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The nullcone in the multi-vector representation of the symplectic group and related combinatorics

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ABSTRACT

We study the nullcone in the multi-vector representation of the symplectic group with respect to a joint action of the general linear group and the symplectic group. By extracting an algebra over a distributive lattice structure from the coordinate ring of the nullcone, we describe a toric degeneration and standard monomial theory of the nullcone in terms of double tableaux and integral points in a convex polyhedral cone.

1. Introduction

Let GL_n and Sp_{2n} denote respectively the general linear group and the symplectic group over the complex number field $\mathbb{C}$. Combinatorics of tableaux provides a unifying scheme to understand representation theory of GL_n and geometry of the flag varieties and the Grassmann varieties. In particular, the theory of double tableaux gives a finite presentation of the coordinate ring of the affine space $M_{n,m} \cong \mathbb{C}^n \otimes \mathbb{C}^m$ which is compatible with the natural action of $GL_n \times GL_m$. Moreover, we can explicitly describe weight bases of representation spaces from the combinatorial structure of tableaux.

In this paper, we develop a parallel theory for the Sp_{2n}-nullcone $\mathcal{N}_{k,2n}$ which is defined by Sp_{2n}-invariant polynomials on the space $M_{k,2n}$ with vanishing constant terms. We begin with a known algebro-combinatorial description of the space $M_{n,m}$ as a cell of the Grassmann variety of n-dimensional spaces in $\mathbb{C}^{m+n}$. Using this observation, we construct a convex polyhedral cone $\mathcal{C}(M_{n,m})$ associated with the space $M_{n,m}$ and study the integral points in the cone. Then we characterize the defining ideal of $\mathcal{N}_{k,2n}$ in terms of integral points in $\mathcal{C}(M_{k,2n})$. This characterization provides a convex polyhedral cone $\mathcal{C}(\mathcal{N}_{k,2n})$ associated with $\mathcal{N}_{k,2n}$ compatible with the action of $GL_k \times Sp_{2n}$. Our construction of the polyhedral cone $\mathcal{C}(\mathcal{N}_{k,2n})$ turns out to be related to a fiber product of the Gelfand–Tsetlin patterns.

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We also describe explicit joint weight vectors of $GL_k \times Sp_{2n}$ in the coordinate ring of $\mathcal{N}_{k,2n}$ in terms of standard double tableaux. As a result, we obtain standard monomial theory for the nullcone and show that the nullcone can be degenerated to an affine toric variety presented by an algebra over a distributive lattice.

The toric degenerations of spherical varieties (e.g., [1,4,8,32]) and standard monomial theory (e.g., [27,29]) have been actively studied. Using classical invariant theory, we can study such combinatorial and geometric results in connection with various representation theoretic questions.

The recent papers [16–18] and their sequels construct algebras encoding branching rules of representations of the classical groups, and then study their standard monomial bases and toric degenerations. With a similar philosophy, [19] and [22] study tensor products of representations for the classical groups with explicit highest weight vectors. By degenerating the multi-homogeneous coordinate rings of the flag varieties, [20] and [21] describe weight vectors of the classical groups in terms of the Gelfand–Tsetlin polyhedral cone. This paper may be understood as an application of such approaches to the nullcone in the multi-vector representation of the symplectic group to obtain explicit combinatorial and representation theoretic descriptions of it. For the nullcone associated with representations of reductive groups and its interesting applications, we refer readers to [24,25].

This paper is arranged as follows: In Section 2, we introduce notations for tableaux and patterns, and review standard monomial theory for the coordinate ring $\mathbb{C}[M_{n,m}]$ of $M_{n,m}$. In Section 3, we define the convex polyhedral cone $\mathcal{C}(M_{n,m})$ associated with a degeneration of $M_{n,m}$, and study its connection to representations of the general linear group. In Section 4, we study the coordinate ring $\mathcal{R}(\mathcal{N}_{k,2n})$ of $\mathcal{N}_{k,2n}$ and show its standard monomial theory and toric degeneration. In Section 5, we describe the integral points in the convex polyhedral cone $\mathcal{C}(\mathcal{N}_{k,2n})$ for $\mathcal{N}_{k,2n}$ and explain its relations to representation theory.

2. Affine space: $M_{n,m}$

In this section, we review some results on the Young tableaux, the Gelfand–Tsetlin patterns, and their applications to representation theory and geometry of the Grassmann varieties.

2.1. Tableaux

Let $M_{n,m} = M_{n,m}(\mathbb{C})$ be the space of complex n by m matrices:

$$M_{n,m} = \big\{(x_{ij}):\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\big\}. \tag{2.1}$$

A *Young diagram* or *shape* is an array of square boxes arranged in left-justified horizontal rows with each row no longer than the one above it (e.g., [7,31]). We identify a shape D with its sequence of row lengths $D = (r_1, r_2, \ldots)$. Then the *transpose* D^t of D is a shape $(c_1, c_2, \ldots)$ where c_i is the length of the i-th column of D. The *length* $\ell(D)$ of shape D is the number of rows in D. For a subset $I = [i_1, \ldots, i_l]$ (respectively $J = [j_1, \ldots, j_l]$) of $\{1, \ldots, n\}$ (respectively $\{1, \ldots, m\}$), which we can think of a filling of shape $(1, \ldots, 1)$ of length l with its elements, we shall call the pair $[I : J]$ an *one-line tableau* of length $\ell([I : J]) = l$. We assume that the entries of I and J are listed in increasing order, i.e., $1 \leqslant i_1 < \cdots < i_l \leqslant n$ and $1 \leqslant j_1 < \cdots < j_l \leqslant m$.

A partial order $\preceq$, called the *tableau order*, can be imposed on the set of one-line tableaux

$$D(n,m) = \big\{[I : J]:\ |I| = |J| \leqslant \min(n,m)\big\}$$

as follows: $[I : J] \preceq [I' : J']$, if the length of $[I : J]$ is not smaller than that of $[I' : J']$, and $i_k \leqslant i'_k$ and $j_k \leqslant j'_k$ for each k not bigger than the length of $[I' : J']$. Then it is easy to see that with respect to the tableau order $D(n,m)$ forms a distributive lattice $(D(n,m), \wedge, \vee)$.

Consider a collection $\{[I_1 : J_1], \ldots, [I_u : J_u]\} \subset D(n,m)$ with $l_k = \ell([I_k : J_k])$ for each k. A concatenation t of its elements is called a *double tableau*, if they are arranged so that $l_k \geqslant l_{k+1}$ for all k. The *shape* $sh(\mathsf{t})$ of t is the Young diagram $(l_1, \ldots, l_u)^t$. We note that by considering first components I_k and the second components J_k separately, we can think of a double tableau of shape D in terms of a pair of fillings of the same shape D.

Let us write $\delta_{[I:J]}$ for the map from $M_{n,m}$ to $\mathbb{C}$ by assigning to a matrix $X \in M_{n,m}$ the determinant of the $l \times l$ submatrix of X with rows and columns indexed by I and J respectively:

$$\delta_{[I:J]} = \det \begin{bmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_l} \\ \vdots & \ddots & \vdots \\ x_{i_l j_1} & \cdots & x_{i_l j_l} \end{bmatrix}.$$

For a double tableau t consisting of $\{[I_k : J_k]\}$, we define the corresponding element in the coordinate ring $\mathbb{C}[M_{n,m}]$ of $M_{n,m}$ to be the following product:

$$\Delta(\mathrm{t}) = \prod_{1 \leqslant k \leqslant u} \delta_{[I_k:J_k]}.$$

Definition 2.1. A double tableau t is called a *standard tableau* if the one-line tableaux in t form a multiple chain in $D(n,m)$, i.e.,

$$[I_1 : J_1] \preceq \cdots \preceq [I_u : J_u].$$

For a standard tableau t, we call $\Delta(\mathrm{t})$ a *standard monomial.*

Now, let us consider the following subposet:

$$\begin{aligned} L(n,m) &= \bigl\{[I:J] \in D(n,m):\ I = \bigl[1,\ldots,|J|\bigr]\bigr\} \\ &\cong \bigl\{[j_1,\ldots,j_l]:\ l \leqslant \min(n,m),\ 1 \leqslant j_1 < \cdots < j_l \leqslant m\bigr\}. \end{aligned}$$

For each $[I:J] \in L(n,m)$, since the first component I is determined by the size of J, we shall write J for $[I:J]$. Similarly, we can also think of the subposet $L'(n,m)$ consisting of $[I:J]$ with $J = [1,\ldots,|I|]$. Then, a *semistandard tableau* as found in the literature (e.g., [7,31]) can be defined as a multiple chain in such posets with respect to the tableau order.

This poset $L(n,m)$ has been extensively studied for the flag varieties, the Grassmann varieties, and the determinantal varieties. For example, the elements of $L(n,m+n)$ with fixed length n, which we shall denote by $Pl(n,m+n)$, may encode the Plücker coordinates for the Grassmann variety $Gr(n,m+n)$ of n-dimensional spaces in $\mathbb{C}^{m+n}$. In fact, this case is general enough to study double tableaux thanks to the following correspondence.

Lemma 2.2. *The following map ξ gives an order isomorphism from $D(n,m)$ to the subposet of $L(n,m+n)$ consisting of all the elements of length n except $[m+1,\ldots,m+n]$: for $[I:J] \in D(n,m)$ with $I = [i_1,i_2,\ldots,i_h]$ and $J = [j_1,j_2,\ldots,j_h]$,*

$$\xi : [I:J] \mapsto [j_1,j_2,\ldots,j_h,m+u_1,m+u_2,\ldots,m+u_{n-h}]$$

where $\{u_k\}$ is defined so that $\{n+1-u_1,\ldots,n+1-u_h\}$ is complement to I in $\{1,2,\ldots,n\}$.

For the proof, see [30, p. 519].

