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Constructing inverse semigroups from category actions

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Abstract

The theory in this paper was motivated by an example of an inverse semigroup important in Girard's 'Geometry of interaction' programme for linear logic. At one level, the theory is a refinement of the Wagner–Preston representation theorem: we show that every inverse semigroup is isomorphic to an inverse semigroup of all partial symmetries (of a specific type) of some structure. At another level, the theory unifies and completes two classical theories: the theory of bisimple inverse monoids created by Clifford and subsequently generalised to all inverse monoids by Leech; and the theory of 0-bisimple inverse semigroups due to Reilly and McAlister. Leech showed that inverse monoids could be described by means of a class of right cancellative categories, whereas Reilly and McAlister showed that 0-bisimple inverse semigroups could be described by means of generalised RP-systems. In this paper, we prove that every inverse semigroup can be constructed from a category acting on a set satisfying what we term the 'orbit condition'. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The ultimate origins of this paper lie in the pioneering work of Clifford [1]. He showed that every bisimple inverse monoid could be described in terms of a right cancellative monoid in which the set of principal left ideals is closed under finite intersections. This result was subsequently generalised to bisimple inverse semigroups by Reilly [17], but to accomplish this, right cancellative monoids were replaced by what Reilly termed 'RP-systems'. These systems were viewed as partial semigroups satisfying certain cancellation conditions. Later, McAlister [11] observed that 0-bisimple inverse semigroups could be described in terms of 'generalised RP-systems'.

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This work was developed in two important ways. Firstly, McAlister showed [12] that arbitrary semigroups in which the intersection of two principal left ideals is either empty or again a principal left ideal can be used to construct inverse monoids; the lack of any cancellation condition is overcome by the use of the $\mathscr{R}^*$-relation, a generalisation of Green's $\mathscr{R}$-relation. Secondly, Leech [8] directly generalised Clifford's result to arbitrary inverse monoids: he showed that inverse monoids could be described by means of right cancellative categories having a weak initial object and possessing pushouts of all pairs of morphisms with a common domain.

In this paper, we complete this process by providing a joint generalisation of McAlister's and Leech's work; in this way we obtain a description of all inverse semigroups. Our work is based on two key observations:

1. The usual description of RP-systems as partial semigroups is not helpful for formulating generalisations. But a little thought reveals them to be nothing other than a special class of monoid actions. In view of Leech's work, this suggests that arbitrary inverse semigroups will arise from category, rather than monoid, actions.

2. The fact that actions would be the key to describing arbitrary inverse semigroups occurred to us whilst reading a paper of Girard [5] on linear logic. Girard introduces an algebraic structure which quickly revealed itself to be an inverse semigroup. Significantly, the multiplication in this semigroup was similar to the multiplication defined in McAlister's paper on 0-bisimple semigroups [12], but the semigroup here was evidently not 0-bisimple. Girard's semigroup, which we call the 'clause semigroup', is defined in terms of the Unification Algorithm. A description of unification in terms of category theory in [18] led us to the correct definition of category actions needed for our generalisation.

The background required to understand this paper is very modest. The relevant inverse semigroup theory may be found in [6]; we need little beyond the basic definitions and properties, including the natural partial order, and Green's relations. Most of the category theory may be found in the first few chapters of [9]. For category actions consult [10].

It is worth pointing out that categories are employed in two different ways in this paper. Firstly, categories are used in the familiar way as 'categories of structures': one considers a collection of objects which are often sets with structure and the morphisms between them. Secondly, categories are regarded as algebraic structures in their own right: as sets equipped with a partial binary operation satisfying certain axioms; in this point of view, categories are generalisations of monoids. The second approach was adopted by Ehresmann [3] in his work on the role of inverse semigroups in differential geometry; it is the basis of a programme enunciated by Lawvere [7] who suggested that the fundamental structures of mathematics were themselves categories; and it has become a vital ingredient in contemporary semigroup theory principally as a result of Tilson's seminal [19].

Bearing these two approaches in mind we can now state the chief object of this paper: we shall show that the category of inverse semigroups is equivalent to a category whose objects are special kinds of category actions.

The theory is developed in seven sections:

Section 2. We show how to construct an inverse semigroup with zero from a category action satisfying what we call 'the orbit condition'. We do this in two, equivalent, ways. Firstly, we show that a natural family of partial bijections of the action (the 'partial symmetries' of the action) forms an inverse semigroup. Then we show that this inverse semigroup may be concretely described by means of equivalence classes of ordered pairs.

Section 3. We show that a category action can be constructed from an inverse semigroup. The actions which arise possess additional properties, which motivate the definition of a special class of category actions called 'systems'.

Section 4. The work of the previous two sections is put on a categorical footing. We define functors between the category of systems and their morphisms and the category of inverse semigroups with zero and their morphisms.

Section 5. An important class of morphisms between systems is introduced called 'equivalences'. We prove that equivalent systems have isomorphic inverse semigroups.

Section 6. We study the composites of the functors introduced in Section 4. We prove that every inverse semigroup is isomorphic to an inverse semigroup arising from a system, whereas every system is *equivalent* to a system arising from an inverse semigroup.

Section 7. The category of inverse semigroups with zero is shown to be equivalent to a suitable quotient of the category of systems.

Section 8. A number of special cases and concrete examples are discussed. In particular, we show that under the equivalence we have established between inverse semigroups and systems, inverse monoids correspond to cyclic systems, and 0-bisimple semigroups correspond to monoid systems. We also obtain natural characterisations of 0-simple and 0-E-unitary inverse semigroups in terms of systems. The paper concludes with a description of Girard's clause semigroup.

Terminology concerning categories of inverse semigroups.

An inverse semigroup might, or might not, have a zero element; if it does, then we can choose to treat the zero as just another element or we can elect to make the zero a distinguished element. An inverse semigroup with a distinguished zero we call an *inverse semigroup with zero*. If S and T are inverse semigroups with zero then a homomorphism θ from S to T is a semigroup homomorphism with the additional property that $\theta(0)=0$. A homomorphism of inverse semigroups with zero is said to be 0-*restricted* if $\theta^{-1}(0)=\{0\}$. The basic category of inverse semigroups considered in this paper is the category of 'inverse-semigroups-with-zero' together with '0-restricted-homomorphisms'.

An *inverse semigroup* means an inverse semigroup pure and simple, which may well have a zero element, but which we choose not to distinguish. Homomorphisms of inverse semigroups are just semigroup homomorphisms; any zero element receives no special treatment.

Although these distinctions may sound academic, they are important in understanding the relationship between our work and that of Leech. This will be fully explained in the relevant parts of the text.

2. A class of category actions

In order to fix notation and terminology we begin with the formal definition of 'category' regarded as a generalisation of a monoid.

Definition. Let C be a set equipped with a partial binary operation which we shall denote by $\cdot$ or by concatenation. If $x, y \in C$ and the product $x \cdot y$ is defined we write $\exists x \cdot y$. An element $e \in C$ is called an *identity* if $\exists e \cdot x$ implies $e \cdot x = x$ and $\exists x \cdot e$ implies $x \cdot e = x$. The set of identities of C is denoted C_O. The pair $(C, \cdot)$ is said to be a *category* if the following axioms hold:

(C1) $x \cdot (y \cdot z)$ exists if, and only if, $(x \cdot y) \cdot z$ exists, in which case they are equal.
(C2) $x \cdot (y \cdot z)$ exists if, and only if, $x \cdot y$ and $y \cdot z$ exist.
(C3) For each $x \in C$ there exist identities e and f such that $\exists x \cdot e$ and $\exists f \cdot x$.

From (C3), it can easily be deduced that the identities e and f are uniquely determined by x. We write $e = \mathbf{d}(x)$ and $f = \mathbf{r}(x)$. Observe that $\exists x \cdot y$ if, and only if, $\mathbf{d}(x) = \mathbf{r}(y)$. If C is a category and e and f identities in C then we put

$$\hom(e, f) = \{x \in C : \mathbf{d}(x) = e \text{ and } \mathbf{r}(x) = f\},$$

the set of all *homomorphisms from e to f*. We also put $\mathrm{end}(e) = \hom(e, e)$, the *endomorphism monoid at e*. We now define what is meant by a 'category acting on a set'.

Definition. Let C be a category, X a set, and $\mathbf{p} : X \to C_O$ a function. Let $C * X$ be the set

$$C * X = \{(a, x) \in C \times X : \mathbf{d}(a) = \mathbf{p}(x)\}.$$

We suppose in addition that there is a function $C * X \to X$, denoted by $(a, x) \mapsto a \cdot x$. We shall write $\exists a \cdot x$ if $(a, x) \in C * X$. We say that C *acts on X (on the left)*, and that X is a *left C-system* if the following axioms hold:

(A1) $\exists \mathbf{p}(x) \cdot x$ and $\mathbf{p}(x) \cdot x = x$ for all $x \in X$.
(A2) If $\exists a \cdot x$ then $\mathbf{p}(a \cdot x) = \mathbf{r}(a)$.
(A3) If $\exists ab$ in C and $\exists (ab) \cdot x$ then $\exists b \cdot x$ and $\exists a \cdot (b \cdot x)$ and $(ab) \cdot x = a \cdot (b \cdot x)$.
We write ${}_C X$ to indicate the fact that X is a set on which C acts on the left.

Let X be a left C-system. For $x \in X$ put

$$C \cdot x = \{a \cdot x : a \in C \text{ and } \exists a \cdot x\}.$$

By (A1), $x \in C \cdot x$. If $X' \subseteq X$ then define

$$C \cdot X' = \bigcup \{C \cdot x : x \in X'\}.$$

A subset $Y \subseteq X$ is said to be *C-invariant* or a *left C-subsystem* if $C \cdot Y \subseteq Y$. It is clear that if Y is C-invariant it can be regarded as a left C-system in its own right. Subsets of the form $C \cdot x$ are always C-invariant and so form a special class of left C-subsystems called *cyclic*. Alternatively, $C \cdot x$ can be regarded as the *orbit* of x under the action of C. We now define morphisms between actions.

Definition. Let $C * X \to X$ and $D * Y \to Y$ be two actions. A *morphism* from ${}_C X$ to ${}_D Y$ is a pair (F, θ) consisting of a functor $F : C \to D$ and a function $\theta : X \to Y$ satisfying the following two axioms:

(M1) $\mathbf{p}(\theta(x)) = F(\mathbf{p}(x))$ for all $x \in X$.

(M2) If $\exists a \cdot x$ in $C * X$ then $\theta(a \cdot x) = F(a) \cdot \theta(x)$.

If $C = D$ and F is the identity functor then we may replace the pair (F, θ) by θ and under these circumstances we say that θ is a *C-homomorphism*. A bijective C-homomorphism is called a *C-isomorphism*.

The definitions above are generalisations of notions famililar in semigroup theory. Let S be a semigroup and X a set. Then S is said to *act on X on the left* if there is a function $S \times X \to X$, given by $(s,x) \mapsto s \cdot x$, satisfying $(st) \cdot x = s \cdot (t \cdot x)$ for all $s, t \in S$ and $x \in X$. If S is a monoid with identity 1 then the pair (S, X) is called a *left S-system* if S acts on X and $1 \cdot x = x$ for all $x \in X$; if C is a category with one identity, then a left C-system in the monoid sense is precisely a left C-system in the category sense. Actions of monoids are discussed in [6], and actions of categories are discussed in [10]. Now let the semigroup S act on the left on the sets X and Y. A function $\theta : X \to Y$ is said to be an *S-homomorphism* if $\theta(s \cdot x) = s \cdot \theta(x)$ for all $s \in S$ and $x \in X$. When S is a monoid and X and Y are left S-systems the monoid definition of S-homomorphism agrees with the category definition.

Lemma 1. (i) *Let X and Y be two left C-systems, and $\theta : X \to Y$ a C-isomorphism. Then $\theta^{-1} : Y \to X$ is a C-isomorphism.*

Let X be a left C-system, and let A and B be C-invariant subsets of X. Let $\theta : A \to B$ be a C-isomorphism. Then

(ii) *If $A' \subseteq A$ is a C-invariant subset of A then $\theta(A')$ is a C-invariant subset of B.*

(iii) *If $B' \subseteq B$ is a C-invariant subset of B then $\theta^{-1}(B')$ is a C-invariant subset of A.*

(iv) *Let $C \cdot x \subseteq A$ be a cyclic C-invariant subset of A. Then $\theta(C \cdot x)$ is a cyclic C-invariant subset of B equal to $C \cdot \theta(x)$.*

(v) *Let $C \cdot y \subseteq B$ be a cyclic C-invariant subset of B. Then $\theta^{-1}(C \cdot y)$ is a cyclic C-invariant subset of A equal to $C \cdot \theta^{-1}(y)$.*

Proof. (i) We show that θ^{-1} satisfies (M1) and (M2).

(M1) holds: let $y \in Y$ and put $x = \theta^{-1}(y)$. Then

$$\mathbf{p}(\theta^{-1}(y)) = \mathbf{p}(x) = \mathbf{p}(\theta(x)) = \mathbf{p}(y),$$

using the fact that θ satisfies (M1).

(M2) holds: let $y \in Y$ and put $x = \theta^{-1}(y)$. Suppose that $\exists a \cdot y$. By definition, $\mathbf{d}(a) = \mathbf{p}(y)$. Since (M1) holds for θ^{-1} we have that $\mathbf{p}(y) = \mathbf{p}(x)$. Thus $\exists a \cdot x$, and so $\exists a \cdot \theta^{-1}(y)$. But θ satisfies (M2), and so $\theta(a \cdot x) = a \cdot \theta(x)$. Hence $\theta(a \cdot \theta^{-1}(y)) = a \cdot y$, consequently $\theta^{-1}(a \cdot y) = a \cdot \theta^{-1}(y)$, as required. (ii) Let $y \in \theta(A')$, and suppose that $\exists a \cdot y$ where $a \in C$. Put $\theta(x) = y$. By definition, $\mathbf{d}(a) = \mathbf{p}(y)$. But θ is a C-homomorphism so that $\mathbf{p}(y) = \mathbf{p}(\theta(x)) = \mathbf{p}(x)$. Hence $\mathbf{d}(a) = \mathbf{p}(x)$, and so $\exists a \cdot x$. But A' is C-invariant and $x \in A'$. Hence $a \cdot x \in A'$, and so $\theta(a \cdot x) \in \theta(A')$. But $\theta(a \cdot x) = a \cdot \theta(x) = a \cdot y$. Hence $a \cdot y \in \theta(A')$. Thus $\theta(A')$ is a C-invariant subset of B.

(iii) Immediate from (i) and (ii).

(iv) By (ii), $\theta(C \cdot x)$ is a C-invariant subset of B. Thus it only remains to show that $\theta(C \cdot x)$ is cyclic. We claim that $\theta(C \cdot x) = C \cdot \theta(x)$. Clearly, $\theta(C \cdot x) \subseteq C \cdot \theta(x)$. Let $a \cdot \theta(x) \in C \cdot \theta(x)$. Then $\mathbf{d}(a) = \mathbf{p}(\theta(x))$. By (M1), $\mathbf{p}(\theta(x)) = \mathbf{p}(x)$. Thus $\exists a \cdot x$. But $a \cdot x \in C \cdot x$ and $\theta(a \cdot x) = a \cdot \theta(x)$, and so $C \cdot \theta(x) \subseteq \theta(C \cdot x)$. Hence $\theta(C \cdot x) = C \cdot \theta(x)$.

(v) Immediate from (i) and (iv) above. □

Let X be a left C-system. Denote by $I({}_C X)$ the set of all C-isomorphisms between C-invariant subsets of X.

Proposition 2. *$I({}_C X)$ is an inverse subsemigroup of $I(X)$, the symmetric inverse monoid on X.*

Proof. By Lemma 1, $I({}_C X)$ is closed under inverses. Let $\theta, \varphi \in I({}_C X)$. Then $\mathrm{dom}(\theta)$ and $\mathrm{im}(\varphi)$ are both C-invariant, and so $A = \mathrm{dom}(\theta) \cap \mathrm{im}(\varphi)$ is C-invariant. By Lemma 1, $\varphi^{-1}(A)$ is C-invariant. The functions

$$(\varphi \mid \varphi^{-1}(A)) : \varphi^{-1}(A) \to A \quad \text{and} \quad (\theta \mid A) : A \to \theta(A)$$

are C-isomorphisms and so

$$\theta \circ \varphi = (\theta \mid A) \circ (\varphi \mid \varphi^{-1}(A))$$

is a C-isomorphism. □

Definition. Let X be a left C-system. We say that it satisfies the *orbit condition* if $C \cdot x \cap C \cdot y$ nonempty implies that $C \cdot x \cap C \cdot y = C \cdot z$ for some $z \in X$.

Notation. We shall denote any element z as above by $x \wedge y$. We denote by $x * y$ and $y * x$ elements of C chosen so that

$$x \wedge y = (x * y) \cdot y = (y * x) \cdot x.$$

We now come to our most important definition.

Definition. $J({}_CX)$ denotes the subset of $I({}_CX)$ consisting of those C-isomorphisms between cyclic C-subsystems of X together with the empty map.

Theorem 3. *Let X be a left C-system. Then $J({}_CX)$ is an inverse subsemigroup of $I({}_CX)$ if, and only if, ${}_CX$ satisfies the orbit condition.*

Proof. Suppose that $J({}_CX)$ is an inverse subsemigroup of $I({}_CX)$. The nonzero idempotents of $J({}_CX)$ are the identity functions on the cyclic C-subsystems. The orbit condition now follows from the fact that the product of two idempotents in $J({}_CX)$ is the identity function on the intersection of their domains. Conversely, suppose that ${}_CX$ satisfies the orbit condition. Let $\theta : C\cdot x \to C\cdot y$ be a C-homomorphism, and let $C\cdot u \subseteq C\cdot x$. From Lemma 1, we have that $\theta(C\cdot u) = C\cdot\theta(u)$. The proof that $J({}_CX)$ is an inverse subsemigroup of $I({}_CX)$ is now straightforward. □

We have succeeded in associating an inverse semigroup with zero $J({}_CX)$ to every left C-system X satisfying the orbit condition. Inverse *semigroups* can be constructed when a strengthened form of the orbit condition holds.

