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Multiple Solutions for Resonant Elliptic Equations via Local Linking Theory and Morse Theory[1]

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We consider two classes of elliptic resonant problems. First, by local linking theory, we study the double-double resonant case and obtain three solutions. Second, we introduce some new conditions and compute the critical groups both at zero and at infinity precisely. Combining Morse theory, we get three solutions for the completely resonant case. © 2001 Academic Press

Key Words: elliptic problem; double-double resonance; local linking; critical group; Morse theory.

1. INTRODUCTION

In this paper we consider the elliptic resonant problem at higher eigenvalue of $-\Delta$ with Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbf{R}^N$, $N \geqslant 1$. More precisely, we will be concerned with the multiple solutions of the problem

$$-\Delta u = g(x, u), \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{P}$$

where $g \in C^1(\bar{\Omega} \times \mathbf{R}, \mathbf{R})$. We denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ the distinct eigenvalues of $-\Delta$ in $H := H_0^1(\Omega)$ and by $H(\lambda_k)$ the corresponding λ_k-eigenspace, respectively. We consider two cases. In the first case, we deal with by using the local linking theory the double-double resonant case (see

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assumptions (A_1) and (A_2)) and obtain at least three solutions. In the second case, by computing the critical groups and by using Morse theory, we study the completely resonant case (see assumption $(\bigstar)$ of subsection 1.2) and obtain at least two nontrivial solution. We also establish some existence results of one nontrivial solution under some very weak conditions.

1.1. *Double-Double Resonance Case.* In order to obtain multiple solutions by using the local linking theory, we first make the following assumptions. From now on, for two functions a, b, we write $a(x) \precnsim b(x)$ (or $a(x) \succnsim b(x)$) to indicate that $a(x) \leqslant b(x)$ (resp. $a(x) \geqslant b(x)$) with strict inequality holding on a set of positive measure.

(A_1) $\lambda_k \leqslant \liminf_{|t| \to \infty} (g(x, t)/t)$ uniformly for a.e. $x \in \Omega$ and there exists $\alpha \in C(\bar{\Omega})$ such that $g'(x, t) \leqslant \alpha(x) \precnsim \lambda_{k+1}$ for a.e. $x \in \Omega$ and $t \in \mathbf{R}$.

(A_2) There exist $m \leqslant k$, $t_0 > 0$ and $\beta \in C(\bar{\Omega})$ such that

$$\lambda_{m-1} \leqslant \frac{2G(x, t)}{t^2} \leqslant \beta(x) \precnsim \lambda_m \qquad \text{for a.e.} \quad x \in \Omega \quad \text{and} \quad 0 < |t| \leqslant t_0,$$

where $G(x, t) = \int_0^t g(x, s)\, ds$.

We see that (A_1) and (A_2) imply that

$$\lambda_k \leqslant \liminf_{|t| \to \infty} \frac{g(x, t)}{t} \leqslant \limsup_{|t| \to \infty} \frac{g(x, t)}{t} \leqslant \alpha(x) \precnsim \lambda_{k+1}$$

and

$$\lambda_{m-1} \leqslant \liminf_{t \to 0} \frac{2G(x, t)}{t^2} \leqslant \limsup_{t \to 0} \frac{2G(x, t)}{t^2} \leqslant \beta(x) \precnsim \lambda_m,$$

which characterize (P) the double resonance at infinity and the double resonance at zero. Let us call (P) a double-double resonance problem. Problems with double resonance at infinity were treated first by Berestycki and deFigueiredo (cf. [1]). Recently, the paper [2] (see also [3] and the references cited therein) studied this problem and obtained one nontrivial solution.

Evidently, (A_1) contains completely resonance at infinity, i.e.,

$$\lim_{|t| \to \infty} \frac{g(x, t)}{t} = \lambda_k.$$

For this problem there are many well-known existence and multiplicity results (see, for example, [4–9, 19–23, 26–29] and the references cited therein). Most of them are under the assumption of the boundedness of nonlinear term, that is, there exists $g_0 \in L^p(\Omega)$ such that $|g(x, t)| \leqslant g_0(x)$ for all $t \in \mathbf{R}$ and a.e. $x \in \Omega$ (see [4–9, 20, 27, 29]). If g is unbounded, one nontrivial solution was obtained in [12, 13, 23, 26]; [10, 11] obtained two solutions under some strong conditions. The main goal of this subsection is to consider the multiple solutions of (P) with double-double resonance and with unbounded nonlinear term. For this end, we introduce a generalized condition of nonquadraticity at infinity (cf. [3]).

(A_3) There exist $\mu \in (0, 2)$ and $\gamma \in C(\bar{\Omega})$ such that

$$\limsup_{|t| \to \infty} \frac{tg(x, t) - 2G(x, t)}{|t|^\mu} \leqslant \gamma(x) \lneqq 0 \qquad \text{uniformly for a.e.} \quad x \in \Omega.$$

Now the first main result stated as:

THEOREM 1.1. *Assume (A_1), (A_2), and (A_3). Then (P) has at least three solutions.*

Remark 1.1. Indeed, there exist some functions satisfying (A_1)–(A_3). For example, let $\varrho\colon \bar{\Omega} \to [0, \frac{\pi}{2}]$ be continuous with $\varrho(x) = 0$ on Ω_1 and $\varrho(x) = \frac{\pi}{2}$ on Ω_2, where Ω_1 and Ω_2 are two subsets of Ω with positive measures. Define

$$g(x, t) = \begin{cases} \lambda_k t + \dfrac{\lambda_{k+1} - \lambda_k}{3} \left(2t \sin\left(\varrho(x) + \dfrac{1}{|t|}\right) - \dfrac{t}{|t|} \cos\left(\varrho(x) + \dfrac{1}{|t|}\right)\right), & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad |t| \geqslant 3, & \\ \leqslant \left(\lambda_k + \dfrac{25}{27} (\lambda_{k+1} - \lambda_k)\right) t, & \text{if} \quad 1 \leqslant |t| \leqslant 3, \\ (\lambda_{m-1} + (\lambda_m - \lambda_{m-1}) \sin \varrho(x))\, t, & \text{if} \quad |t| \leqslant 1. \end{cases}$$

Then it is easy to check that $g(x, t)$ satisfies (A_1)–(A_3) with $\beta(x) = \lambda_{m-1} + (\lambda_m - \lambda_{m-1}) \sin \varrho(x)$, $\gamma(x) = -\cos \varrho(x)$ and $\mu = 1$. Particularly, $g(x, t) - \lambda_k t$ may be linear growth both at infinity and at zero on a subset of positive measure.

Remark 1.2. Assumption (A_3) permits that $\lim_{|t| \to \infty}(tg(x, t) - 2G(x, t)) = \infty$ on a subset of positive measure and at the same time, that $\lim_{|t| \to \infty}(tg(x, t) - 2G(x, t)) = c$ ($c = \textit{constant}$ or $-\infty$) on other subsets of positive measures. (A_3) is a generalization of the condition of nonquadraticity at infinity which was introduced in [3], where $\gamma(x) = \textit{constant} < 0$.

Remark 1.3. Theorem 1.1 generalizes Theorems 2.1 and 2.2 of [10] and Theorem 2 of [5]. In [10] it was supposed that $\lim_{|t| \to \infty}(g(t)/t) = \lambda_k$, $\lambda_{k-1} \leqslant \inf_{t \neq 0}(g(t)/t)$ (a global condition) and that $tg(t)$ is not sign-changing when $|t|$ large. Mizoguchi [5] introduced the so-called density condition with respect to G and obtained only one nontrivial solution by different method. If g is bounded, [5, 7] obtained some similar results by different methods under different conditions.

There is a conjugate result of Theorem 1.1.

THEOREM 1.2. *Suppose that there exist* $m > k$, $t_0 > 0$, $2 > \mu > 0$; $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in C(\bar{\Omega})$ *such that the following conditions hold*:

(A_1') $\limsup_{|t| \to \infty}(g(x, t)/t) \leqslant \lambda_k$ *uniformly for a.e.* $x \in \Omega$ *and* $g'(x, t) \geqslant \bar{\alpha}(x) \geqslant \lambda_{k-1}$ *for a.e.* $x \in \Omega$ *and* $t \in \mathbf{R}$;

(A_2') $\lambda_m \leqslant 2G(x, t)/t^2 \leqslant \bar{\beta}(x) \lessdot \lambda_{m+1}$ *for a.e.* $x \in \Omega$ *and* $0 < |t| < t_0$;

(A_3') $\liminf_{|t| \to \infty}((g(x, t)\, t - 2G(x, t))/|t|^{\mu}) \geqslant \bar{\gamma}(x) \gtrdot 0$.

Then (P) *has at least three solutions.*

Remark 1.4. The proof of Theorems 1.1 and 1.2 is based on the reduction method and the local linking theory (cf. [17, 18]). We prove that the reduction functional defined on a finite dimentional (or infinite dimentional) subspace has the local linking geometry, then the abstract theorem of [17, 18] could be used.

1.2. *The Completely Resonant Case.* We consider

$$\lim_{|t| \to \infty} \frac{g(x, t)}{t} = \lambda_k \qquad \text{and} \qquad \lim_{|t| \to 0} \frac{g(x, t)}{t} = \lambda_m. \tag{$\bigstar$}$$

Obviously, this case is contained in the double-double resonance case. But by introducing some new conditions which enable us to compute the critical groups, we obtain some new results about multiple solutions. For this case, the corresponding functional of (P) is degenerate both at infinity and at zero. Therefore, computing the critical groups becomes the main ingredient when we want to use Morse theory.

Throughout this paper, we write

$$g(x, t) = \lambda_k t + f(x, t) = \lambda_m t + f_0(x, t)$$

and

$$F(x, t) = \int_0^t f(x, s)\, ds, \qquad F_0(x, t) = \int_0^t f_0(x, s)\, ds.$$

First, for computing the critical groups at infinity (cf. [12]), we introduce a control function h_∞ for f.

Let $h_\infty: \mathbf{R}^+ \to \mathbf{R}^+$ be an increasing function and τ_1, τ_2 two constants such that

$$0 < \tau_1 \leqslant \frac{th_\infty(t)}{H_\infty(t)} \leqslant \tau_2 < 2, \qquad h_\infty(s+t) \leqslant c(h_\infty(s) + h_\infty(t)), \quad \forall s, t \in \mathbf{R}^+.$$

Here and in the sequel, the letter c will be indiscriminately used to denote various constants whose exact value is irrelevant. Evidently, $h_\infty(t) = t^\sigma$ with $0 < \sigma < 1$ is a simple example. Now we assume that

(B_1) $|f(x, t)| \leqslant c(1 + h_\infty(|t|))$ for a.e. $x \in \Omega$ and $t \in \mathbf{R}$.

