{L}(X)$, whose domain $\mathscr{M}$ is a σ-algebra of subsets of a set Ω, such that $P(\Omega) = I$. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)\,P(F)$, for every $E \in \mathscr{M}$ and $F \in \mathscr{M}$. We say that P is *equicontinuous* if its range, $\{P(E);\ E \in \mathscr{M}\}$, is an equicontinuous subset of $\mathscr{L}(X)$. It follows from the Orlicz–Pettis lemma that an $\mathscr{L}(X)$-valued function P on a σ-algebra $\mathscr{M}$ is σ-additive if and only if the complex-valued set function

$$\langle Px, x' \rangle: E \to \langle P(E)x, x' \rangle, \qquad E \in \mathscr{M},$$

is σ-additive for each $x \in X$ and $x' \in X'$.

Let $P: \mathscr{M} \to \mathscr{L}(X)$ be a spectral measure. An M-measurable function f

on Ω is said to be *P-integrable* if it is $\langle Px, x'\rangle$-integrable for every $x\in X$ and $x'\in X'$, and for each $E\in \mathscr{M}$ there is an operator $\int_E f\,dP$ in $\mathscr{L}(X)$ such that

$$\left\langle \left(\int_E f\,dP\right) x, x'\right\rangle = \int_E f\,d\langle Px, x'\rangle,$$

for every $x\in X$ and $x'\in X'$. This definition of integrability agrees with that for more general vector measures, [9, Chap. II].

A (*weakly*) *strongly continuous one-parameter group* of operators in X is a homomorphism T: $\mathbb{R}\to\mathscr{L}(X)$, where $\mathbb{R}$ is the additive group of real numbers, such that the mapping $t\to T(t)$, $t\in\mathbb{R}$, is continuous for the (weak) strong operator topology. A group T: $\mathbb{R}\to\mathscr{L}(X)$ is said to be *equicontinuous* if $\{T(t); t\in\mathbb{R}\}$ is an equicontinuous subset of $\mathscr{L}(X)$.

A group of operators T: $\mathbb{R}\to\mathscr{L}(X)$ is said to be a *Stone-type group* if there exists an equicontinuous spectral measure P: $\mathscr{B}\to\mathscr{L}(X)$ whose Fourier–Stieltjes transform is the given group T (i.e., T and P are related by the formula (1)). We remark that since the function $s\to e^{its}$, $s\in\mathbb{R}$, is bounded and $\mathscr{B}$-measurable, for each $t\in\mathbb{R}$, it is necessarily P-integrable, [10, Proposition 1.1], and so the right-hand side of (1) is defined, for each $t\in\mathbb{R}$. Furthermore, the equicontinuity of the range of P implies that the group T is necessarily an equicontinuous group.

Let $S\in\mathscr{L}(X)$. A locally convex Hausdorff space Y is said to be *admissible* for the operator S, [10, p. 275], if there exist a continuous linear injection ι: $X\to Y$ such that Y is the completion or quasicompletion of $\iota(X)$, and an operator S_Y in $\mathscr{L}(Y)$, necessarily unique, such that

$$S_Y\iota x = \iota Sx, \qquad x\in X. \tag{3}$$

In this case the dual space Y' can be identified with the subspace $\{y'\circ\iota;\ y'\in Y'\}$ of X'. Therefore, we write $Y'\subseteq X'$. The subspace Y' of X' separates the points of X. Sets bounded in X remain bounded in Y but, more importantly, sets which are unbounded in X may be bounded in Y.

LEMMA 1.1. *Let X and Z be locally convex Hausdorff spaces and U: $X\to Z$ be a bicontinuous isomorphism of X onto Z. Let $S\in\mathscr{L}(X)$ and $\tilde{S}$ denote the element USU^{-1} of $\mathscr{L}(Z)$. Then a locally convex Hausdorff space is admissible for S if and only if it is admissible for $\tilde{S}$.*

Proof. Suppose that Y is an admissible space for S and ι: $X\to Y$ is the continuous injection such that (3) is satisfied. If κ: $Z\to Y$ denotes the composition $\iota\circ U^{-1}$, then κ is injective, has range dense in Y and

$$S_Y(\kappa z) = S_Y(\iota U^{-1}z) = \iota(SU^{-1}z) = \iota(U^{-1}\tilde{S}z) = \kappa(\tilde{S}z),$$

for each $z \in Z$. This shows that Y is an admissible space for $\tilde{S}$ and the operator $\tilde{S}_Y \in \mathscr{L}(Y)$ satisfying

$$\tilde{S}_Y(\kappa z) = \kappa(\tilde{S}z), \qquad z \in Z,$$

is precisely S_Y.

Since $S = U^{-1}\tilde{S}U$, with $U^{-1}: Z \to X$ a bicontinuous isomorphism of Z onto X, the same argument shows that if Y is admissible space for $\tilde{S}$, then it is also an admissible space for S.

Let T: $\mathbb{R} \to \mathscr{L}(X)$ be a group homomorphism. A locally convex Hausdorff space Y is said to be *T-admissible* if Y is an admissible space for each operator $T(t)$, $t \in \mathbb{R}$, and the continuous linear injection $\iota: X \to Y$ such that

$$(T(t))_Y \iota x = \iota T(t) x, \qquad x \in X, \tag{4}$$

is the same for every $t \in \mathbb{R}$. If the operator $(T(t))_Y \in \mathscr{L}(Y)$ is denoted by $T_Y(t)$, for each $t \in \mathbb{R}$, then $T_Y(\cdot)$: $\mathbb{R} \to \mathscr{L}(Y)$ is again a one-parameter group.

The following definition is the central notion of the paper; it makes precise statement (cf., the introduction) that a group T: $\mathbb{R} \to \mathscr{L}(X)$ which does not satisfy the classical Stone theorem in X, may, nevertheless, satisfy Stone's theorem when interpreted as acting in a suitable larger space containing X.

A one-parameter group T: $\mathbb{R} \to \mathscr{L}(X)$ is said to be an *extended Stone-type group*, if there exists a Hausdorff locally convex T-admissible space Y such that the group $T_Y(\cdot)$: $\mathbb{R} \to \mathscr{L}(Y)$ is a Stone-type group in Y.

The following two simple lemmas show that extended Stone-type groups are stable with respect to certain transfer operations.

LEMMA 1.2. *Let X and Z be locally convex Hausdorff spaces and U: $X \to Z$ be a continuous isomorphism of X onto Z. Let T: $\mathbb{R} \to \mathscr{L}(X)$ be a one-parameter group. For each $t \in \mathbb{R}$, let $\tilde{T}(t)$ denote the element $UT(t)U^{-1}$ of $\mathscr{L}(Z)$. Then the group T: $\mathbb{R} \to \mathscr{L}(X)$ is an extended Stone-type group if and only if the group $\tilde{T}$: $\mathbb{R} \to \mathscr{L}(Z)$ is of extended Stone-type.*

Proof. Suppose that T is an extended Stone-type group in a T-admissible space Y. Then there exist a continuous injection $\iota: X \to Y$, independent of t, such that (4) is satisfied for each $t \in \mathbb{R}$, and an equicontinuous spectral measure P: $\mathscr{B} \to \mathscr{L}(Y)$ such that

$$T_Y(t) = \int_{\mathbb{R}} e^{ist}\, dP(s), \qquad t \in \mathbb{R}. \tag{5}$$

Lemma 1.1 (cf., the proof) shows that, for each $t \in \mathbb{R}$, Y is also an admissible space for $\tilde{T}(t)$ and the continuous injection $\kappa = \iota \circ U^{-1}$ of Z into Y, which is independent of t, satisfies

$$T_Y(t)\,\kappa z = \kappa \tilde{T}(t) z, \qquad z \in Z.$$

Accordingly, Y is $\tilde{T}$-admissible. Furthermore, since $\tilde{T}_Y(t) = T_Y(t)$ for each $t \in \mathbb{R}$, (5) implies that $\tilde{T}$ is an extended Stone-type group in Y.