Definition. Let X be a left C-system. Then ${}_CX$ satisfies the *strong orbit condition* if for all $x, y \in X$ and for some $z \in X$

$$C\cdot x \cap C\cdot y = C\cdot z.$$

Put $J^*({}_CX) = J({}_CX)\setminus\{0\}$. The proof of the following is straightforward.

Theorem 4. *Let the category C act on the set X on the left and satisfy the strong orbit condition. Then the product of any two nonzero elements of $J({}_CX)$ is nonzero. Consequently, $J^*({}_CX)$ is an inverse semigroup.*

To obtain an explicit description of the multiplication in $J({}_CX)$, we need to introduce an equivalence relation $\mathscr{R}^*$ on the set X determined by the action of C.

Definition. Let X be a left C-system. We define a relation $\mathscr{R}^*$ on X as follows: $(x, y) \in \mathscr{R}^*$ if, and only if, $\mathbf{p}(x) = \mathbf{p}(y)$ and for all $a, b \in C$ such that $a\cdot x$ and $b\cdot x$ are defined, we have that

$$a\cdot x = b\cdot x \iff a\cdot y = b\cdot y;$$

both sides of these two equations always exist since from $\mathbf{p}(x) = \mathbf{p}(y)$ we have that $\exists a\cdot x \Leftrightarrow \exists a\cdot y$ and $\exists b\cdot x \Leftrightarrow \exists b\cdot y$.

It is clear that $\mathscr{R}^*$ is an equivalence relation on X, and that if $(x, y) \in \mathscr{R}^*$ and $\exists c\cdot x$ and $\exists c\cdot y$ then $(c\cdot x, c\cdot y) \in \mathscr{R}^*$.

Lemma 5. *Let X be a left C-system and $x, y \in X$. Then the following are equivalent:*
(i) $(x, y) \in \mathscr{R}^*$.
(ii) *There exists $\theta \in I({}_CX)$ such that $\theta(x) = y$.*

Proof. (i) $\Rightarrow$ (ii). Define a function $\theta : C\cdot x \to C\cdot y$ by $\theta(a\cdot x) = a\cdot y$. First, θ is well-defined, for if $a\cdot x = a'\cdot x$ then $a\cdot y = a'\cdot y$ since $(x,y)\in\mathscr{R}^*$. Clearly, $\theta(x) = y$. Next, θ is injective, for suppose that $\theta(a\cdot x) = \theta(b\cdot x)$. Then $a\cdot y = b\cdot y$, and so $a\cdot x = b\cdot x$, since $(x,y)\in\mathscr{R}^*$. To see that θ is surjective, let $a\cdot y\in C\cdot y$. Then $\mathbf{d}(a) = \mathbf{p}(y) = \mathbf{p}(x)$ since $(x,y)\in\mathscr{R}^*$. Thus $\exists a\ \cdot x$ and clearly $\theta(a\cdot x) = a\cdot y$. We finish off by showing that θ is a C-homomorphism by checking that (M1) and (M2) hold.

(M1) holds: let $x'\in C\cdot x$, where $x' = a\cdot x$. Then

$$\mathbf{p}(\theta(x')) = \mathbf{p}(\theta(a\cdot x)) = \mathbf{p}(a\cdot y) = \mathbf{r}(a).$$

But $\mathbf{r}(a) = \mathbf{p}(a\cdot x) = \mathbf{p}(x')$. Hence $\mathbf{p}(\theta(x')) = \mathbf{p}(x')$.

(M2) holds: let $x' = a\cdot x$ and $a'\in C$ such that $\exists a'\cdot x'$. Then

$$\begin{aligned}\theta(a'\cdot x') &= \theta(a'\cdot(ax))\\ &= \theta((a'a)\cdot x) \quad \text{by (A3)}\\ &= (a'a)\cdot y = a'\cdot(a\cdot y) = a'\cdot\theta(a\cdot x) = a'\cdot\theta(x').\end{aligned}$$

(ii) $\Rightarrow$ (i). Let $\theta\in I({}_CX)$ be such that $\theta(x) = y$. Then $\mathbf{p}(x) = \mathbf{p}(y)$ by (M1). Suppose that $a\cdot x = b\cdot x$. Since $\mathrm{dom}(\theta)$ is a left C-system we have that $a\cdot x\in\ \mathrm{dom}(\theta)$. Hence $\theta(a\cdot x) = \theta(b\cdot x)$. But θ is a C-homomorphism and so $a\cdot\theta(x) = b\cdot\theta(x)$. But $\theta(x) = y$ and so $a\cdot y = b\cdot y$. Conversely, $\theta^{-1}(y) = x$ and $\theta^{-1}\in I({}_CX)$ by Lemma 1. Thus $a\cdot y = b\cdot y$ implies $a\cdot x = b\cdot x$. Hence $(x,y)\in\mathscr{R}^*$. $\square$

Let X be a left C-system, and let x and y be a pair of elements such that $\mathbf{p}(x) = \mathbf{p}(y)$. Then if $u\in C$ such that $\exists u\cdot x$ (and so $\exists u\cdot y$) then we write $(u\cdot x, u\cdot y) = u\cdot(x,y)$.

Lemma 6. *Let X be a left C-system satisfying the orbit condition. On the set of ordered pairs $\mathscr{R}^*$ define a relation $\sim$ by*

$$(x,y)\sim(x',y') \Leftrightarrow (x,y) = u\cdot(x',y') \text{ and } (x',y') = v\cdot(x,y)$$

for some $u,v\in C$. Then $\sim$ is an equivalence relation.

Proof. This is almost immediate; the only case that requires any comment is reflexivity, and this follows from the fact that if $(x,y)\in\mathscr{R}^*$ then $\mathbf{p}(x) = \mathbf{p}(y)$ and so $(x,y) = \mathbf{p}(x)\cdot(x,y)$ by (A1). $\square$

Denote by $[x,y]$ the $\sim$-equivalence class containing the pair (x,y). We can now obtain an explicit description of the multiplication in $J({}_CX)$.

Theorem 7. *Let X be a left C-system satisfying the orbit condition. Let S be the set of $\sim$-equivalence classes together with a new symbol 0. Define a product on S as follows:*

$$[x,y]\otimes[w,z] = \begin{cases} [(w*y)\cdot x, (y*w)\cdot z] & \text{if } C\cdot y\cap C\cdot w\neq\emptyset\\ 0 & \text{else}\end{cases}$$

and all other products equal to 0. *Then* $(S, \otimes)$ *is an inverse semigroup isomorphic to* $J({}_CX)$.

In the semigroup $(S, \otimes)$ *we have that*

$$[x,y]^{-1}=[y,x],\quad [x,y]^{-1}\otimes[x,y]=[y,y],\quad \textit{and}\quad [x,y]\otimes[x,y]^{-1}=[x,x],$$

the idempotents are all the elements of the form $[x,x]$ *for some* $x\in X$ *and the natural partial order is given by*

$$[x,y]\leq[w,z]\Leftrightarrow(x,y)=u\cdot(w,z)\quad \textit{for some } u\in C.$$

Proof. Since the proof of the main claim is rather long, we split it into four parts.

1. $(w*y)\cdot x$ *is* $\mathscr{R}^*$*-related to* $(y*w)\cdot z$.

By definition $(w*y)\cdot y=(y*w)\cdot w$, so that $\mathbf{p}((w*y)\cdot y)=\mathbf{p}((y*w)\cdot w)$. Thus $\mathbf{r}(w*y)=\mathbf{r}(y*w)$ by (A2). But $\mathbf{p}((w*y)\cdot x)=\mathbf{r}(w*y)$ and $\mathbf{p}((y*w)\cdot z)=\mathbf{r}(y*w)$ by (A2). Hence

$$\mathbf{p}((w*y)\cdot x)=\mathbf{p}((y*w)\cdot z).$$

Now suppose that

$$a\cdot[(w*y)\cdot x]=b\cdot[(w*y)\cdot x].$$

Then $(a(w*y))\cdot x=(b(w*y))\cdot x$ by (A3). Thus $(a(w*y))\cdot y=(b(w*y))\cdot y$ since $(x,y)\in\mathscr{R}^*$, and so

$$a\cdot[(w*y)\cdot y]=b\cdot[(w*y)\cdot y]$$

by (A3). By definition $(w*y)\cdot y=(y*w)\cdot w$. Thus

$$a\cdot[(y*w)\cdot w]=b\cdot[(y*w)\cdot w].$$

But $(a(y*w))\cdot w=(b(y*w))\cdot w$ by (A3). Hence $(a(y*w))\cdot z=(b(y*w))\cdot z$ since $(w,z)\in\mathscr{R}^*$. We thus have

$$a\cdot[(y*w)\cdot z]=b\cdot[(y*w)\cdot z]$$

by (A3). We may similarly show that

$$a\cdot[(y*w)\cdot z]=b\cdot[(y*w)\cdot z]$$

implies

$$a\cdot[(w*y)\cdot x]=b\cdot[(w*y)\cdot x].$$

Hence $(w*y)\cdot x$ is $\mathscr{R}^*$-related to $(y*w)\cdot z$.

2. $\otimes$ *is a well-defined binary operation.*

Let $[x,y]=[x',y']$ and $[w,z]=[w',z']$. We show that

$$[x,y]\otimes[w,z]=[x',y']\otimes[w',z'].$$

From the definition there are elements $u,v,a,b\in C$ such that

$$(x,y)=u\cdot(x',y') \quad\text{and}\quad (x',y')=v\cdot(x,y)$$

and

$$(w,z)=a\cdot(w',z') \quad\text{and}\quad (w',z')=b\cdot(w,z).$$

Now $y=u\cdot y'$ and $y'=v\cdot y$ imply that $C\cdot y=C\cdot y'$. Similarly $C\cdot w=C\cdot w'$. Thus

$$C\cdot y\cap C\cdot w\neq\emptyset \;\Leftrightarrow\; C\cdot y'\cap C\cdot w'\neq\emptyset.$$

Hence

$$[x,y]\otimes[w,z]=0 \;\Leftrightarrow\; [x',y']\otimes[w',z']=0.$$

We shall consider the case where $C\cdot y\cap C\cdot w\neq\emptyset$. Let

$$C\cdot y\cap C\cdot w=C\cdot(y\wedge w) \text{ and } C\cdot y'\cap C\cdot w'=C\cdot(y'\wedge w')$$

for some $y\wedge w$ and $y'\wedge w'$. Using the notation introduced earlier we have that

$$y\wedge w=(y*w)\cdot w=(w*y)\cdot y$$

and

$$y'\wedge w'=(y'*w')\cdot w'=(w'*y')\cdot y'.$$

From the definition

$$[x,y]\otimes[w,z]=[(w*y)\cdot x,(y*w)\cdot z]$$

and

$$[x',y']\otimes[w',z']=[(w'*y')\cdot x',(y'*w')\cdot z'].$$

Since $C\cdot(y\wedge w)=C\cdot(y'\wedge w')$ there exist elements $c,d\in C$ such that

$$c\cdot(y\wedge w)=y'\wedge w' \quad\text{and}\quad d\cdot(y'\wedge w')=y\wedge w.$$

Now $y\wedge w=(y*w)\cdot w$ and so $c\cdot(y\wedge w)=(c(y*w))\cdot w$. Thus $y'\wedge w'=(c(y*w))\cdot w$. But $w=aw'$ and so $y'\wedge w'=(c(y*w)a)\cdot w'$. Also $y'\wedge w'=(y'*w')\cdot w'$. Hence

$$(y'*w')\cdot w'=(c(y*w)a)\cdot w'.$$

Now $(w',z')\in\mathscr{R}^*$ and so

$$(y'*w')\cdot z'=(c(y*w)a)\cdot z'.$$

We can similarly show that

$$(w' * y') \cdot x' = (c(w * y)u) \cdot x'.$$

But $a \cdot z' = z$ and $u \cdot x' = x$ and so

$$(y' * w') \cdot z' = c \cdot [(y * w) \cdot z] \text{ and } (w' * y') \cdot x' = c \cdot [(w * y) \cdot x],$$

hence

$$((w' * y') \cdot x', (y' * w') \cdot z') = c \cdot ((w * y) \cdot x, (y * w) \cdot z).$$

We may similarly prove that

$$d \cdot ((w' * y') \cdot x', (y' * w') \cdot z') = ((w * y) \cdot x, (y * w) \cdot z).$$

Hence

$$[(w * y) \cdot x, (y * w) \cdot z] = [(w' * y') \cdot x', (y' * w') \cdot z'].$$

3. *For each* $(x, y) \in \mathscr{R}^*$ *define a function* $\theta_{(x,y)} : C \cdot y \to C \cdot x$ *by* $\theta_{(x,y)}(a \cdot y) = a \cdot x$. *From the proof of Lemma* 5 *we have that* $\theta_{(x,y)} \in J({}_C X)$. *We claim that*

$$\theta_{(x,y)} = \theta_{(x',y')} \Leftrightarrow (x, y) \sim (x', y').$$

Suppose that $\theta_{(x,y)} = \theta_{(x',y')}$. Then

$$C \cdot y = C \cdot y', \quad C \cdot x = C \cdot x' \quad \text{and} \quad (x, y), (x', y') \in \mathscr{R}^*.$$

Thus, in particular, $y = a \cdot y'$ and $y' = b \cdot y$ for some $a, b \in C$. Now

$$\theta_{(x,y)}(y) = \theta_{(x,y)}(\mathbf{p}(y) \cdot y) = x$$

and

$$\theta_{(x',y')}(y) = \theta_{(x',y')}(a \cdot y') = a \cdot x'.$$

But $\theta_{(x,y)}(y) = \theta_{(x',y')}(y)$. Thus $x = a \cdot x'$. Similarly, $x' = b \cdot x$. Thus $(x, y) = a \cdot (x', y')$ and $(x', y') = b \cdot (x, y)$, and so $(x, y) \sim (x', y')$.

Now suppose that $(x, y) \sim (x', y')$. We show that $\theta_{(x,y)} = \theta_{(x',y')}$. By assumption,

$$(x, y) = a \cdot (x', y') \quad \text{and} \quad (x', y') = b \cdot (x, y)$$

for some $a, b, \in C$. Clearly, $C \cdot x = C \cdot x'$ and $C \cdot y = C \cdot y'$. We show that the functions $\theta_{(x,y)}$ and $\theta_{(x',y')}$ take the same values. Let $d \cdot y = d' \cdot y' \in C \cdot y = C \cdot y'$. Then $\theta_{(x,y)}(d \cdot y) = d \cdot x$ and $\theta_{(x',y')}(d' \cdot y') = d' \cdot x'$. Now $d \cdot y = d' \cdot y'$ implies $d \cdot y = (d'b) \cdot y$. But $(x, y) \in \mathscr{R}^*$ so that $d \cdot x = (d'b) \cdot x$. Hence $d \cdot x = d' \cdot x'$, as required.

4. *Define a function* $\Theta : S \to J({}_C X)$ *as follows*: $\Theta(0)$ *is just the empty function on* X *and* $\Theta([x, y]) = \theta_{(x,y)}$. *Then* Θ *is an isomorphism.*

Our calculations above show that the definition of $\Theta([x,y])$ is independent of the choice of representative of $[x,y]$. It also follows from our calculations above that Θ is an injective function. To show that Θ is surjective let $\varphi \in J(_C X)$ where $\varphi: C\cdot y \to C\cdot x$. Clearly, $C\cdot\varphi(y) = C\cdot x$, and so by Lemma 6, we have that $(y,\varphi(y)) \in \mathscr{R}^*$. It is now clear that $\varphi = \theta_{(\varphi(y),y)}$ and so $\Theta([\varphi(y),y]) = \varphi$.

To show that Θ is a homomorphism we compute the product $\theta_{(x,y)} \circ \theta_{(w,z)}$. Suppose that $C\cdot y \cap C\cdot w \neq \emptyset$. Then $C\cdot y \cap C\cdot w = C\cdot(y\wedge w)$, and with the notation introduced earlier we have

$$y\wedge w = (y*w)\cdot w = (w*y)\cdot y.$$

Now

$$\theta_{(w,z)}^{-1}(C\cdot(y\wedge w)) = \theta_{(w,z)}^{-1}(C\cdot(y*w)\cdot w) = C\cdot(y*w)\cdot z.$$

Also

$$\theta_{(x,y)}(C\cdot(y\wedge w)) = \theta_{(x,y)}(C\cdot(w*y)\cdot y) = C\cdot(w*y)\cdot x.$$

Thus $\theta_{(x,y)} \circ \theta_{(w,z)}$ has domain $C\cdot(y*w)\cdot z$ and image $C\cdot(w*y)\cdot x$. We now calculate the effect of this composite function:

$$\begin{aligned}(\theta_{(x,y)} \circ \theta_{(w,z)})((a(y*w))\cdot z) &= \theta_{(x,y)}(\theta_{(w,z)}((a(y*w))\cdot z)) \\ &= \theta_{(x,y)}((a(y*w))\cdot w) \\ &= \theta_{(x,y)}(a\cdot(y\wedge w)) \\ &= \theta_{(x,y)}((a(w*y))\cdot y) \\ &= (a(w*y))\cdot x \\ &= \theta_{((w*y)\cdot x,(y*w)\cdot z)}((a(y*w))\cdot z).\end{aligned}$$

We have shown that if $C\cdot y\cap C\cdot w \neq \emptyset$ then

$$\theta_{(x,y)} \circ \theta_{(w,z)} = \theta_{((w*y)\cdot x,(y*w)\cdot z)}.$$

If on the other hand $C\cdot y\cap C\cdot w = \emptyset$ then $\theta_{(x,y)} \circ \theta_{(w,z)}$ is the empty function. It is now immediate that Θ is a homomorphism and so an isomorphism of semigroups.

The remaining assertions are all straightforward to prove. □

3. Category actions from inverse semigroups with zero

In the last section, we showed how to construct an inverse semigroup with zero from a left C-system satisfying the orbit condition. In this section, we show how to construct a category action from an inverse semigroup with zero.

Definition. Let S be an inverse semigroup. Put

$$C'(S) = \{(s,e) \in S\times E(S) : s^{-1}s \leq e\}.$$

Define $\mathbf{d}(s,e)=(e,e), \mathbf{r}(s,e)=(ss^{-1},ss^{-1})$ and a partial product

$$(s,e)\cdot(t,f)=\begin{cases}(st,f) & \text{if } e=tt^{-1},\\ \text{undefined} & \text{else.}\end{cases}$$

If S is an inverse semigroup with zero then put $Z=\{(0,e):e\in E(S)\}$ and $C(S)=C'(S)\setminus Z$.