$(B_2^\pm)$ $\liminf_{|t| \to \infty}(\pm F(x, t)/H_\infty(|t|)) := a^\pm(x) \geqslant 0$, uniformly for a.e. $x \in \Omega$.

We will see that (B_1) and $(B_2^\pm)$ enable us to compute the critical groups at infinity and Betti number precisely.

Since the existence of nontrivial solutions is closely related to the behavior of f_0 at zero, we need some hypotheses around the origin. Similarly, we introduce a control function as follows.

Let $h_0: \mathbf{R}^+ \to \mathbf{R}^+$ be an increasing function and σ_1, σ_2 two constants such that

$$2 < \sigma_1 \leqslant \frac{th_0(t)}{H_0(t)} \leqslant \sigma_2, \qquad h_0(s+t) \leqslant c(h_0(s) + h_0(t)),$$

for $s, t \in \mathbf{R}^+$ and small. A simple example is $h_0(t) = t^\sigma$ with $\sigma > 2$. Now we assume that

(C_1) $|f_0(x, t)| \leqslant ch_0(|t|)$ for a.e. $x \in \Omega$ and $|t|$ small.

$(C_2^\pm)$ $\liminf_{t \to 0}(\pm F_0(x, t)/H_0(|t|)) := b^\pm(x) \geqslant 0$, uniformly for a.e. $x \in \Omega$.

Remark 1.5. We can compute the critical groups at zero precisely under (C_1) and $(C_2^\pm)$. Considering (B_1) (or (C_1)), $(B_2^\pm)$ (resp. $(C_2^\pm)$) are reasonable. Noting that $a^\pm(x)$ and $b^\pm(x)$ are permitted of zero on a positive measure subset of Ω, then $\pm F(x, t)$ and $\pm F_0(x, t)$ may be sign-changing; $F(x, t)$ is allowed to be bounded or unbounded on different subsets of Ω with positive measures.

In order to get multiple solutions, we need a further assumption, that is,

$(D^\pm)$ there exists $\tilde{\alpha} \in C(\bar{\Omega})$ such that $\pm f'(x, t) \leqslant \pm \tilde{\alpha}(x) \lneqq \pm(\lambda_{k\pm 1} - \lambda_k)$ for a.e. $x \in \Omega$.

Set

$$\mu_\infty = \dim(H(\lambda_1) \oplus \cdots \oplus H(\lambda_{k-1})), \qquad \nu_\infty = \dim H(\lambda_k);$$
$$\mu_0 = \dim(H(\lambda_1) \oplus \cdots \oplus H(\lambda_{m-1})), \qquad \nu_0 = \dim H(\lambda_m).$$

Now we are prepared to state the main results in this subsection.

THEOREM 1.3. *Assume* (B_1), (C_1), *and* (D^+). *Then* (P) *has at least two nontrivial solutions in each of the following cases*:

(i) (B_2^+) *and* (C_2^+); $|(\mu_\infty + \nu_\infty) - (\mu_0 + \nu_0)| =$ *odd number*;

(ii) (B_2^+) *and* (C_2^-); $|\mu_\infty + \nu_\infty - \mu_0| =$ *odd number*.

If (D^-) holds, we can estimate the Morse index, therefore we get

THEOREM 1.4. *Assume* (B_1), (C_1), *and* (D^-). *Then* (P) *has at least two nontrivial solutions in each of the following cases*:

(i) (B_2^-) *and* (C_2^+); $\mu_0 + \nu_0 \neq \mu_\infty$;

(ii) (B_2^-) *and* (C_2^-); $\mu_0 \neq \mu_\infty$.

If we drop $(D^\pm)$, we obtain the existence results of one nontrivial solution.

THEOREM 1.5. *Assume* (B_1) *and* (C_1). *Then there exists at least one nontrivial solution in each of the following cases*:

(i) (B_2^+), (C_2^+) *and* $\mu_\infty + \nu_\infty \neq \mu_0 + \nu_0$;

(ii) (B_2^+), (C_2^-) *and* $\mu_\infty + \nu_\infty \neq \mu_0$;

(iii) (B_2^-), (C_2^+) *and* $\mu_\infty \neq \mu_0 + \nu_0$;

(iv) (B_2^-), (C_2^-) *and* $\mu_\infty \neq \mu_0$.

Remark 1.6. Theorem 1.5 extends different results contained in [12, 22]. In [12] it was assumed that $f(x, t)$ is bounded and $F(x, t) \to \infty$ uniformly for $|t| \to \infty$. In [22], $F(x, t)$ is not sign-changing.

Next we consider the following assumptions:

$(E_\infty^\pm)$ There exist $\gamma^\pm \in C(\bar{\Omega})$ such that

$$\liminf_{|t| \to \infty} \frac{\pm(tf(x, t) - 2F(x, t))}{|t|^{\tau+1}} \geqslant \gamma^\pm(x) \geqslant 0.$$

$(E_0^{\pm})$ There exist $\vartheta^{\pm} \in C(\bar{\Omega})$ such that

$$\liminf_{|t| \to 0} \frac{\pm(tf_0(x, t) - 2F_0(x, t))}{|t|^{\sigma+1}} \geqslant \vartheta^{\pm}(x) \succcurlyeq 0.$$

Then we have

THEOREM 1.6. *Assume that there exist $\tau \in (0, 1)$, $\sigma > 2$, such that $|f(x, t)| \leqslant c(1+|t|^{\tau})$ for all $t \in \mathbf{R}$ and a.e. $x \in \Omega$ and that $|f_0(x, t)| \leqslant c\,|t|^{\sigma}$ for $|t|$ small and a.e. $x \in \Omega$. Then (P) has at least one nontrivial solution in each of the following cases*:

(i) (E_∞^-), (E_0^-) *and* $\mu_\infty + \nu_\infty \neq \mu_0 + \nu_0$;

(ii) (E_∞^-), (E_0^+) *and* $\mu_\infty + \nu_\infty \neq \mu_0$;

(iii) (E_∞^+), (E_0^-) *and* $\mu_\infty \neq \mu_0 + \nu_0$;

(iv) (E_∞^+), (E_0^+) *and* $\mu_\infty \neq \mu_0$.

Remark 1.7. We will prove that $(E_\infty^{\pm})$ and $(E_0^{\pm})$ imply completely the critical groups at infinity and at zero, respectively. As we have pointed out in subsection 1.1, $(E_\infty^{\pm})$ generalize the condition of nonquadraticity at infinity (see [3, 32, 33]). But in those papers, no characteristics of the critical groups were obtained under $(E_\infty^{\pm})$. Conditions $(E_0^{\pm})$ seem to be new.

Remark 1.8. It is easy to see that the above theorems contain the case of $\lambda_k = \lambda_m$, which means that the resonance happens both at zero and at infinity simultaneously with the same resonant point. So they extend different results of [5, 7, 10, 26, 27].

Remark 1.9. In Theorems 1.1 to 1.4, condition $g'(x, t) \leqslant \alpha(x) \precneqq \lambda_{k+1}$ (or $g'(x, t) \geqslant \bar{\alpha}(x) \succneqq \lambda_{k-1}$) can be replaced by a weaker version

$$\frac{g(x, t) - g(x, s)}{t - s} \leqslant \alpha(x) \precneqq \lambda_{k+1} \qquad \left(\text{resp. } \frac{g(x, t) - g(x, s)}{t - s} \geqslant \bar{\alpha}(x) \succneqq \lambda_{k-1}\right)$$

for all $t \neq s$. There is no essential difference for proving Theorems 1.1 to 1.4.

2. PROOFS OF THEOREMS 1.1 AND 1.2—BY LOCAL LINKING THEORY

First, we have to establish some lemmas. Let $\|u\| = (\int_\Omega |\nabla u|^2\, dx)^{1/2}$ be the usual norm in H induced by the inner product $\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v\, dx$, $u, v \in H$. $\|\cdot\|_p$ denotes p-norm in $L^p(\Omega)$. Since $H(\lambda_i)$ is the eigenspace corresponding to λ_i, $H(\lambda_i)$ has the unique continuation property.

LEMMA 2.1. *Assume that there exists* $h: \mathbf{R}^+ \to \mathbf{R}^+$, $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1 \leqslant \frac{th(t)}{H(t)} \leqslant c_2 < 2^*,$$

where $2^* = 2N/(N-2)$ *if* $N \geqslant 3$; $2^* = \infty$ *if* $N \leqslant 2$; $H(t) = \int_0^t h(s)\, ds$. *For* $P \in C(\bar{\Omega} \times \mathbf{R}, \mathbf{R})$, *there exists* $\theta^{\pm} \in C(\bar{\Omega})$ *such that*

$$\liminf_{|t| \to \infty} \frac{\pm P(x, t)}{H(|t|)} \geqslant \theta^{\pm}(x) \gneqq 0$$

uniformly for a.e. $x \in \Omega$. *Let* $H = V \oplus W$ *with* $\dim V < \infty$ *and* V *have the unique continuation property. If* $u_n = v_n + w_n$ *with* $v_n \in V$, $w_n \in W$ *and* $w_n / \|u_n\| \to 0$, *then*

$$\liminf_{n \to \infty} \frac{\int_\Omega \pm P(x, u_n)\, dx}{H(\|u_n\|)} > 0.$$

Proof. Evidently, $\dim V < \infty$ implies that there exists $C_0 > 0$ such that

$$|v(x)| \leqslant \sup\{|v(x)|: x \in \Omega\} \leqslant C_0 \|v\|, \qquad \forall v \in V.$$

By the unique continuation property of V, using a similar argument as that in the proof of Lemma 3.2 of [4], we have, for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, that there exist $\delta(\varepsilon_1) \in (0, 1)$ and $\delta(\varepsilon_2) > 0$ such that

$$\operatorname{meas}\{x \in \Omega : |v(x)| < \delta(\varepsilon_1) \|v\|\} < \varepsilon_1, \qquad \forall v \in V \backslash \{0\},$$
$$\operatorname{meas}\{x \in \Omega : |w(x)| > \delta(\varepsilon_2) \|w\|\} < \varepsilon_2, \qquad \forall w \in W.$$

Letting

$$\Omega_{1n} = \{x \in \Omega : |v_n(x)| \geqslant \delta(\varepsilon_1) \|v_n\|\},$$
$$\Omega_{2n} = \{x \in \Omega : |w_n(x)| \leqslant \delta(\varepsilon_2) \|w_n\|\},$$

then $\operatorname{meas}(\Omega \backslash \Omega_{1n}) < \varepsilon_1$, $\operatorname{meas}(\Omega \backslash \Omega_{2n}) < \varepsilon_2$, $\Omega_{1n} \cap \Omega_{2n} \neq \emptyset$ and

$$\int_{\Omega_{1n} \cap \Omega_{2n}} \theta^{\pm}(x)\, dx \geqslant \frac{1}{2} \int_\Omega \theta^{\pm}(x)\, dx > 0,$$

if ε_1 and ε_2 are small enough.