Conversely, since $T(t) = U^{-1}\tilde{T}(t)U$ for each $t \in \mathbb{R}$, with U^{-1}: $Z \to X$ a bicontinuous isomorphism of Z onto X, the same argument shows that T is an extended Stone-type group whenever $\tilde{T}$ is of extended Stone-type.

LEMMA 1.3. *Let T: $\mathbb{R} \to \mathscr{L}(X)$ be a one-parameter group. Let α and β be real numbers with $\alpha \neq 0$. Then T is an extended Stone-type group if and only if the group*

$$S(t) = e^{i\beta t}\,T(\alpha t), \qquad t \in \mathbb{R},$$

is an extended Stone-type group.

Proof. Suppose that T is an extended Stone-type group in a T-admissible space Y. Then there exist a continuous linear injection ι: $X \to Y$, independent of t, such that (4) is satisfied for each $t \in \mathbb{R}$, and an equicontinuous spectral measure P: $\mathscr{B} \to \mathscr{L}(Y)$ satisfying (5). It is easily verified that the same injection ι: $X \to Y$ satisfies

$$(e^{i\beta t}\,T_Y(\alpha t))\,\iota x = \iota S(t) x, \qquad x \in X,$$

for each $t \in \mathbb{R}$. Accordingly, Y is an S-admissible space and $S_Y(t) = e^{i\beta t}\,T_Y(\alpha t)$ for each $t \in \mathbb{R}$. Furthermore, (5) implies that

$$S_Y(t) = e^{i\beta t} T_Y(\alpha t) = \int_{\mathbb{R}} e^{i(\beta + \alpha s)t}\,dP(s) = \int_{\mathbb{R}} e^{iut}\,dQ(u),$$

for each $t \in \mathbb{R}$, where Q: $\mathscr{B} \to L(Y)$ is the equicontinuous spectral measure obtained from P by the obvious change of measure transformation, namely, the change of variable $u(s) = \beta + \alpha s$, $s \in \mathbb{R}$. This shows that S is an extended Stone-type group in Y.

Since $T(t) = e^{i\tilde{\beta} t} S(\tilde{\alpha} t)$ for each $t \in \mathbb{R}$, where $\tilde{\alpha} = 1/\alpha$ and $\tilde{\beta} = -\beta\tilde{\alpha}$, the same argument shows that T is an extended Stone-type group whenever S is of extended Stone-type.

In the remainder of this section we summarize some relevant results concerning groups of isometries in Hardy spaces.

So, let T: $\mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ be a strongly continuous one-parameter group of isometries, where $1 < p < \infty$. For $p \neq 2$, the isometries $\{T(t); t \in \mathbb{R}\}$ can be represented via Forelli's theorem, [6, Theorem 2], in terms of a one-parameter group $\{\phi(t); t\in\mathbb{R}\}$ of Möbius transformations of $\mathbb{D}$, [4, Theorem 2.1]. Recall that a one-parameter group of Möbius transformations of $\mathbb{D}$, say $\{\phi(t); t\in\mathbb{R}\}$, is a homomorphism $t\to\phi(t)$ of the additive group $\mathbb{R}$ into the group of conformal maps of $\mathbb{D}$ onto $\mathbb{D}$, such that for each $z\in\mathbb{D}$ the function $t\to\phi(t)(z)$ is continuous on $\mathbb{R}$. It is known, [5, Proposition 1.5], that for such a group $\{\phi(t); t\in\mathbb{R}\}$ the set of common fixed points in the extended complex plane must be one of the following: (i) a doubleton set consisting of a point of $\mathbb{D}$ and its symmetric image with respect to the unit circle Π (*elliptic* case); (ii) a singleton subset of Π (*parabolic* case); or (iii) a doubleton subset of Π (*hyperbolic* case). An explicit characterization of the groups $\{\phi(t); t\in\mathbb{R}\}$ described above is given by the following result [4, Theorem 1.6].

PROPOSITION 1.4. *Let $\{\phi(t); t\in\mathbb{R}\}$ be a one-parameter group of Möbius transformations of $\mathbb{D}$:*

(i) *If $\{\phi(t); t\in\mathbb{R}\}$ is elliptic, then there are unique constants c and τ, with $c\in\mathbb{R}$, $c\neq 0$, and $\tau\in\mathbb{D}$ such that $\phi(t)(z)=\gamma_\tau(e^{ict}\gamma_\tau(z))$ for each $t\in\mathbb{R}$ and $z\in\mathbb{D}$, where $\gamma_\tau(z)=(z-\tau)/(\bar{\tau}z-1)$.*

(ii) *If $\{\phi(t); t\in\mathbb{R}\}$ is parabolic, there are unique constants c and δ, with $c\in\mathbb{R}$, $c\neq 0$, and $\delta\in\Pi$ such that*

$$\phi(t)(z)=\frac{(1-ict)z+ict\,\delta}{-ic\,\bar{\delta}tz+(1+ict)}$$

for each $t\in\mathbb{R}$ and $z\in\mathbb{D}$.

(iii) *If $\{\phi(t); t\in\mathbb{R}\}$ is hyperbolic, there are unique constants c, τ, δ with $c>0$, $\tau\in\Pi$, $\delta\in\Pi$, and $\tau\neq\delta$, such that for each $t\in\mathbb{R}$ and $z\in\mathbb{D}$,*

$$\phi(t)(z)=\sigma_{\tau,\delta}^{-1}(e^{ct}\sigma_{\tau,\delta}(z)),$$

where $\sigma_{\tau,\delta}(z)=(z-\tau)/(z-\delta)$.

If $\{\phi(t); t\in\mathbb{R}\}$ is a group of Möbius transformations of $\mathbb{D}$, and $1<p<\infty$, one can select in a canonical way a branch of $(\phi'(t))^{1/p}$ for $t\in\mathbb{R}$, where $\phi'(t)$ denotes $\partial\phi(t)(z)/\partial z$, so that $(\phi'(s+t))^{1/p}=[(\phi'(t))^{1/p}\circ\phi(t)][\phi'(t)]^{1/p}$ for each $t,s\in\mathbb{R}$, where "$\circ$" denotes composition of mappings [4, p. 231]. In the following result, [3, Proposition 2.2], the symbol $(\phi'(t))^{1/p}$ will denote this special branch.

PROPOSITION 1.5. *Let T: $\mathbb{R}\to\mathscr{L}(H^p(\mathbb{D}))$ be a strongly continuous group* *of isometries, where $p \in (1, \infty)$, $p \neq 2$. If T is not continuous in the uniform operator topology, then T has a unique representation in the form*

$$T(t)f = e^{iwt}[\phi'(t)]^{1/p} f(\phi(t)), \qquad f \in H^p(\mathbb{D}),$$

for each $t \in \mathbb{R}$, where $w \in \mathbb{R}$ and $\{\phi(t);\ t \in \mathbb{R}\}$ is a one-parameter group of Möbius transformations of $\mathbb{D}$.

So, according to Propositions 1.4 and 1.5, a strongly continuous group of isometries in $H^p(\mathbb{D})$, where $1 < p < \infty$, $p \neq 2$, is either elliptic, parabolic, or hyperbolic. We will examine each of these cases in turn.

2. Elliptic Groups of Isometries

It is assumed throughout this section that T: $\mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ is a strongly continuous one-parameter group of isometries such that the corresponding (unique) group of Möbius transformations $\{\phi(t);\ t \in \mathbb{R}\}$ as given by Proposition 1.5 is elliptic. In this case we will say simply that the group T itself is elliptic.