2.2. Gelfand–Tsetlin patterns

The poset $\widehat{L}(m,m)$, $L(m,m)$ with an extra top element, with respect to the tableau order $\preceq$ turns out to be a distributive lattice whose join-irreducibles form the following poset, which we shall call the *Gelfand–Tsetlin* (*GT*) *poset*:

$$\Gamma_m = \bigl\{z_j^{(i)}:\ 1 \leqslant j \leqslant i \leqslant m\bigr\}$$

satisfying $z_j^{(i+1)} \geqslant z_j^{(i)} \geqslant z_{j+1}^{(i+1)}$ for all i and j [20, Theorem 3.8]. We call $z^{(i)} = (z_1^{(i)}, z_2^{(i)}, \dots, z_i^{(i)})$ the *i-th row* of Γ_m. We will draw it in a reversed triangular array so that $z_j^{(i)}$ are decreasing from left to right along diagonals. For example, Γ_4 can be drawn as

$$\begin{array}{ccccccc}
z_1^{(4)} & & z_2^{(4)} & & z_3^{(4)} & & z_4^{(4)} \\
& z_1^{(3)} & & z_2^{(3)} & & z_3^{(3)} & \\
& & z_1^{(2)} & & z_2^{(2)} & & \\
& & & z_1^{(1)} & & &
\end{array}$$

Definition 2.3. A GT pattern p of GL_m is an order preserving map from Γ_m to the set of non-negative integers:

$$\mathsf{p} : \Gamma_m \to \mathbb{Z}_{\geqslant 0},$$

and the i-th row of p is $\mathsf{p}(z^{(i)}) = (\mathsf{p}(z_1^{(i)}), \mathsf{p}(z_2^{(i)}), \dots, \mathsf{p}(z_i^{(i)}))$.

The m-th row of a GT pattern p of GL_m will be alternatively called the *top row* of p. By identifying p with its values, our definition agrees with the original one in [10].

Note that for each i, since $\mathsf{p}(z_1^{(i)}) \geqslant \mathsf{p}(z_2^{(i)}) \geqslant \cdots \geqslant \mathsf{p}(z_i^{(i)}) \geqslant 0$, the i-th row $\mathsf{p}(z^{(i)})$ of a pattern p can be seen as a Young diagram. There is a well-known conversion procedure between semistandard Young tableaux and GT patterns compatible with successive branching rules of GL_k down to GL_{k-1} for $2 \leqslant k \leqslant m$. See, e.g., [11, Proposition 8.1.6].

Lemma 2.4. *The following procedure gives a bijection from the set of GT patterns of GL_m with a fixed m-th row D and the set of semistandard tableaux of shape D with entries from $\{1, \dots, m\}$: for a given GT pattern p of GL_m with $\mathsf{p}(z^{(m)}) = (r_1, \dots, r_m)$, fill in the cells in Young diagram $(r_1, \dots, r_m)$ corresponding to the skew diagram $\mathsf{p}(z^{(i)})/\mathsf{p}(z^{(i-1)})$ with i for $2 \leqslant i \leqslant m$, then fill in the cells corresponding to $\mathsf{p}(z^{(1)})$ with* 1.

The collection $\mathcal{P}(m)$ of all the GT patterns of GL_m with the function addition forms a semigroup, or more precisely a commutative monoid with the zero function as its identity, which we shall call the *semigroup of patterns* for GL_m:

$$\mathcal{P}(m) = \{\mathsf{p}\colon \Gamma_m \to \mathbb{Z}_{\geqslant 0}\}. \tag{2.2}$$

Then $\mathcal{P}(m)$ is generated by characteristic functions over order increasing subsets of Γ_m and these generators correspond to elements of $L(m, m)$ via Lemma 2.4. In this connection, one can realize the semigroup ring of $\mathcal{P}(m)$ as the Hibi algebra [12] over the distributive lattice $L(m, m)$. Moreover, the semigroup structure is compatible with the combinatorial correspondence given in Lemma 2.4 in the following sense: the GT pattern corresponding to a semistandard Young tableau t or equivalently a multiple chain $J_1 \preceq \cdots \preceq J_u$ in $L(m, m)$ is the product of the corresponding GT patterns p_{J_k} in the semigroup $\mathcal{P}(m)$, i.e.,

$$\mathsf{t} = (J_1 \preceq \cdots \preceq J_u) \mapsto \mathsf{p}_\mathsf{t} = \sum_{k=1}^{u} \mathsf{p}_{J_k} \tag{2.3}$$

and also $\mathsf{p}_J + \mathsf{p}_{J'} = \mathsf{p}_{J \wedge J'} + \mathsf{p}_{J \vee J'}$. We refer readers to [15] and [20] for further details.

2.3. Standard monomials

Let us review standard monomial theory for the Grassmann variety $Gr(n, m+n)$ of n-dimensional spaces in $\mathbb{C}^{m+n}$ and its application to a presentation of the coordinate ring $\mathbb{C}[M_{n,m}]$ of the space $M_{n,m}$. The proofs of the results discussed in this subsection and further details on the structure of $\mathbb{C}[M_{n,m}]$ can be found in [3, §7] and [30, §13].

Note that the Plücker coordinates for $Gr(n, m+n)$ can be matched with the elementary basis elements of $\bigwedge^n \mathbb{C}^{m+n}$. By taking them as elements of $L(n, m+n)$ with fixed length n:

$$Pl(n, m+n) = \{K \in L(n, m+n): |K| = n\},$$

we shall continue to denote by $\delta_K \in \mathbb{C}[M_{n,m+n}]$ the maximum minors over $M_{n,m+n}$ whose columns are indexed by $K \in Pl(n, m+n)$. Then for any incomparable pair $K, K' \in Pl(n, m+n)$, by applying the Plücker relations to $\delta_K \delta_{K'}$, we obtain its standard expression.

Lemma 2.5. *(See [9, pp. 234, 236].)*

i) *For $K, K' \in Pl(n, m+n)$, the corresponding product $\delta_K \delta_{K'} \in \mathbb{C}[M_{n,m+n}]$ can be uniquely expressed as a linear combination of standard monomials, i.e.,*

$$\delta_K \delta_{K'} = \sum_r c_r \delta_{T_r} \delta_{T'_r} \tag{2.4}$$

where $T_r \preceq T'_r$ in $Pl(n, m+n)$ for each r.

ii) *In the right-hand side, $\delta_{K \wedge K'} \delta_{K \vee K'}$ appears with coefficient 1, and $T_r \preceq K \wedge K' \preceq K \vee K' \preceq T'_r$ for all r. Moreover, for each $(T_r, T'_r) \neq (K \wedge K', K \vee K')$, let a be the smallest integer such that the sum s of the a-th entries of T_r and T'_r is different from the sum s_0 of the a-th entries of K and K'. Then $s > s_0$.*

By applying the above identities, one can show that any product $\prod \delta_K \in \mathbb{C}[M_{n,m+n}]$ with $K \in Pl(n, m+n)$ can be uniquely expressed as a linear combination of standard monomials $\Delta(\mathrm{t}) = \prod_j \delta_{K_j}$ with $K_1 \preceq K_2 \preceq \cdots$. See [3,9,13] for further details.

Note that all the maximal minors $\delta_K \in \mathbb{C}[M_{n,m+n}]$ are invariant under the left multiplication of the special linear group SL_n on $\mathbb{C}[M_{n,m+n}]$ and in fact the maximal minors generate the invariant ring $\mathbb{C}[M_{n,m+n}]^{SL_n}$. This shows that the invariant ring $\mathbb{C}[M_{n,m+n}]^{SL_n}$ forms an algebra with straightening laws (ASL) with standard monomials $\Delta(\mathrm{t})$ as its basis.

Now, let us consider the embedding $M_{n,m} \to M_{n,m+n}$ given by $X \mapsto (X' \mid I_n)$ and

$$M^0_{n,m+n} = \{(X' \mid I_n)\} \subset M_{n,m+n}$$

where I_n is the $n \times n$ identity matrix and X' is the matrix obtained by reversing the rows of $X = (x_{i,j})$, i.e.,

$$X' = \begin{pmatrix} x_{n,1} & \cdots & x_{n,m-1} & x_{n,m} \\ x_{n-1,1} & & x_{n-1,m-1} & x_{n-1,m} \\ \vdots & \ddots & & \vdots \\ x_{1,1} & \cdots & x_{1,m-1} & x_{1,m} \end{pmatrix}.$$

For each $K \in Pl(n, m+n)$ which is not $[m+1, \ldots, m+n]$, consider the restriction δ^0_K to $M^0_{n,m+n}$ of the maximal minor $\delta_K \in \mathbb{C}[M_{n,m+n}]$. Then δ^0_K is equal to, up to sign, the minor $\delta_{\xi^{-1}(K)} \in \mathbb{C}[M_{n,m}]$ over $M_{n,m}$ with rows and columns given by the one-line tableau $\xi^{-1}(K) \in D(n, m)$ via the map ξ defined in Lemma 2.2. This provides an algebraic realization of the space $M_{n,m} \cong M^0_{n,m+n}$ as a cell of the Grassmann variety $Gr(n, m+n)$.

Proposition 2.6. *(See [3, Lemma 7.2.6], [30, p. 522].) The map $\widehat{\xi}$ from $\mathbb{C}[M_{n,m+n}]^{SL_n}$ to $\mathbb{C}[M_{n,m}]$:*

$$\widehat{\xi} : \delta_K \mapsto \delta_{\xi^{-1}(K)}$$

gives an isomorphism between $\mathbb{C}[M_{n,m+n}]^{SL_n} / \ker \widehat{\xi}$ and $\mathbb{C}[M_{n,m}]$ where $\ker \widehat{\xi}$ is the ideal generated by $(\delta_{[m+1,\ldots,m+n]} - 1)$.