By our assumptions, for any $\varepsilon > 0$, we have that

$$\frac{|u_n(x)|}{\|u_n\|} \geqslant \delta(\varepsilon_1)\frac{\|v_n\|}{\|u_n\|} - \delta(\varepsilon_2)\frac{\|w_n\|}{\|u_n\|} \geqslant \delta(\varepsilon_1) - \varepsilon,$$

$$\frac{|u_n(x)|}{\|u_n\|} \leqslant C_0\frac{\|v_n\|}{\|u_n\|} + \delta(\varepsilon_2)\frac{\|w_n\|}{\|u_n\|} \leqslant C_0 + \varepsilon,$$

as $x \in \Omega_{1n} \cap \Omega_{2n}$ and $n \to \infty$; and that

$$\frac{|u_n(x)|}{\|u_n\|} \leqslant \delta(\varepsilon_1)\frac{\|v_n\|}{\|u_n\|} + \delta(\varepsilon_2)\frac{\|w_n\|}{\|u_n\|} \leqslant \delta(\varepsilon_1) + \varepsilon$$

as $x \in \Omega_{2n} \backslash \Omega_{1n}$ and $n \to \infty$.

On the other hand, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\frac{\pm P(x, t)}{H(|t|)} \geqslant \theta^{\pm}(x) - \varepsilon \qquad \text{for a.e. } x \in \Omega \quad \text{and} \quad |t| \geqslant T_\varepsilon.$$

Setting $\Omega_n = \{x \in \Omega : |u_n(x)| \geqslant T_\varepsilon\}$, $\Omega_{3n} = \{x \in \Omega_{1n} \cap \Omega_{2n} : |u_n(x)| \geqslant \|u_n\|\}$, $\Omega_{4n} = \{x \in \Omega_{1n} \cap \Omega_{2n} : |u_n(x)| < \|u_n\|\}$, then by the definition of h and for n large enough, we have that

$$\begin{aligned}
&\int_{\Omega_{1n} \cap \Omega_{2n}} \frac{\pm P(x, u_n)}{H(\|u_n\|)}\, dx \\
&\geqslant \int_{\Omega_{1n} \cap \Omega_{2n}} (\theta^{\pm}(x) - \varepsilon)\frac{H(|u_n|)}{H(\|u_n\|)}\, dx \\
&\geqslant \int_{\Omega_{1n} \cap \Omega_{2n}} \theta^{\pm}(x)\frac{H(|u_n|)}{H(\|u_n\|)}\, dx - \varepsilon \int_{\Omega_{1n} \cap \Omega_{2n}} \left(\left(\frac{|u_n|}{\|u_n\|}\right)^{c_1} + \left(\frac{|u_n|}{\|u_n\|}\right)^{c_2}\right) dx \\
&\geqslant \int_{\Omega_{3n}} \theta^{\pm}(x)\frac{H(|u_n|)}{H(\|u_n\|)}\, dx + \int_{\Omega_{4n}} \theta^{\pm}(x)\frac{H(|u_n|)}{H(\|u_n\|)}\, dx - \varepsilon c \\
&\geqslant \int_{\Omega_{3n}} \theta^{\pm}(x)\left(\frac{|u_n|}{\|u_n\|}\right)^{c_1} dx + \int_{\Omega_{4n}} \theta^{\pm}(x)\left(\frac{|u_n|}{\|u_n\|}\right)^{c_2} dx - \varepsilon c \\
&\geqslant \int_{\Omega_{3n}} \theta^{\pm}(x)\, dx + \int_{\Omega_{4n}} \theta^{\pm}(x)(\delta(\varepsilon_1) - \varepsilon)^{c_2}\, dx - \varepsilon c \\
&\geqslant (\delta(\varepsilon_1) - \varepsilon)^{c_2} \int_{\Omega_{1n} \cap \Omega_{2n}} \theta^{\pm}(x)\, dx - \varepsilon c \\
&\geqslant \frac{(\delta(\varepsilon_1) - \varepsilon)^{c_2}}{2} \int_{\Omega} \theta^{\pm}(x)\, dx - \varepsilon c \\
&\geqslant c(\delta(\varepsilon_1) - \varepsilon)^{c_2} - \varepsilon c.
\end{aligned}$$

On the other hand, for n large enough,

$$
\begin{aligned}
\int_{\Omega_{2n}\backslash\Omega_{1n}} \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx
&= \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap(\Omega\backslash\Omega_n)} \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx + \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap\Omega_n} \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx \\
&\geqslant -\varepsilon + \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap\Omega_n} (\theta^{\pm}(x)-\varepsilon)\,\frac{H(x,u_n)}{H(\|u_n\|)}\,dx \\
&\geqslant -\varepsilon c + \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap\Omega_n} \theta^{\pm}(x)\,\frac{H(x,u_n)}{H(\|u_n\|)}\,dx \\
&\geqslant -\varepsilon c + \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap\Omega_n} \theta^{\pm}(x)\left(\frac{|u_n|}{\|u_n\|}\right)^{c_2} dx \\
&\geqslant -\varepsilon c - \int_{(\Omega_{2n}\backslash\Omega_{1n})\cap\Omega_n} (\delta(\varepsilon_1)+\varepsilon)^{c_2}\,\theta^{\pm}(x)\,dx \\
&\geqslant -\varepsilon c - (\delta(\varepsilon_1)+\varepsilon)^{c_2}\,\varepsilon_1 c
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\Omega\backslash\Omega_{2n}} \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx &\geqslant -\varepsilon + \int_{(\Omega\backslash\Omega_{2n})\cap\Omega_n} \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx \\
&\geqslant -\varepsilon + \int_{(\Omega\backslash\Omega_{2n})\cap\Omega_n} (\theta^{\pm}(x)-\varepsilon)\,\frac{H(|u_n|)}{H(\|u_n\|)}\,dx \\
&\geqslant -\varepsilon c - c\int_{\Omega\backslash\Omega_{2n}} \left(\left(\frac{|u_n|}{\|u_n\|}\right)^{c_1} + \left(\frac{|u_n|}{\|u_n\|}\right)^{c_2}\right) dx \\
&\geqslant -c\varepsilon - c\varepsilon_2.
\end{aligned}
$$

Combining the above estimates, we have that

$$
\liminf_{n\to\infty} \int_\Omega \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx \geqslant c(\delta(\varepsilon_1)-\varepsilon)^{c_2} - \varepsilon c - c\varepsilon_1(\delta(\varepsilon_1)+\varepsilon)^{c_2} - \varepsilon_2 c.
$$

Noting that ε_1, ε_2 and ε are arbitrary, we have that

$$
\liminf_{n\to\infty} \int_\Omega \frac{\pm P(x,u_n)}{H(\|u_n\|)}\,dx \geqslant (\delta(\varepsilon_1))^{c_2}\,(c - c\varepsilon_1) > 0. \quad \blacksquare
$$

Setting $H^- = H(\lambda_1) \oplus H(\lambda_2) \oplus \cdots \oplus H(\lambda_{k-1})$, $H^+ = H(\lambda_{k+1}) \oplus H(\lambda_{k+2}) \oplus \cdots$, $H^0 = H(\lambda_k)$, then $H = H^- \oplus H^+ \oplus H^0$.

LEMMA 2.2 [15]. (i) *If $a(x) \leqslant \lambda_{k+1}$ for a.e. $x \in \Omega$, then there exists $\delta > 0$ such that*

$$\|w\|^2 - \int_\Omega a(x)\, w^2\, dx \geqslant \delta\, \|w\|^2, \qquad \text{for all} \quad w \in H^+.$$

(ii) *If $a(x) \geqslant \lambda_k$ for a.e. $x \in \Omega$, then there exists $\delta > 0$ such that*

$$\|v\|^2 - \int_\Omega a(x)\, v^2\, dx \leqslant -\delta\, \|v\|^2, \qquad \text{for all} \quad v \in H^- \oplus H^0.$$

It is well known that the solutions $u \in H$ of (P) are the critical points of C^1 functional

$$I(u) = \tfrac{1}{2} \int_\Omega |\nabla u|^2\, dx - \int_\Omega G(x, u)\, dx.$$

Based on the above lemmas, we show how conditions (A_1) and (A_3) (or (A_1') and (A_3')) imply the compactness condition $(C)_c$ in the Cerami's sense (cf. [14]): any sequence $\{u_n\} \subset H$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\, \|I'(u_n)\| \to 0$ possesses a convergent subsequence. It was shown in [4] that condition $(C)_c$ actually suffices to get a deformation theorem and then, by standard minimax arguments, it allows rather general minimax results.