Proposition 1. (i) $(C'(S),\cdot)$ *is a right cancellative category.*

(ii) *The isomorphisms in* $C'(S)$ *are the elements of the form* $(s,s^{-1}s)$.

(iii) *If* S *is an inverse semigroup with zero then* $(0,0)$ *is a terminal object in* $C'(S)$, *the unique morphism from* (e,e) *to* $(0,0)$ *being* $(0,e)$. *Furthermore, the only morphism with domain* $(0,0)$ *is* $(0,0)$.

(iv) *If* S *is an inverse semigroup with zero then* $C(S)$ *is a full subcategory of* $C'(S)$.

Proof. (i) Observe that $\exists(s,e)\cdot(t,f)$ precisely when $\mathbf{d}(s,e)=\mathbf{r}(t,f)$. It is now a simple matter to show that $(C'(S),\cdot)$ is a category with identities $\{(e,e):e\in E(S)\}$. We now show that $(C'(S),\cdot)$ is right cancellative. Suppose that

$$(s,e)\cdot(t,f)=(u,i)\cdot(t,f).$$

Then $(st,f)=(ut,f)$ so that $st=ut$. Thus $stt^{-1}=utt^{-1}$. But $e=tt^{-1}=i$ and $s^{-1}s$, $u^{-1}u\le e$. Hence $s=u$ and $e=i$, and so $(s,e)=(u,i)$.

(ii) Observe first that the products $(s,s^{-1}s)\cdot(s^{-1},ss^{-1})$ and $(s^{-1},ss^{-1})\cdot(s,s^{-1}s)$ are defined, and that

$$(s,s^{-1}s)\cdot(s^{-1},ss^{-1})=(ss^{-1},ss^{-1})=\mathbf{r}(s,s^{-1}s),$$

and

$$(s^{-1},ss^{-1})\cdot(s,s^{-1}s)=(s^{-1}s,s^{-1}s)=\mathbf{d}(s,s^{-1}s).$$

Thus $(s,s^{-1}s)$ is invertible with inverse (s^{-1},ss^{-1}). Conversely, suppose that (s,e) is an invertible element. Then there exists an element (t,f) such that $(s,e)\cdot(t,f)$ and $(t,f)\cdot(s,e)$ are defined and

$$(s,e)\cdot(t,f)=\mathbf{r}(s,e) \quad \text{and} \quad (t,f)\cdot(s,e)=\mathbf{d}(s,e).$$

Thus

$$(st,f)=(ss^{-1},ss^{-1}) \quad \text{and} \quad (ts,e)=(e,e).$$

Then $e=tt^{-1}$, $f=ss^{-1}$, $st=ss^{-1}$ and $ts=e$. From $st=ss^{-1}$ we obtain $sts=s$. From $ts=e$ we obtain $tst=et=tt^{-1}t=t$. Thus $t=s^{-1}$. Hence $e=s^{-1}s$.

(iii) Clearly, $(0,e)$ is a morphism from (e,e) to $(0,0)$. Now suppose (s,e) is a morphism from (e,e) to $(0,0)$. Then $\mathbf{r}(s,e)=(\mathbf{r}(s),\mathbf{r}(s))=(0,0)$. But this implies that $s=0$. Now suppose $\mathbf{d}(s,e)=(0,0)$. Then $e=0$ and $s^{-1}s\le 0$. Thus $s^{-1}s=0$ and so $s=0$.

(iv) Consider the full subcategory of $C'(S)$ determined by the identities $C'(S)_O \setminus \{(0,0)\}$. We have that (s,e) belongs to this subcategory if, and only if, $\mathbf{d}(s,e)$, $\mathbf{r}(s,e) \neq (0,0)$. It is easy to check that (s,e) belongs to this subcategory if, and only if, $(s,e) \notin Z$. $\square$

The category $C(S)$ is one of the key ingredients in our construction. Observe that if $S = \{0\}$ then $C(S)$ is the empty category.

Definition. We say that a left C-system X satisfies the *right cancellation condition* if whenever $c \cdot x$ and $d \cdot x$ exist and $c \cdot x = d \cdot x$ then $c = d$. We shall call a pair (C,X) a *system* if the following axioms hold:

(S1) C is a right cancellative category acting on the set X on the left.
(S2) The orbit condition holds.
(S3) The function $\mathbf{p}: X \to C_O$ is surjective.
(S4) The right cancellation condition holds.

Definition. Let S be an inverse semigroup with zero. Put $X_S = S \setminus \{0\}$. Define $\mathbf{p}: X_S \to C(S)_O$ by $\mathbf{p}(x) = (xx^{-1}, xx^{-1})$ and define a function $C(S) * X_S \to X_S$ by $(s,e) \cdot x = sx$ if $\mathbf{d}(s,e) = \mathbf{p}(x)$.

Theorem 2. *For every inverse semigroup with zero S the pair $(C(S), X_S)$ is a system.*

Proof. (S1) holds: observe first that the function $C(S) * X_S \to X_S$ is well-defined. For suppose $\exists(s,e) \cdot x$ where $s,e,x \neq 0$ and $(s,e) \cdot x = sx = 0$. Then $sxx^{-1} = 0$. But $e = xx^{-1}$ and $s^{-1}s \leq e$, and so $s = 0$, which contradicts our choice of s. Thus if $\exists(s,e) \cdot x$ then $(s,e) \cdot x \in X_S$. Next we show that X_S is a left $C(S)$-system by checking that the axioms (A1)–(A3) hold.

(A1) holds: by definition, $\mathbf{p}(x) = (xx^{-1}, xx^{-1})$ and $\mathbf{d}(\mathbf{p}(x)) = (xx^{-1}, xx^{-1})$. Thus $\exists \mathbf{p}(x) \cdot x$. By definition $\mathbf{p}(x) \cdot x = xx^{-1}x = x$.

(A2) holds: suppose that $\exists(s,e) \cdot x$. Then by definition $(s,e) \cdot x = sx$. Now

$$\mathbf{p}(sx) = ((sx)(sx)^{-1}, (sx)(sx)^{-1}) = (sxx^{-1}s^{-1}, sxx^{-1}s^{-1}).$$

But $\mathbf{d}(s,e) = \mathbf{p}(x)$ and so $e = xx^{-1}$, also $s^{-1}s \leq e$ and so $ses^{-1} = ss^{-1}$. Thus

$$\mathbf{p}(sx) = (ss^{-1}, ss^{-1}) = \mathbf{r}(s,e).$$

(A3) holds: suppose that $\exists(s,e)(t,f)$ and $\exists((s,e)(t,f)) \cdot x$. Then

$$(s,e)(t,f) = (st,f) \quad \text{and} \quad ((s,e)(t,f)) \cdot x = stx.$$

From the definitions

$$e = tt^{-1}, s^{-1}s \leq e, t^{-1}t \leq f \quad \text{and} \quad f = xx^{-1}.$$

$\exists(t,f)\cdot x$ since $f=xx^{-1}$, and by definition $(t,f)\cdot x=tx$. Now $\mathbf{d}(s,e)=(e,e)$ and

$$\begin{aligned}\mathbf{p}(tx)&=((tx)(tx)^{-1},(tx)(tx)^{-1})=(txx^{-1}t^{-1},txx^{-1}t^{-1})\\&=(tft^{-1},tft^{-1})=(tt^{-1},tt^{-1})=(e,e).\end{aligned}$$

Thus $\exists(s,e)\cdot(tx)$ and $(s,e)\cdot(tx)=stx$. By Proposition 1, $C(S)$ is a right cancellative category.

(S2) holds: we begin by describing the cyclic $C(S)$-subsystems of X_S. By definition,

$$\begin{aligned}C(S)\cdot x&=\{(s,e)\cdot x:(s,e)\in C(S),\ \exists(s,e).x\}\\&=\{sx:s^{-1}s\le e=xx^{-1}\text{ where } s,e,x\ne 0\}.\end{aligned}$$

Clearly, $C(S)\cdot x\subseteq Sx\setminus\{0\}$. Let $sx\in Sx\setminus\{0\}$ and let $e=xx^{-1}$. Then

$$sx=(sxx^{-1})x=(se)x.$$

Consider the ordered pair (se,e). Clearly, $(se)^{-1}se\le e$. Thus $(se,e)\in C(S)$. Furthermore, $(se,e)\cdot x=sx$. Thus $C(S)\cdot x=Sx\setminus\{0\}$. Suppose now that

$$C(S)\cdot x\cap C(S)\cdot y\ne\emptyset.$$

Then there exists $a\in Sx\cap Sy$ with $a\ne 0$. Now $a=sx=ty$ for some $s,t\in S$, and $ax^{-1}x=a$ and $ay^{-1}y=a$. Thus $ax^{-1}xy^{-1}y=a$. It follows that $x^{-1}xy^{-1}y\ne 0$. Clearly, $Sx\cap Sy=Sx^{-1}xy^{-1}y$, and so

$$C(S)\cdot x\cap C(S)\cdot y=C(S)\cdot(x^{-1}xy^{-1}y).$$

(S3) holds: if (e,e) is any identity of $C(S)$ then $e\in S\setminus\{0\}$ and $\mathbf{p}(e)=(e,e)$.

(S4) holds: suppose that

$$(s,e)\cdot x=(t,f)\cdot x$$

Then $sx=tx$, and $e=xx^{-1}=f$. Thus $sxx^{-1}=txx^{-1}$ and so $se=te$. It follows that $s=t$ since $s^{-1}s\le e$ and $t^{-1}t\le f=e$. Hence $(s,e)=(t,f)$. □

The description of the cyclic $C(S)$-systems contained in the proof of the above theorem makes the proof of the following result immediate.

Theorem 3. *Let S be an inverse semigroup. Then $(C(S),X_S)$ is a system satisfying the strong orbit condition.*

Systems are much easier to handle than arbitrary category actions, as the following result indicates.

Lemma 4. *Let (C,X) be a system.*

(i) *For all $x,y\in X$ we have that*

$$(x,y)\in\mathscr{R}^*\ \Leftrightarrow\ \mathbf{p}(x)=\mathbf{p}(y).$$

(ii) *In the construction of Theorem* 2.7, *we have that*

$$(x,y)\sim(x',y') \Leftrightarrow (x,y)=u\cdot(x',y')$$

for some isomorphism u *in* C.

Proof. (i) Suppose that $\mathbf{p}(x)=\mathbf{p}(y)$, and $a\cdot x=b\cdot x$. Then by the cancellation condition $a=b$. Thus $a\cdot y=b\cdot y$. The converse is similar. Hence $(x,y)\in\mathscr{R}^*$.

The converse is immediate from the definition of $\mathscr{R}^*$.

(ii) Suppose that $(x,y)\sim(x',y')$. Then from the definition there are elements u and v of C such that

$$x=u\cdot x',\ \ y=u\cdot y',\ \ x'=v\cdot x \text{ and } y'=v\cdot y.$$

Now $x=u\cdot x'=(uv)\cdot x$. Thus by the cancellation condition $\mathbf{r}(x)=uv$. Similarly, $\mathbf{r}(x')=vu$. Thus u and v are mutually inverse isomorphisms.

The converse is immediate. □

4. Functors between systems and semigroups

We now place the results of the previous two sections in their proper categorical setting.

Definition. Let (C,X) and (D,Y) be systems and let (F,θ) be a morphism from (C,X) to (D,Y). We say that (F,θ) is a *system morphism* if the condition (M3) below holds:

$$C\cdot x\cap C\cdot y=\emptyset \Rightarrow D\cdot\theta(x)\cap D\cdot\theta(y)=\emptyset$$

and

$$C\cdot x\cap C\cdot y=C\cdot z \Rightarrow D\cdot\theta(x)\cap D\cdot\theta(y)=D\cdot\theta(z).$$

The two key categories of this paper are: the category **Sys** of systems and system morphisms; the category **Inv** of inverse semigroups with zero and 0-restricted homomorphisms.

Define a function **J** from **Sys** to **Inv** as follows: if (C,X) is a system then $\mathbf{J}(C,X)$ is the inverse semigroup constructed in Theorem 2.7. If $(F,\theta):(C,X)\to(D,Y)$ is a morphism of systems then $\mathbf{J}(F,\theta):\mathbf{J}(C,X)\to\mathbf{J}(D,Y)$ is defined by $\mathbf{J}(F,\theta)([x,y])=[\theta(x),\theta(y)]$ and $\mathbf{J}(F,\theta)(0)=0$.

Theorem 1. $\mathbf{J}:\mathbf{Sys}\to\mathbf{Inv}$ *is a functor.*

Proof. We begin by checking that $\mathbf{J}(F,\theta)$ is a well-defined function. Let $[x,y]\in\mathbf{J}({}_CX)$. Then $(x,y)\in\mathscr{R}^*$. But by Lemma 3.4(i), this is equivalent to $\mathbf{p}(x)=\mathbf{p}(y)$. But $F(\mathbf{p}(x))=F(\mathbf{p}(y))$ implies that $\mathbf{p}(\theta(x))=\mathbf{p}(\theta(y))$ by (M1). Thus by Lemma 3.4(i), we have that $(\theta(x),\theta(y))\in\mathscr{R}^*$. It follows that $[\theta(x),\theta(y)]\in\mathbf{J}({}_DY)$. Now suppose that

$[x,y]=[x',y']$. Then $(x,y)=u\cdot(x',y')$ for some isomorphism u in C by Lemma 3.4(ii). Clearly, $(\theta(x),\theta(y))=\theta(u)\cdot(\theta x',\theta y')$, and $\theta(u)$ is an isomorphism in D. Thus by Lemma 3.4(ii), we have that $[\theta(x),\theta(y)]=[\theta(x'),\theta(y')]$.

We now show that $\mathbf{J}(F,\theta)$ is a homomorphism in **Inv**. Let $[x,y],[w,z]\in\mathbf{J}({}_CX)$. Suppose first that the product of $[x,y]$ and $[w,z]$ is zero. Then by definition of the product, we have that $C\cdot y\cap C\cdot w$ is empty. But (F,θ) is a system morphism and so $D\cdot\theta(y)\cap D\cdot\theta(w)$ is also empty. It follows that $[\theta(x),\theta(y)]\otimes[\theta(w),\theta(z)]$ is zero. Suppose now that $[x,y]\otimes[w,z]$ is nonzero. Then by assumption $C\cdot y\cap C\cdot w=C\cdot(y\wedge w)$, and so

$$[x,y]\otimes[w,z]=[(w*y)\cdot x,(y*w)\cdot z].$$

Again (F,θ) is a system morphism and so $D\cdot\theta(y)\cap D\cdot\theta(w)=D\cdot\theta(y\wedge w)$. Also $F(w*y)\cdot\theta(y)=\theta(y\wedge w)$ and $F(y*w)\cdot\theta(w)=\theta(y\wedge w)$. It follows that we can take $\theta(w)*\theta(y)$ to be $F(w*y)$ and $\theta(y)*\theta(w)$ to be $F(y*w)$. It is now straightforward to show that $\mathbf{J}(F,\theta)$ is a homomorphism, and that $\mathbf{J}$ is a functor. □

A function $\mathbf{C}$ from **Inv** to **Sys** is defined as follows: if S is an inverse semigroup then $\mathbf{C}(S)=(C(S),X_S)$, as defined in Theorem 3.2. If $\theta:S\to T$ is a homomorphism in **Inv** then $\mathbf{C}(\theta):(C(S),X_S)\to(C(T),X_T)$ is defined to be $\mathbf{C}(\theta)=(F_\theta,\theta)$ where $F_\theta:C(S)\to C(T)$ is defined by $F_\theta(s,e)=(\theta(s),\theta(e))$ and $\theta:X_S\to X_T$ is the restriction of θ to $S\setminus\{0\}$.

Theorem 2. $\mathbf{C}:\mathbf{Inv}\to\mathbf{Sys}$ *is a functor.*

Proof. We show that (F_θ,θ) is a system morphism by checking that the axioms (M1), (M2) and (M3) hold.

(M1) holds: let $x\in X_S$. Then $\mathbf{p}(\theta(x))=(\theta(x)\theta(x)^{-1},\theta(x)\theta(x)^{-1})$. Whereas

$$F_\theta(\mathbf{p}(x))=F_\theta(xx^{-1},xx^{-1})=(\theta(xx^{-1}),\theta(xx^{-1})).$$

Hence $\mathbf{p}(\theta(x))=F_\theta(\mathbf{p}(x))$.

(M2) holds: suppose $\exists(s,e)\cdot x$ in $(C(S),X_S)$. By definition $(s,e)\cdot x=sx$ and so $\theta((s,e)\cdot x)=\theta(sx)$. On the other hand, $F_\theta(s,e)=(\theta(s),\theta(e))$, so that $F_\theta(s,e)\cdot\theta(x)=\theta(s)\theta(x)$.

(M3) holds: suppose that $C(S)\cdot x\cap C(S)\cdot y$ is empty. Then

$$C(S)\cdot(x^{-1}xy^{-1}y)=(Sx^{-1}xy^{-1}y)\setminus\{0\}$$

by the proof of Theorem 3.2. Hence $x^{-1}xy^{-1}y=0$. It follows that

$$\theta(x)^{-1}\theta(x)\theta(y)^{-1}\theta(y)=0.$$

Thus $C(T)\cdot\theta(x)\cap C(T)\cdot\theta(y)$ is also empty. A similar argument shows that

$$C(S)\cdot x\cap C(S)\cdot y=C(S)\cdot z$$

implies

$$C(T)\cdot\theta(x)\cap C(T)\cdot\theta(y)=C(T)\cdot\theta(z).$$

It is now easy to check that C is a functor. □

5. Equivalent systems

In this section, we introduce a special class of system morphisms called 'equivalences'. We shall prove that if there is an equivalence between two systems then their associated inverse semigroups are isomorphic. Isomorphisms between systems will be equivalences, but the point of the definition is that equivalences form a much broader class of morphisms than isomorphisms. In order to define equivalences, we need to describe some extra structures which the categories **Sys** and **Inv** possess. We begin by defining 'transformations' between system morphisms; these arise since system morphisms are essentially functors and so we can consider natural transformations between them.

Definition. Let (F,θ) and (G,φ) be system morphisms from (C,X) to (D,Y). A *transformation* τ from (F,θ) to (G,φ) is defined by the following two conditions:

(T1) $\tau:F\rightarrow G$ is a natural transformation.