Theorem 2.1. *Let $1 < p < \infty$, $p \neq 2$, and T: $\mathbb{R} \to \mathscr{L}(H^p(\mathbb{D})$ be a strongly continuous, elliptic group of isometries. Then T is an extended Stone-type group.*

Proof. It is shown in the proof of Theorem 3.6 in [3] that there exists an isometry U of $H^p(\mathbb{D})$ onto $H^p(\mathbb{D})$ satisfying $U^2 = I$ such that the group S given by $S(t) = UT(t)U$ for each $t \in \mathbb{R}$, is of the form

$$S(f)f \colon z \to e^{iwt}e^{ict/p}f(e^{ict}z), \qquad z \in \mathbb{D},$$

for each $f \in H^p(\mathbb{D})$ and $t \in \mathbb{R}$, where $c, w \in \mathbb{R}$ are constants with $c \neq 0$. So, by Lemmas 1.2 and 1.3, it suffices to show that the group defined by

$$R(t)f \colon z \to f(e^{ict}z), \qquad z \in \mathbb{D}, \tag{6}$$

for each $f \in H^p(\mathbb{D})$ and $t \in \mathbb{R}$, is an extended Stone-type group. At this stage it is necessary to split the proof into a consideration of two seperate cases.

Case (i). Suppose that $1 < p < 2$. If we pass to the boundary in (6), thereby representing R as a group in $H^p(\Pi)$, and note that this operation is a bicontinuous isomorphism, then if suffices by Lemma 1.2 to show that the group specified by

$$V(t)f \colon z \to f(e^{ict}z), \qquad z \in \Pi,$$

for each $f \in H^p(\Pi)$ and $t \in \mathbb{R}$, is an extended Stone-type group.

Identifying $H^p(\Pi)$ with the usual closed subspace of $L^p(\Pi)$ we note that the Fourier transform map κ: $L^p(\Pi) \to l^q(\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $q > 0$ is the number satisfying $p^{-1} + q^{-1} = 1$, maps $H^p(\Pi)$ into the closed subspace $Y = \{\xi \in l^q(\mathbb{Z});\ \xi(n) = 0,\ n < 0\}$ of $l^q(\mathbb{Z})$. Furthermore, the restriction, ι, of κ to $H^p(\Pi)$, is continuous, injective, and has dense range in Y (since $H^p(\Pi)$ contains all trigonometric polynomials of the form $\sum_{n=0}^{N} a_n z^n$, $N \geq 0$, for example).

For each $t \in \mathbb{R}$, let $V_Y(t)$ denote the operator in $\mathscr{L}(Y)$ given by

$$V_Y(t)\xi: n \to e^{icnt}\xi(n), \qquad n \in \mathbb{Z},$$

for each $\xi \in Y$. If $f \in H^p(\Pi)$ and $t \in \mathbb{R}$ we note that $V(t) f$ is the translation of f by the element e^{ict} of the group Π, from which it follows via properties of the Fourier transform that

$$V_Y(t)\,\iota f = \iota V(t) f.$$

This shows that the Banach space Y is V-admissible.

For each integer $n \geq 0$, let $P(\{cn\})$: $Y \to Y$ denote the projection defined by $P(\{cn\})\xi = v$ for each $\xi \in Y$, where $v(m) = 0$ if $m \in \mathbb{Z}$, $m \neq n$, and $v(n) = \xi(n)$. Then the (necessarily equicontinuous) spectral measure P: $\mathscr{B} \to \mathscr{L}(Y)$ defined by

$$P(E) = \sum_{n=0}^{\infty} \chi_E(cn)\, P(\{cn\}),$$

for each $E \in \mathscr{B}$, satisfies

$$V_Y(t) = \int_{\mathbb{R}} e^{its}\, dP(s), \qquad t \in \mathbb{R}.$$

This shows that V is an extended Stone-type group in Y which completes the proof of the theorem for the case when $1 < p < 2$.

Case (ii) Suppose now that $2 < p < \infty$. As remarked earlier in the proof, it suffices to show that the group (6) is an extended Stone-type group.

Let Y denote the Hilbert space $H^2(\mathbb{D})$ and ι: $H^p(\mathbb{D}) \to Y$ be the natural inclusion of $H^p(\mathbb{D})$ into $H^2(\mathbb{D})$. Then ι is injective, continuous, and its range is dense in Y. For each $t \in \mathbb{R}$, the element $R_Y(t)$ of $\mathscr{L}(Y)$ given by

$$R_Y(t) f: z \to f(e^{ict}z), \qquad z \in \mathbb{D},$$

for each $f \in Y$, clearly satisfies

$$R_Y(t)\,\iota f = \iota R(t) f, \qquad f \in H^p(\mathbb{D}).$$

Accordingly, Y is an R-admissible space.

Since the strongly continuous group $R_Y(\cdot)\colon \mathbb{R} \to \mathscr{L}(Y)$ consists of unitary operators in Y it follows from the classical Stone theorem in Hilbert space that R_Y is the Fourier–Stieltjes transform of some (unique) spectral measure $P\colon \mathscr{B} \to \mathscr{L}(Y)$. That is, R is an extended Stone-type group which completes the proof for $2 < p < \infty$.

3. Parabolic Groups of Isometries

For strongly continuous, elliptic groups of isometries in $H^p(\mathbb{D})$, $1 < p < \infty$, it was shown in Section 2 that the state of affairs is very satisfactory; all such groups are of extended Stone-type. For $1 < p < 2$, the situation is similar in the case of parabolic groups.

Theorem 3.1. *Let $1 < p < 2$ and $T\colon \mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ be a strongly continuous, parabolic group of isometries. Then T is an extended Stone-type group.*

The proof will be via a series of lemmas. To begin with we allow p to belong to the set $(1, 2) \cup (2, \infty)$.

It is shown in the proof of Theorem 3.6 in [3] that there exist an isometry W of $H^p(\mathbb{D})$ onto itself and a particular strongly continuous group of isometries $V\colon \mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ such that

$$WT(t)\, W^{-1} = e^{iwt} V(2ct), \qquad t \in \mathbb{R}, \tag{7}$$

where $c, w \in \mathbb{R}$ are constants with $c \neq 0$. So, by Lemmas 1.2 and 1.3 it suffices to show that V is an extended Stone-type group.

Let $U\colon \mathbb{R} \to \mathscr{L}(L^p(\mathbb{R}))$ denote the usual group of translations in $L^p(\mathbb{R})$, that is,

$$U(t) f\colon s \to f(s+t), \qquad s \in \mathbb{R},$$

for each $f \in L^p(\mathbb{R})$ and $t \in \mathbb{R}$. Then there exists an isometry S of $H^p(\mathbb{R})$ onto $H^p(\mathbb{D})$ such that

$$S^{-1} V(t) S = U^{(p)}(t), \qquad t \in \mathbb{R}, \tag{8}$$

where, for each $t \in \mathbb{R}$, $U^{(p)}(t)$ denotes the restriction of the operator $U(t)$ to the closed invariant subspace $H^p(\mathbb{R})$ of $L^p(\mathbb{R})$; see again the proof of Theorem 3.6 in [3]. Furthermore, if for each $\lambda \in \mathbb{R}$, $E(\lambda)$ denotes the Riesz projection in $L^p(\mathbb{R})$ corresponding to the p-multiplier $\chi_{(-\infty, \lambda]}$, then $H^p(\mathbb{R})$ is invariant for $E(\lambda)$ and the family of projections

$$F(\lambda) = SE^{(p)}(\lambda) S^{-1}, \qquad \lambda \in \mathbb{R}, \tag{9}$$

where $E^{(p)}(\lambda)$ denotes the restriction of $E(\lambda)$ to $H^p(\mathbb{R})$, is the (unique) spectral family for the group V [3, Corollary 3.18].