LEMMA 2.3. *Assume (A_1) and (A_3) (or (A_1') and (A_3')). Then I satisfies the compactness condition $(C)_c$ for every $c \in \mathbf{R}$.*

Proof. We suppose that the first alternative holds. The proof with the second alternative is similar. Assume $\{u_n\}$ is such that $I(u_n) \to c$ and that $(1 + \|u_n\|)\, \|I'(u_n)\| \to 0$ as $n \to \infty$, then it is enough to prove that $\{u_n\}$ is bounded. By negation, assume that $\|u_n\| \to \infty$, and write $u_n = u_n^+ + u_n^- + u_n^0 \in H^+ \oplus H^- \oplus H^0 = H$. For any $\varepsilon > 0$, by (A_1) and (A_2) we have that

$$-\varepsilon - \frac{C_\varepsilon}{|t|} \leqslant \frac{f(x, t)}{t} \leqslant \alpha(x) - \lambda_k + \varepsilon + \frac{C_\varepsilon}{|t|}, \qquad \text{for} \quad t \neq 0.$$

If $|u_n^+| \geqslant |u_n^- + u_n^0|$, then

$$f(x, u_n)(u_n^+ - u_n^- - u_n^0) \leqslant (\alpha(x) - \lambda_k + \varepsilon)(u_n^+)^2 - (\alpha(x) - \lambda_k + \varepsilon)(u_n^- + u_n^0)^2 + C_\varepsilon\, |u_n^+ - u_n^- - u_n^0|.$$

If $|u_n^+| < |u_n^- + u_n^0|$, then

$$f(x, u_n)(u_n^+ - u_n^- - u_n^0) \leqslant -\varepsilon(u_n^+)^2 + \varepsilon(u_n^- + u_n^0)^2 + C_\varepsilon\, |u_n^+ - u_n^- - u_n^0|.$$

Consequently,

$$\begin{aligned} &f(x, u_n)(u_n^+ - u_n^- - u_n^0) \\ &\quad \leqslant (\alpha(x) - \lambda_k + \varepsilon)(u_n^+)^2 + \varepsilon(u_n^- + u_n^0)^2 + C_\varepsilon\, |u_n^+ - u_n^- - u_n^0|. \end{aligned}$$

Combining Lemma 2.2, we have the estimates,

$$\begin{aligned} &\langle I'(u_n), u_n^+ - u_n^- - u_n^0 \rangle \\ &\quad \geqslant \|u_n^+\|^2 - \|u_n^-\|^2 - \lambda_k\, \|u_n^+\|_2^2 + \lambda_k\, \|u_n^-\|_2^2 - \int_\Omega (\alpha(x) - \lambda_k + \varepsilon)(u_n^+)^2\, dx \\ &\quad -\varepsilon \int_\Omega (u_n^- + u_n^0)^2\, dx - C_\varepsilon \int_\Omega (u_n^+ - u_n^- - u_n^0)\, dx \\ &\quad \geqslant \delta\, \|u_n^+\|^2 + \left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right) \|u_n^-\|^2 - \varepsilon c\, \|u_n^+\|^2 \\ &\quad -\varepsilon c\, \|u_n^-\|^2 - \varepsilon c\, \|u_n^0\|^2 - c\, \|u_n\|, \end{aligned}$$

it follows that $\|u_n^\pm\|/\|u_n\| \to 0$. By Lemma 2.1 and (A_3), we have that

$$\limsup_{n\to\infty} \frac{\int_\Omega (u_n f(x, u_n) - 2F(x, u_n))\, dx}{\|u_n\|^\mu} < 0,$$

which contradicts the fact that

$$\int_\Omega (f(x, u_n)\, u_n - 2F(x, u_n))\, dx = 2I(u_n) - \langle I'(u_n), u_n \rangle \to 2c, \qquad \text{as} \quad n \to \infty.$$

Hence $\|u_n\|$ is bounded. ∎

LEMMA 2.4. *Assume* (A_1). *Let* $H = V \oplus W$ *with* $\dim V < \infty$ *and* V *have the unique continuation property. If* $u_n = v_n + w_n$ *with* $\|u_n\| \to \infty$ *and* $\|v_n\|/\|u_n\| \to 1$, *then*

(a) (A_3) *implies that* $\liminf_{n\to\infty} (\int_\Omega F(x, u_n)\, dx/\|u_n\|^\mu) > 0$;

(b) (A_3') *implies that* $\limsup_{n\to\infty} (\int_\Omega F(x, u_n)\, dx/\|u_n\|^\mu) < 0$.

Proof. (a) For $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$tf(x, t) - 2F(x, t) \leqslant (\gamma(x) + \varepsilon)\, |t|^\mu,$$

hence

$$\frac{d}{dt}\left(\frac{F(x,t)}{t^2}\right) \leqslant \frac{(\gamma(x)+\varepsilon)\,|t|^{\mu}}{t^3} \qquad \text{for} \quad |t| \geqslant T_\varepsilon.$$

Integrating the above inequality over an interval $[t, T] \subset [T_\varepsilon, \infty)$ yields the estimate

$$\frac{F(x,T)}{T^2} - \frac{F(x,t)}{t^2} \leqslant \int_t^T \frac{(\gamma(x)+\varepsilon)\,|t|^{\mu}}{t^3}\,dt.$$

Noting that (A_1) implies that $0 \leqslant \liminf_{|t|\to\infty}(F(x,t)/t^2) \leqslant (1/2)(\lambda_{k+1}-\lambda_k)$, then $F(x,t)/|t|^{\mu} \geqslant -(\gamma(x)+\varepsilon)/(2-\mu)$ for $t \geqslant T_\varepsilon$ and a.e. $x \in \Omega$. By the same way, we can prove that it is also true for $t \leqslant -T_\varepsilon$ and a.e. $x \in \Omega$. Hence

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^{\mu}} \geqslant -\frac{\gamma(x)}{2-\mu} \geqslant 0.$$

By Lemma 2.1, we have that

$$\liminf_{n\to\infty} \frac{\int_\Omega F(x,u_n)\,dx}{\|u_n\|^{\mu}} > 0.$$

The proof of (b) is similar and will be omitted. ∎

Before proving Theorems 1.1 and 1.2, we recall a global version of the Lyapunov–Schmidt method (cf. Lemma 2.1 of [16]). Let H be a real separable Hilbert space and X and Y be two closed subspaces of H such that $H = X \oplus Y$. Assume that $I \in C^1(H, \mathbf{R})$. If there are $m > 0$ and $\tau > 1$ such that

$$\langle I'(u+v) - I'(u+w), v-w \rangle \geqslant m\,\|v-w\|^{\tau} \qquad \text{for all} \quad u \in X, \quad v, w \in Y,$$

then there exists $\psi \in C(X, Y)$ such that

$$I(u+\psi(u)) = \min_{v\in Y} I(u+v).$$

Moreover, $\psi(u)$ is the unique member of Y such that $\langle I'(u+\psi(u)), v\rangle = 0$ for all $v \in Y$. Furthermore, if we define $\bar{I}(u) = I(u+\psi(u))$, then $\bar{I} \in C^1(X, \mathbf{R})$ and

$$\langle \bar{I}'(u), u_1 \rangle = \langle I'(u+\psi(u)), u_1 \rangle \qquad \text{for all} \quad u, u_1 \in X.$$

An element $u \in X$ is a critical point of $\bar{I}$ if and only if $u + \psi(u)$ is a critical point of I. Now we have to prove the following lemma.

LEMMA 2.5. *Assume that* $\|I'(u)\| \leqslant c(1+\|u\|^{\tau-1})$ *for* $u \in H$ *and that* I *satisfies the compactness condition* $(C)_c$. *Then* $\bar{I}$ *satisfies the compactness condition* $(C)_c$.

Proof. Let $u_n \in X$ be such that $\bar{I}(u_n) \to c$ and that $(1+\|u_n\|)\,\bar{I}'(u_n) \to 0$, that is

$$I(u_n + \psi(u_n)) \to c, \qquad (1+\|u_n\|)(P_X I'(u_n+\psi(u_n)) \to 0,$$

here and then, we denote by $P_X\colon H \to X$ (or $P_Y\colon H \to Y$) the projection onto X along Y (resp. onto Y along X). By the definition of ψ, we know that $P_Y I'(u_n + \psi(u_n)) = 0$; therefore $(1+\|u_n\|)\, I'(u_n+\psi(u_n)) \to 0$.

On the other hand, we have done if there exists a subsequence, which is denoted by the same way, $\|u_n\| \to 0$ as $n \to \infty$. Otherwise we suppose that $\|u_n\| \geqslant c_0$ for all n large enough, hence

$$\begin{aligned} m\,\|\psi(u_n)\|^{\tau} &\leqslant \langle I'(u_n+\psi(u_n)) - I'(u_n), \psi(u_n)\rangle \\ &= -\langle I'(u_n), \psi(u_n)\rangle \\ &\leqslant c(1+\|u_n\|^{\tau-1})\,\|\psi(u_n)\|. \end{aligned}$$

It follows that $\|\psi(u_n)\|/\|u_n\| \leqslant c$ and that

$$\begin{aligned} \|\psi(u_n)\|\; I'(u_n+\psi(u_n)) = \frac{\|\psi(u_n)\|}{\|u_n\|}\,\|u_n\|\; I'(u_n+\psi(u_n)) \to 0, \\ (1+\|u_n+\psi(u_n)\|)(I'(u_n+\psi(u_n)) \to 0, \end{aligned}$$

as $n \to \infty$. Therefore, up to a subsequence, $u_n + \psi(u_n) \to u^* + w^*$ for some $u^* \in X$, $w^* \in Y$. Hence, we have that $u_n \to u^*$ and $w^* = \psi(u^*)$. ∎

Remark 2.1. If I satisfies the usual (PS) condition, then so does $\bar{I}$ (cf. [7]).

Remark 2.2. Under the assumption of (A_1), we will find that $\tau = 2$ and that $\|I'(u_n)\| \leqslant c(1+\|u_n\|)$ holds for all $u \in H$.

Proof of Theorem 1.1. We divide the proof into steps.

Step 1. For any $u \in H^0 \oplus H^-$, $v, w \in H^+$, by (A_1) and Lemma 2.2,

$$\langle I'(u+v) - I'(u+w), v-w\rangle \geqslant \|v-w\|^2 - \int_\Omega \alpha(x)(v-w)^2\,dx \geqslant \delta\,\|v-w\|^2.$$

Therefore, there exists $\psi\colon H^0 \oplus H^- \to H^+$ such that $I(u+\psi(u)) = \min_{w\in H^+} I(u+w)$. The functional $\bar{I}\colon H^0 \oplus H^- \to \mathbf{R}$ defined by $\bar{I}(u) = I(u+\psi(u))$ is of class C^1 and an element $u \in H^0 \oplus H^-$ is a critical point of $\bar{I}$ if and only if $u+\psi(u)$ is a critical point of I.

