A note on precompact uniform frames

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Abstract

Precompactness or total boundedness for uniform frames is usually distinguished by a cover approach. In this note, we provide alternate characterizations of precompact uniform frames. In particular, we formulate pointfree filter analogues of various classical topological results on precompactness. We also revisit the notion of convergence and clustering of filters in a frame and introduce weakly Cauchy filters and strong Cauchy completeness in the setting of uniform frames.

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1. Introduction

The 1957 Paris Séminaire Ehresmann conceived a novel approach to topology from a lattice theoretic point of view. The standard resource for this development of topology independent of points, the theory of frames (locales), is the text by Johnstone [8]. In this pointfree setting, using covers of a frame, Hong in [5] introduced a notion of clustering and convergence of filters and described compact regular frames through convergence of maximal filters. Filters have been effectively used to provide completeness criteria for structured...
frames (see for example [1,2] or [3]). The uniform structure for a locale was defined by Isbell in [7] and the cover approach to uniformity was developed by Pultr in [10,11]. The articles [1–3,5,6] together with the book [8] provide the notation and terminology of this paper.

We continue this section by previewing some basic facts concerning frames, nearness frames and uniform frames. The next section provides the details concerning convergence in frames. Thereafter we define weakly Cauchy filters and strongly Cauchy complete uniform frames and then in Section 3 provide pointfree versions of classical topological results concerning precompact uniform frames.

1.1. Frames and uniform frames

We present the pertinent definitions for a frame $L$. For an extensive treatment of frames see [8]. A frame $L$ is a complete lattice (with top and bottom denoted by 1 and 0, respectively) satisfying the distributive law

$$x \land \bigvee S = \bigvee_{s \in S} (x \land s) \quad \text{for all } x \in L \text{ and any } S \subseteq L.$$ 

For frames $L$ and $M$, a frame homomorphism is a map $h : L \rightarrow M$ which preserves top and bottom, finite meets and all joins. Compositions of frame homomorphisms are again frame homomorphisms, so we have the category Frm of frames and frame homomorphisms.

In the frame $L$, for $x, y \in L$ we say that $y$ is rather below $x$ (written as $y \prec x$) if there is $t \in L$ such that $y \land t = 0$ and $t \lor x = 1$. This can also be expressed as $y^* \lor x = 1$, where $y^* = \bigvee \{ t \in L : t \land y = 0 \}$ is the pseudocomplement of $y$. $L$ is called a regular frame if for each $x \in L$, $x$ can be expressed as $x = \bigvee \{ y \in L : y \prec x \}$. RegFrm denotes the category of regular frames and frame homomorphisms.

The element $c \in L$ is a compact element if $c = \bigvee X$ for $X \subseteq L$ implies that $c = \bigvee F$ for some $F$ a finite subset of $X$.

We will use the symbol “$\subseteq f$” in the sequel to denote finite subsets.

$L$ is called a compact frame provided that the unit in $L$ is a compact element.

Any $A \subseteq L$ is a cover on $L$ if $\bigvee A = 1$. cov$L$ will denote the collection of all covers on the frame $L$. For $A, B \in \text{cov } L$ we say that $A$ refines $B$ (written as $A \leq B$) if for each $a \in A$, $a \leq b$ for some $b \in B$. The meet of $A$ and $B$ is the set $A \land B = \{ a \land b : a \in A \text{ and } b \in B \}$. For any element $x \in L$ the set $Ax = \bigvee \{ a \in A : a \land x \neq 0 \}$ is the star of $x$ with respect to the cover $A$.

If $A$ is a cover of the frame $L$ the set $A^* = AA = \{ Aa : a \in A \}$ is the star of $A$ which is also a cover of $L$ as $A \leq A^*$. We say that $A$ star refines $B$ (written as $A \leq^* B$) if $A^* \leq B$.