LEMMA 3.2. *Let $1 < p < 2$. Then there exists a Banach space Y which is an admissible space for each projection $F(\lambda)$, $\lambda \in \mathbb{R}$. Furthermore, the projection-valued set function of intervals defined by*

$$(u, v] \to F_Y(v) - F_Y(u),$$

whenever $u, v \in \mathbb{R}$ and $u \leqslant v$, has an extension to an $\mathscr{L}(Y)$-valued spectral measure on $\mathscr{B}$.

Proof. Let κ denote the restriction of the Fourier transform map (denoted by $\hat{}$) from $L^p(\mathbb{R})$ to $H^p(\mathbb{R})$. If Y denotes the closed subspace of $L^q(\mathbb{R})$ given by

$$\{g \in L^q(\mathbb{R});\ g(\lambda) = 0 \text{ for a.e. } \lambda < 0\},$$

where $q > 0$ is the number such that $p^{-1} + q^{-1} = 1$, then the Paley–Wiener theorem implies that the continuous, injective mapping κ assumes its values in Y. In addition, the range of κ is dense in Y. To see this notice that f actually belongs to $H^p(\mathbb{R})$, whenever $f \in L^p(\mathbb{R})$ satisfies $\hat{f}(\lambda) = 0$ for a.e. $\lambda < 0$ (by the Paley–Wiener theorem).

Now, for each $\lambda \in \mathbb{R}$, let $Q(\lambda)$: $L^q(\mathbb{R}) \to L^q(\mathbb{R})$ denote the operator of pointwise multiplication by $\chi_{(-\infty, \lambda]}$ and $F_Y(\lambda) \in \mathscr{L}(Y)$ denote the restriction of $Q(\lambda)$ to the invariant subspace Y. Then it follows from properties of the Fourier transform and the definition of the Riesz projections that

$$F_Y(\lambda)\, \kappa f = \kappa E^{(p)}(\lambda) f, \qquad f \in H^p(\mathbb{R}), \tag{10}$$

for each $\lambda \in \mathbb{R}$. If S: $H^p(\mathbb{R}) \to H^p(\mathbb{D})$ is the isometry satisfying (8) and (9), and $\iota = \kappa \circ S^{-1}$: $H^p(\mathbb{D}) \to Y$, then it follows from (9) and (10) that

$$F_Y(\lambda)\, \iota f = \iota F(\lambda) f, \qquad f \in H^p(\mathbb{D}),$$

for each $\lambda \in \mathbb{R}$. Since ι is injective, continuous, and has dense range in Y it is then clear that Y is an admissible space for each projection $F(\lambda)$, $\lambda \in \mathbb{R}$.

Finally, since for each interval $(u, v] \subseteq \mathbb{R}$, the operator $F_Y(v) - F_Y(u)$ is the operator in Y of pointwise multiplication by $\chi_{(u,v]}$ it is clear that the so defined set function of intervals is extendable to an $\mathscr{L}(Y)$-valued spectral measure P on $\mathscr{B}$, namely that of pointwise multiplication by Borel subsets of $\mathbb{R}$. This completes the proof of the lemma.

LEMMA 3.3. *Let $1 < p < 2$ and V:$\mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ be the group given by (7). Then, in the notation of the proof of Lemma 3.2, the Banach space Y is V-admissible.*

Proof. It was noted earlier that the spectral family of V is the function $F: \mathbb{R} \to \mathcal{L}(H^p(\mathbb{D}))$ given by (9). So, if $t \in \mathbb{R}$ is fixed, then it follows from [3, Theorems 2.5(i) and 3.6] that, for the strong operator topology in $H^p(\mathbb{D})$,

$$V(t) = \lim_{a \to \infty} \int_{-a}^{a} e^{it\lambda}\, dF(\lambda), \tag{11}$$

where, for each $a > 0$, the operator $V(t; a) = \int_{-a}^{a} e^{it\lambda}\, dF(\lambda)$ is defined as a strong limit of Riemann–Stieltjes sums of the form

$$\sum_{n=1}^{k} e^{its_n}(F(\lambda_n) - F(\lambda_{n-1}))$$

(with $s_n \in (\lambda_{n-1}, \lambda_n]$ for each $1 \leqslant n \leqslant k$), where $\{\lambda_n\}_{n=1}^{k}$ is a partition of $[-a, a]$. It is then clear from Lemma 3.2 that Y is an admissible space for each operator $V(t; a)$, $a > 0$ (with respect to the imbedding ι: $H^p(\mathbb{D}) \to Y$ specified in the proof of Lemma 3.2), and

$$(V(t; a))_Y = \int_{-a}^{a} e^{its}\, dP(s) \tag{12}$$

where the integral in (12) is now with respect to the *σ-additive* spectral measure $P: \mathcal{B} \to \mathcal{L}(Y)$. By the dominated convergence theorem for σ-additive vector measures, [9, II Theorem 2], it follows that the limit

$$\lim_{a \to \infty} (V(t; a))_Y = \int_{\mathbb{R}} e^{its}\, dP(s), \tag{13}$$

which we denote by $V_Y(t)$, exists in the space $\mathcal{L}(Y)$. Using (11) and the continuity of the mapping ι it follows that

$$V_Y(t)\,\iota f = \lim_{a \to \infty} (V(t; a))_Y\, \iota f = \lim_{a \to \infty} \iota V(t; a) f = \iota V(t) f,$$

for each $f \in H^p(\mathbb{D})$. This shows that Y is a V-admissible space and completes the proof of the lemma.

The proof of Theorem 3.1 is now an immediate consequence of Lemma 3.3 and (13) since the P in that formula is a σ-additive, $\mathcal{L}(Y)$-valued spectral measure on $\mathcal{B}$ (cf., proof of Lemma 3.2).

In contrast to Theorem 3.1, it turns out that the situation is fundamentally different when $2 < p < \infty$.

THEOREM 3.4. *Let $2 < p < \infty$ and T: $\mathbb{R} \to \mathcal{L}(H^p(\mathbb{D}))$ be a strongly continuous, parabolic group of isometries. Then T is not an extended Stone-type group in any T-admissible space.*

Proof. The idea of the proof, which is by contradiction, follows the lines of the proof of Theorem 3.3 in [7].

So, suppose that T is an extended Stone-type group. Then it follows from Lemmas 1.2 and 1.3 that the group (7) and, hence, also the group $U^{(p)}$ given by (8), is an extended Stone-type group in some $U^{(p)}$-admissible space, say Y. Accordingly, there exists an equicontinuous spectral measure $\tilde{Q}: \mathscr{B} \to \mathscr{L}(Y)$ such that

$$U_Y^{(p)}(t) = \int_{\mathbb{R}} e^{its}\, d\tilde{Q}(s), \qquad t \in \mathbb{R}.$$

In particular, substituting $t = 1$ gives

$$U_Y^{(p)}(1) = \int_{\mathbb{R}} e^{is}\, d\tilde{Q}(s) = \int_{\Pi} z\, dQ(z), \tag{14}$$

where Q is the equicontinuous $\mathscr{L}(Y)$-valued spectral measure defined on the σ-algebra, $\mathscr{B}(\Pi)$, of Borel subsets of the unit circle Π by

$$Q(E) = \tilde{Q}(\{s \in \mathbb{R}; e^{is} \in E\}), \qquad E \in \mathscr{B}(\Pi). \tag{15}$$

Accordingly, $U^{(p)}(1)$ is an *extended pseudo-unitary* operator in Y, in the sense of [7, Section 1]. Our first aim is to show that the measure Q arises in a special way.