Since $[m+1, \ldots, m+n]$ is the largest element in $Pl(n, m+n)$, standard monomial basis elements $\prod_r \delta_{K_r}$ which do not end with $\delta_{[m+1,\ldots,m+n]}$ form a $\mathbb{C}$-basis for $\mathbb{C}[M_{m,n}]^{SL_n} / \ker \widehat{\xi}$. Using this map $\widehat{\xi}$, we can transfer the multiplicative structure of $\mathbb{C}[M_{n,m+n}]^{SL_n}$ to $\mathbb{C}[M_{n,m}]$. In particular, since ξ in Lemma 2.2 is an order isomorphism, the properties of straightening laws given in Lemma 2.5 can be also transferred to any product $\delta_{[I:J]}\delta_{[I':J']}$ in $\mathbb{C}[M_{n,m}]$.

Corollary 2.7. *Let A, B be elements in $D(n,m)$. Then in the algebra $\mathbb{C}[M_{n,m}]$, the corresponding product $\delta_A\delta_B$ can be expressed as a linear combination of standard monomials*

$$\delta_A\delta_B = \sum_r c_r \delta_{X_r}\delta_{Y_r} \tag{2.5}$$

such that for each r we have either $X_r \preceq A \wedge B \preceq A \vee B \preceq Y_r$ or $X_r \preceq A \wedge B$ with $\delta_{Y_r} = 1$, and $\delta_{A\wedge B}\delta_{A\vee B}$ appears in the right-hand side with coefficient 1.

Let us consider a non-standard monomial $\prod_r \delta_{[I_r:J_r]}$ with $l_r = \ell([I_r : J_r])$ and $l_1 \geqslant l_2 \geqslant \cdots$. Then, by applying the above relations as many times as necessary, we can express $\prod_r \delta_{[I_r:J_r]}$ as a linear combination of standard monomials with shapes $(l'_1, l'_2, \ldots)^t$ such that $\sum_{r=1}^k l'_r \geqslant \sum_{r=1}^k l_r$ for each $k \geqslant 1$. Hence we can impose a filtration $\mathsf{F}^{sh} = \{\mathsf{F}^{sh}_D\}$ by shapes on the algebra $\mathbb{C}[M_{n,m}]$, and then consider its associated graded algebra:

$$\begin{aligned}
\mathbb{C}[M_{n,m}] &= \sum_{\ell(D)\leqslant \min(n,m)} \mathsf{F}^{sh}_D\big(\mathbb{C}[M_{n,m}]\big), \\
\mathsf{gr}^{sh}\big(\mathbb{C}[M_{n,m}]\big) &= \sum_{\ell(D)\leqslant \min(n,m)} \mathsf{gr}^{sh}_D\big(\mathbb{C}[M_{n,m}]\big).
\end{aligned} \tag{2.6}$$

Finally, we obtain standard monomial theory for $\mathbb{C}[M_{n,m}]$.

Corollary 2.8. *(See [3, Theorem 7.2.7], [30, p. 530].) Standard monomials $\Delta(\mathrm{t})$ form a $\mathbb{C}$-basis of $\mathbb{C}[M_{n,m}]$. For its associated graded algebra, the D-graded component $\mathsf{gr}^{sh}_D(\mathbb{C}[M_{n,m}])$ is spanned by standard monomials of shape D.*

We refer readers to [3,30] and [5,6] for further details on the ASL structure of $\mathbb{C}[M_{n,m}]$ and double tableaux.

3. Lattice cone for $M_{n,m}$

A *lattice cone* is the intersection of a convex polyhedral cone in $\mathbb{R}^l$ for some l with $\mathbb{Z}^l$. We shall construct a lattice cone associated with the space $M_{n,m}$ in terms of order preserving maps on a subposet of the GT poset Γ_{m+n} for GL_{m+n}. Once $\mathbb{C}[M_{n,m}]$ is shown to be a flat deformation of the Hibi algebra $\mathcal{H}_{D(n,m)}$ over $D(n,m)$, the lattice cone for $M_{n,m}$ can be understood as a cone encoding the affine toric variety $Spec(\mathcal{H}_{D(n,m)})$. The lattice cone for $M_{n,m}$ is related to the Gelfand–Tsetlin cones attached to the flag varieties studied in [20,23,28].

3.1. The semigroup $\mathcal{P}_{n,m}$

Let us write $\widehat{1}$ for $[m+1,\ldots,m+n] \in Pl(n,m+n)$. For the semigroup $\mathcal{P}(m+n)$ defined in (2.2), let $\mathcal{P}(n,m)$ denote its subsemigroup generated by the GT patterns p_J corresponding to $J \in Pl(n,m+n)$. Recall that for a subtractive subset B of a monoid $(A,+)$, the quotient monoid A/B is defined by the following equivalence relation: $a \sim a'$ if there are b and b' in B such that $a+b = a'+b'$.

Definition 3.1. The semigroup $\mathcal{P}_{n,m}$ of patterns for $M_{n,m}$ is the quotient of $\mathcal{P}(n,m)$ by the multiples of the pattern $\mathsf{p}_{\widehat{1}}$ corresponding to $\widehat{1}$:

$$\mathcal{P}_{n,m} = \mathcal{P}(n,m)/\langle \mathsf{p}_{\widehat{1}}\rangle.$$

Let $\mathbb{C}[\mathcal{P}_{n,m}]$ denote the semigroup ring of $\mathcal{P}_{n,m}$.

We can obtain a more explicit description for the elements of $\mathcal{P}_{n,m}$. Simple computations of the bijection given in Lemma 2.4 can easily show the following.

Lemma 3.2.

i) *For every* $\mathsf{p} \in \mathcal{P}(n,m)$, *the support of* p *is contained in*

$$\Gamma^n_{m+n} = \{z^{(i)}_j \in \Gamma_{m+n}\colon\ j \leqslant n\}.$$

ii) *If* $\mathsf{p} \in \mathcal{P}(n,m)$, *then for all* $z^{(a)}_b \in \Gamma_{m+n}$ *with* $z^{(a)}_b \geqslant z^{(m+n)}_n$ *we have* $\mathsf{p}(z^{(a)}_b) = \mathsf{p}(z^{(m+n)}_n)$.

iii) *If* $\mathsf{p} = \mathsf{p}_{\widehat{1}}$, *then* $\mathsf{p}(z^{(a)}_b) = 1$ *for* $z^{(a)}_b \geqslant z^{(m+n)}_n$ *and* $\mathsf{p}(z^{(a)}_b) = 0$ *otherwise.*

Example 3.3. The ordered elements $[1,2,3]$, $[1,3,4]$, $[2,5,6]$, $\widehat{1} = [5,6,7]$ of $Pl(3,7)$ form the following semistandard tableau t of shape $(4,4,4)$:

1	1	2	5
2	3	5	6
3	4	6	7

and we can visualize its corresponding GT pattern p_t for GL_7 by listing its values as

$$\begin{array}{ccccccccccccc}
4 & & 4 & & 4 & & 0 & & 0 & & 0 & & 0 \\
& 4 & & 4 & & 3 & & 0 & & 0 & & 0 & \\
& & 4 & & 3 & & 2 & & 0 & & 0 & & \\
& & & 3 & & 2 & & 2 & & 0 & & & \\
& & & & 3 & & 2 & & 1 & & & & \\
& & & & & 3 & & 1 & & & & & \\
& & & & & & 2 & & & & & &
\end{array}$$

Note that its support is exactly $\Gamma^3_7 \subset \Gamma_7$, and $\mathsf{p}(z^{(a)}_b) = 4$ for all $z^{(a)}_b \geqslant z^{(7)}_3$.

Then, from the description of supports for $\mathsf{p} \in \mathcal{P}(n,m)$ and $\mathsf{p}_{\widehat{1}}$, we obtain the following characterization for the elements of the quotient $\mathcal{P}_{n,m}$.

Corollary 3.4. *Every element of the semigroup* $\mathcal{P}_{n,m}$ *of patterns for* $M_{n,m}$ *can be uniquely represented by an order preserving map to the set of non-negative integers from*

$$\Gamma_{n,m} = \Gamma^n_{m+n} \backslash \{z^{(a)}_b\colon\ z^{(a)}_b \geqslant z^{(m+n)}_n\}.$$

Then the GT pattern for GL_7 shown in Example 3.3 corresponds to the following order preserving map defined on $\Gamma_{3,4}$:

$$\begin{array}{ccccc}
& & 3 & & \\
& 3 & & 2 & \\
3 & & 2 & & 2 \\
& 3 & & 2 & & 1 \\
& & 3 & & 1 & \\
& & & 2 & &
\end{array} \tag{3.1}$$

Now we define the convex polyhedral cone associated with the space $M_{n,m}$ as the collection of all non-negative real valued order preserving maps over $\Gamma_{n,m}$:

$$\mathcal{C}(M_{n,m}) = \{f\colon\ \Gamma_{n,m} \to \mathbb{R}_{\geqslant 0}\}.$$

Then, by identifying f with its values $(f(z^{(a)}_b)) \in \mathbb{R}^{nm}$ for $z^{(a)}_b \in \Gamma_{n,m}$, we can realize the semigroup $\mathcal{P}_{n,m}$ of patterns for $M_{n,m}$ as the set of integral points in $\mathcal{C}(M_{n,m})$, i.e., the *lattice cone for* $M_{n,m}$:

$$\mathcal{P}_{n,m} = \mathcal{C}(M_{n,m}) \cap \mathbb{Z}^{nm}.$$

3.2. Degeneration of $\mathbb{C}[M_{n,m}]$

We want to consider a degeneration of the algebra $\mathbb{C}[M_{n,m}]$. We shall use basically the same degeneration technique shown in [8].

Recall that the Hibi algebra $\mathcal{H}_L$ over a distributive lattice L is the quotient ring

$$\mathcal{H}_L \cong \mathbb{C}[z_x\colon x \in L]/(z_x z_y - z_{x\wedge y} z_{x\vee y})$$

of the polynomial ring by the ideal generated by $\{z_x z_y - z_{x\wedge y} z_{x\vee y}\}$. Then the Hibi algebra defines an affine toric variety [12].