Let $\mu \subseteq \text{cov } L$ and $x, y \in L$. The element $x$ is said to be uniformly below the element $y \in L$ (expressed as $x \ll^\mu y$ or for brevity $x \ll y$) if $Ax \subseteq y$ for some $A \in \mu$. A nearness on the frame $L$ is any non-empty collection $\mu$ of covers of $L$ satisfying:

(N-I) For any $A, B \in \mu$, $A \land B \in \mu$.
(N-II) If $C \in \mu$ and $C \leq D$, then $D \in \mu$.
(N-III) For each $x \in L$, $x = \bigvee \{ y \in L : y \ll x \}$, i.e. $\mu$ is an admissible system of covers.
The members of the nearness \( \mu \) on the frame \( L \) are called uniform covers. The frame \( L \) together with the nearness structure \( \mu \) (written as \((L, \mu)\)) is called a nearness frame. For nearness frames \((L, \mu)\) and \((N, \nu)\) a frame homomorphism \( h : (L, \mu) \to (N, \nu) \) is called a uniform homomorphism if \( h \) preserves uniform covers, i.e. \( h(A) = \{ h(a) : a \in A \} \in \nu \) whenever \( A \in \mu \). The category \( \text{NFrm} \) denotes the category of nearness frames and uniform homomorphisms. If the nearness \( \mu \) on a frame \( L \) satisfies the additional condition:

\[(N\text{-IV}) \text{ For each } A \in \mu \text{ there is } B \in \mu \text{ such that } B \leq^* A,\]

then \( \mu \) is called a uniformity. The resulting pair \((L, \mu)\) is then a uniform frame. Uniform frames and uniform frame homomorphisms are the objects and maps in the category \( \text{UFrm} \).

It is well known that any compact regular frame \( L \) has a unique nearness namely, \( \text{cov} \ L \), which is a uniformity (see \([1]\) or \([3]\)).

A nearness frame \((L, \mu)\) is strong if for each \( A \in \mu \), the cover \( \check{A} \in \mu \) where \( \check{A} = \{ x \in L : x \prec a \text{ for some } a \in A \} \). For such nearness frames, for each \( A \in \mu \) the cover \( A^\prec = \{ x \in L : x \prec a \text{ for some } a \in A \} \) is also uniform as \( \check{A} \leq A^\prec \). Note also that every uniform frame is a strong nearness frame.

2. Convergence in frames

Let \((L, \mu)\) be a uniform frame. For a filter \( F \) in \((L, \mu)\), let

\[ \text{sec} \ F = \{ y \in L : y \land x \neq 0 \text{ for each } x \in F \}. \]

We say that \( F \) is:

- completely prime if, whenever \( \bigvee \ L S \in F \) for any \( S \subseteq L \), then \( S \cap F \neq \emptyset \);
- Cauchy if \( F \) meets every uniform cover;
- convergent or \( F \) converges if \( F \) meets every cover of \( L \);
- clustered or \( F \) clusters in case \( \text{sec} \ F \) meets every cover of \( L \);
- maximal in case \( F = G \) whenever \( G \) is a filter with \( F \subseteq G \).

For maximal \( F \), \( \text{sec} \ F = F \) (see \([5]\)). It is clear that every convergent filter in a uniform frame is a Cauchy filter. We will need the results of the following lemma all of which can be found in \([5]\).

**Lemma 2.1.**

1. Every completely prime filter clusters.
2. Every convergent filter clusters.
3. Any filter contained in a clustered filter also clusters.
4. Every maximal clustered filter converges.
5. If \( L \) is regular, then \( L \) is compact \( \iff \) every filter in \( L \) is clustered.
6. Any filter containing a completely prime filter is convergent.
7. Any filter that contains a convergent filter also converges.
8. If \( L \) is regular, then \( L \) is compact \( \iff \) every maximal filter in \( L \) converges.
The following theorem is recognizable as the pointfree filter version of the classical folklore if a Cauchy net clusters to a point \( y \) in a uniform space then it also converges to \( y \).

**Theorem 2.1.** Every clustered Cauchy filter in a uniform frame converges.

**Proof.** Let \( F \) be a clustered Cauchy filter in the uniform frame \((L, \mu)\). Also, let \( A \in \text{cov} L \). Then \( \tilde{A} = \{x \in L : x \triangleleft a \text{ for some } a \in A\} \in \text{cov} L \). Since \( F \) is clustered, \( \text{sec} F \cap \tilde{A} \neq \emptyset \).