Recall that $U(1)$ is the operator of translation by 1 in $L^p(\mathbb{R})$. Now, $U(1) = e^{iA_1}$, where A_1 is a well-bounded operator of type (B) in $L^p(\mathbb{R})$, $\sigma(A_1) \subseteq [0, 2\pi]$ and 2π is not an eigenvalue of A_1 [8]. Since $H^p(\mathbb{R})$ is invariant for $U(1)$ and $U(1)^{-1}$ (cf., discussion prior to Corollary 3.18 in [3]) it follows that $H^p(\mathbb{R})$ is invariant for A_1 [3, Theorem 3.2], and hence, is invariant for the spectral family of A_1 [3, Theorem 3.1].

For $\lambda \in [0, 2\pi]$, let $\chi_\lambda: \Pi \to \mathbb{C}$ denote the characteristic function of the arc $\{e^{is}; 0 \leqslant s \leqslant \lambda\}$. Then, for each $\lambda \in [0, 2\pi]$, the idempotent function $s \to \chi_\lambda(e^{is})$, $s \in \mathbb{R}$, is a p-multiplier for $\mathbb{R}$; see [7, Lemma 1.4; or 8, Lemma 6], for example. The corresponding multiplier operators $S_\lambda \in \mathscr{L}(L^p(\mathbb{R}))$, all projections, induce the spectral family of A_1 (cf., proof of Theorem 1 in [8]). Accordingly, from the remarks in the previous paragraph it follows that $H^p(\mathbb{R})$ is invariant for each operator S_λ, $\lambda \in [0, 2\pi]$, and hence, also invariant for all differences $S_\lambda - S_\mu$. Of course, $S_\lambda - S_\mu$ is the projection corresponding to the p-multiplier $\chi_{(\mu,\lambda]}(e^{i(\cdot)})$, where $\chi_{(\mu,\lambda]}: \Pi \to \mathbb{C}$ denotes the characteristic function of the arc $\{e^{is}; \mu < s \leqslant \lambda\}$.

So, with each arc $E \subseteq \Pi$ of the form $\{e^{is}; \mu < s \leqslant \lambda\}$, we have associated a projection $P(E) \in \mathscr{L}(L^p(\mathbb{R}))$ which necessarily commutes with $U(1)$, as it commutes with A_1, and such that $H^p(\mathbb{R})$ is invariant for $P(E)$. Furthermore, it follows from properties of p-multipliers that $P(\cdot)$ is multiplicative and finitely additive as a set function of arcs and hence, has a multiplicative and finitely additive extension to the ring of subsets, $\mathscr{R}$, of Π, generated by the semiring of all arcs of the form $\{e^{is}; \mu < s \leqslant \lambda\}$. If this extension to $\mathscr{R}$ is again denoted by P, then it is clear that $H^p(\mathbb{R})$ invariant for each projection $P(E)$, $E \in \mathscr{R}$. Accordingly, we can consider, for each $E \in \mathscr{R}$, the restriction, $P^{(p)}(E)$, of $P(E)$, to the closed invariant subspace $H^p(\mathbb{R})$.

The crucial observation is that, whereas $\{P(E);\ E \subseteq \Pi,\ E \text{ an arc}\}$ is a uniformly bounded set of operators this is generally *not* the case for $\{P(E);\ E \in \mathscr{R}\}$. It is this fact which will eventually give us our desired contradiction. For the moment, however, we wish to show that the resolution of the identity for the operator $U_Y^{(p)}(1)$, namely the spectral measure (15), must arise by extension of the set function $P^{(p)}(\cdot)$ from the ring of sets $\mathscr{R}$ to the σ-algebra $\mathscr{B}(\Pi)$.

LEMMA 3.5. *Let* $\iota\colon H^p(\mathbb{R}) \to Y$ *be the continuous imbedding such that*

$$U_Y^{(p)}(1)\,\iota f = \iota U^{(p)}(1)\,f, \qquad f \in H^p(\mathbb{R}).$$

Then, for each $E \in \mathscr{R}$,

$$Q(E)\,\iota f = \iota P^{(p)}(E)\,f, \qquad f \in H^p(\mathbb{R}),$$

where Q is the measure (15). *In particular, Y is necessarily an admissible space for each operator* $P^{(p)}(E)$, $E \in \mathscr{R}$, *and* $P_Y^{(p)}(E) = Q(E)$.

Proof. Since both Q and $P^{(p)}$ are finitely additive it suffices to verify the result for a single arc $E = \{e^{is}; s \in (a, b]\}$.

If $\Phi(z) = \sum_{-N}^{N} \alpha_n z^n$ is any trigonometric polynomial on Π, then it follows from (14) and the functional calculus for Q that

$$\Phi(U_Y^{(p)}(1)) = \int_\Pi \Phi(z)\, dQ(z). \tag{16}$$

By [7, Lemma 1.1] we have also that

$$\Phi(U_Y^{(p)}(1)) = \Phi(U^{(p)}(1))_Y = \sum_{N}^{N} \alpha_n (U_Y^{(p)}(1))^n. \tag{17}$$

Fix $f \in H^p(\mathbb{R})$. If $y' \in Y'$, then

$$\langle Q(E)\,\iota f, y'\rangle = \int_\Pi \chi_E(z)\, d\langle Q(z)\,\iota f, y'\rangle. \tag{18}$$

Let $\{\mathscr{F}_r\}_{r=1}^{\infty}$ denote the sequence of Fejer kernels on Π. Since the uniformly bounded sequence of trigonometric polynomials

$$\Phi_r = \chi_E * \mathscr{F}_r, \qquad r = 1, 2, \ldots,$$

converges pointwise on Π to the bounded function

$$\psi = \chi_E + \tfrac{1}{2}\chi_{\{e^{ib}\}} - \tfrac{1}{2}\chi_{\{e^{ia}\}},$$

which by (18) satisfies

$$\langle Q(E)\,\imath f, y'\rangle + \tfrac{1}{2}\langle Q(\{e^{ib}\})\,\imath f, y'\rangle - \tfrac{1}{2}\langle Q(\{e^{ia}\})\,\imath f, y'\rangle = \int_{\Pi} \psi \, d\langle Q\imath f, y'\rangle, \tag{19}$$

it follows from (16), (17), and the dominated convergence theorem that the right-hand side of (19) is equal to

$$\begin{aligned} \lim_{r\to\infty} \int_{\Pi} \Phi_r \, d\langle Q\imath f, y'\rangle &= \lim_{r\to\infty} \langle \Phi_r(U_Y^{(p)}(1))\,\imath f, y'\rangle \\ &= \lim_{r\to\infty} \langle \Phi_r(U^{(p)}(1))\, f, y' \circ \imath\rangle. \end{aligned} \tag{20}$$

Consider now the elements $\Phi_r(U^{(p)}(1))\, f = \Phi_r(U(1))\, f$, $r = 1, 2, \ldots$, of $H^p(\mathbb{R})$, as belonging to $L^p(\mathbb{R})$ and extend $y' \circ \imath \in (H^p(\mathbb{R}))'$, via the Hahn–Banach theorem, to a continuous linear functional on $L^p(\mathbb{R})$, say ξ. Then it is shown in the proof of Lemma 3.1 in [7], provided we take there G to be the line group $\mathbb{R}$ and $g = 1$, that the last limit in (20) is equal to

$$\langle P(E)\, f, \xi\rangle + \tfrac{1}{2}\langle S[\phi_{\{e^{ib}\}}]\, f, \xi\rangle - \tfrac{1}{2}\langle S[\Phi_{\{e^{ia}\}}]\, f, \xi\rangle, \tag{21}$$

where $P(E)$ is precisely the projection in $L^p(\mathbb{R})$ corresponding to the arc E, as defined earlier, and $S[\phi_{\{z\}}]$, $z \in \Pi$, denotes the multiplier operator in $L^p(\mathbb{R})$ corresponding to the p-multiplier $s \to \chi_{\{z\}}(e^{is})$, $s \in \mathbb{R}$. Since singleton subsets of $\mathbb{R}$ are null for Lebesgue measure it follows that $S[\phi_{\{z\}}] \equiv 0$ for each $z \in \Pi$. So, (21) is actually equal to $\langle P(E)\, f, \xi\rangle$ which in turn equals $\langle P^{(p)}(E)\, f, y' \circ \imath\rangle$ since $f \in H^p(\mathbb{R})$. Accordingly, the left-hand side of (19) is equal to $\langle P^{(p)}(E)\, f, y' \circ \imath\rangle$.