(T2) $\varphi(x)=\tau_{\mathbf{p}(x)}\cdot\theta(x)$ for all $x\in X$.

The right-hand side of (T2) makes sense since $\tau_e\in\hom(F(e),G(e))$ for each identity e in C. Thus

$$\mathbf{d}(\tau_{\mathbf{p}(x)})=F(\mathbf{p}(x))=\mathbf{p}(\theta(x)),$$

by (M1), and so $\tau_{\mathbf{p}(x)}\cdot\theta(x)$ is defined.

Definitions. The *identity transformation* from (F,θ) to (F,θ) is the identity natural transformation 1_F from F to itself; condition (T2) holds automatically. The transformation τ is said to be an *isomorphism* if $\tau:F\rightarrow G$ is a natural isomorphism. In this case τ^{-1} denotes the transformation from G to F defined by $\tau_e^{-1}=(\tau_e)^{-1}$ for each identity e in C.

Lemma 1. *Let (C,X) and (D,Y) be systems.*

(i) *The transformations between the system morphisms from (C,X) to (D,Y) form a category which is a preorder.*

(ii) *If $\tau:(F,\theta)\rightarrow(G,\varphi)$ and $\sigma:(G,\varphi)\rightarrow(F,\theta)$ are transformations then τ is an isomorphism and $\sigma=\tau^{-1}$.*

Proof. (i) Let μ be a transformation from (F,θ) to (G,φ) and let ν be a transformation from (G,φ) to (H,ψ). By (T1), μ is a natural transformation from F to G and ν is a natural transformation from G to H. Thus $\nu\mu$ is a natural transformation from F to H. For each $x \in X$ we have by (T2) that

$$\varphi(x) = \mu_{\mathbf{p}(x)} \cdot \theta(x) \quad \text{and} \quad \psi(x) = \nu_{\mathbf{p}(x)} \cdot \varphi(x).$$

Thus

$$\psi(x) = (\nu\mu)_{\mathbf{p}(x)} \cdot \theta(x).$$

Hence $\nu\mu$ is a transformation. It is easy to check that if τ is a transformation from (F, θ) to (G, ϕ) then $\tau 1_F = \tau = 1_G \tau$. It is now evident that the transformations form a category. To prove that this category is a preorder, suppose that μ and ν are both transformations from (F, θ) to (G, ϕ). Then by (T2)

$$\mu_{\mathbf{p}(x)} \cdot \theta(x) = \nu_{\mathbf{p}(x)} \cdot \theta(x)$$

for all $x \in X$. Thus by the right cancellation condition, $\mu_{\mathbf{p}(x)} = \nu_{\mathbf{p}(x)}$. But $\mathbf{p} : X \to C_o$ is a surjection, and so $\mu_e = \nu_e$ for every $e \in C_o$. Thus $\mu = \nu$.

(ii) Immediate from (i). □

The category **Inv** also has some extra structure, which arises from the fact that every inverse semigroup comes equipped with a natural partial order. Let $\theta, \varphi : S \to T$ be two homomorphisms in **Inv**. We write $\varphi \leq \theta$ if $\varphi(s) \leq \theta(s)$ for all $s \in S$. Thus the set of all homomorphisms from S to T is a partially ordered set. The link between transformations in **Sys** and the order relation between homomorphisms in **Inv** is provided by the following result.

Lemma 2. (i) *Let $(F, \theta), (G, \varphi) : (C, X) \to (D, Y)$ be morphisms of systems. If $\tau : (F, \theta) \to (G, \varphi)$ is a transformation then $\mathbf{J}(F, \theta) \leq \mathbf{J}(G, \varphi)$ in* **Inv**.

(ii) *Let $\theta, \varphi : S \to T$ be homomorphisms in* **Inv** *such that $\varphi \leq \theta$. Then there is a transformation $\tau : \mathbf{C}(\varphi) \to \mathbf{C}(\theta)$.*

Proof. (i) By Theorem 4.2,

$$\mathbf{J}(F, \theta)([x, y]) = [\theta(x), \theta(y)] \text{ and } \mathbf{J}(G, \varphi)([x, y]) = [\varphi(x), \varphi(y)].$$

By (T2), we have that

$$\varphi(x) = \tau_{\mathbf{p}(x)} \cdot \theta(x) \quad \text{and} \quad \varphi(y) = \tau_{\mathbf{p}(y)} \cdot \theta(y).$$

But $\mathbf{p}(x) = \mathbf{p}(y)$. Thus by Theorem 2.7, we have that

$$[\varphi(x), \varphi(y)] \leq [\theta(x), \theta(y)].$$

Hence $\mathbf{J}(F, \theta) \leq \mathbf{J}(G, \phi)$.

(ii) Recall that $\mathbf{C}(\theta) = (F_\theta, \theta)$ and $\mathbf{C}(\varphi) = (F_\varphi, \varphi)$ where

$$F_\theta(s, e) = (\theta(s), \theta(e)) \quad \text{and} \quad \theta : X_S \to X_T$$

and

$$F_\varphi(s,e)=(\varphi(s),\varphi(e)) \quad \text{and} \quad \varphi:X_S\to X_T.$$

For each $(e,e)\in C(S)_O$ put $\tau_{(e,e)}=(\varphi(e),\theta(e))$. Then $\tau_{(e,e)}\in C(S)$ since $\varphi(e)\leq\theta(e)$ by assumption. Clearly,

$$\mathbf{d}(\tau_{(e,e)})=F_\theta(e,e) \quad \text{and} \quad \mathbf{r}(\tau_{(e,e)})=F_\varphi(e,e).$$

Thus

$$\tau_{(e,e)}\in \hom(F_\theta(e,e),F_\varphi(e,e)).$$

We show that τ is a transformation of system morphisms by showing that the axioms (T1) and (T2) hold.

(T1) holds: let $(e,e),(f,f)\in C(S)_O$ and $(s,e)\in\hom((e,e),(f,f))$. Then

$$F_\varphi(s,e)\tau_{(e,e)}=(\varphi(s),\varphi(e))(\varphi(e),\theta(e))=(\varphi(s)\varphi(e),\varphi(e))$$

which is equal to $(\varphi(s),\theta(e))$ since $s^{-1}s\leq e$. Whereas

$$\tau_{(f,f)}F_\theta(s,e)=(\varphi(f),\theta(f))(\theta(s),\theta(e))=(\varphi(f)\theta(s),\theta(e))$$

which is equal to $(\varphi(s),\theta(e))$, since $\varphi(s)\leq\theta(s)$ and $\varphi(s)=\varphi(ss^{-1})\theta(s)=\varphi(f)\theta(s)$.

(T2) holds: let $x\in X_S$. Then

$$\mathbf{p}(x)=(xx^{-1},xx^{-1}) \quad \text{and} \quad \tau_{\mathbf{p}(x)}=(\varphi(xx^{-1}),\theta(xx^{-1})).$$

Now

$$\tau_{\mathbf{p}(x)}\cdot\theta(x)=(\varphi(xx^{-1}),\theta(xx^{-1}))\cdot\theta(x)=\varphi(xx^{-1})\theta(x)$$

which is equal to $\varphi(x)$ since $\varphi(x)\leq\theta(x)$. □

We now single out a special class of system morphisms. In the definition below, we use the following notation: if (C,X) is a system then $(1_C,1_X)$ is the identity system morphism at (C,X), where 1_C is the identity functor on C and 1_X is the identity function on X.

Definition. Let (C,X) and (D,Y) be systems. A system morphism (F,θ) from (C,X) to (D,Y) is said to be an *equivalence* (*of systems*) if there is a system morphism (G,φ) from (D,Y) to (C,X) and isomorphisms

$$\sigma:(1_D,1_Y)\to(F\circ G,\theta\circ\varphi) \quad \text{and} \quad \tau:(1_C,1_X)\to(G\circ F,\varphi\circ\theta).$$

The key properties of equivalences are described in our next result.

Theorem 3. (i) *If (C,X) is a system then $(1_C,1_X)$ is an equivalence.*

(ii) *The composition of equivalences is an equivalence.*

(iii) *Let* $(F,\theta):(C,X) \rightarrow (D,Y)$ *be an equivalence. Then* $\mathbf{J}(F,\theta):\mathbf{J}(C,X) \rightarrow \mathbf{J}(D,Y)$ *is an isomorphism.*

Proof. The proofs of (i) and (ii) are straightforward.

(iii) By definition there is a system morphism (G,φ) from (D,Y) to (C,X) and isomorphisms σ and τ such that

$$\sigma:(1_D,1_Y)\rightarrow(F\circ G,\theta\circ\varphi) \quad \text{and} \quad \tau:(1_C,1_X)\rightarrow(G\circ F,\varphi\circ\theta).$$

Since $\mathbf{J}$ is a functor we have that

$$\mathbf{J}(F,\theta)\mathbf{J}(G,\phi)=\mathbf{J}(F\circ G,\theta\circ\varphi) \quad \text{and} \quad \mathbf{J}(G,\phi)\mathbf{J}(F,\theta)=\mathbf{J}(G\circ F,\varphi\circ\theta).$$

Now $\mathbf{J}(1_D,1_Y)$ is the identity homomorphism on $\mathbf{J}(D,Y)$ and $\mathbf{J}(1_C,1_X)$ is the identity homomorphism on $\mathbf{J}(C,X)$. Thus by Lemma 2, $\mathbf{J}(F\circ G,\theta\circ\varphi)$ is an identity homomorphism, as is $\mathbf{J}(G\circ F,\varphi\circ\theta)$. It is now immediate that $\mathbf{J}(F,\theta)$ is an isomorphism. $\square$

Two systems are said to be *equivalent* if there is an equivalence of systems between them. The above result implies that equivalent systems give rise to isomorphic inverse semigroups. An explicit description of equivalences is contained in the following result.

Proposition 4. *A system morphism* (F,θ) *from* (C,X) *to* (D,Y) *is an equivalence of systems if, and only if, the following axioms hold*:

(ES1) F *is an equivalence of categories.*

(ES2) *For each* $y\in Y$ *there exists an isomorphism* $u\in D$ *and an element* $x\in X$ *such that* $y=u\cdot\theta(x)$.

(ES3) *If* $y_1=a\cdot y_2$ *in* ${}_DY$ *and* $\theta(x_1)=y_1$ *and* $\theta(x_2)=y_2$ *then there exists* $a'\in C$ *such that* $x_1=a'\cdot x_2$.

Proof. Let (F,θ) be an equivalence of systems from (C,X) to (D,Y). Then there is a system morphism (G,φ) from (D,Y) to (C,X) and isomorphisms σ and τ such that

$$\sigma:(1_D,1_Y)\rightarrow(F\circ G,\theta\circ\varphi) \quad \text{and} \quad \tau:(1_C,1_X)\rightarrow(G\circ F,\varphi\circ\theta).$$

Thus

$$\sigma:1_D\rightarrow F\circ G \quad \text{and} \quad \tau:1_C\rightarrow G\circ F$$

are natural isomorphisms. In particular, F is an equivalence of categories, and so (ES1) holds. To show that (ES2) holds: let $y\in Y$. Since σ is an isomorphism from $(1_D,1_Y)$ to $(F\circ G,\theta\circ\phi)$, we have that $\theta(\varphi(y))=\sigma_{\mathbf{p}(y)}\cdot y$ by (T2). Put $x=\varphi(y)$ and $u=(\sigma_{\mathbf{p}(y)})^{-1}$. Then $y=u\cdot\theta(x)$, as required. To show that (ES3) holds: let $y_1=a\cdot y_2$ in ${}_DY$ and let $\theta(x_1)=y_1$ and $\theta(x_2)=y_2$. Since there is an isomorphism τ from $(1_C,1_X)$ to $(G\circ F,\phi\circ\theta)$, we have that $x=\tau_{\mathbf{p}(x)}\cdot(\varphi\circ\theta)(x)$ for all $x\in X$ by (T2). Thus

$$x_1=\tau_{\mathbf{p}(x_1)}\cdot(\varphi\circ\theta)(x_1) \quad \text{and} \quad x_2=\tau_{\mathbf{p}(x_2)}\cdot(\varphi\circ\theta)(x_2).$$

Now

$$x_1 = \tau_{\mathbf{p}(x_1)} \cdot (\varphi \circ \theta)(x_1) = \tau_{\mathbf{p}(x_1)} \cdot \varphi(\theta(x_1)).$$

However $\theta(x_1) = y_1$ and so

$$x_1 = \tau_{\mathbf{p}(x_1)} \cdot \varphi(y_1).$$

Also $\varphi(y_1) = G(a) \cdot \varphi(y_2)$. Thus

$$x_1 = \tau_{\mathbf{p}(x_1)} \cdot (G(a) \cdot \varphi(y_2)).$$

Since $\varphi(y_2) = \varphi(\theta(x_2))$, we have that

$$x_1 = \tau_{\mathbf{p}(x_1)} \cdot (G(a) \cdot \varphi(\theta(x_2))).$$

But

$$(\varphi \circ \theta)(x_2) = \tau^{-1}_{\mathbf{p}(x_2)} \cdot x_2.$$

Put $a' = \tau_{\mathbf{p}(x_1)} G(a) \tau^{-1}_{\mathbf{p}(x_2)}$. Then $x_1 = a' \cdot x_2$.

To prove the converse, let $(F, \theta) : (C, X) \to (D, Y)$ be a system morphism satisfying (ES1), (ES2) and (ES3). We shall prove that it is an equivalence of systems. For each identity e in D there exists, by (ES1), an identity $G(e)$ in C and isomorphism $\sigma_e \in \hom(e, F(G(e)))$. In the usual way (see [9]), this information may be used to construct a functor $G : D \to C$. Now let $y \in Y$. By (ES2), there is an isomorphism u in D and element x of X, such that $y = u \cdot \theta(x)$. Now $\sigma_{\mathbf{p}(y)} \cdot y$ is defined and so

$$\sigma_{\mathbf{p}(y)} \cdot y = (\sigma_{\mathbf{p}(y)} u) \cdot \theta(x).$$

Now

$$\sigma_{\mathbf{p}(y)} u \in \hom(F(\mathbf{p}(x)), F(G(\mathbf{p}(y))))$$

and so since F is an equivalence of categories there exists a unique isomorphism u' in C such that $u' \in \hom(\mathbf{p}(x), G(\mathbf{p}(y)))$ and $F(u') = \sigma_{\mathbf{p}(y)} u$. Now

$$\sigma_{\mathbf{p}(y)} \cdot y = F(u') \cdot \theta(x) = \theta(u' \cdot x)$$

by (M2). Thus $y = \sigma^{-1}_{\mathbf{p}(y)} \cdot \theta(u' \cdot x)$. Define $\varphi(y) = u' \cdot x$. It is now straightforward to check that (G, φ) is an equivalence of systems.

By (ii) above, $(F \circ G, \theta \circ \varphi)$ is an equivalence from (D, Y) to itself. We have seen that there is an isomorphism $\sigma_e : e \to F(G(e))$ for each identity e in D. Together these isomorphisms are the components of a natural isomorphism σ from 1_D to $F \circ G$. It is also easy to check that (T2) holds. Thus σ is an isomorphism from $(1_D, 1_Y)$ to $(F \circ G, \theta \circ \varphi)$.

We shall now define an isomorphism τ from $(1_C, 1_X)$ to $(G \circ F, \varphi \circ \theta)$. We begin by constructing a natural isomorphism from 1_C to $G \circ F$. Let f be an identity in C. Then by the above $\sigma_{F(f)}$ is an isomorphism from $F(f)$ to $F(G(F(f)))$. Since F is an equivalence of categories, there exists a unique isomorphism τ_f from f to $G(F(f))$ such that $F(\tau_f)=\sigma_{F(f)}$. It is now easy to check that the τ_f are the components of a natural isomorphism from 1_C to $G\circ F$. We finish off by showing that τ satisfies (T2). Let x_1 be an arbitrary element of X. Put $y=\theta(x_1)$. With the notation as before, there is an isomorphism u in D and element x in X such that $\theta(x_1)=y=u\cdot\theta(x)$. As above, there exists a unique isomorphism u' from $\mathbf{p}(x)$ to $G(\mathbf{p}(y))$ such that $F(u')=\sigma_{\mathbf{p}(y)}u$. Now, $y=\theta(x_1)=u\cdot\theta(x)$, thus by (ES3), there is a unique morphism v from $\mathbf{p}(x)$ to $\mathbf{p}(x_1)$ such that $F(v)=u$ and $x_1=v\cdot x$. In fact, v is an isomorphism since F is an equivalence. Now,

$$F(u')u^{-1}=F(\tau_{\mathbf{p}(x_1)}),$$

and $u^{-1}=F(v)^{-1}$. The category product $u'v^{-1}$ is defined and so

$$F(u'v^{-1})=F(\tau_{\mathbf{p}(x_1)}).$$

Both $u'v^{-1}$ and $\tau_{\mathbf{p}(x_1)}$ are morphisms from $\mathbf{p}(x_1)$ to $G(F(\mathbf{p}(x_1)))$, and so $\tau_{\mathbf{p}(x_1)}=u'v^{-1}$ since F is an equivalence. By definition $\varphi(y)=u'\cdot x$ and so

$$\varphi(y)=\varphi(\theta(x_1))=u'\cdot x=(\tau_{\mathbf{p}(x_1)}v)\cdot x,$$

which is equal to $\tau_{\mathbf{p}(x_1)}\cdot x_1$, as required. □

6. Composing the functors

In this section, we shall justify the whole approach we have been adopting. In Section 4, we constructed functors $\mathbf{J}:\mathbf{Sys}\to\mathbf{Inv}$ and $\mathbf{C}:\mathbf{Inv}\to\mathbf{Sys}$. We shall now compare S with $(\mathbf{J}\circ\mathbf{C})(S)$, and (C,X) with $(\mathbf{C}\circ\mathbf{J})(C,X)$.

We start by comparing S with $(\mathbf{J}\circ\mathbf{C})(S)$. Let S be any inverse semigroup with zero. We shall denote by $\check{S}$ the inverse semigroup of all S-isomorphisms between principal left ideals of S. Define $\Theta:S\to\check{S}$ by $\Theta(s):Ss^{-1}s\to Sss^{-1}$ and $\Theta(s)=as^{-1}$ for each $a\in Ss^{-1}s$. Then Θ is an injective homomorphism, since it is a restriction of the familiar Wagner–Preston representation (see Theorem V.1.10 of [6]).