Thus \( \exists t \in \tilde{A} \) such that \( t \wedge y \neq 0 \) for each \( y \in F \). Since \( t \in \tilde{A} \), \( t \triangleleft a \) for some \( a \in A \). Thus \( \exists B \in \mu \) such that \( Bt \leq a \). Then \( B \leq \{t^*, a\} \) and hence \( \{t^*, a\} \in \mu \). Since \( F \) is Cauchy, \( \{t^*, a\} \cap F \neq \emptyset \). If \( t^* \in F \), then \( t \wedge t^* \neq 0 \) which is a contradiction. Thus \( t^* \notin F \). Hence \( a \in F \). Thus \( A \cap F \neq \emptyset \) and hence \( F \) converges. \( \Box \)

We call a filter in a uniform frame weakly Cauchy if \( \text{sec} F \) meets every uniform cover. Clearly every filter that clusters in a uniform frame is weakly Cauchy. The following lemma is apparent from the fact that \( F \subseteq \text{sec} F \).

**Lemma 2.2.** Every Cauchy filter in a uniform frame is weakly Cauchy.

A uniform frame \((L, \mu)\) is Cauchy complete if every Cauchy filter in \((L, \mu)\) converges. We say that the uniform frame \((L, \mu)\) is strongly Cauchy complete if every weakly Cauchy filter in \((L, \mu)\) clusters. As a result of the above lemma together with Theorem 2.1 we have the following theorem.

**Theorem 2.2.** Every strongly Cauchy complete uniform frame is Cauchy complete.

**Proof.** Let \( F \) be any Cauchy filter in a strongly Cauchy complete uniform frame \((L, \mu)\). By Lemma 2.2, \( F \) is weakly Cauchy in \( L \) and so clusters since \((L, \mu)\) is strongly Cauchy complete. Then \( F \) is a clustered Cauchy filter in the uniform frame \((L, \mu)\) and by Theorem 2.1, \( F \) converges. Hence, \((L, \mu)\) is Cauchy complete. \( \Box \)

**Theorem 2.3.** Every compact regular frame is strongly Cauchy complete.

**Proof.** Let \( L \) be a compact regular frame. Also let \( F \) be any weakly Cauchy filter on \( L \). By Lemma 2.1(5), \( F \) clusters. Hence \( L \) is strongly Cauchy complete. \( \Box \)

As a consequence of the above two theorems we have the following result proved differently in [2] where it is shown that any Cauchy filter in a compact regular frame contains a completely prime filter. From Lemma 2.1(6), [2, Lemma 10] then concludes the corollary that follows.

**Lemma 2.3.** Let \((L, \mu)\) and \((M, \nu)\) be uniform frames. If \( h : (L, \mu) \to (M, \nu) \) is a uniform frame homomorphism and \( F \) is any weakly Cauchy filter in \( M \), then \( h^{-1}(F) = \{x \in L : h(x) \in F\} \) is a weakly Cauchy filter in \( L \).
Proof. Let $F$ be a weakly Cauchy filter in $M$. It is straightforward to show that $h^{-1}(F)$ is a filter in $L$. Now let $A \in \mu$. Since $h$ is uniform, $h(A) \in \nu$. Since $F$ is weakly Cauchy in $M$, sec $F \cap h(A) \neq \emptyset$. Thus $\exists a \in A$ such that $h(a) \land y \neq 0$ for each $y \in F$. Then for each $x \in h^{-1}(F)$ as $h(x) \in F$, $h(a) \land x = h(a) \land h(x) \neq 0$. As $h$ is a frame homomorphism, $a \land x \neq 0$. This is valid for every $x \in h^{-1}(F)$ and so $\text{sec}(h^{-1}(F)) \cap A \neq \emptyset$. Hence $h^{-1}(F)$ is weakly Cauchy in $L$.