To complete the proof of the lemma it suffices to show that $Q(\{\lambda\}) \equiv 0$ for each $\lambda \in \Pi$. For, if this is the case, then it follows from (19) that

$$\langle Q(E)\,\imath f, y'\rangle = \langle P^{(p)}(E)\, f, y' \circ \imath\rangle = \langle \imath P^{(p)}(E)\, f, y'\rangle,$$

for each $y' \in Y'$, and hence, $Q(E)\,\imath f = \imath P^{(p)}(E)\, f$, as required.

So, it remains to show that $Q(\{\lambda\}) \equiv 0$ for each $\lambda \in \Pi$. For simplicity we consider the case $\lambda = 1$. Since $\iota(H^p(\mathbb{R}))$ is dense in Y it suffices to show that $Q(\{1\})\,\iota h = 0$ for each $h \in H^p(\mathbb{R})$.

Fix an element $h \in H^p(\mathbb{R})$. By the dominated convergence theorem for vector measures we have

$$Q(\{1\})\,\iota h = \lim_{N \to \infty} \int_{\Pi} (N+1)^{-1} \mathscr{F}_N(z)\, dQ(z)\, \iota h,$$

in the topology of the space Y. It follows from the functional calculus for Q and [7, Lemma 1.1] that

$$\begin{aligned}(N+1)^{-1} &\int_{\Pi} \mathscr{F}_N(z)\, dQ(z)\, \iota h \\ &= (N+1)^{-1} \mathscr{F}_N(U_Y^{(p)}(1))\, \iota h = \iota((N+1)^{-1} \mathscr{F}_N(U^{(p)}(1))h),\end{aligned}$$

for each $N = 1, 2, \ldots$. Since ι is continuous if suffices to show that $(N+1)^{-1} F_N(U^{(p)}(1))h \to 0$ in $H^p(\mathbb{R})$ as $N \to \infty$ or equivalently, since $H^p(\mathbb{R})$ is invariant for each operator $\mathscr{F}_N(U(1))$, $N = 1, 2, \ldots$ (as each $\mathscr{F}_N(U(1))$ is a trigonometric polynomial in $U(1)$), that

$$\lim_{N \to \infty} (N+1)^{-1} \mathscr{F}_N(U(1))h = 0 \tag{22}$$

in $L^p(\mathbb{R})$. Now, it is shown in the proof of Lemma 3.1 in [7], provided again we substitute there G to be the group $\mathbb{R}$ and $g = 1$, that

$$\lim_{N \to \infty} (N+1)^{-1} \mathscr{F}_N(U(1))\varphi = S[\phi_{\{1\}}]\varphi, \tag{23}$$

in $L^p(\mathbb{R})$, whenever $\varphi \in C_c * C_c$, where C_c denotes the space of continuous functions on $\mathbb{R}$ with compact support. Since the L^p-operator norms of $\{(N+1)^{-1} \mathscr{F}_N(U(1))\}_{N=1}^{\infty}$ are uniformly bounded it follows that (23) actually holds for all $\varphi \in L^p(\mathbb{R})$. But, as noted earlier $S[\phi_{\{z\}}] \equiv 0$ for each $z \in \Pi$ and so (22) follows. This completes the proof of the lemma.

It is via Lemma 3.5 that we can now achieve our desired contradiction. For, let y' be any non-zero element of Y', assumed fixed from now on. Then $\xi = y' \circ \iota$ is a non-zero element of $(H^p(\mathbb{R}))'$. Since the Riesz projection $E(0)$ corresponding to the p-multiplier $\chi_{(-\infty,0]}$ is a continuous projection of $L^p(\mathbb{R})$ onto $H^p(\mathbb{R})$ we can define a continuous linear functional $\tilde{\xi}$ on $L^p(\mathbb{R})$ by the formula

$$\langle h, \tilde{\xi} \rangle = \langle E(0)h, \xi \rangle, \qquad h \in L^p(\mathbb{R}). \tag{24}$$

Since the projections $P(F)$, $F \in \mathscr{R}$, commute with $E(0)$ it is easily verified that

$$\langle P(F)h, \xi \rangle = \langle E(0)\, P(F)h, \xi \rangle = \langle P^{(p)}(F)\, E(0)h, \xi \rangle, \qquad F \in \mathscr{R}, \quad (25)$$

for each $h \in L^p(\mathbb{R})$. By Lemma 3.5 the right-hand side of (25) is equal to the set function $\langle Q(\cdot)\, \iota E(0)h, y' \rangle$ on $\mathscr{R}$, for each $h \in L^p(\mathbb{R})$. Hence, if $\{F_j\}_{j=1}^{\infty}$ is any sequence of sets in $\mathscr{R}$ and $\{h_j\}_{j=1}^{\infty}$ any sequence in the unit ball of $L^p(\mathbb{R})$, then it follows from the equicontinuity of Q and the boundedness of $\{\iota E(0)h_j\}_{j=1}^{\infty}$ in Y that

$$\sup_j |\langle P(F_j)h_j, \xi \rangle| = \sup_j |\langle Q(F_j)\, \iota E(0)h_j, y' \rangle| < \infty.$$

Accordingly, ξ is the zero functional on $L^p(\mathbb{R})$ [7, Lemma 3.4], and so, by (24), $\langle E(0)h, \xi \rangle = 0$ for all $h \in L^p(\mathbb{R})$. Since $E(0)(L^p(\mathbb{R})) = H^p(\mathbb{R})$ it follows that ξ is the zero element of $(H^p(\mathbb{R}))'$. This gives the desired contradiction and the proof of the theorem is complete.

4. Hyperbolic Groups of Isometries

In this final section we consider the remaining case of hyperbolic groups of isometries in $H^p(\mathbb{D})$. It turns out, for the case $1 < p < 2$, that the question of whether or not the group is of extended Stone-type can be reduced to considering a certain translation group in $L^p(G)$, where G is the group of non-zero real numbers with respect to multiplication (its Haar measure is $dx/|x|$). Since it will be needed later we begin by calculating the dual group, $\hat{G}$, of G.

Let C_2 denote the 2-element cyclic group, realized as the numbers $\{-1, 1\}$ with respect to multiplication, and $\mathbb{R}^{\oplus}$ denote the multiplicative group of positive real numbers. If sgn denotes the function on $\mathbb{R}\setminus\{0\}$ defined by $\operatorname{sgn}(x) = x/|x|$, $x \in \mathbb{R}\setminus\{0\}$, then the mapping

$$\gamma \to (\operatorname{sgn}(\gamma), |\gamma|), \qquad \gamma \in G, \quad (26)$$

is a bicontinuous group isomorphism of G onto the direct product group $C_2 \times \mathbb{R}^{\oplus}$. Now, the dual group of C_2 consists of the two characters $\{\mathrm{Sgn}, I_1\}$, where I_1 denotes the function constantly equal to 1 on C_2 and Sgn denotes the restriction of sgn to $\{-1, 1\}$. Since the logarithm ln: $\mathbb{R}^{\oplus} \to \mathbb{R}$ is a bicontinuous group isomorphism of $\mathbb{R}^{\oplus}$ onto the additive group of reals $\mathbb{R}$, it follows that the dual group $\hat{\mathbb{R}}^{\oplus}$ can be identified with

$\hat{\mathbb{R}}$ which is, of course, just $\mathbb{R}$ again. It follows from (26) that $\hat{G}$ consists of precisely those characters on G of the form

$$\gamma \to e^{is \ln|\gamma|}, \qquad \gamma \in G, \tag{27.1}$$

for some $s \in \mathbb{R}$, or of the form

$$\gamma \to e^{is \ln|\gamma|} \operatorname{sgn}(\gamma), \qquad \gamma \in G, \tag{27.2}$$

for some $s \in \mathbb{R}$.