Step 2. $-I$ hence $-\bar{I}$ is bounded below on $H^-\oplus H^0$. By negation, if there exists $u_n\in H^0\oplus H^-$ such that $-I(u_n)\leqslant -n$, then $I(u_n)\geqslant n$ and $\|u_n\|\to\infty$ as $n\to\infty$. By (A_1) we know that

$$\liminf_{|t|\to\infty}\frac{2G(x,t)}{t^2}\geqslant\lambda_k \qquad \text{uniformly for a.e.} \quad x\in\Omega,$$

hence, for sufficiently small ε, there exists $C_\varepsilon>0$ such that $G(x,t)-\lambda_k t^2\geqslant -t^2\varepsilon-C_\varepsilon$ for all $t\in\mathbf{R}$.

Write $u_n=u_n^0+u_n^-$. If $\|u_n^-\|/\|u_n\|\to c_0\neq 0$, then we have that

$$\begin{aligned}I(u_n)&\leqslant\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k-1}}\right)\|u_n^-\|^2-\int_\Omega\left(G(x,u_n)-\frac{1}{2}\lambda_k u_n^2\right)dx\\&\leqslant\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k-1}}\right)\|u_n^-\|^2+\frac{1}{2}\|u_n\|_2^2\,\varepsilon+c-\varepsilon\\&\leqslant\frac{1}{2}\|u_n\|^2\left(\left(1-\frac{\lambda_k}{\lambda_{k-1}}\right)\frac{\|u_n^-\|^2}{\|u_n\|^2}+\frac{\varepsilon}{\lambda_1}\right)+c_\varepsilon,\end{aligned}$$

it follows that $I(u_n)\to-\infty$ as $n\to\infty$. If $c_0=0$, then $\lim_{n\to\infty}(\|u_n^0\|/\|u_n\|)=1$. It follows from (A_3) and Lemma 2.4 that

$$\liminf_{n\to\infty}\frac{\int_\Omega F(x,u_n)\,dx}{\|u_n\|^\mu}>0.$$

Consequently,

$$I(u_n)\leqslant\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k-1}}\right)\|u_n^-\|^2-\int_\Omega F(x,u_n)\,dx\to-\infty.$$

Step 3. Letting $H^1=H(\lambda_1)\oplus H(\lambda_2)\oplus\cdots\oplus H(\lambda_{m-1})$ and $H^2=H(\lambda_m)\oplus\cdots\oplus H(\lambda_k)$, then $H^0\oplus H^-=H^1\oplus H^2$. The functional $-\bar{I}$ satisfies the local linking condition on $H^1\oplus H^2$, i.e.,

$$\begin{aligned}-\bar{I}(u)&\geqslant 0 \qquad \text{for} \quad u\in H^1 \quad \text{with } \|u\|\leqslant\delta_0;\\-\bar{I}(u)&\leqslant 0 \qquad \text{for} \quad u\in H^2 \quad \text{with } \|u\|\leqslant\delta_0.\end{aligned}$$

In fact, since $\dim H^1<\infty$, condition (A_2) implies that there exists δ_1 such that

$$-\bar{I}(u)\geqslant -I(u)\geqslant -\tfrac{1}{2}\|u\|^2+\tfrac{1}{2}\lambda_{m-1}\|u\|_2^2\geqslant 0 \qquad \text{for} \quad u\in H^1 \quad \text{with } \|u\|\leqslant\delta_1.$$

Furthermore, since (A_2) implies that $G(x, t) \leqslant \frac{1}{2}\beta(x)\, t^2 + c\, |t|^p$ for all $t \in \mathbf{R}$, where $2 < p < 2^*$, we have by Lemma 2.2 that

$$
\begin{aligned}
-\bar{I}(u) &= -I(u+\psi(u)) \\
&\leqslant -\tfrac{1}{2}\, \|u+\psi(u)\|^2 + \tfrac{1}{2} \int_\Omega \beta(x)\, |u+\psi(u)|^2\, dx + c \int_\Omega |u+\psi(u)|^p\, dx \\
&\leqslant -c\, \|u+\psi(u)\|^2 + c\, \|u+\psi(u)\|^p.
\end{aligned}
$$

Noting that $p > 2$ and $\psi \in C^1(H^- \oplus H^0, H^+)$, we have $\delta_2 > 0$ such that $-\bar{I}(u) \leqslant 0$ for $u \in H^2$ with $\|u\| \leqslant \delta_2$.

Step 4. By Lemmas 2.3 and 2.5, $\bar{I}$ satisfies the compactness condition $(C)_c$ and evidently, $\inf_{u \in H^1 \oplus H^2}(-\bar{I}(u)) < 0$. Therefore, combining Steps 1–3 and Local Linking Theorem (cf. [17, 18]), we know that $-\bar{I}$ has at least three critical points hence (P) has at least three solutions. ▌

Proof of Theorem 1.2. For this case, we have to consider $\Phi = -I$. Then $\langle \Phi'(v+w) - \Phi'(v+w_1), w-w_1 \rangle \geqslant c\, \|w-w_1\|^2$ for $w, w_1 \in H^-$ and $v \in H^0 \oplus H^+$. It follows that there exists $\phi \in C(H^0 \oplus H^+, H^-)$ such that $-I(v+\phi(v)) = \min_{w \in H^-}(-I(v+w)) := \tilde{I}(v)$ for $v \in H^0 \oplus H^+$. By a similar argument, $-\tilde{I}$ is bounded below and satisfies the local linking geometry with respect to $H^1 = H(\lambda_k) \oplus \cdots \oplus H(\lambda_m)$, $H^2 = H(\lambda_{m+1}) \oplus H(\lambda_{m+2}) \oplus \cdots$, we omit the details. ▌

3. PROOFS OF THEOREMS 1.3–1.6—BY MORSE THEORY

In this section, we will deal with (B_1), $(B_2^\pm)$, (C_1), and $(C_2^\pm)$. First of all, we show how conditions $(B_2^\pm)$ imply the usual (PS) condition.

Lemma 3.1. *Assume (B_1) and $(B_2^\pm)$. Then I satisfies (PS) condition.*

Proof. We just consider (B_2^-), since the proof with the other alternative is similar.

Let $\{u_n\}$ be the (PS)-sequence. We write $u_n = w_n + z_n + v_n$ with $w_n \in H^+$, $z_n \in H^-$ and $v_n \in H^0$. It is enough to prove the boundedness of $\{u_n\}$. Since

$$
\begin{aligned}
&\langle I'(u_n), w_n - z_n \rangle \\
&\quad = \int_\Omega \nabla u_n \cdot \nabla (w_n - z_n)\, dx - \lambda_k \int_\Omega u_n (w_n - z_n)\, dx \\
&\qquad - \int_\Omega f(x, u_n)(w_n - z_n)\, dx
\end{aligned}
$$

$$\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\| - c\,\|w_n+z_n\|\left(\int_\Omega h_\infty{}^2(|u_n|)\,dx\right)$$

$$\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\|$$

$$-c\,\|w_n+z_n\|\left(\int_{|u_n|\geqslant\|u_n\|} h_\infty^2(|u_n|)\,dx + \int_{|u_n|\leqslant\|u_n\|} h_\infty^2(|u_n|)\,dx\right)^{1/2}$$

$$\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\|$$

$$-c\,\|w_n+z_n\|\left(\int_{|u_n|\geqslant\|u_n\|}\left(\frac{|u_n|}{\|u_n\|}\right)^{\tau_2} h_\infty^2(\|u_n\|)\,dx\right.$$

$$\left.+\int_{|u_n|\leqslant\|u_n\|} h_\infty^2(\|u_n\|)\,dx\right)^{1/2}$$

$$\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\| - c\,\|w_n+z_n\|\,(1+h_\infty(\|u_n\|))$$

$$\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\|$$

$$-c\,\|w_n+z_n\|\,(1+h_\infty(\|v_n\|)+\|w_n\|^{\tau_2-1}+\|z_n\|^{\tau_2-1}),$$

it follows that $c\,\|w_n\|^2+c\,\|z_n\|^2\leqslant cH_\infty^2(\|v_n\|)$. Noting that

$$I(u_n)\geqslant c\,\|w_n+z_n\|^2 - c\,\|w_n+z_n\| - \int_\Omega F(x,v_n)\,dx$$

$$-c\,\|w_n+z_n\|\,(1+h_\infty(\|v_n\|)+\|w_n\|^{\tau_2-1}+\|z_n\|^{\tau_2-1})$$

$$\geqslant c\,\|w_n+z_n\|^2 - ch_\infty^2(\|v_n\|) - \int_\Omega F(x,v_n)\,dx,$$

then, if $\{\|v_n\|\}$ is unbounded, we have that

$$\frac{I(u_n)}{h_\infty^2(\|v_n\|)}\geqslant -c+\frac{\int_\Omega -F(x,v_n)\,dx}{H_\infty(\|v_n\|)}\,\|v_n\|^{2-\tau_2}\,c,$$

which implies that $I(u_n)\to\infty$, a contradiction! ∎

By the next lemma, we compute the homology groups $H_q(H, I^a)$, where $H_q(\cdot,\cdot)$ denotes the homology group with coefficients in a field $\mathscr{F}$, $I^a=\{u\colon I(u)\leqslant a\}$. We will denote such $H_*(H, I^{-a})$ by $C_*(I,\infty)$ when a large enough and call them the critical groups at infinity (cf. [12]).

LEMMA 3.2. *Assume* (B_1) *and* $(B_2^{\pm})$. *Then*

(i) (B_2^+) *implies that* $C_q(I,\infty)=\delta_{q,\mu_\infty+\nu_\infty}\mathscr{F}$;

(ii) (B_2^-) *implies that* $C_q(I,\infty)=\delta_{q,\mu_\infty}\mathscr{F}$.