**Corollary 2.1.** Every compact regular frame is Cauchy complete.

### 3. Precompact uniform frames

A uniform frame $(L, \mu)$ is precompact if $\mu$ is generated by its finite members. For a uniform frame $(L, \mu)$, let $\mu_P = \{ A \in \mu : B \subseteq A \text{ for some finite } B \in \mu \}$. It is shown in [12] that $(L, \mu_P)$ is a precompact uniform frame and that precompact uniform frames form a coreflective subcategory of $\text{U Frm}$ with coreflection map the identity $\text{id}_L : (L, \mu_P) \to (L, \mu)$. Dube [4] provides the corresponding results for precompactness in $\text{N Frm}$.

A frame $L$ is called almost compact if for each $A \in \text{cov} L$ there is $B \subseteq_f A$ such that $(\bigvee B)^* = 0$. For further treatment of almost compact frames, [6,9] are suggested. It is shown in [6] that for regular frames compactness and almost compactness are equivalent conditions.

We now confine the notion of almost compactness to uniformities and we call a uniform frame $(L, \mu)$ uniformly almost compact provided that for each uniform cover $A$ there is $B \subseteq_f A$ such that $(\bigvee B)^* = 0$. Clearly, whenever the underlying frame of a uniform frame is almost compact the uniform frame itself is uniformly almost compact. Certainly, for regular frames compactness, almost compactness and uniformly almost compactness are equivalent. Recall that a frame homomorphism $h : L \to M$ is dense if $h(x) = 0$ implies $x = 0$.

**Lemma 3.1.** If $h : (L, \mu) \to (M, \nu)$ is a dense uniform homomorphism and $(M, \nu)$ is uniformly almost compact, then so is $(L, \mu)$.

**Proof.** Let $h : (L, \mu) \to (M, \nu)$ be a dense uniform homomorphism with $(M, \nu)$ uniformly almost compact. Let $A \in \mu$. Since $h$ is uniform, $h(A) \in \nu$. As $(M, \nu)$ is uniformly almost compact $\exists D \subseteq_f h(A)$ such that $(\bigvee D)^* = 0$. Since $D$ is finite $\exists B \subseteq_f A$ such that $D = \{ h(b) : b \in B \}$. Then

$$
(\bigvee B)^* = \left( \bigvee_{b \in B} h(b) \right)^* = (\bigvee D)^* = 0.
$$

Since $h((\bigvee B)^*) \land h(\bigvee B) = 0$, $h((\bigvee B)^*) \subseteq (h(\bigvee B))^* = 0$. Since $h$ is dense, $(\bigvee B)^* = 0$. Hence, $(L, \mu)$ is uniformly almost compact.

**Theorem 3.1.** A uniform frame is precompact $\iff$ it is uniformly almost compact.
Proof. Suppose that \((L, \mu)\) is precompact and let \(A \in \mu\). Then \(\exists \) a finite \(B \in \mu\) such that \(B \leq A\). Then for each \(b \in B\) there is \(a_b \in A\) such that \(b \leq a_b\). Then easily \(\{a_b \mid b \in B\}\) is a finite (uniform) subcover of \(A\) and the result follows.

For the converse, under the given hypothesis let \(A \in \mu\). Then \(A^c \in \mu\). By the hypothesis there is \(Y = \{y_1, y_2, \ldots, y_k\} \subseteq A^c\) such that \((\bigwedge Y)^* = 0\). Then for each \(1 \leq i \leq k\), \(y_i < a_i\), i.e., \(y_i^* \vee a_i = 1\) for some \(a_i \in A\). Then \(\{a_1, a_2, \ldots, a_k\}\) is a finite subcover of \(A\) since

\[
\bigvee_{i=1}^{k} a_i = \bigvee_{i=1}^{k} a_i \vee 0 = \bigvee_{i=1}^{k} a_i \vee (\bigwedge Y)^* \\
= \bigvee_{i=1}^{k} a_i \vee \bigwedge_{i=1}^{k} y_i^* = \bigwedge_{i=1}^{k} \left( y_i^* \vee a_i \right) \\
= \bigwedge_{i=1}^{k} \left( y_i^* \vee a_i \vee a_j \right) = 1, \quad \text{since } y_i^* \vee a_i = 1 \text{ for each } i.
\]