THEOREM 4.1. *Let $1 < p < 2$ and T: $\mathbb{R} \to \mathscr{L}(H^p(\mathbb{D}))$ be a strongly continuous, hyperbolic group of isometries. Then T is an extended Stone-type group.*

Proof. It is shown in the proof of Theorem 3.6 in [3] that there exists a linear isometry U of $H^p(\mathbb{D})$ onto $H^p(\mathbb{R})$ and real numbers δ and ρ, with $\rho > 0$, such that

$$UT(t)U^{-1} = e^{i\delta t} V(\rho t), \qquad t \in \mathbb{R},$$

where V: $\mathbb{R} \to \mathscr{L}(H^p(\mathbb{R}))$ is the group of isometries given by

$$V(t)f\colon w \to e^{t/p} f(e^t w), \qquad w \in \mathbb{R},$$

for each $f \in H^p(\mathbb{R})$. So, by Lemmas 1.2 and 1.3 it suffices to show that V is an extended Stone-type group.

The linear operator C: $L^p(\mathbb{R}) \to L^p(G)$ defined by

$$Cf\colon s \to |s|^{1/p} f(s), \qquad s \in \mathbb{R},$$

for each $f \in L^p(\mathbb{R})$, is an isometry of $L^p(\mathbb{R})$ onto $L^p(G)$. If $C^{(p)}$ denotes the restriction of C to $H^p(\mathbb{R})$ and $Z = C^{(p)}(H^p(\mathbb{R}))$ is the range of $C^{(p)}$, then Z is a closed subspace of $L^{(p)}(G)$ and $C^{(p)}$ is an isometry of $H^p(\mathbb{R})$ onto Z. So, by Lemma 1.2 it suffices to show that the group $\mathscr{H}$: $\mathbb{R} \to \mathscr{L}(Z)$ given by

$$\mathscr{H}(t) = C^{(p)} V(t)(C^{(p)})^{-1}, \qquad t \in \mathbb{R},$$

is an extended Stone-type group. Define a group $\tilde{V}$: $\mathbb{R} \to \mathscr{L}(L^p(\mathbb{R}))$ by

$$\tilde{V}(t)f\colon w \to e^{t/p} f(e^t w), \qquad w \in \mathbb{R},\ f \in L^p(\mathbb{R}),$$

for each $t \in \mathbb{R}$. Then Z is an invariant subspace for each operator $\tilde{\mathscr{H}}(t) = C\tilde{V}(t)C^{-1}$, $t \in \mathbb{R}$, and the restriction of $\tilde{\mathscr{H}}(t)$ to Z is precisely the operator

$\mathscr{H}(t)$, for each $t \in \mathbb{R}$. By direct calculation it can be shown that, for each $t \in \mathbb{R}$, $\tilde{\mathscr{H}}(t)$ is the operator given by

$$\tilde{\mathscr{H}}(t) f: \gamma \to f(e^t \gamma), \qquad \gamma \in G,$$

for each $f \in L^p(G)$.

Let κ denote the Fourier transform map of $L^p(G)$ into $L^q(\hat{G})$, where $q > 0$ is the number such that $p^{-1} + q^{-1} = 1$. Then κ is continuous, injective, and has dense range in $L^q(\hat{G})$. For each $t \in \mathbb{R}$, let $\tilde{\mathscr{H}}_q(t)$ be the operator in $\mathscr{L}(L^q(\hat{G}))$ defined by

$$\tilde{\mathscr{H}}_q(t) \psi: v \to [e^t, v]\, \psi(v), \qquad v \in \hat{G}, \tag{28}$$

for each $\psi \in L^q(\hat{G})$, where $[e^t, v]$ denotes the evaluation of the character $v \in \hat{G}$ at the element $e^t \in G$. Noting that for each $t \in \mathbb{R}$, $\tilde{\mathscr{H}}(t)$ is the operator in $L^p(G)$ of translation by the element e^t of G, it follows from properties of the Fourier transform that

$$\tilde{\mathscr{H}}_q(t)\, \kappa f = \kappa \tilde{\mathscr{H}}(t) f, \qquad f \in L^p(G),$$

for each $t \in \mathbb{R}$. Accordingly, $L^q(\hat{G})$ is an $\tilde{\mathscr{H}}$-admissible space. We claim that the extended group $\tilde{\mathscr{H}}_q: \mathbb{R} \to \mathscr{L}(L^q(\hat{G}))$ is actually a Stone-type group.

By the discussion prior to Theorem 4.1 we can identify $\hat{G}$ with $\hat{C}_2 \times \hat{\mathbb{R}}^{\oplus}$. Then each element of $L^q(\hat{G})$ may be interpreted as a function of two variables (u, s) with $u \in \{-1, 1\}$ and $s \in \mathbb{R}$, where s is identified with the character $e^{is \ln|\cdot|}$ in $\hat{\mathbb{R}}^{\oplus}$. That is, characters of the form (27.1) are identified with $(1, s)$, $s \in \mathbb{R}$, and characters of the form (27.2) are identified with $(-1, s)$, $s \in \mathbb{R}$. The Haar measure λ on $\hat{G}$ is the product of the Haar measures λ_2 on $\hat{C}_2$ and λ_1 on $\hat{\mathbb{R}}^{\oplus}$ (which equals Lebesgue measure when $\hat{\mathbb{R}}^{\oplus}$ is identified with $\mathbb{R}$).

If $g \in L^q(\hat{G})$ and $h \in L^q(\hat{G}))'$, then it follows from (28) that

$$\langle \tilde{\mathscr{H}}_q(t)\, g, h \rangle = \int_{\hat{G}} [e^t, (u, s)]\, g(u, s)\, h(u, s)\, d\lambda(u, s), \tag{29}$$

for each $t \in \mathbb{R}$. Since $e^t > 0$ it follows from (27.1) and (27.2) that for each $t \in \mathbb{R}$, $[e^t, (u, s)] = e^{its}$ for all $u \in \{-1, 1\}$, $s \in \mathbb{R}$. Substitute this into (29) and use Fubini's theorem gives

$$\langle \tilde{\mathscr{H}}_q(t)\, g, h \rangle = \int_{\mathbb{R}} e^{ist} (g(1, s)\, h(1, s) + g(-1, s)\, h(-1, s))\, d\lambda_1(s), \tag{30}$$

for each $t \in \mathbb{R}$.

For each Borel set $E \in \mathscr{B}$, define a projection operator $P(E) \in \mathscr{L}(L^q(\hat{G}))$ by the formula

$$P(E) f: (u, s) \to \chi_E(s) f(u, s), \qquad f \in L^q(\hat{G}).$$

The so-defined set function $P(\cdot)$ is a spectral measure. If g and h are as above, then it follows, by Fubini's theorem again, that the measure $\langle P(\cdot) g, h \rangle$ is given by

$$\langle P(E) g, h \rangle = \int_E (g(1, s) h(1, s) + g(-1, s) h(-1, s)) \, d\lambda_1(s), \qquad E \in \mathscr{B}.$$

Hence, for each $t \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} e^{ist} \, d\langle P(s) g, h \rangle = \int_{\mathbb{R}} e^{ist}(g(1, s) h(1, s) + g(-1, s) h(-1, s)) \, d\lambda_1(s).$$

It follows from (30) that $\tilde{\mathscr{H}}_q$ is the Fourier–Stieltjes transform of P, that is, $\tilde{\mathscr{H}}_q$ is a Stone-type group in $L^q(\hat{G})$.