Proof. (i) For any $u \in H$, we write $u = w + z + v$ with $w \in H^+$, $z \in H^-$ and $v \in H^0$. Let $\delta = \min\{1 - \lambda_k/\lambda_{k+1}, \lambda_k/\lambda_{k-1} - 1\}$ and consider the field

$$\mathscr{D} = \left\{ u \in H : \|w\|^2 - \frac{\delta\lambda_1}{8(\lambda_1 + \lambda_k)} \|z\|^2 - \frac{\lambda H_\infty^2(\|v\|)}{1 + \|v\|^2} \leqslant M \right\},$$

where parameters $\lambda > 0$, $M > 0$ will be determined later. Then the normal vector on the boundary $\partial\mathscr{D}$ of $\mathscr{D}$ is given by $\nu = \nu(u) = w - dz - \lambda\xi'(\|v\|)(v/\|v\|)$, where $d = \delta\lambda_1/8(\lambda_1 + \lambda_k)$, $\xi(t) = H_\infty^2(t)/(1 + t^2)$. Then for $u \in \partial\mathscr{D}$ and ε small enough,

$$\begin{aligned}
\langle I'(u), \nu\rangle &\geqslant \delta\,\|w\|^2 + \delta d\,\|z\|^2 - c\int_\Omega (1 + h_\infty(\|u\|))\,|\nu|\,dx \\
&\geqslant \delta\,\|w\|^2 + \delta d\,\|z\|^2 - c\,\|\nu\|\,(1 + h_\infty(\|v\|) + \|w\|^{\tau_2 - 1} + \|z\|^{\tau_2 - 1}) \\
&\geqslant \delta\,\|w\|^2 + \delta d\,\|z\|^2 \\
&\quad - c(\|w\| + d\,\|z\| + \lambda\xi'(\|v\|))(1 + h_\infty(\|v\|) + \|w\|^{\tau_2 - 1} + \|z\|^{\tau_2 - 1}) \\
&\geqslant \tfrac{1}{2}\min\{\delta, \delta d\}(\|w + z\|^2 - (1 + d)\,c\varepsilon^{-1}h_\infty^2(\|v\|)) \\
&\quad - c\lambda(1 + h_\infty(\|v\|)\,\xi'(\|v\|)) - c\lambda(1 + \varepsilon^{-1})(\xi'(\|v\|))^2 - c.
\end{aligned}$$

On the other hand, it is easy to check that

$$h_\infty^2(t) \leqslant c + 2\tau_2^2\xi(t), \qquad (\xi'(t))^2 \leqslant \frac{4H_\infty^2(t)}{(1 + t^2)^4}\left(\frac{\tau_2}{t} + t(\tau_2 - 1)\right)^2, \qquad \text{for} \quad t > 0.$$

Then there exist $c_d > 0$, $c_\lambda > 0$ and for $\lambda > 6c_d$, $M > 2c_\lambda/\min\{\delta, \delta d\}$, we have that

$$\begin{aligned}
\langle I'(u), \nu\rangle &\geqslant \frac{1}{2}\min\{\delta, \delta d\}\left(\|w\|^2 - d\,\|z\|^2 - 2\left(c_d + \frac{\lambda}{3}\right)\xi(\|v\|)\right) - c_\lambda \\
&\geqslant \frac{1}{2}\min\{\delta, \delta d\}(\|w\|^2 - d\,\|z\|^2 - \lambda\xi(\|v\|)) - c_\lambda \\
&= \frac{1}{2}\min\{\delta, \delta d\}\,M - c_\lambda > 0.
\end{aligned}$$

It implies that I has no critical point outside $\mathscr{D}$ and that the negative gradient of $-I'(u)$ points inward to $\mathscr{D}$ on $\partial\mathscr{D}$. Furthermore, for $u \in \mathscr{D}$,

$$I(u) \leqslant \frac{1}{2}\left(1+\frac{\lambda_k}{\lambda_1}\right)\|w\|^2 - \frac{1}{2}\,\delta\,\|z\|^2$$

$$-\int_\Omega F(x, v)\,dx + c(1+h_\infty(\|v\|)+\|w\|^{\tau_2-1}+\|z\|^{\tau_2-1})\,\|w+z\|$$

$$\leqslant -\frac{\delta}{8}\,\|z\|^2 + c\lambda\xi(\|v\|) + ch_\infty^2(\|v\|) + c - \int_\Omega F(x, v)\,dx$$

$$\leqslant -\frac{\delta}{8}\,\|z\|^2 + ch_\infty^2(\|v\|) + c - \int_\Omega F(x, v)\,dx.$$

By the definition of h_∞ and Lemma 2.1, we have that

$$\frac{h_\infty^2}{H_\infty(t)} \leqslant c\,|t|^{\tau_2-2} \qquad \text{and} \qquad \liminf_{\substack{\|v\|\to\infty \\ v\in H^0}} \frac{\int_\Omega F(x, v)\,dx}{H_\infty(\|v\|)} > 0,$$

it follows that

$$\liminf_{\substack{\|v\|\to\infty \\ v\in H^0}} \frac{\int_\Omega F(x, v)\,dx}{h_\infty^2(\|v\|)} = \infty,$$

hence $I(u) \to -\infty$ as $\|v+z\| \to \infty$. On the contrary, it is easy to see that

$$I(u) \geqslant -c\left(\|z\|^2 + \int_\Omega F(x, v)\,dx\right) - c,$$

which implies that $\|z+v\| \to \infty$ whenever $I(u) \to -\infty$. Now we choose $a > 0$ such that $K := \{u \in H : I'(u) = 0\} \subset \{u \in H : |I(u)| < a\}$. Hence there exists $R_2 = R_2(a)$ such that $\mathscr{D}_2 := \{u \in \mathscr{D} : \|z+v\| \geqslant R_2\} \subset I^{-a} \cap \mathscr{D}$. Setting $\bar{b} = \max\{|I(u)|: \|z+v\| \leqslant R_2, \quad \|w\|^2 \leqslant \delta\lambda_1/8(\lambda_1+\lambda_k)\,\|z\|^2 + \lambda\xi(\|v\|)\}$ and $b > \max\{a, \bar{b}\}$, then $I^{-b} \cap \mathscr{D} \subset \mathscr{D}_2$. Finally, choose $R_1 \gg R_2 > 0$ such that $\mathscr{D}_1 := \{u \in \mathscr{D} : \|v+z\| \geqslant R_1\} \subset I^{-b} \cap \mathscr{D}$. Since $\langle I'(u), v\rangle > 0$ on $\partial\mathscr{D}$, then we can prove that $(\mathscr{D}, \mathscr{D} \cap I^{-a})$ is a strong deformation retraction of the topological pair (H, I^{-a}). On the other hand, there exists a geometric deformation ζ of $\mathscr{D}_2$ onto $\mathscr{D}_1$ and by the second deformation theorem (cf. [21]), there is a strong deformation retraction η of $I^{-a} \cap \mathscr{D}$ onto $I^{-b} \cap \mathscr{D}$. Hence, $\zeta \circ \eta$ is a strong deformation retraction of $I^{-a} \cap \mathscr{D}$ onto $\mathscr{D}_1$. It follows that

$$\begin{aligned} H_q(H, I^{-a}) &\cong H_q(\mathscr{D}, \mathscr{D} \cap I^{-a}) \\ &\cong H_q(\mathscr{D}, \mathscr{D}_1) \\ &\cong H_q(H^0 \oplus H^-, \{u \in (H^0 \oplus H^-) : \|u\| \leqslant R_1\}) \\ &\cong \delta_{q, \mu_\infty + \nu_\infty} \mathscr{F}. \end{aligned}$$

(ii) Setting

$$\mathscr{O} = \left\{ u \in H : \|z\|^2 - \frac{\delta\lambda_1}{8(\lambda_1 + \lambda_k)} \|w\|^2 - \frac{\lambda H_\infty^2(\|v\|)}{1 + \|v\|^2} \leqslant M \right\}.$$

Then the normal vector on the boundary $\partial\mathscr{O}$ of $\mathscr{O}$ is given by $\nu = \nu(u) = z - (\delta\lambda_1/8(\lambda_1 + \lambda_k))\, w - \lambda\xi'(\|v\|)(v/\|v\|)$, where $\xi(t) = H_\infty^2(t)/(1 + t^2)$. By a similar argument, there exist appropriately large λ and M such that

$$\begin{aligned} \langle I'(u), \nu\rangle &\leqslant -\frac{1}{2} \max \left\{ \delta, \frac{\delta^2\lambda_1}{8(\lambda_1 + \lambda_k)} \left(\|z\|^2 - d \frac{\delta\lambda_1}{8(\lambda_1 + \lambda_k)} \|w\|^2 - \lambda\xi(\|v\|) \right) \right\} + c \\ &\leqslant -\frac{1}{2} cM + c < 0, \end{aligned}$$

it follows that I has no critical point in $H \backslash \mathscr{O}$ and that the negative gradient flow of $I'(u)$ outwards to $\mathscr{O}$ on $\partial\mathscr{O}$. On the other hand, for $u \in \mathscr{O}$,

$$\begin{aligned} I(u) \geqslant &\frac{1}{2} \delta \|w\|^2 - \frac{1}{2} \left(1 + \frac{\lambda_k}{\lambda_1}\right) \|z\|^2 - \int_\Omega F(x, v)\, dx \\ &- c(1 + h_\infty(\|v\|) + \|w\|^{\tau_2 - 1} + \|z\|^{\tau_2 - 1}) \|w + z\| \\ \geqslant &\frac{\delta}{8} \|w\|^2 - ch_\infty^2(\|v\|) - \int_\Omega F(x, v)\, dx - c. \end{aligned}$$

By (B_2^-) and Lemma 2.1,

$$\lim_{\|v\| \to \infty} \frac{\int_\Omega - F(x, v)\, dx}{h_\infty^2(\|v\|)} = \infty.$$

It follows that $I(u) \to \infty$ as $\|v + w\| \to \infty$. Similarly, $\|v + w\| \to \infty$ as $I(u) \to \infty$. Therefore, by the definition of $\mathscr{O}$ and the above arguments, we can find a large enough such that $K := \{u \in H : I'(u) = 0\} \subset \{u \in H : |I(u)| < a\}$ and $I^{-a} \subset H \backslash \mathscr{O}$. Since $K \subset \mathscr{O} \backslash I^{-1}[a, \infty)$, the flow of the negative gradient vector provides a strong deformation retraction of $H \backslash \mathscr{O}$ onto I^{-a}. Then

$$H_q(H, I^{-a}) \cong H_q(H, H \backslash \mathscr{O}) \cong H_{q, \mu_\infty} \mathscr{F}. \quad \blacksquare$$

Next we compute the critical groups at zero (cf. [21, 25]). Let $H_0^0 = H(\lambda_m)$, $H_0^+ = H(\lambda_{m+1}) \oplus H(\lambda_{m+2}) \oplus \cdots$, $H_0^- = H(\lambda_1) \oplus \cdots \oplus H(\lambda_{m-1})$, then $H = H_0^0 \oplus H_0^- \oplus H_0^+$. We first prove the following auxiliary result.