Lemma 1. *Θ is an isomorphism from S to $\check{S}$.*

Proof. It remains only to prove that Θ is a surjection. Let $\theta:Sx\to Sy$ be an S-isomorphism. Since S is inverse $x^{-1}x$ is the unique idempotent generator of Sx and $y^{-1}y$ is the unique idempotent generator of Sy. Put $a=\theta(x^{-1}x)$. Now

$$\theta(sx)=\theta(sxx^{-1}x)=\theta(sx(x^{-1}x))=sx\theta(x^{-1}x)=sxa.$$

Thus $\theta=(\rho_a\mid Sx)$, where ρ_a is right multiplication by a. Observe that

$$a=\theta(x^{-1}x)=\theta((x^{-1}x)(x^{-1}x))=x^{-1}xa.$$

Also $a \in Sy$ and so $ay^{-1}y = a$. Hence $a \in x^{-1}xSy^{-1}y$. There exists $a' \in Sx$ such that $\theta(a') = y^{-1}y$ since θ is surjective. Now

$$y^{-1}y = \theta(a') = \theta(a'x^{-1}x) = a'\theta(x^{-1}x) = a'a.$$

Thus $a'a = y^{-1}y$. It follows that

$$aa'a = ay^{-1}y = a.$$

Also $a' \in Sx^{-1}x$ and so $a'aa' \in Sx^{-1}x$. We calculate $\theta(a'aa')$:

$$\theta(a'aa') = a'a\theta(a') = y^{-1}yy^{-1}y = \theta(a').$$

But θ is injective and so $a' = a'aa'$. Since S is inverse we must have that $a' = a^{-1}$ and so $a^{-1}a = y^{-1}y$. Now $a^{-1}, aa^{-1} \in Sx$. Thus we may calculate

$$\theta(aa^{-1}) = a\theta(a^{-1}) = a\theta(a') = ay^{-1}y = a.$$

But $\theta(x^{-1}x) = a$. Hence $aa^{-1} = x^{-1}x$ since θ is injective.

We have proved that for every $\theta: Sx \to Sy$ an S-isomorphism, the element $a = \theta(x^{-1}x)$ is such that $a^{-1} = \theta^{-1}(y^{-1}y)$, $\theta = (\rho_a \mid Sx)$ and $x^{-1}x\mathscr{R}a\mathscr{L}y^{-1}y$. Thus $\Theta(a^{-1}) = \theta$. $\square$

We may now prove that every inverse semigroup with zero is isomorphic to an inverse semigroup arising from a category acting on a set satisfying the orbit condition.

Theorem 2. *Let S be an inverse semigroup with zero.*

(i) *The $C(S)$-isomorphisms from $C(S) \cdot x$ to $C(S) \cdot y$ induce, and are induced by, S-isomorphisms from Sx to Sy.*

(ii) *S and $(\mathbf{J} \circ \mathbf{C})(S)$ are isomorphic.*

Proof. (i) Let $\theta: C(S) \cdot x \to C(S) \cdot y$ be a $C(S)$-isomorphism. By the proof of Theorem 3.2, we have that $C(S) \cdot x = Sx \backslash \{0\}$ and $C(S) \cdot y = Sy \backslash \{0\}$. Extend θ to a function from Sx to Sy by defining $\theta(0) = 0$. Clearly, $\theta: Sx \to Sy$ is a bijection. We show that θ is an S-isomorphism. As a first step, we prove that $sx\mathscr{R}\theta(sx)$ for all $s \in S$. We consider the cases where sx is nonzero and zero separately. Suppose that sx is nonzero. Then $sx \in C(S) \cdot x$. By (M1), $\mathbf{p}(\theta(sx)) = \mathbf{p}(sx)$. Then $\theta(sx)\theta(sx)^{-1} = (sx)(sx)^{-1}$ and so $sx\mathscr{R}\theta(sx)$. If $sx = 0$ then $\theta(sx) = 0$ by definition and $sx\mathscr{R}\theta(sx)$ is immediate.

We can now show that $\theta: Sx \to Sy$ is an S-homomorphism. We need to show for all $t \in S$ and $sx \in Sx$ that $\theta(t(sx)) = t\theta(sx)$. Suppose that $t(sx) = 0$ but $t, sx \neq 0$. From the result above $\theta(sx)\mathscr{R}sx$ and so since $\mathscr{R}$ is a left congruence, $t\theta(sx)\mathscr{R}tsx = 0$. Thus $t\theta(sx) = 0$ and so $\theta(t(sx)) = t\theta(sx)$. Thus we may suppose that $t(sx) \neq 0$. Put $e = (sx)(sx)^{-1}$. Then $t(sx) = t(e(sx))$. Consider the ordered pair (te, e). Observe that $e \neq 0$, for if $e = 0$ then $sx = esx = 0$ which is a contradiction. Also, $te \neq 0$ since $te = 0$ implies that $tesx = tsx = 0$ which is a contradiction. Thus $te, e \neq 0$. Also

$$(te)^{-1}te = et^{-1}te \leq e.$$

It follows that $(te,e) \in C(S)$. Now

$$\mathbf{d}(te,e) = (e,e) = \mathbf{p}(sx).$$

Thus $\exists (te,e)\cdot(sx)$ and

$$(te,e)\cdot(sx) = tesx = tsx.$$

Since θ is a $C(S)$-homomorphism, we have that

$$\theta((te,e)\cdot(sx)) = (te,e)\cdot\theta(sx).$$

Hence $\theta(tsx) = te\theta(sx)$. Now

$$e = (sx)(sx)^{-1}\mathscr{R}\theta(sx)$$

by the result above. Thus $te\theta(sx) = t\theta(sx)$. It follows that $\theta(t(sx)) = t\theta(sx)$.

Conversely, let $\theta : Sx \to Sy$ be an S-isomorphism. We can define a $C(S)$-isomorphism θ' from $C(S)\cdot x$ to $C(S)\cdot y$ by $\theta'((s,e)\cdot x) = \theta(sx)$. Suppose $(t,f)\cdot x = (s,e)\cdot x$. Then by Theorem 3.2, $(t,f) = (s,e)$ and so θ' is well-defined. It is easy to check that $\theta'((t,f)\cdot((s,e)\cdot x)) = (t,f)\cdot\theta'((s,e)\cdot x)$.

(ii) Define a function $\imath : S \to J({}_{C(S)}X_S)$ by

$$\imath(s) = \rho_{s^{-1}} : C(S)\cdot(s^{-1}s) \to C(S)\cdot(ss^{-1})$$

if s is nonzero, and $\imath(0) = 0$. By (i) and Lemma 1 $\imath(s) \in J({}_{C(S)}X_S)$, and $\imath$ is an isomorphism. □

An immediate corollary of the above theorem is that every inverse *semigroup* is isomorphic to an inverse semigroup arising from a category acting on a set satisfying the strong orbit condition.

We now compare (C,X) with $(\mathbf{C}\circ\mathbf{J})(C,X)$; it is here that the notion of equivalence comes into play.

Theorem 3. (i) *Let (C,X) be a system. For each function $\mathbf{q} : C_O \to X$, such that $\mathbf{p}(\mathbf{q}(e)) = e$ for each $e \in C_O$, there exists an equivalence of systems $(F_{\mathbf{q}}, \theta_{\mathbf{q}}) : (C,X) \to (\mathbf{C}\circ\mathbf{J})(C,X)$.*

(ii) *Let $\mathbf{q}, \mathbf{q}' : C_O \to X$ be functions from C_O to X such that*

$$\mathbf{p}(\mathbf{q}(e)) = e = \mathbf{p}(\mathbf{q}'(e))$$

for each $e \in C_O$. Then there is an isomorphic transformation from $(F_{\mathbf{q}}, \theta_{\mathbf{q}})$ to $(F_{\mathbf{q}'}, \theta_{\mathbf{q}'})$.

Proof. (i) Define $(F,\theta) = (F_{\mathbf{q}}, \theta_{\mathbf{q}})$ as follows: $F : C \to C(J({}_CX))$ is the function defined by

$$F(s) = ([\mathbf{q}(\mathbf{r}(s)), s\cdot\mathbf{q}(\mathbf{d}(s))], [\mathbf{q}(\mathbf{d}(s)), \mathbf{q}(\mathbf{d}(s))])$$

and $\theta : X \to X_{J({}_CX)}$ is the function defined by

$$\theta(x) = [\mathbf{q}(\mathbf{p}(x)), x].$$

In what follows we shall use the following notation:

$$e_s = \mathbf{q}(\mathbf{d}(s)) \quad \text{and} \quad f_s = \mathbf{q}(\mathbf{r}(s)).$$

The fact that θ and F are well-defined functions is straightforward to check from the definitions. The proof of the theorem consists of a series of verifications. We begin by showing that F is a functor. It is straightforward to check that F maps identities to identities. Now suppose that $\exists st$ in C. Then by definition

$$F(s) = ([f_s, s \cdot e_s], [e_s, e_s]) \quad \text{and} \quad F(t) = ([f_t, t \cdot e_t], [e_t, e_t]).$$

Now

$$\mathbf{d}(F(s)) = ([e_s, e_s], [e_s, e_s]) \quad \text{and} \quad \mathbf{r}(F(t)) = ([f_t, f_t], [f_t, f_t]).$$

But by assumption $\mathbf{d}(s) = \mathbf{r}(t)$. Thus $e_s = f_t$ and so $\mathbf{d}(F(s)) = \mathbf{r}(F(t))$. We now compute $F(s)F(t)$. By definition

$$F(s)F(t) = ([f_s, s \cdot e_s] \otimes [f_t, t \cdot e_t], [e_t, e_t]).$$

Now

$$C \cdot (s \cdot e_s) \cap C \cdot f_t = C \cdot (s \cdot e_s).$$

Thus

$$e_s * (s \cdot e_s) = \mathbf{r}(s) \quad \text{and} \quad (s \cdot e_s) * f_t = s.$$

Hence

$$F(s)F(t) = ([f_s, (s \cdot t) \cdot e_t], [e_t, e_t]),$$

which is equal to $F(st)$. Thus F is a functor.

To show that (F, θ) is a morphism of systems, we have to check that (M1), (M2) and (M3) hold. Both (M1) and (M2) are straightforward to check. We show that (M3) holds. From the proof of Theorem 3.2,

$$C(J({}_CX)) \cdot \theta(w) = J({}_CX) \otimes [w, w] \setminus \{0\} = \{[d, u \cdot w] \in J({}_CX) \setminus \{0\} : u \in C\}.$$

If

$$C(J({}_CX)) \cdot \theta(x) \cap C(J({}_CX)) \cdot \theta(y)$$

is nonempty, then we can find elements in the intersection such that $[a, u \cdot x] = [b, v \cdot y]$. But then $u \cdot x = p \cdot (v \cdot y)$ for some $p \in C$, and so $C \cdot x \cap C \cdot y$ is nonempty. Now suppose that $C \cdot x \cap C \cdot y = C \cdot z$. It is straightforward to check that

$$C(J({}_CX)) \cdot \theta(x) \cap C(J({}_CX)) \cdot \theta(y) = C(J({}_CX)) \cdot \theta(z).$$

To show that (F,θ) is an equivalence of systems, we have to check that (ES1), (ES2) and (ES3) hold.

(ES1) holds: to prove this we have to show that F is full, faithful and dense.

F is full: let e and f be identities in C and let $(\mathbf{s},\mathbf{e})$ be a morphism in $C(J({}_CX))$ such that

$$\mathbf{d}(\mathbf{s},\mathbf{e})=F(e) \quad \text{and} \quad \mathbf{r}(\mathbf{s},\mathbf{e})=F(f).$$

Let $(\mathbf{s},\mathbf{e})=([x,y],[z,z])$. Then from $\mathbf{s}^{-1}\mathbf{s}\leq\mathbf{e}$ we obtain $y=u\cdot z$ for some $u\in C$. From $\mathbf{d}(\mathbf{s},\mathbf{e})=(\mathbf{e},\mathbf{e})$ we obtain $\mathbf{e}=[\mathbf{q}(e),\mathbf{q}(e)]$, and so $[z,z]=[\mathbf{q}(e),\mathbf{q}(e)]$, from which we have that $z=a\cdot\mathbf{q}(e)$ for some isomorphism a. From

$$([x,x],[x,x])=\mathbf{r}(s,e)=([\mathbf{q}(f),\mathbf{q}(f)],[\mathbf{q}(f),\mathbf{q}(f)]),$$

we obtain $x=b\cdot\mathbf{q}(f)$ for some isomorphism b. Thus

$$(\mathbf{s},\mathbf{e})=([b\cdot\mathbf{q}(f),(ua)\cdot\mathbf{q}(e)],[a\cdot\mathbf{q}(e),a\cdot\mathbf{q}(e)]).$$

Now

$$\mathbf{d}(b^{-1}ua)=\mathbf{d}(a)=\mathbf{p}(\mathbf{q}(e))=e,$$

and

$$\mathbf{r}(b^{-1}ua)=\mathbf{r}(b^{-1})=\mathbf{d}(b)=\mathbf{p}(\mathbf{q}(f))=f,$$

and $F(b^{-1}ua)=(\mathbf{s},\mathbf{e})$.

F is faithful: Suppose that $F(s)=F(t)$ and

$$\mathbf{r}(s)=\mathbf{r}(t)=f \quad \text{and} \quad \mathbf{d}(s)=\mathbf{d}(t)=e.$$

We show that $s=t$. By assumption,

$$[\mathbf{q}(\mathbf{r}(s)),s\cdot\mathbf{q}(\mathbf{d}(s))]=[\mathbf{q}(\mathbf{r}(t)),t\cdot\mathbf{q}(\mathbf{d}(t))].$$

Thus from the definition of $\sim$-equivalence we have that

$$\mathbf{q}(f)=u\cdot\mathbf{q}(f) \quad \text{and} \quad s\cdot\mathbf{q}(e)=u\cdot(t\cdot\mathbf{q}(e))$$

for some isomorphism u in C. By the cancellation condition, $u=f$ and so $s=t$.

F is dense: Let $([x,x],[x,x])$ be any identity in $C(J({}_CX))$. Consider the ordered pair

$$([x,\mathbf{q}(\mathbf{p}(x))],[\mathbf{q}(\mathbf{p}(x)),\mathbf{q}(\mathbf{p}(x))]).$$

It is easy to check that it is a well-defined isomorphism in $C(J({}_CX))$, and that

$$\mathbf{r}([x,\mathbf{q}(\mathbf{p}(x))],[\mathbf{q}(\mathbf{p}(x)),\mathbf{q}(\mathbf{p}(x))])=([x,x],[x,x]),$$

and

$$\mathbf{d}([x,\mathbf{q}(\mathbf{p}(x))],[\mathbf{q}(\mathbf{p}(x)),\mathbf{q}(\mathbf{p}(x))])=F(\mathbf{p}(x)).$$

(ES2) *holds*: Let $[x, y]$ be any element of $X_{J({}_C X)}$. Then the ordered pair

$$\mathbf{a} = ([x, \mathbf{q}(\mathbf{p}(y))], [\mathbf{q}(\mathbf{p}(y)), \mathbf{q}(\mathbf{p}(y))]),$$

is a well-defined isomorphism in $C(J({}_C X))$ and

$$[x, y] = \mathbf{a} \cdot \theta(y).$$

(ES3) *holds*: Let

$$\theta(x) = (\mathbf{s}, \mathbf{e}) \cdot \theta(w).$$

By assumption,

$$\mathbf{d}(\mathbf{s}, \mathbf{e}) = \mathbf{p}([\mathbf{q}(\mathbf{p}(w)), w])$$

and so $\mathbf{e} = [\mathbf{q}(\mathbf{p}(w)), \mathbf{q}(\mathbf{p}(w))]$. Let $\mathbf{s} = [a, b]$. Then $[b, b] \leq \mathbf{e}$ and so $b = r \cdot \mathbf{q}(\mathbf{p}(w))$ for some $r \in C$. Now

$$[\mathbf{q}(\mathbf{p}(x)), x] = (\mathbf{s}, \mathbf{e}) \cdot [\mathbf{q}(\mathbf{p}(w)), w]$$

and so

$$[\mathbf{q}(\mathbf{p}(x)), x] = \mathbf{s} \otimes [\mathbf{q}(\mathbf{p}(w)), w].$$

Now

$$\mathbf{s} \otimes [\mathbf{q}(\mathbf{p}(w)), w] = [a, b] \otimes [\mathbf{q}(\mathbf{p}(w)), w]$$

which is equal to

$$[(\mathbf{q}(\mathbf{p}(w)) * b) \cdot a, (b * \mathbf{q}(\mathbf{p}(w))) \cdot w].$$

But

$$C \cdot b \cap C \cdot \mathbf{q}(\mathbf{p}(w)) = C \cdot b,$$

and so

$$b * \mathbf{q}(\mathbf{p}(w)) = r \quad \text{and} \quad \mathbf{q}(\mathbf{p}(w)) * b = \mathbf{p}(b).$$

Thus

$$[\mathbf{q}(\mathbf{p}(x)), x] = [\mathbf{p}(b) \cdot a, r \cdot w].$$

Therefore, for some isomorphism u in C we have that $x = (ur) \cdot w$. Now we compute $F(ur)$. By definition,

$$F(ur) = ([\mathbf{q}(\mathbf{r}(ur)), (ur) \cdot \mathbf{q}(\mathbf{d}(ur))], [\mathbf{q}(\mathbf{d}(r)), \mathbf{q}(\mathbf{d}(r))]).$$

But

$$\mathbf{p}(x) = \mathbf{r}(u) \quad \text{and} \quad \mathbf{d}(r) = \mathbf{p}(w),$$

and

$$b = r \cdot \mathbf{q}(\mathbf{p}(w)).$$

Thus

$$F(ur) = ([\mathbf{q}(\mathbf{p}(x)), u \cdot b], \mathbf{e}).$$

But $\mathbf{q}(\mathbf{p}(x)) = u \cdot a$. Hence

$$F(ur) = ([a, b], \mathbf{e}) = (\mathbf{s}, \mathbf{e}).$$

(ii) Define $\tau : F_{\mathbf{q}} \to F_{\mathbf{q}'}$ by

$$\tau_e = ([\mathbf{q}'(e), \mathbf{q}(e)], [\mathbf{q}(e), \mathbf{q}(e)])$$

for each $e \in C_O$. It is easy to check that τ_e is a well-defined isomorphism in $\mathbf{C}(\mathbf{J}(C, X))$ and that $\tau_e \in \hom(F_{\mathbf{q}}(e), F_{\mathbf{q}'}(e))$.