Let ι denote the restriction of κ to the closed subspace Z of $L^p(G)$ and Y denote the closure of $\iota(Z)$ in $L^q(\hat{G})$. Then ι is continuous, injective and has dense range in Y.

CLAIM 1. *The Banach space Y is invariant for each operator $\tilde{\mathscr{H}}_q(t)$, $t \in \mathbb{R}$.*

Proof. Fix $t \in \mathbb{R}$. If $\psi \in \iota(Z)$, then $\psi = \iota f = \kappa f$ for some $f \in Z$ and so it follows from (28) and properties of the Fourier transform that

$$\tilde{\mathscr{H}}_q(t) \psi = [e^t, \cdot\,] \psi = [e^t, \cdot\,] k f = \kappa f(e^t, \cdot) = \kappa \tilde{\mathscr{H}}(t) f.$$

Since $f \in Z$ and Z is an invariant subspace for the operator $\tilde{\mathscr{H}}(t)$ it follows that $\tilde{\mathscr{H}}_q(t) \psi = \kappa \tilde{\mathscr{H}}(t) f = \iota \tilde{\mathscr{H}}(t) f \in \iota(Z) \subseteq Y$. That is, $\tilde{\mathscr{H}}_q(t) \psi \in Y$ whenever $\psi \in \iota(Z)$. Since Y is the closure of $\iota(Z)$ it follows from the continuity of $\tilde{\mathscr{H}}_q(t)$ that $\tilde{\mathscr{H}}_q(t)(Y) \subseteq Y$.

CLAIM 2. *The space Y is invariant for each projection $P(E)$, $E \in \mathscr{B}$.*

Proof. If we define $F(\lambda) = P((-\infty, \lambda])$, for each $\lambda \in \mathbb{R}$, then the so defined function $F: \mathbb{R} \to \mathscr{L}(L^q(\hat{G}))$ is the (unique) spectral family for the strongly continuous group $\tilde{\mathscr{H}}_q$. Since Y is invariant for each operator $\tilde{\mathscr{H}}_q(t)$, $t \in \mathbb{R}$ (cf., Claim 1), it follows that Y is invariant for each operator $F(\lambda)$, $\lambda \in \mathbb{R}$, [3, Theorem 3.4], and hence, is invariant for each projection $P((u, v]) = F(v) - F(u)$, whenever $u \leqslant v$ are real numbers. If $\mathscr{R}$ denotes the ring of sets generated by the semiring of intervals $\{(u, v]; u, v \in \mathbb{R}, u \leqslant v\}$, then it is clear by additivity of P that Y is invariant for each projection $P(E)$, $E \in \mathscr{R}$. Let $\mathscr{M} = \{E \in \mathscr{B}; P(E)(Y) \subseteq Y\}$. If $\{E_n\}_{n=1}^{\infty}$ is a monotone sequence of elements of $\mathcal{M}$, with limit E say, then it follows from the (strong) σ-additivity of P that $E \in \mathcal{M}$. Accordingly, $\mathcal{M}$ is a monotone class of sets containing $\mathcal{R}$ and so $\mathcal{M} = \mathcal{B}$. This completes the proof of the claim.

It follows from Claim 2 that the restriction, Q, of P to Y, defines a spectral measure $Q: \mathcal{B} \to \mathcal{L}(Y)$ whose Fourier–Stieltjes transform is, of course, the group $\tilde{\mathcal{H}}_q$ restricted to the closed invariant subspace Y. Denote this restricted group by $\mathcal{H}_Y$, that is, for each $t \in \mathbb{R}$, $\mathcal{H}_Y(t)$ is the restriction to Y of the operator $\tilde{\mathcal{H}}_q(t)$. Then

$$\mathcal{H}_Y(t) = \int_{\mathbb{R}} e^{its}\, dQ(s), \qquad t \in \mathbb{R},$$

that is, $\mathcal{H}_Y$ is a Stone-type group in Y. But, if $f \in Z$, then it follows from the definition of ι and the fact, for each $t \in \mathbb{R}$, that $\mathcal{H}(t)$ is the restriction to Z of the operator $\tilde{\mathcal{H}}(t)$, that

$$\mathcal{H}_Y(t)\, \iota f = \tilde{\mathcal{H}}_q(t)\, \kappa f = \kappa \tilde{\mathcal{H}}(t) f = \iota \mathcal{H}(t) f,$$

for each $t \in \mathbb{R}$. This shows that Y is an $\mathcal{H}$-admissible space and hence, that $\mathcal{H}$ is an extended Stone-type group in the space Y. The proof of Theorem 4.1 is thereby complete.

There remains the case of strongly continuous, hyperbolic groups of isometries $T: \mathbb{R} \to \mathcal{L}(H^p(\mathbb{D}))$ for p in the interval $(2, \infty)$. It was noted in the proof of Theorem 4.1 that such a group T is of extended Stone-type if and only if the associated group $\mathcal{H}: \mathbb{R} \to \mathcal{L}(Z)$ given by

$$\mathcal{H}(t) f: \gamma \to f(e^t \gamma), \qquad \gamma \in G,\ f \in Z,$$

for each $t \in \mathbb{R}$, is an extended Stone-type group, where Z is a certain closed subspace of $L^p(G)$; note that the arguments used to deduce this fact did not require the restriction $1 < p < 2$. Observing that $\mathcal{H}$ is the restriction to Z of a certain group of translations $\tilde{\mathcal{H}}$ in $L^p(G)$ and that *none* of the operators $\tilde{\mathcal{H}}(t)$, $t \in \mathbb{R} \setminus \{0\}$, is an extended pseudo-unitary operator [7, Theorem 4.2], we would very much expect that an argument along the lines of the proof of Theorem 3.4 would show that $\mathcal{H}$, hence also T, is not an extended Stone-type group in any $\mathcal{H}$-admissible space. However, to carry out such an argument would require an explicit description of the spectral family for $\mathcal{H}$ or equivalently, of the spectral family for the group V given in the proof of Theorem 4.1. Unfortunately, unlike for parabolic groups, there is no such description available in the hyperbolic case (unless $p = 2$ [11]). So, we can only suggest a conjecture, namely, that for $2 < p < \infty$, a strongly continuous, hyperbolic group of isometries T: $\mathbb{R} \to \mathcal{L}(H^p(\mathbb{D}))$ is not an extended Stone-type group in any T-admissible space.

Final Remark. It is worth noting that any one-parameter strongly continuous group of isometries in $H^\infty(\mathbb{D})$ necessarily satisfies Stone's theorem in the space $H^\infty(\mathbb{D})$ itself. For, if T: $\mathbb{R} \to \mathscr{L}(H^\infty(\mathbb{D}))$ is such a group, then it follows from a result of L. A. Rubel (see [5], for example) that there exists $\alpha \in \mathbb{R}$ such that

$$T(t) = e^{i\alpha t} I, \qquad t \in \mathbb{R},$$

and hence, T is the Fourier–Stieltjes transform of the spectral measure P: $\mathscr{B} \to \mathscr{L}(H^\infty(\mathbb{D}))$ given by $P(E) = \chi_E(\{\alpha\}) I$, for each $E \in \mathscr{B}$.

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