LEMMA 3.3. *Assume* $(C_2^\pm)$. *Then*

$$\liminf_{\substack{\|v\| \to \infty \\ v \in H_0^0}} \frac{\pm \int_\Omega F_0(x, v)\, dx}{H_0(\|v\|)} > 0.$$

Proof. By the definition of h_0, we have that

$$\begin{aligned}
\frac{\int_\Omega H_0(|v|)\, dx}{H_0(\|v\|)} &\leqslant \frac{\sigma_2}{\sigma_1} \int_\Omega \frac{|v|\, h_0(|v|)}{\|v\|\, h_0(\|v\|)}\, dx \\
&\leqslant c \left(\int_{\|v\| \leqslant |v|} \left(\frac{|v|}{\|v\|} \right)^{2(\sigma_2 - 1)} dx + \int_{\|v\| \geqslant |v|} \left(\frac{|v|}{\|v\|} \right)^{2(\sigma_1 - 1)} dx \right)^{1/2} \\
&\leqslant c.
\end{aligned}$$

Noting that $|v(x)| \to 0$ uniformly for a.e. $x \in \Omega$ as $v \in H_0^0$ with $\|v\| \to 0$ and that, for any $\varepsilon_1 > 0$ there exists $\delta(\varepsilon_1) > 0$ such that $\operatorname{meas}(\Omega \backslash \Omega_{\varepsilon_1}) < \varepsilon_1$, where $\Omega_{\varepsilon_1} := \{x \in \Omega : |v(x)| \geqslant \delta(\varepsilon_1)\, \|v\|\}$ (cf. [4]), it follows that

$$\begin{aligned}
\frac{\pm \int_\Omega F_0(x, v)\, dx}{H_0(\|v\|)} &\geqslant \frac{\int_\Omega (b^\pm(x) - \varepsilon)\, H_0(|v|)\, dx}{H_0(\|v\|)} \\
&\geqslant \int_{\Omega_{\varepsilon_1}} b^\pm(x)\, \frac{H_0(|v|)}{H_0(\|v\|)}\, dx - c\varepsilon \\
&\geqslant \int_{\Omega_{\varepsilon_1}\ |v| \geqslant \|v\|} b^\pm(x) \left(\frac{|v|}{\|v\|} \right)^{\sigma_1} dx \\
&\quad + \int_{\Omega_{\varepsilon_1}\ |v| < \|v\|} b^\pm(x) \left(\frac{|v|}{\|v\|} \right)^{\sigma_1} dx - c\varepsilon \\
&\geqslant (\delta(\varepsilon_1))^{\delta_1} \int_{\Omega_{\varepsilon_1}} b^\pm(x)\, dx - c\varepsilon,
\end{aligned}$$

which implies the conclusion. ∎

LEMMA 3.4. *Assume* (C_1) *and* $(C_2^\pm)$. *Then*

(i) (C_2^+) *implies that* $C_q(I, 0) \cong \delta_{q, \mu_0 + \nu_0} \mathscr{F}$;

(ii) (C_2^-) *implies that* $C_q(I, 0) \cong \delta_{q, \mu_0} \mathscr{F}$.

Proof. (i) We write $u \in H$ as $u = w + z + v$ with $w \in H_0^+$, $z \in H_0^-$, $v \in H_0^0$ and consider a neighborhood $\mathcal{N}$ of zero defined by

$$\mathcal{N} := \left\{ u \in H : \|w\|^2 - d\,\|z\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} \leqslant r_1^2,\ \|z\|^2 + \|v\|^2 \leqslant r_2^2 \right\},$$

where $d \ll (\lambda_1 / 8(\lambda_1 + \lambda_m)) \min\{(\lambda_m/\lambda_{m-1}) - 1, 1 - \lambda_m/\lambda_{m+1}\}$ is fixed; λ, r_1, r_2 will be determined later. Then the boundary of $\mathcal{N}$ consists of

$$\Gamma_1 = \left\{ u\colon \|w\|^2 - d\,\|z\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} = r_1^2,\ \|z\|^2 + \|v\|^2 \leqslant r_2^2 \right\},$$

$$\Gamma_2 = \left\{ u\colon \|w\|^2 - d\,\|z\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} \leqslant r_1^2,\ \|z\|^2 + \|v\|^2 = r_2^2 \right\}.$$

The normal vector on $\partial \Gamma_1$ is

$$\nu = \nu(u) = w - dz - \lambda \xi'(\|v\|) \frac{v}{\|v\|}, \qquad \text{where} \quad \xi(t) = \frac{H_0^2(t)}{1 + t^2}.$$

Let $\omega = \min\{\lambda_m/\lambda_{m-1} - 1, 1 - \lambda_m/\lambda_{m+1}\}$. Then for $u \in \partial \Gamma_1$

$$\begin{aligned}
\langle I'(u), \nu \rangle &\geqslant \omega\,\|w\|^2 + d\omega\,\|z\|^2 - c\,\|v\| \left(\int_\Omega h_0^2(|u|)\,dx \right)^{1/2} \\
&\geqslant \omega\,\|w\|^2 + d\omega\,\|z\|^2 \\
&\quad - c\,\|v\| \left(\int_{|u| \geqslant \|u\|} h_0^2(\|u\|) \left(\frac{\sigma_2}{\sigma_1} \right)^2 \left(\frac{|u|}{\|u\|} \right)^{2(\sigma_2 - 1)} dx \right. \\
&\quad \left. + \int_{|u| < \|u\|} h_0^2(\|u\|)\,dx \right)^{1/2} \\
&\geqslant \omega\,\|w\|^2 + d\omega\,\|z\|^2 - c\,\|v\|\,(h_0(\|v\|) + \|w\|^{\sigma_2 - 1} + \|z\|^{\sigma_2 - 1}) \\
&\geqslant \omega\,\|w\|^2 - c\varepsilon\,\|w\|^2 - c\,\|w\|^{\sigma_2} - c(d\varepsilon^{-1} + \lambda\varepsilon)\,\|w\|^{2(\sigma_2 - 1)} \\
&\quad + d\omega\,\|z\|^2 - cd\varepsilon\,\|z\|^2 - cd\,\|z\|^{\sigma_2} - c(\varepsilon^{-1} + \lambda\varepsilon)\,\|z\|^{2(\sigma_2 - 1)} \\
&\quad - (cd\varepsilon^{-1} h_0^2(\|v\|) + c\lambda\xi'(\|v\|)\,h_0(\|v\|) + c\lambda\varepsilon^{-1}(\xi'(\|v\|))^2).
\end{aligned}$$

Noting that $\sigma_2 > 2$, and by the definition of h_0, we may find $\lambda = \lambda(d)$ large enough, ε and r_1 small enough, such that

$$\langle I'(u), v\rangle \geqslant \frac{\omega}{3}\left(\|w\|^2 - d\,\|z\|^2 - 3\left(c+\frac{\lambda}{4}\right)\xi(\|v\|)\right)$$

$$\geqslant \frac{\omega}{3}\,(\|w\|^2 - d\,\|z\|^2 - \lambda\xi(\|v\|))$$

$$= r_1^2,$$

which means that the negative gradient flow is inward on $\partial\Gamma_1$. Next, we estimate the value of I around Γ_2. Let $A_0 = \mathbf{id} - \lambda_m(-\Delta)^{-1}$, then there exists $\varepsilon, \theta \in (0, 1)$ such that

$$I(u) \leqslant \frac{1}{2}\,\|A_0\|\,\|w\|^2 - \frac{1}{2}\,\omega\,\|z\|^2 - \int_\Omega F_0(x, v)\,dx$$

$$+ c\,\|w+z\|\int_\Omega |h_0(v+\theta(w+z))|\,dx$$

$$\leqslant \frac{1}{2}\,\|A_0\|\,\|w\|^2 - \frac{1}{2}\,\omega\,\|z\|^2 - \int_\Omega F_0(x, v)\,dx$$

$$+ c\,\|w+z\|\,(h_0(\|v\|) + \|w\|^{\sigma_2 - 1} + \|z\|^{\sigma_2 - 1})$$

$$\leqslant \|A_0\|\,\|w\|^2 - \frac{1}{4}\,\omega\,\|z\|^2 - \int_\Omega F_0(x, v)\,dx + c\varepsilon^{-1}h_0(\|v\|)$$

$$\leqslant \|A_0\|\,r_1^2 + \left(d\,\|A_0\| - \frac{1}{4}\,\omega\right)\|z\|^2 - \int_\Omega F_0(x, v)\,dx$$

$$+ \|A_0\|\,\lambda\,\frac{H_0^2(\|v\|)}{1+\|v\|^2} + c\varepsilon^{-1}h_0^2(\|v\|)$$

$$\leqslant \|A_0\|\,r_1^2 + \left(d\,\|A_0\| - \frac{1}{4}\,\omega\right)\|z\|^2 + h_0^2(\|v\|)\left(c - \frac{\int_\Omega F_0(x, v)\,dx}{h_0^2(\|v\|)}\right).$$

In view of Lemma 3.3, for r_2 small, there exist $\varepsilon_0 > 0$, $0 < r_3 < r_2$, such that

$$I(u) \geqslant -\frac{\varepsilon_0}{2} \qquad \text{for} \quad \|v\|^2 + \|z\|^2 \leqslant r_3^2, \qquad u \in \mathcal{N};$$

$$I(u) < 0 \qquad \text{for} \quad \|v\|^2 + \|z\|^2 \geqslant r_3^2, \qquad u \in \mathcal{N};$$

$$I(u) \leqslant -\varepsilon_0 \qquad \text{for} \quad \|v\|^2 + \|z\|^2 = r_2^2, \qquad u \in \mathcal{N}.$$

The remainder of the proof is similar to that of [22], we just give the sketch. First, the inwardness of the negative gradient flow on $\partial\Gamma_1$ and the estimates of $I(u)$ around Γ_2 imply that $I^0 \cap \mathcal{N}$ is a strong deformation retraction of $\mathcal{N}$ and that $I^0 \cap \mathcal{N} \backslash \{0\}$ can be deformed to $I^{-3\varepsilon_0 \backslash 4} \cap \mathcal{N}$, where

$$I^{-3\varepsilon_0 \backslash 4} \cap \mathcal{N} \subset \mathcal{N}_1 := \{u \in \mathcal{N} : \|v\|^2 + \|z\|^2 \geqslant r_3^2\} \subset I^0 \cap \mathcal{N} \backslash \{0\}.$$