(T1) *holds*: let $s \in \hom(e, f)$. Straightforward, if somewhat unwieldy, calculations show that

$$F_{\mathbf{q}'}(s)\tau_e = ([\mathbf{q}'(f), s \cdot \mathbf{q}(e)], [\mathbf{q}(e), \mathbf{q}(e)]) = \tau_f F_{\mathbf{q}}(s).$$

(T2) *holds*: for each $x \in X$ we have that $\tau_{\mathbf{p}(x)} \cdot \theta_{\mathbf{q}}(x) = \theta_{\mathbf{q}'}(x)$. $\square$

7. An equivalence of categories

The results of the last section show that the categories **Sys** and **Inv** are close to being equivalent; that they are not is due to the choice we had to make in order to prove Theorem 6.3: (C, X) and $(\mathbf{C} \circ \mathbf{J})(C, X)$ are equivalent but not canonically so. In this section, we shall get around this difficulty by showing that a suitable quotient category of **Sys** is equivalent to **Inv**.

Definition. Define a relation $\cong$ on **Sys** as follows: $(F, \theta) \cong (G, \varphi)$ if, and only if, (F, θ) and (G, φ) are between the same systems and there is an isomorphism from (F, θ) to (G, φ).

The proof of the following is immediate from the properties of natural transformations (see [9]).

Lemma 1. *The relation* $\cong$ *is a congruence on* **Sys**.

If $(F, \theta) \cong (G, \varphi)$ then there is an isomorphism $\tau : (F, \theta) \to (G, \varphi)$, and so $\mathbf{J}(F, \theta) = \mathbf{J}(G, \varphi)$ by Lemma 5.2. Thus the functor $\mathbf{J} : \mathbf{Sys} \to \mathbf{Inv}$ induces a functor $\mathbf{J}'$ from $\mathbf{Sys}/\cong$ to **Inv**. Also, the functor $\mathbf{C} : \mathbf{Inv} \to \mathbf{Sys}$ induces a functor $\mathbf{C}' : \mathbf{Inv} \to \mathbf{Sys}/\cong$ by composition with the natural functor from **Sys** to $\mathbf{Sys}/\cong$. By Theorem 6.2, we have that $(\mathbf{J}\circ\mathbf{C})(S)\cong S$. Thus $(\mathbf{J}'\circ\mathbf{C}')(S)\cong S$. By Theorem 6.3, there is an equivalence $(F_{\mathbf{q}},\theta_{\mathbf{q}}):(C,X)\to(\mathbf{C}\circ\mathbf{J})(C,X)$ for each $\mathbf{q}:C_O\to X$, and all such equivalences are isomorphic. Thus they all represent the same morphism in $\mathbf{Sys}/\cong$. Finally, the definition of equivalences implies that $\cong$-classes containing equivalences are isomorphisms in the category $\mathbf{Sys}/\cong$. We have thus proved the following result.

Theorem 2. *The functors* $\mathbf{J}'$ *and* $\mathbf{C}'$ *induce an equivalence of categories between* $\mathbf{Sys}/\cong$ *and* $\mathbf{Inv}$.

8. Special cases

The general theory we have developed will now be specialised. In particular, we will describe how the classical theory of 0-bisimple inverse semigroups and the Leech theory of inverse monoids are special cases.

8.1. Right cancellative categories

We show first that finding examples of systems in category theory is not difficult.

Definition. Let C be a category. Then C acts on itself on the left as follows: define $\mathbf{p}:C\to C_O$ by $\mathbf{p}(x)=\mathbf{r}(x)$, and define $s\cdot x=sx$ if $\mathbf{d}(s)=\mathbf{p}(x)$, the usual category product in C.

Proposition 1. *Let C be a right cancellative category considered as a left C-system.*
(i) *C satisfies the orbit condition if, and only if, any two morphisms $s,t\in C$ such that $us=vt$ for some $u,v\in C$ have a pushout in C.*
(ii) *If C satisfies the condition in* (i), *it is a system.*

Proof. (i) Let C be a right cancellative category satisfying the orbit condition. Let $s,t\in C$ such that $us=vt$ for some $u,v\in C$. We show that s and t have a pushout. Since $C\cdot s\cap C\cdot t$ is nonempty we have that $C\cdot s\cap C\cdot t=C\cdot p$ for some $p\in C$. Let $p=as=bt$ for some $a,b\in C$. Now let h and k be any elements of C such that $hs=kt$. Then $hs=kt\in C\cdot p$ and so $hs=kt=cp$ for some $c\in C$. But then $hs=cp=cas$ and so $h=ca$ by right cancellativity. Similarly, $kt=cp=cbt$ and so $k=cb$. The element c is unique by right cancellativity. Consequently, (a,b) is the pushout of (s,t).

Conversely, suppose that C is right cancellative and that C has pushouts of pairs of morphisms which can be completed to a commutative square. Suppose that $C\cdot s\cap C\cdot t$ is nonempty. Then s and t can be completed to a commutative square and so, by assumption, have a pushout. Let $p=as=bt$ where (a,b) is the pushout of (s,t). We claim that $C\cdot s\cap C\cdot t=C\cdot p$. Let $z\in C\cdot s\cap C\cdot t$. Then $z=us=vt$ for some $u,v\in C$. But then by the property of pushouts there exists $c\in C$ such that $u=ca$ and $v=cb$. Thus $z=us=cas=cp$. Hence $z\in C\cdot p$. Conversely, let $z\in C\cdot p$. Then $z=cp$ for some $c\in C$, and so $z=cas=cbt$. Hence $z\in C\cdot s\cap C\cdot t$.

(ii) By definition (S1) holds, and (S2) holds by assumption. (S3) holds from the definition of $\mathbf{p}$ and (S4) holds since the action is just the category product and the category is right cancellative. $\square$

We now give three examples of this construction of increasing generality.

Examples. 1. *The bicyclic monoid.* Consider the monoid $(\mathbb{N},+)$ as a one-object category acting on itself on the left. Clearly, $(\mathbb{N},+)$ is right cancellative. Observe that

$$(\mathbb{N}+m)\cap(\mathbb{N}+n)=\mathbb{N}+\max\{m,n\}.$$

Thus ${}_{\mathbb{N}}\mathbb{N}$ satisfies the strong orbit condition. Hence $J^*({}_{\mathbb{N}}\mathbb{N})$ is an inverse semigroup by Theorem 1.7. Define

$$m \dot{-} n=\begin{cases} m-n & \text{if } m\geq n,\\ 0 & \text{else.}\end{cases}$$

Then we can write

$$\max\{m,n\}=(n\dot{-}m)+m=(m\dot{-}n)+n.$$

Clearly the only isomorphism in $\mathbb{N}$ is 0. Thus $J^*({}_{\mathbb{N}}\mathbb{N})=\mathbb{N}\times\mathbb{N}$ as underlying set. The multiplication is given by

$$(a,b)\otimes(c,d)=((c\dot{-}b)+a,(b\dot{-}c)+d).$$

Thus $J^*({}_{\mathbb{N}}\mathbb{N})$ is just the *bicyclic monoid* [16].

2. *The polycyclic monoids.* Let X be any nonempty set. Denote by X^* the free monoid on X. When X contains only one element, X^* is isomorphic to $\mathbb{N}$; we shall assume in this example that X has at least two elements. X^* is a cancellative monoid with identity element λ, the empty string. It is easy to check that $X^*u\cap X^*v\neq\emptyset$, for any strings $u,v\in X^*$, if, and only if, u is a suffix of v or v is a suffix of u. Define the string u/v as follows:

$$u/v=\begin{cases} h & \text{if } u=hv,\\ \lambda & \text{else.}\end{cases}$$

If $X^*u\cap X^*v$ is nonempty then

$$X^*u\cap X^*v=X^*z,$$

where $z=(v/u)u=(u/v)v$. Thus ${}_{X^*}X^*$ satisfies the orbit condition and so $J({}_{X^*}X^*)$ is an inverse semigroup with zero. Clearly, the only isomorphism in X^* is λ and so the underlying set of $J({}_{X^*}X^*)$ is just $(X^*\times X^*)\cup\{0\}$. The product in $J({}_{X^*}X^*)$ is given by

$$(u,v)\otimes(x,y)=\begin{cases} ((x/v)u,(v/x)y) & \text{if } X^*v\cap X^*x\neq\emptyset,\\ 0 & \text{else.}\end{cases}$$

This is just the polycyclic monoid on $|X|$ generators [15].

3. *The generalised polycyclic semigroups.* The bicyclic monoid and the polycyclic monoids are special cases of the following more general construction.

A (*directed*) *graph* $\mathscr{G}$ consists of a set of *arrows* G and a set of *vertices* G_o together with two functions $\alpha, \omega : G \to G_o$ called the *source* and *target* respectively. If $x, y \in G$ then x and y are said to be *composable* if $\alpha(x) = \omega(y)$. A sequence of arrows $(x_1, \ldots, x_n)$ is called a *composable sequence* if (x_i, x_{i+1}) are composable for $i = 1, \ldots, n-1$. The *free category* G^* generated by G is defined to be the set of all composable sequences together with the set $\{1_e : e \in G_o\}$. We define $\mathbf{d}(1_e) = 1_e = \mathbf{r}(1_e)$ and

$$\mathbf{d}(x_1, \ldots, x_n) = 1_{\alpha(x_n)} \quad \text{and} \quad \mathbf{r}(x_1, \ldots, x_n) = 1_{\omega(x_1)}.$$

A partial multiplication is defined in G^* as follows: if

$$\mathbf{d}(x_1, \ldots, x_m) = \mathbf{r}(y_1, \ldots, y_n)$$

then

$$(x_1, \ldots, x_m)(y_1, \ldots, y_n) = (x_1, \ldots, x_m, y_1, \ldots, y_n),$$

and the elements $\{1_e : e \in G_o\}$ act as identities. In this way, G^* is a category. It is easy to see that G^* is a right cancellative category (in fact, it is cancellative).

We let G^* act on itself on the left. To show that the orbit condition holds, we first define a new partial binary operation on G^*. Let $u, v \in G^*$ with $\mathbf{d}(u) = \mathbf{d}(v)$. Define

$$u/v = \begin{cases} h & \text{if } u = hv, \\ \mathbf{r}(v) & \text{else.} \end{cases}$$

Suppose that

$$G^*(x_1, \ldots, x_m) \cap G^*(y_1, \ldots, y_n)$$

is nonempty. Then either $(x_1, \ldots, x_m)$ is a suffix of $(y_1, \ldots, y_n)$, in which case

$$G^*(x_1, \ldots, x_m) \cap G^*(y_1, \ldots, y_n) = G^*(y_1, \ldots, y_n),$$

or $(y_1, \ldots, y_n)$ is a suffix of $(x_1, \ldots, x_m)$, in which case

$$G^*(x_1, \ldots, x_m) \cap G^*(y_1, \ldots, y_n) = G^*(x_1, \ldots, x_m).$$

Observe that in both cases $\alpha(x_m) = \alpha(y_n)$. Now suppose that the intersection

$$G^* 1_e \cap G^*(x_1, \ldots, x_n)$$

is nonempty. The set $G^* 1_e$ consists of all strings $(y_1, \ldots, y_m)$ with $\mathbf{d}(y_1, \ldots, y_m) = 1_e$. But $\mathbf{d}(x_1, \ldots, x_n) = 1_e$. Hence

$$G^* 1_e \cap G^*(x_1, \ldots, x_n) = G^*(x_1, \ldots, x_n).$$

Finally, if the intersection

$$G^* 1_e \cap G^* 1_f$$

is nonempty then $e=f$. It is now straightforward to check that if $G^*u \cap G^*v$ is nonempty then

$$G^*u \cap G^*v = G^*(v/u)u = G^*(u/v)v$$

for all $u, v \in G^*$. Thus $P(\mathcal{G}) = J({}_{G^*}G^*)$ is an inverse semigroup. It is clear that the only isomorphisms in G^* are the identities. Thus the underlying set of $P(\mathcal{G})$ is just

$$\{(x,y) \in G^* \times G^* : \mathbf{r}(x) = \mathbf{r}(y)\} \cup \{0\}.$$

The multiplication is given by

$$(u,v) \otimes (x,y) = \begin{cases} ((x/v)u, (v/x)y) & \text{if } G^*v \cap G^*x \neq \emptyset, \\ 0 & \text{else.} \end{cases}$$

The inverse semigroups $P(\mathcal{G})$ are called *generalised polycyclic semigroups*; they occur naturally in the study of Cuntz–Krieger C^*-algebras. The polycyclic monoids are special cases: they arise as the semigroups $P(\mathcal{G})$ when $\mathcal{G}$ is the graph which is a ‘bouquet of circles’.

8.2. Cyclic systems and inverse monoids

In this section, we shall show that Leech’s theory of inverse monoids [8] is a special case of our general construction.

Definition. A system ${}_CX$ is said to be *cyclic* if $X = C \cdot x_0$ for some $x_0 \in X$.

Lemma 1. *If S is an inverse monoid with zero then $(C(S), X_S)$ is a cyclic $C(S)$-system.*

Proof. We claim that $X_S = C(S) \cdot 1$. Let $x \in X_S$. Then $x \in S \backslash \{0\}$. Now $(x,1) \in C(S)$ since $x^{-1}x \leq 1$. Also,

$$\mathbf{d}(x,1) = (1,1) \quad \text{and} \quad \mathbf{p}(1) = (1,1),$$

so that $\exists (x,1) \cdot 1$. But $(x,1) \cdot 1 = x1 = x$. □

The converse of the above result is proved below.

Proposition 2. *Let ${}_CX$ be a cyclic system such that $X = C \cdot x_0$. Put $1 = \mathbf{p}(x_0)$. Then*
(i) *$J({}_CX)$ is a monoid.*
(ii) *For every $e \in C_O$, $hom(1,e) \neq \emptyset$.*
(iii) *For all $s,t \in C$, $us = vt$ for some $u,v \in C$ implies that s and t have a pushout.*
(iv) *Put $C' = \{a \in C : \mathbf{d}(a) = 1\}$. Then C acts on C' on the left by restricting the action by left multiplication of C on itself, ${}_CC'$ is a system, and ${}_CC'$ is equivalent to ${}_CX$.*

Proof. (i) We show that $[x_0, x_0]$ is the identity of $J({}_C X)$. Let $[x,y]$ be any nonzero element of $J({}_C X)$. Then

$$[x,y] \otimes [x_0, x_0] = [(x_0 * y) \cdot x, (y * x_0) \cdot x_0].$$

Now $C \cdot y \cap C \cdot x_0 = C \cdot y$ and so $x_0 * y = \mathbf{p}(y)$ and $(y * x_0) \cdot x_0 = y$. Hence $[x,y] \otimes [x_0, x_0] = [x,y]$. We may similarly prove that $[x_0, x_0] \otimes [x,y] = [x,y]$.

(ii) Let $e \in C_O$. Since $\mathbf{p}$ is surjective there exists $x \in X$ such that $\mathbf{p}(x) = e$. Now $x \in C \cdot x_0$ and so $x = a \cdot x_0$ for some $a \in C$. But then

$$\mathbf{d}(a) = \mathbf{p}(x_0) = 1 \quad \text{and} \quad e = \mathbf{p}(x) = \mathbf{p}(a \cdot x_0) = \mathbf{r}(a).$$

Thus $a \in \hom(1, e)$.

(iii) By assumption $us = vt$ for some $u, v \in C$. Thus $\mathbf{d}(s) = \mathbf{d}(t)$. Let $z \in X$ such that $\mathbf{p}(z) = \mathbf{d}(s) = \mathbf{d}(t)$. Thus $\exists s \cdot z$ and $\exists t \cdot z$. Put $x = s \cdot z$ and $y = t \cdot z$. Since $z \in C \cdot x_0$ then $z = a \cdot x_0$ for some $a \in C$. Thus $x = (sa) \cdot x_0$ and $y = (ta) \cdot x_0$ and so $u \cdot x = v \cdot y$. Hence $C \cdot x \cap C \cdot y$ is nonempty. By assumption $C \cdot x \cap C \cdot y = C \cdot w$ for some w. Let $w = b \cdot x = c \cdot y$ for some $b, c \in C$. Now

$$w = b \cdot x = (bsa) \cdot x_0 \quad \text{and} \quad w = c \cdot y = (cta) \cdot x_0.$$

Thus $(bsa) \cdot x_0 = (cta) \cdot x_0$. By the right cancellation condition we have that $bsa = cta$. By right cancellation in C, we have that $bs = ct$.

We now show that (b,c) is a pushout of (s,t). Suppose that $hs = gt$ for some $g, h \in C$. Then $(hs) \cdot z = (gt) \cdot z$, so that $h \cdot (s \cdot z) = g \cdot (t \cdot z)$. But $s \cdot z = x$ and $t \cdot z = y$ and so $h \cdot x = g \cdot y$. Now

$$h \cdot x = g \cdot y \in C \cdot x \cap C \cdot y = C \cdot w$$

and so $h \cdot x = g \cdot y = d \cdot w$ for some $d \in C$. Now $h \cdot x = (hsa) \cdot x_0$ and $d \cdot w = (dbsa) \cdot x_0$. Thus $(hsa) \cdot x_0 = (dbsa) \cdot x_0$. By the right cancellation condition $hsa = dbsa$, and by right cancellation $h = db$. Also $g \cdot y = (gta) \cdot x_0$ and $d \cdot w = (dcta) \cdot x_0$. Thus $(gta) \cdot x_0 = (dcta) \cdot x_0$. By the right cancellation condition $gta = dcta$, and by right cancellation $g = dc$. Thus $(h,g) = d(b,c)$. The element d is unique by right cancellation.

(iv) We begin by showing that ${}_C C'$ is a system by checking that the axioms (S1), (S2), (S3) and (S4) hold.

(S1) holds: the action of C on C' is the restriction of the action of C on itself described in Section 8.1. The function $\mathbf{p}: C' \to C_O$ is defined by $\mathbf{p}(x) = \mathbf{r}(x)$. C' is C-invariant: let $x \in C'$ and $a \in C$ be such that $\exists ax$. Then $\mathbf{d}(ax) = \mathbf{d}(x) = 1$ so that $ax \in C'$.