To extract the Hibi algebra structure from the algebra $\mathbb{C}[M_{n,m}]$, we will use the following weight defined via the correspondence given in Lemma 2.2.

Definition 3.5. Let us fix an integer N greater than $2(n+m)$. For $[I:J] \in D(n,m)$ and $\xi([I:J]) = [q_1,\dots,q_n]$, we define their weights $wt([I:J]) = wt([q_1,\dots,q_n])$ as

$$wt\bigl([I:J]\bigr) = \sum_{r\geqslant 1}(m+r-q_r)N^{n-r}.$$

The weight of a double tableau t consisting of $\{[I_k:J_k]\}$ is defined to be the sum of individual weights, i.e., $wt(\mathrm{t}) = \sum_k wt([I_k:J_k])$.

We shall assume that, by extending the above formula, the weight of $[m+1,\dots,m+n]$ is equal to zero. We define the weight of $\Delta(\mathrm{t})$ to be the weight of the corresponding double tableau t.

Proposition 3.6. *For $A, B \in D(n,m)$, let*

$$\delta_A\delta_B = \sum_r c_r \delta_{X_r}\delta_{Y_r}$$

be the standard expression of $\delta_A\delta_B$ given in (2.5)*. Then, $wt(A)+wt(B) \geqslant wt(X_r)+wt(Y_r)$ for all r and the equality holds only for $(X_r,Y_r) = (A\wedge B, A\vee B)$.*

It follows directly from Lemma 2.5 and Corollary 2.7. Let $\xi(A) = K$, $\xi(B) = K'$, $\xi(X_r) = T_r$, and $\xi(Y_r) = T'_r$. If $K = [k_1,\dots,k_n]$ and $K' = [k'_1,\dots,k'_n]$, then the i-th entry of $K\wedge K'$ is $\min\{k_i,k'_i\}$ and the i-th entry of $K\vee K'$ is $\max\{k_i,k'_i\}$ for $1 \leqslant i \leqslant n$. Therefore, if $(T_r,T'_r) = (K\wedge K', K\vee K)$, then we have $wt(A)+wt(B) = wt(X_r)+wt(Y_r)$. Otherwise, from the second statement of Lemma 2.5 we have $wt(A)+wt(B) > wt(X_r)+wt(Y_r)$. Note that if δ_{Y_r} is 1, then as discussed after Definition 3.5 the weight of the corresponding element $[m+1,\dots,m+n]$ is 0, and therefore we still have the inequality $wt(A)+wt(B) > wt(X_r)+wt(Y_r) = wt(X_r)$.

Theorem 3.7. *The algebra $\mathbb{C}[M_{n,m}]$ is a flat deformation of the Hibi algebra $\mathcal{H}_{D(n,m)}$ over $D(n,m)$. More precisely, there is a flat $\mathbb{C}[t]$ module whose general fiber is isomorphic to $\mathbb{C}[M_{n,m}]$ and special fiber is isomorphic to the Hibi algebra over $D(n,m)$.*

Proof. Let us define a $\mathbb{Z}$-filtration $\mathsf{F}^{wt} = \{\mathsf{F}^{wt}_d\}$ on $\mathbb{C}[M_{n,m}]$ with respect to the weight wt, i.e., $\mathsf{F}^{wt}_d(\mathbb{C}[M_{n,m}])$ is the $\mathbb{C}$-span of the set

$$\bigl\{\Delta(\mathrm{t})\colon wt(\mathrm{t}) \leqslant d\bigr\}.$$

The filtration F^{wt} is well defined, since every product $\prod \delta_A$ can be expressed as a linear combination of standard monomials with smaller weights by the above proposition. For all pairs $A, B \in D(n,m)$, since $wt(A)+wt(B) = wt(A\wedge B)+wt(A\vee B)$, $\delta_A\delta_B$ and $\delta_{A\wedge B}\delta_{A\vee B}$ belong to the same associated graded space. Therefore, we have $s_A \cdot_{gr} s_B = s_{A\wedge B} \cdot_{gr} s_{A\vee B}$ where s_C are elements corresponding to δ_C in the associated graded ring $\mathrm{gr}^{wt}(\mathbb{C}[M_{n,m}])$ of $\mathbb{C}[M_{n,m}]$ with respect to the filtration F^{wt}. Then it is straightforward to show that the associated graded ring $\mathrm{gr}^{wt}(\mathbb{C}[M_{n,m}])$ forms the Hibi algebra over

$D(n,m)$. From the general properties of the Rees algebras (e.g., [1]), the Rees algebra $\mathcal{R}^t$ of $\mathbb{C}[M_{n,m}]$ with respect to F^{wt}:

$$\mathcal{R}^t = \bigoplus_{d \geqslant 0} \mathsf{F}_d^{wt}\big(\mathbb{C}[M_{n,m}]\big)t^d$$

is flat over $\mathbb{C}[t]$ with its general fiber isomorphic to $\mathbb{C}[M_{n,m}]$ and special fiber isomorphic to the associated graded ring which is $\mathcal{H}_{D(n,m)}$. □

3.3. Representations

Every irreducible polynomial representation of GL_k is, via the correspondence between dominant weights and Young diagrams, uniquely labeled by a Young diagram with no more than k rows. Let ρ_k^D denote the irreducible representation of GL_k labeled by Young diagram D. Now we impose an action of $GL_n \times GL_m$ on the space $M_{n,m}$ by

$$(g_1, g_2)Q = \big(g_1^t\big)^{-1} Q\, g_2^{-1} \tag{3.2}$$

for $g_1 \in GL_n$, $g_2 \in GL_m$, and $Q \in M_{n,m}$. Then we have the following decomposition of the coordinate ring $\mathbb{C}[M_{n,m}]$ with respect to the action:

$$\mathbb{C}[M_{n,m}] = \sum_{\ell(D) \leqslant \min(n,m)} \rho_n^D \otimes \rho_m^D$$

where the summation runs over all D of length $\ell(D)$ less than or equal to $\min(n,m)$. This result is known as GL_n-GL_m duality. See [11, Corollary 4.5.19] and [14, Theorem 2.1.2].

Every minor $\delta_{[I:J]}$ over $M_{n,m}$ with $[I:J] \in D(n,m)$ is scaled under the action of the diagonal subgroups of GL_n and GL_m. Therefore, standard monomials $\Delta(\mathsf{t}) = \prod_{k=1}^r \delta_{[I_k:J_k]}$ can be seen as joint weight vectors for the irreducible $GL_n \times GL_m$ representation $\rho_n^D \otimes \rho_m^D$ where D is equal to $sh(\mathsf{t})$, i.e., $D = (\ell([I_1:J_1]), \ldots, \ell([I_r:J_r]))^t$. Then, by Corollary 2.8 the above representation decomposition is compatible with the associated graded algebra in (2.6) in that $\mathsf{gr}_D^{sh}(\mathbb{C}[M_{n,m}]) = \rho_n^D \otimes \rho_m^D$ and

$$\mathsf{gr}^{sh}\big(\mathbb{C}[M_{n,m}]\big) = \sum_{\ell(D) \leqslant \min(n,m)} \rho_n^D \otimes \rho_m^D.$$

On the other hand, if we write $\mathcal{P}_{n,m}^D$ for the collection of elements p in the lattice cone $\mathcal{P}_{n,m}$ for $M_{n,m}$ such that $\mathsf{p}(z^{(m)}) = D$, then $\mathcal{P}_{n,m}$ can be expressed as the following disjoint union:

$$\mathcal{P}_{n,m} = \bigcup_{\ell(D) \leqslant \min(n,m)} \mathcal{P}_{n,m}^D$$

over Young diagrams D of length not more than $\min(n,m)$.

Recall that the GT patterns p of GL_k with fixed top row $\mathsf{p}(z^{(k)}) = D$ encode weight basis elements for the irreducible representation ρ_k^D of GL_k with highest weight D [10]. Hence the joint weight vectors of $\rho_n^D \otimes \rho_m^D$ can be encoded by pairs of GT patterns of GL_n and GL_m with the same top row D.

Proposition 3.8. *The Hibi algebra $\mathcal{H}_{D(n,m)}$ over $D(n,m)$ is isomorphic to the semigroup ring $\mathbb{C}[\mathcal{P}_{n,m}]$.*

Proof. Since the multiple chains in $D(n,m)$ provide a $\mathbb{C}$-basis of the Hibi algebra over $D(n,m)$ [12], let us find a bijection between the set of standard tableaux of shape D and $\mathcal{P}_{n,m}^D$. For a standard tableau t of shape D consisting of $\{[I_k:J_k]\}$, consider the GT pattern p of GL_{n+m} corresponding to the semistandard tableau whose columns are $\{\xi([I_k:J_k])\}$ where ξ is the bijection given in Lemma 2.2. Then this correspondence is injective, and it is straightforward to check that p is an element of $\mathcal{P}(n,m)$ satisfying $\mathsf{p}(z^{(m)}) = D$. To see surjectivity, note that p as an element of $\mathcal{P}_{n,m} = \mathcal{P}(n,m)/\langle \mathsf{p}_{\hat{1}} \rangle$ can be decomposed into two parts having supports in $\{z_b^{(a)} \in \Gamma_{n,m}\colon a \geqslant m\}$ and in $\{z_b^{(a)} \in \Gamma_{n,m}\colon a \leqslant m\}$ respectively, and therefore GT patterns of GL_n and GL_m with the same top rows D. So we have a one-to-one correspondence between $\mathcal{P}_{n,m}^D$ and the set of standard tableaux of shape D. This bijection provides an algebra isomorphism from our discussion (2.3). $\square$

In the proof we used the following observation: $\Gamma_{n,m}$ can be seen as a gluing of two GT posets Γ_n and Γ_m along their top rows. For instance, the pattern in the quotient $\mathcal{P}_{n,m}$ given in (3.1) can be seen as a fiber product of GT patterns for GL_3 and GL_4 over their top rows:

$$\begin{array}{ccccc} 3 & & 2 & & 2 \\ & 3 & & 2 & \\ & & 3 & & \end{array} \quad \text{and} \quad \begin{array}{ccccccc} 3 & & 2 & & 2 & & 0 \\ & 3 & & 2 & & 1 & \\ & & 3 & & 1 & & \\ & & & 2 & & & \end{array}$$

Remark 3.9. This is a GT pattern version of the correspondence given in Lemma 2.2 for tableaux. That is, for a standard monomial $\Delta(\mathrm{t}) = \prod_k \delta_{[I_k:J_k]}$, let T^- and T^+ be the semistandard tableaux whose columns are $\{I_k\}$ and $\{J_k\}$ respectively, and let $\xi([I_k : J_k]) = K_k \in Pl(n, m+n)$ for each k. Then the pattern $\mathrm{p} \in \mathcal{P}(n+m)$ corresponding to the semistandard tableau with columns $\{K_k\}$ can be, as an element of $\mathcal{P}_{n,m}$, identified with the gluing of p_- and p_+ along their top rows where $\mathrm{p}_- \in \mathcal{P}(n)$ and $\mathrm{p}_+ \in \mathcal{P}(m)$ are the GT patterns corresponding to T^- and T^+ respectively.