Evidently, $\mathcal{N}_1$ can be deformed to Γ_2, hence,

$$\begin{aligned} C_q(I, 0) &\cong H_q(I^0 \cap \mathcal{N}, I^0 \cap \mathcal{N} \backslash \{0\}) \\ &\cong H_q(\mathcal{N}, I^0 \cap \mathcal{N} \backslash \{0\}) \\ &\cong H_q(\mathcal{N}, \Gamma_2) \\ &\cong \delta_{q, \mu_0 + \nu_0} \mathcal{F}. \end{aligned}$$

(ii) Define

$$\mathcal{M} := \left\{ u \colon \|z\|^2 - d\,\|w\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} \leqslant r_1^2,\ \|w\|^2 + \|v\|^2 \leqslant r_2^2 \right\},$$

$$\Gamma_1 = \left\{ u \colon \|z\|^2 - d\,\|w\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} = r_1^2,\ \|w\|^2 + \|v\|^2 \leqslant r_2^2 \right\},$$

$$\Gamma_r = \left\{ u \colon \|z\|^2 - d\,\|w\|^2 - \lambda \frac{H_0^2(\|v\|)}{1 + \|v\|^2} \leqslant r_1^2,\ \|w\|^2 + \|v\|^2 = r^2 \right\}.$$

Then $\partial \mathcal{M} = \Gamma_1 \cap \Gamma_{r_2}$ and the normal vector of Γ_1 is $\nu = \nu(u) = z - dw - \lambda \xi'(\|v\|)(v/\|v\|)$ with $\xi(t) = H_0^2(t)/(1 + t^2)$. By a similar computation, we can determine λ, r_1 and r_2 such that $\langle I'(u), \nu \rangle < 0$ on $\partial \Gamma_1$, which implies that the negative gradient of I is outward on $\partial \Gamma_1$. Moreover, there exist $\varepsilon_0 > 0$, $r_4 < r_3 < r_2$ such that

$$\begin{aligned} I(u) &\leqslant \frac{\varepsilon_0}{2} && \text{for} \quad \|w\|^2 + \|v\|^2 \leqslant r_4^2, \quad u \in \mathcal{M}, \\ I(u) &> 0 && \text{for} \quad \|w\|^2 + \|v\|^2 \geqslant r_4^2, \quad u \in \mathcal{M}, \\ I(u) &\geqslant \varepsilon_0 && \text{for} \quad \|w\|^2 + \|v\|^2 \geqslant r_3^2, \quad u \in \mathcal{M}. \end{aligned}$$

Let $\mathcal{M}_1 := \mathcal{M} \cap \{u \colon \|w\|^2 + \|v\|^2 \leqslant r_4^2\} \cup \Gamma_{r_3}$, then it is evident by a geometric deformation, that there exists a strong deformation retraction of $\mathcal{M}$ onto $\mathcal{M}_1$. On the other hand, let $\eta(t, u)$ be the negative gradient flow, $\tilde{\tau}_1(u)$ be the time of reaching Γ_1, and $\tilde{\tau}_2(u)$ be the time of reaching I^0. Then $\tilde{\tau}(u) = \min\{\tilde{\tau}_1(u), \tilde{\tau}_2(u)\}$ induces a flow as

$$\tilde{\sigma}_1(u, t) = \begin{cases} \eta(u, t\tilde{\tau}(u)), & \text{if} \quad u \in \mathcal{M}_1, \quad \tilde{\tau}(u) > 0, \\ u, & \text{if} \quad u \in \mathcal{M}_1, \quad \tilde{\tau}(u) = 0. \end{cases}$$

By the estimates of $I(u)$ around Γ_{r_3}, $\tilde{\sigma}_1$ provides a strong deformation retraction of $\mathscr{M}_1$ to $\mathscr{M}_2 := \mathscr{M} \cap I^0 \cup \Gamma_{r_3}$. Finally, the flow $\tilde{\sigma}_2(u, t) := \eta(u, t\tilde{\tau}_1(u))$ shows that Γ_{r_3} is a strong deformation retraction of $\mathscr{M}_2 \backslash \{0\}$. Combining the above arguments,

$$\begin{aligned} C_q(I, 0) &\cong H_q(I^0 \cap \mathscr{M}, I^0 \cap \mathscr{M} \backslash \{0\}) \\ &\cong H_q((I^0 \cap \mathscr{M}) \cap \Gamma_{r_3}, \Gamma_{r_3} \cap (I^0 \cap \mathscr{M} \backslash \{0\})) \\ &\cong H_q(\mathscr{M}_1, \Gamma_{r_3} \cap (I^0 \cap \mathscr{M} \backslash \{0\})) \\ &\cong H_q(\mathscr{M}, \Gamma_{r_3} \cap (I^0 \cap \mathscr{M} \backslash \{0\})) \\ &\cong H_q(\mathscr{M}, \Gamma_{r_3}) \\ &\cong \delta_{q, \mu_0} \mathscr{F}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 1.3. By the proof of Theorem 1.1, there exists $\psi: H^0 \oplus H^- \to H^+$ such that $I(u + \psi(u)) = \min_{w \in H^+} I(u + w)$. Consider reduction functional $\bar{I}: H^0 \oplus H^- \to \mathbf{R}$ defined by $\bar{I}(u) = I(u + \psi(u))$. Noting that $\lim_{|t| \to \infty}(F(x, t)/t^2) = 0$ and that

$$\liminf_{n \to \infty} \frac{\int_\Omega F(x, u_n)\, dx}{H_\infty(\|u_n\|)} > 0$$

for $u_n = z_n + v_n$ with $z_n \in H^-$, $v_n \in H^0$, $\|u_n\| \to \infty$ and $z_n/\|u_n\| \to 0$, then it is easy to prove that $-I$, hence $-\bar{I}$, is bounded below on $H^- \oplus H^0$. Since I satisfies (PS) condition then so does $-\bar{I}$ (see Remark 2.1), there exists $u^* \in H^- \oplus H^0$ which is a minimum of $-\bar{I}$ on $H^- \oplus H^0$. Suppose now that u^* is an isolated critical point of $-\bar{I}$, hence $u^* + \psi(u^*)$ is an isolated critical point of I (see Lemma 2.1 of [16]) and there exist two neighborhood U_1 and U_2 of u^* and $u^* + \psi(u^*)$, respectively, such that

$$\deg(I, U_2, 0) = \deg(\bar{I}, U_1, 0) = (-1)^{\mu_\infty + \nu_\infty},$$

where deg denotes the Leray–Schauder degree. By the relation between the critical groups and the degree (cf. [21]),

$$\sum_{q=0}^{\infty} (-1)^q \dim C_q(I, u^* + \psi(u^*)) = (-1)^{\mu_\infty + \nu_\infty}.$$

Therefore, the hypotheses of Theorem 1.3 imply that $u^* + \psi(u^*) \neq 0$. Assume that (B_2^+) holds and that there is no other critical point, then combining Lemma 3.2, Lemma 3.4, and Morse relations, we have that

$$(C_2^+) \qquad \text{implies that} \quad (-1)^{\mu_0+\nu_0} + (-1)^{\mu_\infty+\nu_\infty} = (-1)^{\mu_\infty+\nu_\infty};$$

$$(C_2^-) \qquad \text{implies that} \quad (-1)^{\mu_0} + (-1)^{\mu_\infty+\nu_\infty} = (-1)^{\mu_\infty+\nu_\infty};$$

these are impossible! ▮

Proof of Theorem 1.4. For $u = w + v \in H^+ \oplus H^0$ with $w \in H^+$, $v \in H^0$ we see that

$$I(u) \geqslant \|u\|^2 \left(c \left(\frac{\|w\|}{\|u\|} \right)^2 - \frac{\int_\Omega F(x, u)\, dx}{\|u\|^2} \right).$$

Evidently, $I(u) \to \infty$ as $\|u\| \to \infty$ with $\|w\|/\|u\| \to c \neq 0$. But, if $\|w\|/\|u\| \to c = 0$, it follows by Lemma 2.1 that $\lim_{\|u\| \to \infty}(-\int_\Omega F(x, u)\, dx) = \infty$, hence $I(u) \to \infty$, which implies that I is bounded below on $H^+ \oplus H^0$. On the other hand, it is obvious that $I(u) \to -\infty$ as $u \in H^-$ with $\|u\| \to \infty$. By Saddle Point Theorem and its characteristics of the critical groups (see [24, 30]), there is a critical point u_1 such that $C_{\mu_\infty}(I, u_1) \neq 0$. By (D^-) and Lemma 2.2, we can prove that the Morse index $m(u_1)$ of u_1 is great or equal to μ_∞. By Gromoll-Meyer Theorem (cf. [31]), $m(u_1) = \mu_\infty$. Shifting Theorem (cf. [21, 25]) implies that $C_q(I, u_1) = C_{q-\mu_\infty}(I_0, 0)$, where I_0 is defined on the null space of $I''(u_1)$. Then $C_0(I_0, 0) \neq 0$ means that 0 is a minimizer of I_0. Consequently, $C_q(I, u_1) = \delta_{q, \mu_\infty} \mathscr{F}$. Now (C_2^+) and Lemma 3.4 imply that $u_1 \neq 0$ if $\mu_0 + \nu_0 \neq \mu_\infty$. Furthermore, if there is no other critical point, Morse relation reads as $(-1)^{\mu_0+\nu_0} + (-1)^{\mu_\infty} = (-1)^{\mu_\infty}$, a contradiction! Similarly we can prove the case of (C_2^-). ▮

Proof of Theorem 1.5. By combining Lemma 3.2 and Lemma 3.4, it is easy to check that each case of Theorem 1.5 implies that $C_q(I, 0) \neq C_q(I, \infty)$ for some q. Then Morse inequalities imply the existence of one nontrivial solution. ▮

Proof of Theorem 1.6. Combining Lemma 2.1, Lemma 3.1, Lemma 3.2, and Lemma 3.4 and using a similar argument as that in the proof of Lemma 2.4, we can prove that

$$(E_\infty^\pm) \qquad \text{implies that} \quad C_q(I, \infty) = \delta_{q, \mu_\infty + (\delta_{1, \mp 1})\nu_\infty} \mathscr{F};$$

$$(E_0^\pm) \qquad \text{implies that} \quad C_q(I, 0) = \delta_{q, \mu_0 + (\delta_{1, \mp 1})\nu_0} \mathscr{F}.$$

Then the conclusion follows immediately from Morse inequalities. ▮

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