(S2) holds: in order to show that the orbit condition holds, we shall define a function $\theta: X \to C'$ as follows: let $x \in X$. Then $x \in C \cdot x_0$ so that $x = a \cdot x_0$ for some $a \in C$. But a is uniquely determined by the right cancellation condition. Thus we may define θ by the condition $x = \theta(x) \cdot x_0$. In fact, θ is a bijection. It is easy to see that it is injective. To prove that it is surjective, let $a \in C'$. Then $\mathbf{d}(a) = 1$, so that $\exists a \cdot x_0$. Put $x = a \cdot x_0$. Then clearly, $\theta(x) = a$.

We show first that

$$C\cdot x\cap C\cdot y=\emptyset \Leftrightarrow C\theta(x)\cap C\theta(y)=\emptyset.$$

If $C\theta(x)\cap C\theta(y)$ is nonempty, then $a\theta(x)=b\theta(y)$ for some $a,b\in C$. Thus

$$a\cdot x=(a\theta(x))\cdot x_0=(b\theta(y))\cdot x_0=b\cdot y\in C\cdot x\cap C\cdot y.$$

Hence $C\cdot x\cap C\cdot y$ is nonempty. Conversely, suppose that $C\cdot x\cap C\cdot y$ is nonempty. Then $a\cdot x=b\cdot y$ for some $a,b\in C$. Thus $(a\theta(x))\cdot x_0=(b\theta(y))\cdot x_0$, and so $a\theta(x)=b\theta(y)$. Hence $C\theta(x)\cap C\theta(y)$ is nonempty.

We now show that

$$C\cdot x\cap C\cdot y=C\cdot z \Leftrightarrow C\theta(x)\cap C\theta(y)=C\cdot\theta(z).$$

Suppose that $C\cdot x\cap C\cdot y=C\cdot z$. Let $w\in C\theta(x)\cap C\theta(y)$. Then $w=a\theta(x)=b\theta(y)$ for some $a,b\in C$. Thus

$$(a\theta(x))\cdot x_0=(b\theta(y))\cdot x_0=w\cdot x_0$$

and so

$$w\cdot x_0=a\cdot x=b\cdot y.$$

Hence, by assumption, $w\cdot x_0=c\cdot z$ for some $c\in C$. Thus $w\cdot x_0=(c\theta(z))\cdot x_0$. Hence $w=c\theta(z)$ and so $w\in C\theta(x)\cap C\theta(y)$ and

$$C\theta(x)\cap C\theta(y)\subseteq C\theta(z).$$

Now suppose that $c\theta(z)\in C\theta(z)$. Then $c\theta(z)\cdot x_0=c\cdot z\in C\cdot x\cap C\cdot y$. Thus $c\cdot z=d\cdot x$ for some $d\in C$, and so $(c\theta(z))\cdot x_0=(d\theta(x))\cdot x_0$. Hence $c\theta(z)=d\theta(x)$ and so $c\theta(z)\in C\theta(x)$. Similarly, we can show that $c\theta(z)\in C\theta(y)$. Hence

$$C\theta(x)\cap C\theta(y)=C\theta(z).$$

The converse is proved similarly.

We can now prove that the orbit condition holds. Suppose that $Ca\cap Cb$ is nonempty, where $a,b\in C'$. By the result above $C\cdot\theta^{-1}(a)\cap C\cdot\theta^{-1}(b)$ is nonempty. Thus $C\cdot\theta^{-1}(a)\cap C\cdot\theta^{-1}(b)=C\cdot z$ for some $z\in X$. Hence $Ca\cap Cb=C\theta(z)$.

(S3) holds: for if $e\in C_O$ then $\hom(1,e)$ is nonempty, so that there is an element $c\in C'$ such that $\mathbf{d}(c)=1$ and $\mathbf{r}(c)=e$. Thus $c\in C'$ and $\mathbf{p}(c)=e$.

(S4) holds: since C is right cancellative.

We now prove that the systems (C,C') and (C,X) are equivalent. Let $I:C\to C$ be the identity functor.

We show first that (I,θ) is a system morphism by checking that (M1), (M2) and (M3) hold.

(M1) holds: $\mathbf{p}(\theta(x))=\mathbf{r}(\theta(x))=\mathbf{p}(x)$, since $x=\theta(x)\cdot x_0$.

(M2) holds: let $a\in C$ and $x\in X$ be such that $a\cdot x$ exists. Now by definition $a\cdot x=(a\theta(x))\cdot x_0$. Hence $\theta(a\cdot x)=a\theta(x)$.

(M3) holds: we have already shown this above.

We finish off by checking that (I,θ) is an equivalence of systems by showing that (ES1),(ES2) and (ES3) hold.

(ES1) holds: immediate.

(ES2) holds: θ is a bijection.

(ES3) holds: straightforward. □

The above result shows that *inverse monoids with zero* are determined by categories C satisfying the following conditions:

(L1) There is an identity $1 \in C$, called a *weak initial object*, such that $\hom(1,e) \neq \emptyset$ for every identity e.

(L2) C is right cancellative.

(L3) If $s,t \in C$ are such that $us = vt$ for some $u,v \in C$, then s and t have a pushout.

The inverse semigroup is constructed by taking $X = \{a \in C : \mathbf{d}(a) = 1\}$ and letting C act on X on the left by left multiplication.

This is essentially Leech's construction [8]. The only difference is condition (L3); Leech has the condition:

(L3)* Any two elements $s,t \in C$ with $\mathbf{d}(s) = \mathbf{d}(t)$ have a pushout.

The reason for this is that we take 'inverse monoids with zero' as our basic class of semigroups, whereas Leech takes just the class of inverse monoids: the zero is not a distinguished element.

Leech's original construction of inverse monoids from categories satisfying (L1), (L2) and (L3)* is in fact a special case of a general construction in category theory: the construction of categories of partial functions from categories of functions. However, Leech discovered two important facts: first, that Clifford's original construction of bisimple inverse monoids [1] could be interpreted categorically (in the algebraic sense described in the introduction); and secondly, that every inverse monoid could be constructed from a suitable category.

8.3. *Monoid systems and 0-bisimple semigroups*

In this section, we show that the classical theory of 0-bisimple inverse semigroups due to Clifford [1], Reilly [17] and McAlister [11] is a special case of our theory.

We begin by constructing a system from a 0-bisimple inverse semigroup which is different from the usual one.

Proposition 1. *Let S be a 0-bisimple inverse semigroup, and let e be any nonzero idempotent in S. Put $X_e = R_e$ and $C_e = R_e \cap eSe$. Then C_e is a right cancellative monoid, $C_e \times X_e \to X_e$ defined by $(a,x) \mapsto ax$ is a monoid action and (C_e, X_e) is a system when $\mathbf{p} : X_e \to C_e$ is defined by $\mathbf{p}(x) = e$.*

Proof. This is essentially proved in [17, 11]. We simply note that for $x \in X_e$ we have that $C_e \cdot x = Sx \cap R_e$. It is easy to check that $Sx \cap Sy = Sx^{-1}xy^{-1}y$. Thus

$$C_e \cdot x \cap C_e \cdot y = S(x^{-1}xy^{-1}y) \cap R_e.$$

The right-hand side is empty precisely when $x^{-1}xy^{-1}y=0$; to see this, suppose first that $x^{-1}xy^{-1}y$ is nonzero. Then there exists $z\in L_{x^{-1}xy^{-1}y}\cap R_e$, since S is 0-bisimple; in which case

$$C_e\cdot x\cap C_e\cdot y=Sz\cap R_e=C_e\cdot z,$$

and is nonempty. Conversely, suppose that $w\in Sx^{-1}xy^{-1}y\cap R_e$. Then $w(x^{-1}xy^{-1}y)=w$ and $w\mathscr{R}e$. Since e is nonzero, w must be nonzero. But then $x^{-1}xy^{-1}y$ must be nonzero. □

The relationship between the above system and the standard one is described below.

Proposition 2. *Let S be a 0-bisimple inverse semigroup, and e a nonzero idempotent. Then (C_e,X_e) is equivalent to $(C(S),X_S)$.*

Proof. Define (F,θ) as follows: $F:C_e\to C(S)$ is defined by $F(a)=(a,e)$, and $\theta(x)=x$. F is well-defined since $a\in R_e\cap eSe$ and so $aa^{-1}=e$ and $a^{-1}a\leq e$. Clearly, $\mathbf{r}(a,e)=(e,e)$ and so $F(C_e)=\mathrm{end}((e,e))$. We show first that (F,θ) is a morphism of systems by checking that (M1), (M2) and (M3) hold.

(M1) holds:

$$\mathbf{p}(\theta(a))=\mathbf{p}(a,e)=(aa^{-1},aa^{-1})=(e,e)$$

and

$$F(\mathbf{p}(a))=F(aa^{-1})=F(e)=(e,e).$$

(M2) holds: $\theta(ax)=ax$, $F(a)=(a,e)$, $\theta(x)=x$ and $F(a)\cdot\theta(x)=ax$.

(M3) holds: suppose that $C_e\cdot x\cap C_e\cdot y$ is empty. Then by Proposition 1, $x^{-1}xy^{-1}y=0$ and so $Sx\cap Sy=0$. But

$$C(S)\cdot x=Sx\backslash\{0\}\quad\text{and}\quad C(S)\cdot y=Sy\backslash\{0\}$$

by the proof of Theorem 3.2. Thus $C(S)\cdot x\cap C(S)\cdot y$ is empty. Now suppose that $C_e\cdot x\cap C_e\cdot y=C_e\cdot z$. Then by Proposition 1, the idempotent $x^{-1}xy^{-1}y$ is nonzero and $C_e\cdot z=Sz\cap R_e$ where $z\in L_{x^{-1}xy^{-1}y}\cap R_e$. But $Sx\cap Sy=Sx^{-1}xy^{-1}y$ and $z\mathscr{L}x^{-1}xy^{-1}y$ implies that $Sx\cap Sy=Sz$. It follows that $C(S)\cdot x\cap C(S)\cdot y=C(S)\cdot z$.

We now show that (F,θ) is an equivalence of systems by showing that (ES1), (ES2) and (ES3) hold.

(ES1) holds: let (f,f) be any identity of $C(S)$. Then since S is 0-bisimple there exists $a\in S$ such that $e=a^{-1}a$ and $aa^{-1}=f$. Now, $(a,e)\in C(S)$, and $\mathbf{d}(a,e)=(e,e)$ and $\mathbf{r}(a,e)=(f,f)$. Thus (a,e) is an isomorphism.

(ES2) holds: Let $x\in X_S$, so that $x\in S\backslash\{0\}$. Since S is 0-bisimple there exists $y\in S$ such that $y^{-1}y=x^{-1}x$ and $yy^{-1}=e$. Thus $y\in R_e=X_e$. Put $s=xy^{-1}$. Then $s^{-1}s=e$. Thus (s,e) is an isomorphism in $C(S)$. Also, $\exists(s,e)\cdot y$ since $\mathbf{d}(s,e)=(e,e)=\mathbf{p}(y)$, and $(s,e)\cdot y=sy=xy^{-1}y=x$. But $y=\theta(y)$. Thus $x=(s,e)\cdot\theta(y)$.

(ES3) holds: suppose that $\theta(x)=(s,e)\cdot\theta(y)$. Then $x,y\in X_e=R_e$ and $x=sy$ where $s^{-1}s\leq e$. Also $\mathbf{r}(s,e)=\mathbf{p}(\theta(x))$ and so $ss^{-1}=xx^{-1}$. Thus $e=xx^{-1}=ss^{-1}$ and so $s\in R_e$. Also $s^{-1}s\leq e$ and so $se=s$. Hence $s\in R_e\cap eSe$. □

Systems in which the acting category is in fact a monoid we shall term *monoid systems*. We can now prove that 0-bisimple inverse semigroups can be precisely described by monoid systems.

Theorem 3. (i) *$J(_CX)$ is 0-bisimple if, and only if, for all $x,y\in X$ there exist isomorphisms $u,v\in C$ such that $\mathbf{p}(u\cdot x)=\mathbf{p}(v\cdot y)$.*

(ii) *If C is a monoid and (C,X) a system then $J(_CX)$ is 0-bisimple.*

(iii) *$J(_CX)$ is 0-bisimple if, and only if, $_CX$ is equivalent to a system of the form $_{C'}X'$ where C' is a monoid.*

Proof. (i) Suppose that $J(_CX)$ is 0-bisimple and let $x,y\in X$. Then $[x,x]$ and $[y,y]$ are nonzero idempotents of $J(_CX)$. Since $J(_CX)$ is 0-bisimple there is an element $[a,b]$ of $J(_CX)$ such that

$$[b,b]=[y,y] \quad\text{and}\quad [a,a]=[x,x].$$

Thus $b=v\cdot y$ and $a=u\cdot x$ for some u and v isomorphisms. Since $[a,b]\in J(_CX)$, we have that $\mathbf{p}(a)=\mathbf{p}(b)$ and so $\mathbf{p}(v\cdot y)=\mathbf{p}(u\cdot x)$.

Conversely, let $[x,x]$ and $[y,y]$ be two nonzero idempotents in $J(_CX)$. By assumption, there are isomorphisms u and v such that $\mathbf{p}(u\cdot x)=\mathbf{p}(v\cdot y)$. Hence $[u\cdot x,v\cdot y]$ is a well-defined element of $J(_CX)$. But

$$[v\cdot y,v\cdot y]=[y,y] \quad\text{and}\quad [u\cdot x,u\cdot x]=[x,x].$$

Thus $J(_CX)$ is 0-bisimple.

(ii) The function $\mathbf{p}$ maps X to $\{1\}$ the set containing the identity of the monoid. Thus for all $x,y\in X$ we have that $\mathbf{p}(x)=\mathbf{p}(y)$. Thus (i) is trivially satisfied, and so $J(_CX)$ is 0-bisimple.

(iii) Suppose that $J(_CX)$ is 0-bisimple. Let e be any identity of C. Put $C'=\text{end}\ (e)$. Let f be any identity. Since $\mathbf{p}$ is surjective there are elements x and y in X such that $\mathbf{p}(x)=f$ and $\mathbf{p}(y)=e$. By (i), there exist isomorphisms u and v in C such that $u\cdot x=v\cdot y$. But $\mathbf{d}(u)=\mathbf{p}(x)=f$, $\mathbf{r}(u)=\mathbf{r}(v)$ and $\mathbf{d}(v)=\mathbf{p}(y)=e$. Thus $v^{-1}u\in\hom(f,e)$ is a well-defined isomorphism. Thus C' is a dense subcategory of C. Let $X'=\{x\in X:\mathbf{p}(x)=e\}$. It is straightforward to check that $_{C'}X'$ is a system equivalent to $_CX$.

Conversely, suppose that (C,X) is equivalent to (C',X'), a monoid system. Then $J(_CX)$ is isomorphic to $J(_{C'}X')$ by Theorem 5.3. By (ii) the latter is 0-bisimple. □

Thus monoid systems completely characterise 0-bisimple inverse semigroups. Such systems are determined by the following set of axioms: let C be a monoid with identity 1 and let X be a set. Then

(MS1) There is a monoid action $C\times X\to X$.

(MS2) C is right cancellative.

(MS3) If $a \cdot x = b \cdot x$ then $a = b$.

(MS4) $C \cdot x \cap C \cdot y \neq \emptyset$ implies $C \cdot x \cap C \cdot y = C \cdot z$ for some $z \in X$.

Such monoid systems are easily seen to be equivalent to the 'generalised RP-systems' of McAlister [11].

8.4. 0-simple inverse semigroups

The following result may be deduced from Corollary 2.5 of [14].

Proposition 1. *Let S be an inverse semigroup with zero. Then S is 0-simple if, and only if, for any nonzero idempotents e and f there exists an idempotent e' such that $e\mathscr{D}e' \leq f$.*

Proposition 2. *$J(_CX)$ is 0-simple if, and only if, for all $x, y \in X$ there exists an element $a \in C$ such that $\mathbf{p}(x) = \mathbf{p}(a \cdot y)$.*

Proof. Suppose that $J(_CX)$ is 0-simple, and let $x, y \in X$. Then $[x,x]$ and $[y,y]$ are nonzero idempotents in $J(_CX)$. By Proposition 1, there exists an idempotent $[y', y']$ such that

$$[x,x]\mathscr{D}[y',y'] \leq [y,y].$$

Thus there are isomorphisms u and v in C such that $[u \cdot x, v \cdot y']$ is defined. Since $[y',y'] \leq [y,y]$ there exists $b \in C$ such that $y' = b \cdot y$. But then

$$\mathbf{p}(x) = \mathbf{p}((u^{-1}v) \cdot y') = \mathbf{p}((u^{-1}vb) \cdot y)$$

so that we can put $a = (u^{-1}vb)$.

Conversely, suppose that the condition holds, and that $[x,x]$ and $[y,y]$ are two nonzero idempotents in $J(_CX)$. By assumption there exists $a \in C$ such that $\mathbf{p}(x) = \mathbf{p}(a \cdot y)$. Thus $[x, a \cdot y]$ is an element of $J(_CX)$. But

$$[a \cdot y, a \cdot y] = [a \cdot y, a \cdot y] \leq [y,y] \quad \text{and} \quad [x,x] = [x,x],$$

thus by Proposition 1, the semigroup $J(_CX)$ is 0-simple. $\square$

We consider a simple application of the above result to generalised polycyclic semigroups. A directed graph $\mathscr{G}$ is said to be *strongly connected* if for any two vertices e and f there exists a composable sequence of arrows $(x_1, \ldots, x_n)$ such that

$$\alpha(x_n) = e \quad \text{and} \quad \omega(x_1) = f.$$

Proposition 3. *Let $\mathscr{G}$ be a directed graph. Then $P(\mathscr{G})$ is 0-simple if, and only if, $\mathscr{G}$ is strongly connected.*

Proof. Suppose that $P(\mathscr{G})$ is 0-simple. Let e and f be any two vertices of $\mathscr{G}$. Then in G^* there are identities 1_e and 1_f. By Proposition 2, there exists an element a

in G^* such that $\mathbf{p}(1_e)=a\cdot 1_f$. Thus $1_e=\mathbf{r}(a)$ and $\mathbf{d}(p)=1_f$. But a is a sequence of composable arrows of G from f to e. Thus G is strongly connected.