4. Standard monomial theory for $\mathcal{N}_{k,2n}$

In this section, we define the nullcone $\mathcal{N}_{k,2n}$ in the multi-vector representation of the symplectic group and consider the $GL_k \times Sp_{2n}$ action on it. Then we study standard monomial theory and a toric degeneration of $\mathcal{N}_{k,2n}$. Having an explicit description of the standard monomials for $\mathbb{C}[M_{k,2n}]$, we develop a relative theory to $M_{k,2n}$ for $\mathcal{N}_{k,2n}$ by investigating the defining ideal of $\mathcal{N}_{k,2n}$.

4.1. Nullcone for Sp_{2n}

For the space $\mathbb{C}^{2n}$ with the elementary basis $\{e_i\}$, let us fix our skew symmetric bilinear form $\langle\, , \rangle$ on it such that for every i, e_{2i-1} and e_{2i} form an isotropic pair with $\langle e_{2i-1}, e_{2i}\rangle = 1$. We can consider the space $M_{k,2n}$ of $k \times 2n$ complex matrices as k copies of $\mathbb{C}^{2n}$ with the natural action of the symplectic group Sp_{2n}. Then by the first fundamental theorem of invariant theory (e.g., [11, Theorem 4.2.2], [14, Theorem 3.8.3.2]), the Sp_{2n}-invariants of $\mathbb{C}[M_{k,2n}]$ are generated by the basic invariants $r_{ij} = \langle v_i, v_j\rangle$ obtained from row vectors v_i and v_j, or in terms of the coordinates specified in (2.1),

$$r_{ij} = \sum_{u=1}^{n} (x_{i,2u-1}x_{j,2u} - x_{j,2u-1}x_{i,2u})$$

for $1 \leqslant i < j \leqslant k$.

Definition 4.1. The *nullcone $\mathcal{N}_{k,2n}$ for Sp_{2n}* is the subvariety of $M_{k,2n}$ defined by the Sp_{2n}-invariants with vanishing constant terms.

If we let $\mathcal{I}$ denote the ideal of $\mathbb{C}[M_{k,2n}]$ generated by $\{r_{ij}\colon 1 \leqslant i < j \leqslant k\}$, then it is a radical ideal and the coordinate ring of $\mathcal{N}_{k,2n}$ is

$$\mathcal{R}(\mathcal{N}_{k,2n}) = \mathbb{C}[M_{k,2n}]/\mathcal{I}.$$

See [14, Theorem 3.8.6.2]. One can also study the nullcone $\mathcal{N}_{k,2n}$ as the zero fiber $\pi^{-1}(0)$ of the quotient $\pi : M_{k,2n} \to M_{k,2n}//Sp_{2n}$ and investigate the orbit structure. See [26] for this direction.

From the action of $GL_k \times GL_{2n}$ on $M_{k,2n}$ given in (3.2), by taking Sp_{2n} as a subgroup of GL_{2n}, we can consider the action of $GL_k \times Sp_{2n}$ on $\mathbb{C}[M_{k,2n}]$. Moreover, since GL_k and Sp_{2n} commute with each other in this action, the ideal $\mathcal{I}$ is stable under $GL_k \times Sp_{2n}$. Therefore, we can regard $\mathcal{R}(\mathcal{N}_{k,2n})$ as a $GL_k \times Sp_{2n}$ stable complement of $\mathcal{I}$.

Recall that by highest weight theory, every polynomial representation of GL_k and Sp_{2n} can be uniquely labeled by a Young diagram with no more than k and n rows respectively. We let ρ_k^D and σ_{2n}^D denote the irreducible representations of GL_k and Sp_{2n} respectively labeled by Young diagram D.

Proposition 4.2. *(See [14, Theorem 3.8.6.2].) Under the action of $GL_k \times Sp_{2n}$, we have the following decomposition*:

$$\mathcal{R}(\mathcal{N}_{k,2n}) = \sum_{r(D) \leqslant \min(n,k)} \rho_k^D \otimes \sigma_{2n}^D$$

where the summation runs over all Young diagrams D with length no more than $\min(k, n)$.

We remark that the space $\mathcal{H}(M_{k,2n})$ of Sp_{2n}*-harmonics* in $\mathbb{C}[M_{k,2n}]$ can be defined by the kernel of the symplectic analogs of Laplacian differential operators. Then, as is the case for $\mathcal{R}(\mathcal{N}_{k,2n})$, the space of harmonics is stable under the action of $GL_k \times Sp_{2n}$. In fact, $\mathcal{H}(M_{k,2n})$ and $\mathcal{R}(\mathcal{N}_{k,2n})$ share the same decomposition under the action of $GL_k \times Sp_{2n}$ [11,14]. Therefore, our results may be used to study the space of harmonics.

4.2. Standard monomials for $\mathcal{N}_{k,2n}$

Let us fix some notations. We write ω for the following Sp_{2n}-invariant element in $\bigwedge^2 \mathbb{C}^{2n}$:

$$\omega = \sum_{u=1}^{n} e_{2u-1} \wedge e_{2u} \in \bigwedge^2 \mathbb{C}^{2n}.$$

For $J = [j_1, \ldots, j_p] \in L(2n, 2n)$, write e_J for the elementary basis element $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_p} \in \bigwedge^p \mathbb{C}^{2n}$.

Definition 4.3. An ω-sum of Sp_{2n} is a linear combination $\sum_{d=1}^r c_d J_d$ of elements from $L(2n, 2n)$ such that

$$\sum_{d=1}^{r} c_d e_{J_d} \in \omega \wedge \left(\bigwedge^{p-2} \mathbb{C}^{2n}\right)$$

for some $p \geqslant 2$. We denote by Ω_{2n} the collection of ω-sums of Sp_{2n}.

Proposition 4.4. *The following set generates the ideal $\mathcal{I} \subset \mathbb{C}[M_{k,2n}]$ of the nullcone $\mathcal{N}_{k,2n}$*:

$$\Theta = \left\{ \sum_d c_d \delta_{[I:J_d]}\colon \sum_d c_d J_d \in \Omega_{2n} \right\}.$$

Proof. The ideal generated by Θ contains $\mathcal{I}$, because the basic Sp_{2n}-invariants r_{ij} are elements of Θ with $I = [i, j]$, $J_d = [2d-1, 2d]$ and $c_d = 1$ for $1 \leqslant d \leqslant n$. For $e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{p-2}} \in \bigwedge^{p-2} \mathbb{C}^{2n}$, let us consider the following elements in $\omega \wedge (\bigwedge^{p-2} \mathbb{C}^{2n})$:

$$\begin{aligned}
\omega \wedge (e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{p-2}}) &= \sum_{u=1}^{n} e_{2u-1} \wedge e_{2u} \wedge e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{p-2}} \\
&= \sum_{u=1}^{n} \sigma_u (e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_p}) \\
&= \sum_{u=1}^{n} \sigma_u e_{J_u}
\end{aligned}$$

where $\{j_1, \dots, j_p\} = \{2u-1, 2u, k_1, \dots, k_{p-2}\}$ with $j_1 \leqslant \cdots \leqslant j_p$. If there is a repetition in $\{2u-1, 2u, k_1, \dots, k_{p-2}\}$, then $\sigma_u = 0$. If there is no repetition in $\{2u-1, 2u, k_1, \dots, k_{p-2}\}$, then σ_u is the signature of the permutation sorting $2u-1, 2u, k_1, \dots, k_{p-2}$ in increasing order. Since $\omega \wedge (\bigwedge^{p-2} \mathbb{C}^{2n})$ is spanned by these elements, the elements of Θ are linear combinations of their associated elements $\sum_{u=1}^{n} \sigma_u \delta_{[I:J_u]}$. The column expansions for the determinants $\delta_{[I:J_u]}$ show that $\sum_u \sigma_u \delta_{[I:J_u]}$ is an element of the ideal generated by the basic Sp_{2n}-invariants $\{r_{ij}\}$. This shows that Θ is contained in $\mathcal{I}$, and therefore Θ generates the ideal $\mathcal{I}$. $\square$

Next, we characterize standard monomials of $\mathbb{C}[M_{k,2n}]$ associated with the elements of the ideal $\mathcal{I}$, and then we define standard monomials for the quotient $\mathcal{R}(\mathcal{N}_{k,2n})$. Let us impose the lexicographic order on the elements of the same length in $L(2n, 2n)$. We say $[i_1, \dots, i_p] >_{lex} [j_1, \dots, j_p]$ if the left-most non-zero entry of $(i_1 - j_1, \dots, i_p - j_p) \in \mathbb{Z}^p$ is positive. We fix the element $\widetilde{J} = [1, 3, \dots, 2n-1]$ of length n having $2d-1$ as its d-th smallest entry for $1 \leqslant d \leqslant n$.