Thus each uniform cover has a finite subcover under the hypothesis, in particular, for \(\tilde{A} \in \mu\) there is a finite cover \(S = \{s_1, s_2, \ldots, s_k\} \subseteq \tilde{A}\). Then for each \(1 \leq i \leq k\), \(s_i \leq b_i\) for some \(b_i \in A\). Thus there is \(W \in \mu\) such that \(W_s \leq b_i\) for each \(i\). If \(0 \neq w \in W\), then \(0 \neq w = w \land 1 = w \land \bigwedge s_i = \bigvee_{i=1}^{k} (w \land s_i)\). Thus \(w \land s_j \neq 0\) for some \(1 \leq j \leq k\) and so \(w \leq Ws_j \leq b_j\). Thus \(W \leq \{b_1, b_2, \ldots, b_k\} = B\). Hence \(B \in \mu\) and is a finite refinement of \(A\) concluding \((L, \mu)\) to be precompact. \(\square\)

It is also folklore that Cauchy nets play a significant role in the characterization of precompact uniform spaces where a uniform space is precompact provided that each net has a Cauchy subnet. We now give a pointfree analogue of this result using the elegant filter approach.

**Theorem 3.2.** A uniform frame \((L, \mu)\) is precompact \(\iff\) each filter in \(L\) is contained in a Cauchy filter.

Proof. Suppose that \((L, \mu)\) is precompact and let \(F\) be any filter in \(L\). Then \(F \subseteq G\) for some maximal filter \(G\) in \(L\). \(G\) is then Cauchy if for \(A \in \mu\), there is \(B \in \mu\) such that \(B \leq A\) and \(B\) is finite. Since \(B\) is a finite cover and \(G\) is maximal, \(\exists b \in B\) such that \(b \in G\). Since \(B \leq A\), \(b \leq a\) for some \(a \in A\). Consequently, \(a \in G\) and so \(G \cap A \neq \emptyset\).

For the converse, suppose that each filter in \(L\) is contained in a Cauchy filter but \((L, \mu)\) is not precompact. By Theorem 3.1, \(\exists A \in \mu\) such that for each \(B \subseteq f A\), \((\bigvee B)^* \neq 0\). Then \(\{ (\bigvee B)^* \mid B \subseteq f A\} \) generates a (proper) filter \(F\) in \(L\). By the hypothesis, \(F \subseteq G\) for some Cauchy filter \(G\) in \(L\). Consequently, \(A \cap G \neq \emptyset\). Thus \(\exists a \in A\) such that \(a \in G\). Since \(\{a\} \subseteq f A\), \(a^c \in G\) which is a contradiction. Thus \((L, \mu)\) must be precompact. \(\square\)

As a consequence of the above theorem we have the following characterization of precompact uniform frames.
Corollary 3.1. A uniform frame \((L, \mu)\) is precompact \(\iff\) each maximal filter in \(L\) is Cauchy.

As an immediate consequence we have the following theorem in pointfree form of the corresponding result for uniform spaces (see for example [13]).

Theorem 3.3. A uniform frame is compact if and only if it is Cauchy complete and precompact.

Proof. Suppose that \((L, \mu)\) is compact. Then \(\mu = \text{cov} L\) and easily \((L, \mu)\) is Cauchy complete and precompact.

Conversely, suppose that \((L, \mu)\) is Cauchy complete and precompact. Let \(F\) be any filter in \(L\). By Theorem 3.2, since \((L, \mu)\) is precompact, \(F \subseteq G\) for some Cauchy filter \(G\) in \(L\). As \((L, \mu)\) is Cauchy complete we have that \(G\) converges. By Lemma 2.2(2) \(G\) clusters and hence by Lemma 2.2(3) \(F\) clusters. Thus every filter in \(L\) clusters. Accordingly, by Lemma 2.2(5), \(L\) is compact.

It should be noted that the notion of a weakly Cauchy filter and strong Cauchy completeness can also be introduced for nearness frames. All of the above results are also true for nearness frames, save those on precompactness which are true for strong nearness frames.

References