Conversely, suppose that G is strongly connected. Let x and y be any elements of G^*. Let $\mathbf{r}(x)=1_e$ and $\mathbf{r}(y)=1_f$. Since G is strongly connected there exists a sequence of composable arrows, a, such that

$$\mathbf{r}(a)=1_e \quad \text{and} \quad \mathbf{d}(a)=1_f.$$

Thus $a\cdot y$ exists and

$$\mathbf{p}(x)=\mathbf{r}(x)=1_e=\mathbf{r}(a\cdot y)=\mathbf{p}(a\cdot y).$$

Hence $P(\mathscr{G})$ is 0-simple by Proposition 2. □

8.5. 0-E-unitary inverse semigroups

Definition. An inverse semigroup S with zero is said to be *0-E-unitary* if $e\leq s$ with e a nonzero idempotent implies that s is an idempotent.

Definition. A system (C,X) satisfies the *left cancellation condition* if $a\cdot x=a\cdot y$ implies $x=y$ for all $a\in C$ and $x,y\in X$.

Proposition 1. *If S is 0-E-unitary then $(C(S),X_S)$ satisfies the left cancellation condition.*

Proof. Let $(s,e)\cdot x=(s,e)\cdot y$. Then

$$sx=sy, \qquad e=xx^{-1}=yy^{-1} \quad \text{and} \quad s^{-1}s\leq e.$$

Put $w=s^{-1}sx=s^{-1}sy$. Then $w\leq x,y$ and so $w^{-1}w\leq x^{-1}y$. Suppose that $w^{-1}w=0$. Then $x^{-1}s^{-1}sx=0$ and so $sx=0$. But this is a contradiction. Thus $w^{-1}w$ is a nonzero idempotent. Hence $x^{-1}y$ is an idempotent. But $x^{-1}x\mathscr{R}x^{-1}y$ and so $x^{-1}y=x^{-1}x$. Thus $x=y$, and so the left cancellation condition holds. □

Proposition 2. *$J({}_CX)$ is 0-E-unitary if, and only if, (C,X) satisfies the left cancellation condition.*

Proof. Suppose that $J({}_CX)$ is 0-E-unitary, and that $x=a\cdot y=a\cdot z$. Then $\mathbf{d}(a)=\mathbf{p}(y)$ and $\mathbf{d}(a)=\mathbf{p}(z)$. Thus $[y,z]$ is an element of $J({}_CX)$. Now $[x,x]\leq[y,z]$, and $[x,x]$ is a nonzero idempotent. Thus, by assumption, $[y,z]$ is an idempotent and so $y=z$. Thus the left cancellation condition holds.

Conversely, suppose that the left cancellation condition holds, and let $[x,x]\leq[y,z]$ in $J({}_CX)$. Then $(x,x)=a\cdot(y,z)$ for some $a\in C$. Hence $x=a\cdot y=a\cdot z$. Thus by the left cancellation condition we must have $y=z$. Thus $[y,z]$ is an idempotent. □

We may apply the above result to the generalised polycyclic semigroups. Because G^* is cancellative ${}_{G^*}G^*$ satisfies the left cancellation condition. The following is now immediate.

Proposition 3. *Let $\mathcal{G}$ be a directed graph. Then $P(\mathcal{G})$ is 0-E-unitary.*

8.6. The clause semigroup

The clause semigroup was originally introduced by Girard in [5]. His definition was made without the use of categories and without a full formalisation of the semigroup structure. The process of trying to understand his construction in the light of McAlister's work [12] led directly to the main constructions of this paper.

For this example we shall need some standard definitions from universal algebra.

Definitions. An *operator domain* Ω is a set of function symbols indexed by their arities. We denote by Ω_n the set of operators in Ω whose arity is n. The set of *terms*, $T_\Omega(X)$, in a set X of *variables* over the operator domain Ω, is the smallest set satisfying the following two conditions:

(1) $x \in X$ implies $\langle x \rangle \in T_\Omega(X)$.

(2) If $\rho \in \Omega_n$ and $t_1, \dots t_n \in T_\Omega(X)$ then $\rho(t_1, \dots t_n) \in T_\Omega(X)$.

The term $\langle x \rangle$ is usually written x.

Definitions. A *term substitution* is a function $f : X \to T_\Omega(X)$. The *support of f*, $\mathrm{supp}(f)$, is the set $\{x \in X : f(x) \neq x\}$. We shall assume that this set is always finite. If $t \in T_\Omega(X)$ then $\mathrm{var}(t)$ is the set of all variables appearing in t. Substitutions can be *applied* to terms as follows:

(1) $f(\langle x \rangle) = f(x)$ for all $x \in X$.

(2) $f(\rho(t_1, \dots, t_n)) = \rho(f(t_1), \dots, f(t_n))$.

A *renaming substitution* is a bijective substitution, and thus a bijection on X.

We now give some definitions and state some results from 'Unification theory'. Consult [2, 4]; Ref. [4] contains proofs in Section 8.4.

Definitions. Let s and t be terms. If $\sigma(s) = t$ for some substitution σ, then we write $s \leq t$ and say that s *subsumes* t. Let σ and ρ be substitutions. If $\tau \circ \sigma = \rho$ for some substitution τ, then we write $\sigma \leq \rho$ and say that σ is *more general than* ρ.

Theorem 1. (i) *Let s and t be terms and let ρ and σ be term substitutions such that $\mathrm{supp}(\rho) \subseteq \mathrm{var}(s)$ and $\mathrm{supp}(\sigma) \subseteq \mathrm{var}(t)$. Suppose that $\rho(s) = t$ and $\sigma(t) = s$ then ρ and σ are mutually inverse renaming substitutions.*

(ii) *Let ρ and σ be substitutions. Then $\sigma \leq \rho$ and $\rho \leq \sigma$ if, and only if, $\tau \circ \sigma = \rho$ for some renaming substitution τ.*

Definitions. Let s and t be terms. Let σ be a substitution with $\mathrm{supp}(\sigma) = \mathrm{var}(s) \cup \mathrm{var}(t)$. Then σ is said to *unify s and t* if $\sigma(s) = \sigma(t)$. The substitution σ is called a *most general* *unifier* (*mgu*) *of* s *and* t if the following two conditions hold:

1. σ unifies s and t.
2. If σ' unifies s and t then $\sigma \leq \sigma'$.

We write $\sigma = \text{mgu}(s,t)$ to mean that σ is a most general unifier of the terms s and t. If $\sigma = \text{mgu}(s,t)$ then $\sigma(s) = \sigma(t)$ is called a *most common instance of* s *and* t.

The key result is the following called the 'Unification Algorithm'.

Theorem 2. *Given terms* s *and* t *then either*:

(i) s *and* t *have no unifier*;

or

(ii) s *and* t *are unifiable, and they have an mgu.*

We shall now define a category action which satisfies the orbit condition.

Definitions. Let $\mathscr{X} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. Let Ω be a fixed operator domain. Define a category C_Ω as follows: the identities of C_O are the identity functions on the finite subsets of $\mathscr{X}$. A morphism $f : 1_X \to 1_Y$ is a term substitution $f : X \to T_\Omega(Y)$ such that $Y = \bigcup \{\text{var}(f(x)) : x \in X\}$. If $g : 1_Y \to 1_Z$ is another morphism of C_Ω, then we write $\exists gf$ and define $gf : 1_X \to 1_Z$ by $(gf)(x) = g(f(x))$ for each $x \in X$. Let $X_\Omega = T_\Omega(\mathscr{X})$ and define $\mathbf{p} : X_\Omega \to (C_\Omega)_O$ by $\mathbf{p}(t) = 1_{\text{var}(t)}$. Define a function $C_\Omega * X_\Omega \to X_\Omega$ by $\sigma \cdot t = \sigma(t)$.

Proposition 3. *With the definition above* C_Ω *acts on* X_Ω *and satisfies the orbit condition. For* $s, t \in X_\Omega$, *we have that* $(s,t) \in \mathscr{R}^* \Leftrightarrow \mathbf{p}(s) = \mathbf{p}(t)$.

Proof. It is clear that C_Ω is a category. To show that we have an action, we have to check that the axioms (A1), (A2) and (A3) hold. Of these, only (A2) needs any comment. Suppose that $\exists \sigma \cdot t$ where $\sigma : 1_X \to 1_Y$. Then $\mathbf{p}(t) = 1_{\text{var}(t)}$ and so $X = \text{var}(t)$. Now $\mathbf{p}(\sigma \cdot t) = 1_{\text{var}(\sigma(t))}$. But $Y = \text{var}(\sigma(t))$ since t contains all the variables of X, and every variable $y \in Y$ appears as a variable in $\sigma(x)$ for some $x \in X$, by assumption. Hence $\mathbf{p}(\sigma \cdot t) = \mathbf{r}(\sigma)$.

We now show that the orbit condition holds. Let s_1 and s_2 be two terms and suppose that $C_\Omega \cdot s_1 \cap C_\Omega \cdot s_2$ is nonempty. We may assume without loss of generality that $\text{var}(s_1) \cap \text{var}(s_2)$ is empty; for otherwise, we can find renaming substitutions γ and δ such that $\exists \gamma \cdot s_1, \exists \delta \cdot s_2$ and $\gamma(s_1)$ and $\delta(s_2)$ have no variables in common. Since γ and δ are bijections we have that

$$C_\Omega \cdot s_1 \cap C_\Omega \cdot s_2 = C_\Omega \cdot \gamma(s_1) \cap C_\Omega \cdot \delta(s_2).$$

By assumption, there exist morphisms $\alpha_1, \alpha_2 \in C_\Omega$ such that $\alpha_1 \cdot s_1 = \alpha_2 \cdot s_2$. Thus from the definitions

$$\mathbf{d}(\alpha_1) = \text{var}(s_1), \qquad \mathbf{d}(\alpha_2) = \text{var}(s_2) \quad \text{and} \quad \alpha_1(s_1) = \alpha_2(s_2).$$

Define a substitution $\alpha : \mathscr{X} \to T_\Omega(\mathscr{X})$ by

$$\alpha(x) = \begin{cases} \alpha_1(x) & \text{if } x \in \mathrm{var}(s_1) \\ \alpha_2(x) & \text{if } x \in \mathrm{var}(s_2) \\ x & \text{else.} \end{cases}$$

Clearly, $\alpha(s_1) = \alpha(s_2)$ and so s_1 and s_2 are unifiable. Let $\mu = \mathrm{mgu}(s_1, s_2)$ and put $t = \mu(s_1) = \mu(s_2)$. We shall prove that

$$C_\Omega \cdot s_1 \cap C_\Omega \cdot s_2 = C_\Omega \cdot t.$$

We show first that $C_\Omega \cdot s_1 \cap C_\Omega \cdot s_2 \subseteq C_\Omega \cdot t$. Let $\beta_1 \cdot s_1 = \beta_2 \cdot s_2$. Define a substitution $\beta : \mathscr{X} \to T_\Omega(\mathscr{X})$ by

$$\beta(x) = \begin{cases} \beta_1(x) & \text{if } x \in \mathrm{var}(s_1), \\ \beta_2(x) & \text{if } x \in \mathrm{var}(s_2), \\ x & \text{else.} \end{cases}$$

Clearly, $\beta(s_1) = \beta(s_2)$. But $\mu = \mathrm{mgu}(s_1, s_2)$ and so $\mu \leq \beta$ by Theorem 2. Let $\theta\mu = \beta$ for some substitution θ. Now

$$\beta_1(s_1) = \beta(s_1) = (\theta\mu)(s_1) = \theta(\mu(s_1)) = \theta(t).$$

Define a morphism $\varphi : 1_{\mathrm{var}(t)} \to 1_{\mathrm{var}(\beta_1(s_1))}$ by $\varphi(x) = \theta(x)$. Then $\beta_1 \cdot s_1 = \varphi \cdot t$.

Now we prove that $C_\Omega \cdot t \subseteq C_\Omega \cdot s_1 \cap C_\Omega \cdot s_2$. Let $\alpha \cdot t \in C_\Omega \cdot t$. Define $\mu_1 : 1_{\mathrm{var}(s_1)} \to 1_{\mathrm{var}(\mu(s_1))}$ by $\mu_1(x) = \mu(x)$, and define $\mu_2 : 1_{\mathrm{var}(s_2)} \to 1_{\mathrm{var}(\mu(s_2))}$ by $\mu_2(x) = \mu(x)$. Then $\mu(s_1) = \mu_1 \cdot s_1$ and $\mu(s_2) = \mu_2 \cdot s_2$. Thus $\alpha \cdot t = (\alpha\mu_1) \cdot s_1 = (\alpha\mu_2) \cdot s_2$.

Finally, we compute $\mathscr{R}^*$. Let $\mathbf{p}(s) = \mathbf{p}(t)$ and suppose that $\sigma \cdot s = \tau \cdot s$. Now $\mathrm{var}(s) = \mathrm{var}(t)$, so that the variables appearing in s and t are the same. From $\sigma \cdot s = \tau \cdot s$, we have that σ and τ take the same values on each of the variables of s. Hence, σ and τ take the same values on each of the variables of t. Thus $\sigma \cdot t = \tau \cdot t$, and conversely. □

We may now provide an explicit description of $J({}_{C_\Omega}X_\Omega)$.

Theorem 4. *The underlying set of* $J({}_{C_\Omega}X_\Omega)$ *is*

$$\{[s,t] : s, t \in X_\Omega, \text{ and } var(s) = var(t)\} \cup \{0\}.$$

Furthermore, the equivalence relation $\sim$ *is given by*

$$(s,t) \sim (s',t') \Leftrightarrow (s,t) = \sigma \cdot (s',t')$$

for some renaming substitution σ. *The binary operation in* $J({}_{C_\Omega}X_\Omega)$ *is given by the following expression, where we assume that* $var(t) \cap var(u) = \emptyset$:

$$[s,t] \otimes [u,v] = [\mu(s), \mu(v)]$$

if $\mu = mgu(t,u)$ *and* 0 *otherwise.*

Proof. By Theorem 2, we have already seen that

$$(s,t)\in\mathscr{R}^* \;\Leftrightarrow\; \mathrm{var}(s)=\mathrm{var}(t).$$

Suppose that $(s,t)\sim(s',t')$. Then $(s,t)=\sigma\cdot(s',t')$ and $(s',t')=\tau\cdot(s,t)$. Thus in particular

$$\sigma(s')=s \quad\text{and}\quad \tau(s)=t.$$

The conditions of Theorem 1(i) are satisfied and so σ and τ are mutually inverse relabelling substitutions.

We now compute $[s,t]\otimes[u,v]$. Recall that t and u have no variables in common. Suppose that $C_\Omega\cdot t\cap C_\Omega\cdot u$ is nonempty. By definition

$$[s,t]\otimes[u,v]=[(u*t)\cdot s,(t*u)\cdot v].$$

Let $\mu=\mathrm{mgu}(t,u)$. Then as we have seen

$$C_\Omega\cdot t\cap C_\Omega\cdot u=C_\Omega\cdot\mu(t)=C_\Omega\cdot\mu(u).$$

By definition

$$(u*t)\cdot t=\mu(t) \quad\text{and}\quad (t*u)\cdot u=\mu(u).$$

By assumption s and t share the same variables. Likewise, u and v share the same variables. Thus

$$\mu(s)=(u*t)\cdot s \quad\text{and}\quad \mu(v)=(t*u)\cdot v.$$

Hence

$$[s,t]\otimes[u,v]=[\mu(s),\mu(v)]. \qquad\square$$

The semigroup $J({}_{C_\Omega}X_\Omega)$ is the *clause semigroup*.

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References

[1] A.H. Clifford, A class of d-simple semigroups, Amer. J. Math. 75 (1953) 547–556.
[2] N. Dershowitz, J.-P. Jouannaud, Rewrite systems, in: Handbook of Theoretical Computer Science, vol. B, Elsevier Science, Amsterdam, 1992, pp. 243–320.

[3] A. Ehresmann, Charles Ehresmann: Oeuvres Complètes et Commentées, 7 volumes, Amiens, 1984.
[4] J.H. Gallier, Logic for Computer Science, Wiley, New York, 1987.
[5] J-Y. Girard, Geometry of interaction III: the general case, in: J.-Y. Girard, Y. Lafont, L. Regnier (Eds.), Proc. Workshop on Linear Logic, Cornell, 1993, Advances in Linear Logic, Cambridge University Press, Cambridge, 1993.
[6] J. Howie, Introduction to Semigroup Theory, Academic Press, New York, 1976.
[7] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Rend. Sem. Mat. e Fis. di Milano 43 (1973) 135–166.
[8] J. Leech, Constructing inverse monoids from small categories, Semigroup Forum 36 (1987) 89–116.
[9] S. MacLane, Categories for the Working Mathematician, Springer, Berlin, 1971.
[10] S. MacLane, I. Moerdijk, Sheaves in Geometry and Logic, Springer, Berlin, 1992.
[11] D.B. McAlister, 0-bisimple inverse semigroups, Proc. London Mathematical Society (3) 28 (1974) 193–221.
[12] D.B. McAlister, One-to-one partial right translations of a right cancellative semigroup, J. Algebra 43 (1976) 231–251.
[13] J. Meakin, M. Sapir, Congruences on free monoids and submonoids of polycyclic monoids, J. Austral. Math. Soc. (Ser. A) 54 (1993) 236–254.
[14] K.S.S. Nambooripad, The natural partial order on a regular semigroup, Proc. Edinburgh Math. Soc. 23 (1980) 249–260.
[15] M. Nivat, Perrot, Une généralisation du monoide bicyclique, C.R. Acad. Sci. Paris 271A (1970) 824–827.
[16] M. Petrich, Inverse Semigroups, Wiley, New York, 1984.
[17] N.R. Reilly, Bisimple inverse semigroups, Trans. Amer. Math. Soc. 132 (1968) 101–114.
[18] D.E. Rydeheard, J.G. Stell, Foundations of equational deduction: a categorical treatment of equational proofs and unifications algorithms, Lecture Notes in Computer Science, vol. 283, Springer, Berlin, 1987, pp. 114–139.
[19] B. Tilson, Categories as algebra: an essential ingredient in the theory of monoids, J. Pure Appl. Algebra 48 (1987) 83–198.