Lemma 4.5. *Let $\sum_{d=1} c_d J_d$ be an ω-sum of Sp_{2n}. Then the smallest element J_1 among $\{J_d\}$ with respect to the lexicographic order satisfies $J_1 \not\succeq \widetilde{J}$. Conversely, if $J_1 \in L(2n, 2n)$ satisfies $J_1 \not\succeq \widetilde{J}$, there is an ω-sum of Sp_{2n} whose smallest non-zero term with respect to the lexicographic order is J_1.*

In particular, note that if $\ell(J) > n$, then $J \not\succeq \widetilde{J}$. The above lemma is from computations of the fundamental representations of Sp_{2n}, which can be realized in the quotient of $\bigwedge \mathbb{C}^{2n}$ by the ideal generated by ω. Its proof can be found in [20, Propositions 5.6, 5.9] and [7, §17]. See also [2] for a combinatorial description of such computations.

Definition 4.6. Let us define a distributive lattice $D(\mathcal{N})$ for $\mathcal{N} = \mathcal{N}_{k,2n}$ as

$$D(\mathcal{N}) = \bigl\{[I:J] \in D(k, 2n) \colon \ell\bigl([I:J]\bigr) \leqslant \min(k, n) \text{ and } J \succeq \widetilde{J}\bigr\}.$$

A multiple chain $\mathrm{t} = (X_1 \preceq X_2 \preceq \cdots)$ in the poset $D(\mathcal{N})$ is called an $\mathcal{N}$-standard tableau, and the corresponding monomial $\Delta(\mathrm{t}) = \prod_r \delta_{X_r} \in \mathbb{C}[M_{k,2n}]$ is called an $\mathcal{N}$-standard monomial.

Proposition 4.7.

i) *For $A, B \in D(\mathcal{N})$, the corresponding product $\delta_A \delta_B$ in $\mathcal{R}(\mathcal{N}_{k,2n})$ can be expressed as a linear combination of $\mathcal{N}$-standard monomials*:

$$\delta_A \delta_B = \sum_r c_r \delta_{X_r} \delta_{Y_r}$$

where δ_{Y_r} can possibly be 1, *and $\delta_{A\wedge B}\delta_{A\vee B}$ appears in the right-hand side with coefficient* 1.

ii) *Moreover, in the above expression, $wt(A) + wt(B) \geqslant wt(X_r) + wt(Y_r)$ and the equality holds only for $(X_r, Y_r) = (A \wedge B, A \vee B)$.*

Proof. Note that for $A, B \in D(\mathcal{N})$, $A \wedge B$ and $A \vee B$ belong to $D(\mathcal{N})$ and the corresponding element $\delta_{A\wedge B}\delta_{A\vee B}$ appears in the standard expression of $\delta_A\delta_B$ in $\mathbb{C}[M_{k,2n}]$ by Proposition 3.6. Then the first statement follows easily from the following computation: starting from the standard expression $\sum_r c_r \delta_{X_r}\delta_{Y_r}$ of $\delta_A\delta_B$ in $\mathbb{C}[M_{k,2n}]$ given in Proposition 3.6, if there is $X_r = [I:J]$ which is in $D(k, 2n) \backslash D(\mathcal{N})$ then we can obtain the $\mathcal{N}$-standard expression of $\delta_{X_r}\delta_{Y_r}$ by successive applications of the elements $(\delta_{[I:J]} - \sum_d s_d \delta_{[I:J_d]})$ in the generating set Θ of the ideal $\mathcal{I}$ such that $[I:J_d] >_{lex} [I:J]$ for all d, combined with the relations in Proposition 3.6 if necessary. We can always find such elements in Θ by Lemma 4.5. For the second statement, note that after replacing a non-$\mathcal{N}$-standard term $\delta_{X_r}\delta_{Y_r}$ by $(\sum_d s_d \delta_{[I:J_d]})\delta_{Y_r}$, the weights of new terms $wt([I:J_d]) + wt(Y_r)$ are strictly smaller than $wt(X_r) + wt(Y_r)$. If a term $\delta_{X_r}\delta_{Y_r}$ is already $\mathcal{N}$-standard, then the inequality of the weight directly follows from Proposition 3.6. $\square$

Now we state standard monomial theory for the nullcone $\mathcal{N}_{k,2n}$ and show its degeneration by the same method used for $\mathbb{C}[M_{n,m}]$ in Section 3.

Theorem 4.8. *The $\mathcal{N}$-standard monomials in $\mathbb{C}[M_{k,2n}]$ project to a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{N}_{k,2n})$. In particular, the $\mathcal{N}$-standard monomials $\Delta(\mathrm{t})$ of shape $sh(\mathrm{t}) = D$ project to a basis of $\rho_k^D \otimes \sigma_{2n}^D$.*

Proof. For a standard monomial $\prod_r \delta_{[I_r:J_r]}$ of $\mathbb{C}[M_{k,2n}]$, if $J_s \not\geq \widetilde{J}$ for some s, then we find $\delta_{[I_s:J_s]} - \sum_d c_d \delta_{[I_s:J_{d,s}]} \in \Theta$ with $J_{d,s} >_{lex} J_s$ for all d by Lemma 4.5. Then

$$\left(\delta_{[I_s:J_s]} - \sum_d c_d \delta_{[I_s:J_{d,s}]}\right) \prod_{r \neq s} \delta_{[I_r:J_r]}$$

is in the ideal $\mathcal{I}$ having the monomial $\prod_r \delta_{[I_r:J_r]}$ as its initial term. By repeating this procedure, combined with the relation in Proposition 4.7 if necessary, we can express $\prod_r \delta_{[I_r:J_r]}$ as a linear combination of $\mathcal{N}$-standard monomials. This implies that $\mathcal{N}$-standard monomials project to a spanning set of the space $\mathcal{R}(\mathcal{N}_{k,2n})$. Moreover, since the subspace of $\mathcal{R}(\mathcal{N}_{k,2n})$ spanned by $\mathcal{N}$-standard monomials of shape D is stable under $GL_k \times Sp_{2n}$, the dimension of the space $\rho_k^D \otimes \sigma_{2n}^D$ is less than or equal to the number of $\mathcal{N}$-standard monomials of shape D.

Now we claim that the number of $\mathcal{N}$-standard monomials of shape D is exactly the dimension of the space $\rho_k^D \otimes \sigma_{2n}^D$. For an $\mathcal{N}$-standard monomial $\prod_r \delta_{[I_r:J_r]}$ of shape D, the row indices $\{I_r\}$ form a semistandard tableau T^- of shape D with entries from $\{1, \ldots, k\}$. Then, the number of such semistandard tableaux is equal to the dimension of ρ_k^D (e.g., [7,11]). On the other hand, the column indices $\{J_r\}$ form a semistandard tableau T^+ of shape D such that each column is greater than or equal to $\widetilde{J}$. The set of all possible such semistandard tableaux T^+ with entries from $\{1, \ldots, 2n\}$ labels the weight basis of σ_{2n}^D (e.g., [2,20]). Therefore, the number of all the $\mathcal{N}$-standard monomials $\Delta(\mathrm{t})$ with $sh(\mathrm{t}) = D$ is equal to the dimension of $\rho_k^D \otimes \sigma_{2n}^D$. This finally shows that the $\mathcal{N}$-standard monomials with shape D project to a $\mathbb{C}$-basis of $\rho_k^D \otimes \sigma_{2n}^D$. $\square$

Theorem 4.9. *The algebra $\mathcal{R}(\mathcal{N}_{k,2n})$ is a flat deformation of the Hibi algebra over $D(\mathcal{N})$. More precisely, there is a flat $\mathbb{C}[t]$ module whose general fiber is $\mathcal{R}(\mathcal{N}_{k,2n})$ and special fiber is isomorphic to the Hibi algebra $\mathcal{H}_{D(\mathcal{N})}$ over $D(\mathcal{N})$.*

Proof. From Theorem 4.8, every element of $\mathcal{R}(\mathcal{N}_{k,2n})$ can be uniquely expressed as a linear combination of $\mathcal{N}$-standard monomials $\Delta(\mathrm{t})$. Hence, we can impose the same filtration F^{wt} of $\mathbb{C}[M_{k,2n}]$ on $\mathcal{R}(\mathcal{N}_{k,2n})$ via the weights wt on $\mathcal{N}$-standard monomials (Definition 3.5 with $D(k,2n)$): $\mathsf{F}_d^{wt}(\mathcal{R}(\mathcal{N}_{k,2n}))$ is the $\mathbb{C}$-span of the set

$$\left\{\Delta(\mathrm{t}) \colon wt(\mathrm{t}) \leqslant d\right\}.$$

This filtration is well defined, since in the standard expression $\sum_r c_r \Delta(\mathrm{t}_r)$ of any product $\prod \delta_A$, the weights $wt(\mathrm{t}_r)$ are smaller than the weight of $\prod \delta_A$ by Proposition 4.7. Moreover, since the equality holds only for $(X_r, Y_r) = (A \wedge B, A \vee B)$, as in the case for the space $M_{k,2n}$, we have the relation $s_A \cdot_{gr} s_B = s_{A \wedge B} \cdot_{gr} s_{A \vee B}$ where s_C are elements corresponding to δ_C in the associated graded algebra $\mathsf{gr}^{wt}(\mathcal{R}(\mathcal{N}_{k,2n}))$ with respect to F^{wt}. Therefore it is easy to see that the associated graded algebra forms the Hibi algebra over $D(\mathcal{N})$. Now for the flat degeneration, we can construct the Rees algebra $\mathcal{R}^t$:

$$\mathcal{R}^t = \bigoplus_{d \geqslant 0} \mathsf{F}_d^{wt}\big(\mathcal{R}(\mathcal{N}_{k,2n})\big) t^d$$

with respect to F^{wt}, then from the general properties of the Rees algebras (e.g., [1]), $\mathcal{R}^t$ is flat over $\mathbb{C}[t]$ with general fiber isomorphic to $\mathcal{R}(\mathcal{N}_{k,2n})$ and special fiber isomorphic to the associated graded algebra which is $\mathcal{H}_{D(\mathcal{N})}$. $\square$

5. Lattice cone for $\mathcal{N}_{k,2n}$

In this section we study a lattice cone associated with $\mathcal{N}_{k,2n}$. As is the case for $M_{n,m}$, it turns out that each point in the lattice cone for $\mathcal{N}_{k,2n}$ can be identified with a pair of Gelfand–Tsetlin patterns.

Proposition 5.1. *Let $\Delta(\mathrm{t}) = \prod_r \delta_{[I_r:J_r]}$ be an $\mathcal{N}$-standard monomial and $\mathsf{p}_{\mathrm{t}} \in \mathcal{P}_{k,2n}$ be the pattern corresponding to $\Delta(\mathrm{t})$. Then, p_{t} has its support in the following subposet of $\Gamma_{k,2n}$:*

$$\digamma_{k,2n} = \Gamma_{k,2n} \setminus (A \cup B) \tag{5.1}$$

where the subsets A and B of Γ_{2n+k} are defined as

$$A = \{ z_b^{(a)} \in \Gamma_{2n+k}\colon\ a \leqslant 2n \text{ and } b > (a+1)/2 \};$$
$$B = \{ z_b^{(a)} \in \Gamma_{2n+k}\colon\ z^{(2n)}_{\min(k,n)+1} \geqslant z_b^{(a)} \}.$$

We shall prove it in a few steps. First, note that if a GT pattern $\mathsf{p} \in \mathcal{P}(k+2n)$ of GL_{2n+k} corresponds to an $\mathcal{N}$-standard monomial, then this proposition says that the length of the $2n$-th row is at most $\min(k,n)$ and the support of p corresponding to the bottom $2n$ rows is contained in the "left half" of $\Gamma_{2n} \subset \Gamma_{2n+k}$.

Example 5.2. For $k = 4$ and $n = 3$, let us consider the following $\mathcal{N}$-standard monomial for $\mathcal{N}_{4,6}$:

$$\delta_{[123:135]}\delta_{[124:136]}\delta_{[12:24]}\delta_{[13:35]}\delta_{[1:4]}.$$

Then the corresponding semistandard tableau with respect to ξ given in Lemma 2.2 is the following chain in $Pl(4,10)$:

1	1	2	3	4
3	3	4	5	7
5	6	7	7	8
7	8	8	9	9

and we can visualize its corresponding GT pattern for GL_{10} by listing its function values as

$$\begin{array}{ccccccccccccccccccc}
5 & & 5 & & 5 & & 5 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 5 & & 5 & & 5 & & 5 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 5 & & 5 & & 5 & & 3 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 5 & & 5 & & 4 & & 1 & & 0 & & 0 & & 0 & & & \\
 & & & & 5 & & 4 & & 2 & & 0 & & 0 & & 0 & & & & \\
 & & & & & 5 & & 4 & & 1 & & 0 & & 0 & & & & & \\
 & & & & & & 5 & & 3 & & 0 & & 0 & & & & & & \\
 & & & & & & & 4 & & 2 & & 0 & & & & & & & \\
 & & & & & & & & 3 & & 0 & & & & & & & & \\
 & & & & & & & & & 2 & & & & & & & & &
\end{array}$$

Note that the length of the 6-th row $(5,4,2)$ is 3 and the support corresponding to the bottom six rows is contained only in the "left half" of the poset of $\Gamma_6 \subset \Gamma_{10}$.

Let us write $\widehat{1}$ for $[2n+1,\ldots,2n+k] \in Pl(k,2n+k)$, and write $\mathsf{p}_{\widehat{1}}$ for the GT pattern of GL_{2n+k} corresponding to $\widehat{1}$ via Lemma 2.4. Recall that $\widetilde{J} = [1,3,\ldots,2n-1] \in L(2n,2n)$.

Lemma 5.3. *Let $[I:J] \in D(k,2n)$ be an one-line tableau and $K = \xi([I:J])$ be the element of $Pl(k,2n+k)$ corresponding to $[I:J]$ via Lemma 2.2. Let $supp(\mathsf{p}_K)$ denote the support of the GT pattern p_K corresponding to K. If $J \not\succeq \widetilde{J}$, then the following intersection*

$$supp(\mathsf{p}_K) \cap \{ z_b^{(a)} \in \Gamma_{2n+k}\colon\ a \leqslant 2n \text{ and } b > (a+1)/2 \}$$

is non-empty. Conversely, if $J \succeq \widetilde{J}$, then the intersection is empty.

Proof. This is an easy computation similar to [20, Lemma 5.11]. $\square$

Recall that for each standard monomial $\Delta(\mathrm{t}) = \prod_r \delta_{[I_r:J_r]}$ of $\mathbb{C}[M_{k,2n}]$, by the bijection constructed in the proof of Proposition 3.8, we can find its corresponding pattern $\mathsf{p}_\mathrm{t} \in \mathcal{P}_{k,2n}$ for $M_{k,2n}$. More precisely, p_t as an element of $\mathcal{P}(k,2n)$ is the sum of GT patterns p_r of GL_{2n+k} corresponding to $\xi([I_r : J_r])$ where ξ is the bijection given in Lemma 2.2. If $\Delta(\mathrm{t}) = \prod_r \delta_{[I_r:J_r]}$ is an $\mathcal{N}$-standard monomial, then $J_r \succeq \widetilde{J}$ for all r. Therefore, by the above lemma, the support of $\mathsf{p}_\mathrm{t} \in \mathcal{P}_{k,2n}$ does not intersect with $\{z_b^{(a)} \in \Gamma_{k+2n}\colon a \leqslant 2n$ and $b > (a+1)/2\}$. Also, note that $J \not\succeq \widetilde{J}$ if $\ell(J) > n$ and that $\ell(I) \leqslant k$. Then the condition that p_t is supported in $\Gamma_{k,2n} \backslash \{z_b^{(a)} \in \Gamma_{k+2n}\colon z_{\min(k,n)+1}^{(2n)} \geqslant z_b^{(a)}\}$ follows from the fact $\ell([I:J]) \leqslant \min(k,n)$ for $[I:J] \in D(\mathcal{N})$. This finishes the proof of Proposition 5.1.

Now by using the poset identified in (5.1), we can define the semigroup and the cone for $\mathcal{N}_{k,2n}$.

Definition 5.4. The semigroup $\mathcal{P}(\mathcal{N}_{k,2n})$ of patterns for $\mathcal{N}_{k,2n}$ is the set of order preserving maps from $\digamma_{k,2n}$ to the set of non-negative integers with the usual function addition as its product. We let $\mathbb{C}[\mathcal{P}(\mathcal{N}_{k,2n})]$ denote the semigroup ring of $\mathcal{P}(\mathcal{N}_{k,2n})$.

We can define the convex polyhedral cone associated with the space $\mathcal{N}_{k,2n}$ as the collection of all non-negative real valued order preserving maps on $\digamma_{k,2n}$:

$$\mathcal{C}(\mathcal{N}_{k,2n}) = \{f\colon \digamma_{k,2n} \to \mathbb{R}_{\geqslant 0}\}.$$

Then, by identifying f with its values $(f(z_b^{(a)})) \in \mathbb{R}^N$ for $z_b^{(a)} \in \digamma_{k,2n}$, we can realize the semigroup $\mathcal{P}(\mathcal{N}_{k,2n})$ of patterns for $\mathcal{N}_{k,2n}$ as the intersection of $\mathcal{C}(\mathcal{N}_{k,2n})$ with $\mathbb{Z}^N$, i.e., the *lattice cone for* $N_{k,2n}$:

$$\mathcal{P}(\mathcal{N}_{k,2n}) = \mathcal{C}(\mathcal{N}_{k,2n}) \cap \mathbb{Z}^N$$

where N is equal to the number of elements in the poset $\digamma_{k,2n}$. Let us denote by $\mathcal{P}(\mathcal{N}_{k,2n})_D$ the collection of $\mathsf{p} \in \mathcal{P}(\mathcal{N}_{k,2n})$ whose $2n$-th row is equal to Young diagram D, i.e., $\mathsf{p}(z^{(2n)}) = D$. Then the lattice cone for $\mathcal{N}_{k,2n}$ can be expressed as the disjoint union

$$\mathcal{P}(\mathcal{N}_{k,2n}) = \bigcup_D \mathcal{P}(\mathcal{N}_{k,2n})_D$$

over all D with $\ell(D) \leqslant \min(k,n)$.

The following is a lattice cone version of Theorem 4.8.

Proposition 5.5.

i) *For $\mathcal{N} = \mathcal{N}_{k,2n}$, the Hibi algebra $\mathcal{H}_{D(\mathcal{N})}$ over $D(\mathcal{N})$ is isomorphic to the semigroup ring $\mathbb{C}[\mathcal{P}(\mathcal{N}_{k,2n})]$.*
ii) *There is an one-to-one correspondence between $\mathcal{P}(\mathcal{N}_{k,2n})_D$ and the set of weight vectors for $\rho_k^D \otimes \sigma_{2n}^D$ in $\mathcal{R}(\mathcal{N}_{k,2n})$.*

Proof. The proof of the first statement is parallel to that of Proposition 3.8. Note that the multiple chains in $D(\mathcal{N})$ provide a $\mathbb{C}$-basis for the Hibi algebra $\mathcal{H}_{D(\mathcal{N})}$ over $D(\mathcal{N})$ [12]. The pattern corresponding to an $\mathcal{N}$-standard tableau consists of two parts having supports in $\{z_b^{(a)} \in \digamma_{k,2n}\colon a \geqslant 2n\}$ and in $\{z_b^{(a)} \in \digamma_{k,2n}\colon a \leqslant 2n\}$ respectively, and therefore GT patterns of GL_k and GL_{2n} with the same top row D. Furthermore, by Proposition 5.1, such GT patterns of GL_{2n} have their supports in $\{z_b^{(a)} \in \digamma_{k,2n}\colon a \leqslant 2n$ and $b \leqslant (a+1)/2\}$, and then they represent GT patterns for Sp_{2n} (e.g., [20]). Hence the correspondence given in Proposition 3.8 provides a bijection between the set of $\mathcal{N}$-standard monomials of shape D and the set of pairs of patterns $(\mathsf{p}_-, \mathsf{p}_+)$ where p_- and p_+ are GT patterns for GL_k and Sp_{2n} respectively with the same top row D. Therefore, we obtain the bijection between the set of weight vectors for $\rho_k^D \otimes \sigma_{2n}^D$ and $\mathcal{P}(\mathcal{N}_{k,2n})_D$. $\square$

Example 5.6. The GT pattern of GL_{10} given in Example 5.2, considered as an element of $\mathcal{P}_{4,6} = \mathcal{P}(4,6)/\langle \mathsf{p}_{\widehat{1}} \rangle$, can be visualized as a pattern over $\Gamma_{4,6}$ as follows:

$$
\begin{array}{ccccccccc}
 & & & 5 & & & & & \\
 & & 5 & & 3 & & & & \\
 & 5 & & 4 & & 1 & & & \\
5 & & 4 & & 2 & & 0 & & \\
 & 5 & & 4 & & 1 & & 0 & \\
 & & 5 & & 3 & & 0 & & 0 \\
 & & & 4 & & 2 & & 0 & \\
 & & & & 3 & & 0 & & \\
 & & & & & 2 & & &
\end{array}
$$

Note that the non-zero entries are corresponding to the poset $\mathcal{F}_{4,6}$. As we discussed in Remark 3.9 for $M_{n,m}$, it can also be seen as the fiber product of two GT patterns, one for GL_4 and the other for Sp_6, along their top rows:

$$
\begin{array}{ccccccc}
5 & & 4 & & 2 & & 0 \\
 & 5 & & 4 & & 1 & \\
 & & 5 & & 3 & & \\
 & & & 5 & & &
\end{array}
\quad \text{and} \quad
\begin{array}{cccccc}
5 & & 4 & & 2 & \\
 & 5 & & 4 & & 1 \\
 & & 5 & & 3 & \\
 & & & 4 & & 2 \\
 & & & & 3 & \\
 & & & & & 2
\end{array}
$$

The above GT patterns correspond to the following semistandard tableaux of shape $(5,4,2)$ via the conversion procedure given in Lemma 2.4:

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 \\
\hline
2 & 2 & 2 & 3 & \multicolumn{1}{c}{} \\
\cline{1-4}
3 & 4 & \multicolumn{3}{c}{} \\
\cline{1-2}
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 4 \\
\hline
3 & 3 & 4 & 5 & \multicolumn{1}{c}{} \\
\cline{1-4}
5 & 6 & \multicolumn{3}{c}{} \\
\cline{1-2}
\end{array}
$$

They were denoted by T^- and T^+ respectively in the proof of Theorem 4.8, and they can be easily read from the row indices and the column indices of the $\mathcal{N}$-standard monomial in Example 5.2.

Finally, we remark that the discussion in [15] on a simplicial decomposition of a polyhedral cone and its relation with an algebra decomposition can be directly applied to our case, and then we can interpret standard monomial theory of $\mathcal{R}(\mathcal{N}_{k,2n})$ in terms of a simplicial decomposition of $\mathcal{C}(\mathcal{N}_{k,2n})$. More precisely, if we take a maximal linearly ordered subset $\mathcal{S}$ of $D(\mathcal{N})$, then all the products of elements from

$$\widehat{\mathcal{S}} = \{\delta_{[I:J]} \in \mathcal{R}(\mathcal{N}_{k,2n}) \colon [I:J] \in \mathcal{S}\}$$

are $\mathcal{N}$-standard monomials. Therefore, elements in $\widehat{\mathcal{S}}$ are algebraically independent and generate a polynomial subring of $\mathcal{R}(\mathcal{N}_{k,2n})$. On the other hand, a computation of Lemma 2.4 shows that $\mathcal{S}$ must be induced from a linearization of the poset $\mathcal{F}_{k,2n}$. Moreover, from our construction of $\mathcal{C}(\mathcal{N}_{k,2n})$ in terms of $\mathcal{F}_{k,2n}$, all possible linearizations of $\mathcal{F}_{k,2n}$ give rise to a simplicial decomposition of the cone $\mathcal{C}(\mathcal{N}_{k,2n})$. Consequently, we can obtain a decomposition of $\mathcal{R}(\mathcal{N}_{k,2n})$ into polynomial rings. This decomposition is not disjoint, however, it is compatible with a simplicial decomposition of $\mathcal{C}(\mathcal{N}_{k,2n})$ induced from linearizations of $\mathcal{F}_{k,2n}$. For more details in this direction, we refer readers to [15].

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References

[1] V. Alexeev, M. Brion, Toric degenerations of spherical varieties, Selecta Math. (N.S.) 10 (4) (2004) 453–478.

[2] A. Berele, Construction of Sp-modules by Tableaux, Linear Multilinear Algebra 19 (1986) 299–307.

[3] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.

[4] P. Caldero, Toric degenerations of Schubert varieties, Transform. Groups 7 (1) (2002) 51–60.

[5] C. De Concini, D. Eisenbud, C. Procesi, Hodge Algebras, Astérisque 91 (1982).

[6] P. Doubilet, G.-C. Rota, J. Stein, On the foundations of combinatorial theory IX: Combinatorial methods in invariant theory, Stud. Appl. Math. 53 (1976) 185–216.

[7] W. Fulton, J. Harris, Representation Theory, a First Course, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1991.

[8] N. Gonciulea, V. Lakshmibai, Degenerations of flag and Schubert varieties to toric varieties, Transform. Groups 1 (3) (1996) 215–248.

[9] V. Lakshmibai, N. Gonciulea, Flag Varieties, Acutalities Math., Hermann, 2001.

[10] I.M. Gelfand, M.L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Dokl. Akad. Nauk SSSR (N.S.) 71 (1950) 825–828 (in Russian); English translation in: I.M. Gelfand Collected Papers, vol. II, Springer-Verlag, Berlin, 1988, pp. 653–656.

[11] R. Goodman, N.R. Wallach, Representations and Invariants of the Classical Groups, Encyclopedia Math. Appl., vol. 68, Cambridge University Press, Cambridge, 1998.

[12] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws, in: Commutative Algebra and Combinatorics, Kyoto, 1985, in: Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109.

[13] W. Hodge, Some enumerative results in the theory of forms, Proc. Cambridge Philos. Soc. 39 (1943) 22–30.

[14] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur Lectures, Tel Aviv, 1992, in: Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.

[15] R. Howe, Weyl chambers and standard monomial theory for poset lattice cones, Q. J. Pure Appl. Math. 1 (1) (2005) 227–239.

[16] R. Howe, S. Jackson, S.T. Lee, E.-C. Tan, J. Willenbring, Toric degeneration of branching algebras, Adv. Math. 220 (2009) 1809–1841.

[17] R. Howe, S.T. Lee, Bases for some reciprocity algebras I, Trans. Amer. Math. Soc. 359 (2007) 4359–4387.

[18] R. Howe, E.-C. Tan, J. Willenbring, Stable branching rules for classical symmetric pairs, Trans. Amer. Math. Soc. 357 (2005) 1601–1626.

[19] R. Howe, E.-C. Tan, J. Willenbring, A basis for GL_n tensor product algebra, Adv. Math. 196 (2005) 531–564.

[20] S. Kim, Standard monomial theory for flag algebras of $GL(n,\mathbb{C})$ and $Sp(2n,\mathbb{C})$, J. Algebra 320 (2) (2008) 534–568.

[21] S. Kim, Standard monomial theory and degeneration of $SO(m,\mathbb{C})$ representations, J. Algebra 322 (11) (2009) 3896–3911.

[22] S. Kim, S.T. Lee, Pieri algebras for the orthogonal and symplectic groups, arXiv:0907.1336.

[23] M. Kogan, E. Miller, Toric degeneration of Schubert varieties and Gelfand–Tsetlin polytopes, Adv. Math. 193 (1) (2005) 1–17.

[24] H. Kraft, N.R. Wallach, On the nullcone of representations of reductive groups, Pacific J. Math. 224 (1) (2006) 119–139.

[25] H. Kraft, N.R. Wallach, Polarizations and nullcone of representations of reductive groups, in: Symmetry and Spaces: In Honor of Gerry Schwarz, in: Progr. Math., vol. 278, Birkhäuser, Boston, 2010, pp. 153–167.

[26] H. Kraft, Geometrische Methoden in der Invariantentheorie, Aspects Math., vol. D1, Vieweg-Verlag, 1985.

[27] V. Lakshmibai, The development of standard monomial theory. II, in: A Tribute to C.S. Seshadri, Chennai, 2002, in: Trends Math., Birkhäuser, Basel, 2003, pp. 283–309.

[28] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, New York, 2005.

[29] C. Musili, The development of standard monomial theory. I, in: A Tribute to C.S. Seshadri, Chennai, 2002, in: Trends Math., Birkhäuser, Basel, 2003, pp. 385–420.

[30] C. Procesi, Lie Groups. An Approach through Invariants and Representations, Universitext, Springer, New York, 2007.

[31] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.

[32] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, American Mathematical Society, Providence, RI, 1996.